MATH 322 Final Project

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Problem 6

If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2}$$

if $(x,y) \neq (0,0)$, prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

Proof. First, we can show that f is not continuous at $(0,0)$ with respect to the standard Euclidean metric on $\mathbb{R}^2$ and $\mathbb{R}$ respectively.

Let $\delta > 0$ be given. Let x be a real number such that

$$|x| < \frac{\delta}{\sqrt{2}} \tag{1}$$

If we let $y = x$, then it follows that

$$\begin{aligned}
||(x,y) - (0,0)||_2 &= \sqrt{(x-0)^2 + (y-0)^2} && (2)\\
&= \sqrt{x^2+y^2} && (3)\\
&= \sqrt{x^2+x^2} && (4)\\
&= \sqrt{2}|x| && (5)\\
&< \sqrt{2}\cdot\frac{\delta}{\sqrt{2}} && (6)\\
&= \delta && (7)
\end{aligned}$$

where (6) follows from (1). Observe that

$$\begin{aligned}
||f(x,y) - f(0,0)||_1 &= |\frac{xy}{x^2+y^2} - 0| && (8)\\
&= |\frac{x^2}{x^2+x^2}| && (9)\\
&= \frac{1}{2} && (10)\\
&\geq \frac{1}{2} && (11)
\end{aligned}$$

Thus, directly from the definition, we see that f is not continuous at $(0,0)$.

Now I will show that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$. If $(x,y) \neq (0,0)$ then $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist and are defined as usual with

$$(D_1 f)(x,y) = \frac{\partial f}{\partial x}(x,y) \tag{12}$$
$$= \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} \tag{13}$$
$$= \frac{y^3 - x^2 y}{(x^2+y^2)^2} \tag{14}$$

and

$$(D_2 f)(x,y) = \frac{\partial f}{\partial y}(x,y) \tag{15}$$
$$= \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} \tag{16}$$
$$= \frac{x^3 - xy^2}{(x^2+y^2)^2} \tag{17}$$

In the case that $(x,y) = 0$, we can look at what f the two cases when $x = 0$ is fixed and $y = 0$ is fixed. If $x = 0$ is fixed then, $f(x,y) = \frac{0}{y^2} = 0$ at all y. Thus

$$(D_2 f)(x,y) = \frac{\partial f}{\partial y}(x,y) = 0 \tag{18}$$

If $y = 0$ is fixed then, $f(x,y) = \frac{0}{x^2} = 0$ at all x. Thus

$$(D_1 f)(x,y) = \frac{\partial f}{\partial x}(x,y) = 0 \tag{19}$$

(18) and (19) imply

$$(D_2 f)(0,0) = 0 \tag{20}$$

and

$$(D_1 f)(0,0) = 0 \tag{21}$$

respectivly.

$\square$