Calculus BC Problem Set 1

Josh Davis

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0 Preface

I wrote this collection of problems in order to help student's master their ability to solve BC Calculus Problems. All problems should be solved without a calculator unless otherwise specified.

1 Limits and Continuity

Problem 1.1

Evaluate the following limits analytically (this means with algebra by hand). Do not use L'Hôpital's rule.:

i)

$$\lim_{x\to 3} \frac{x^2+9}{x-3}$$

ii)

$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$

iii)

$$\lim_{x\to -3} \frac{x^2-9}{x-3}$$

iv)

$$\lim_{x\to -3} \frac{x^4-81}{x+3}$$

Problem 1.2

Evaluate the following limit with the squeeze theorem:

$$\lim_{x\to 0} x\sin(x)$$

Problem 1.3

Evaluate the following limit analytically:

$$\lim_{x \to \pi^-} \frac{x^2 - \pi^2}{|x - \pi|}$$

Problem 1.4

Evaluate the following limit analytically:

$$\lim_{x \to \infty} \left(\frac{x^2 + 7x + 9}{x - 2} - x \right)$$

Problem 1.5

Consider a function $f(x)$ that has the following property

$$-|x + 5| \leq f(x) \leq |x + 5|$$

for all real numbers. Evaluate the following limit:

$$\lim_{x \to -5} f(x)$$

Problem 1.6

Evaluate the following limits by using the squeeze theorem:

i)

$$\lim_{x \to 0} \frac{\sin(x)}{\ln(x^{-1})}$$

ii)

$$\lim_{x \to -\infty} \frac{\cos(x)\sin(3x)}{x^3}$$

iii)

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Problem 1.7

Determine whether the following functions are continuous at $x = 3$.

i)

$$f(x) = \frac{(x - 4)(x - 3)}{x}$$

ii)

$$f(x) = \begin{cases} \frac{x}{\ln\left(\frac{x}{3}\right)} & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases}$$

iii)

$$f(x) = \begin{cases} 2^x & \text{if } x \leq 3 \\ 7 + \sin(\frac{3\pi}{2}x) & \text{if } x > 3 \end{cases}$$

Problem 1.8

Come up with functions with the following properties:

i) A removable discontinuity at $x = 8$.

ii) A non-piece wise function, $f(x)$, which satisfies (a) and (b). An example of a non-piece wise function is $f(x) = 2x$. An example of a piece wise function is the function in Problem 1.7, part *ii*.

(a) $\lim_{x\to\infty} f(x) = 7$

(b) $\lim_{x\to-\infty} f(x) = -7$

iii) A function with vertical asymptotes at $x = 1$ and $x = -5$.

iv) A function with the horizontal asymptotes $y = 0$ and $y = 1$.

Problem 1.9

Does the equation $x^2 + e^{\frac{x}{10}} = 102$ have a solution on the interval $[0, 10]$.

Problem 1.10

Does the equation $2\sin(\frac{\pi}{2}x) + x = 3$ have a solution on $[0, 2]$

2 Differentiation: Definition and Fundamental Properties

Problem 2.1

Use the limit of the difference quotient (limit definition of the derivative) to find the derivative of the following functions

i) $f(x) = x$

ii) $f(x) = 2x^2$

iii) $f(x) = \sin(x)$

iv) $f(x) = \sqrt{x+1}$

Problem 2.2

Write the equation of the tangent line for each of the following functions at $x = 2$.

i)

$$f(x) = x^2 + x$$

ii)

$$f(x) = 3\sin(x) + 2$$

iii)

$$f(x) = e^x + 2^x$$

iv)

$$f(x) = \sqrt{x+1} + \ln(x)$$

Problem 2.3

Find all values of a and b such that $f(x)$ is differentiable at $x = \lambda \neq 0$.

$$f(x) = \begin{cases} ax^2 + bx - 1 & \text{if } x \leq \lambda \\ ax & \text{if } x > \lambda \end{cases}$$

If $\lambda = 0$, are there values of a and b which make $f(x)$ differentiable at $x = 0$.

Problem 2.4

i) If $f(x) = \cos(x)$, what is $f'(x)$?

ii) If $f(x) = \sin(x)$, what is $f'(x)$?

iii) If $f(x) = \cos(x) + i\sin(x)$ what is $f'(x)$? What is $\frac{f'(x)}{i}$? Write an equation that relates $f(x)$ to $f'(x)$.[1]

Problem 2.5

Find the first and second derivatives of the following functions. If you are familiar with the chain rule, do not use it.

i) $f(x) = x\sin(x)$

ii) $f(x) = x\ln(x) - x$

iii) $f(x) = e^x \sin(x)$

iv) $f(x) = \sin(2x)$ *first derivative only*

v) $f(x) = \ln(x^2)$

vi) $f(x) = \ln(x^n)$ for $n \in \mathbb{N}$.

Problem 2.6

Find the derivatives of the following functions with the product rule or the quotient rule.

i) $f(x) = \tan(x)$

ii) $f(x) = \sec(x)$

iii) $f(x) = \cot(x)$

iv) $f(x) = \csc(x)$

v) $f(x) = \sin^2(x)$

vi) $f(x) = \sec^2(x)$

vii) $f(x) = \tan(x)\sec(x)$

[1] $i = \sqrt{-1}$

Problem 2.7

Find the derivatives of the following functions with the product rule or the quotient rule.

i) $f(x) = \ln(xe^x)$

ii) $f(x) = x\sin(x)\cos(x)$

iii) $f(x) = \frac{2x^2+3x+7}{4x+2}$

iv) $f(x) = \frac{e^x}{1+e^x}$

Problem 2.8

Derive the product rule by evaluating the following limit:[2]

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Problem 2.9

The position of a particle on a line is given by $p(t) = 3t^3 - t^2 + 4t$. What is the average rate of change of the position of the particle from $t = 1$ to $t = 4$.

Problem 2.10

For $n > 1$, find the derivative of e^{nx} with the power rule, given that $\frac{d}{dx}(e^{(n-1)x}) = (n-1)e^{(n-1)x}$.

[2] Hint: You first have to add and subtract $f(x)g(x+h)$ to the numerator.