15.1 Double Integrals Over Rectangles

Iterated Integrals

Fubini's Theorem allows us to switch the order of integration. For $a \leq x \leq b, c \leq y \leq d$,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_c^d f(x,y)\,dx\,dy \tag{1}$$

15.2 Double Integrals Over General Regions

Integrals Between Curves

Type I Region lies between two functions of x, that is

$$a \leq x \leq b, \qquad g_1(x) \leq y \leq g_2(x)$$

Type II Region lies between two functions of y, that is

$$c \leq y \leq d, \qquad h_1(x) \leq x \leq h_2(x)$$

To solve these, make sure the function bounds are in the inner integral. See textbook for images.
Area is defined as $\iint 1\,dA$.

Switching Order of Integration

Integrals can be switched as long as the region is the same. For example, the region defined by

$$0 \leq x \leq 4, \sqrt{x} \leq y \leq 2$$

is the same region as

$$0 \leq y \leq 2, 0 \leq x \leq y^2$$

Draw a picture!

15.3 Double Integrals in Polar Coordinates

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\cos\theta \tag{1}$$

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\cos\theta) r\,dr\,d\theta \tag{2}$$

The "infinitesimal rectangle" $dA = dx\,dy = r\,dr\,d\theta$. **Don't forget the r!**

15.4 Applications of Double Integrals

Omitted.

15.5 Surface Area

$$A = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA \tag{1}$$

Note the similarity to the arc length formula.

15.6 Triple Integrals

Iterated Integrals

Fubini's Theorem Allows us to switch the order of integration. There are different types of regions as well, defined between two functions. Make sure the functions are in the inner integrals.

15.7 Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates Points are given as (r, θ, z). r and θ are the polar coordinates and z is the height.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{1}$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \tag{2}$$

$$\iiint_E f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2} \int_{f_1}^{f_2} \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \tag{3}$$

15.8 Triple Integrals in Spherical Coordinates

Spherical Coordinates Points are given as (ρ, θ, ϕ). ρ is the distance from the point to the origin, θ is the same angle as in polar coordinates (xy-plane), and ϕ is the angle between the positive z-axis and the line segment to the point.

Note

$$\rho \geq 0 \qquad \text{and} \qquad 0 \leq \phi \leq \pi$$

A sphere is given as $\rho = r$ and a cone is given as $\phi = c$.

Conversion

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi \tag{1}$$

Remember since θ corresponds to polar coordinates, so x and y are just multiplied by $\rho \sin\phi$. See diagram in textbook.

$$\rho^2 = x^2 + y^2 + z^2 \tag{2}$$

For $f(x, y, z) = g(\rho, \theta, \phi)$,

$$\iiint\limits_E f(x, y, z)\, dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} g(\rho, \theta, \phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \tag{3}$$

Where $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

15.9 Change of Variables in Multiple Integrals

Boundaries

To change coordinates, we consider the transformation T from the uv-plane region S to the xy-plane R:

$$T(u, v) = (x, y)$$

The inverse transformation T^{-1} converts $(x, y) \to (u, v)$.

The Jacobian

To actually calculate the integral, we have to convert $dA = \mathbf{J}\, du\, dv$, where the **Jacobian** is defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{1}$$

To integrate a region S in the uv-plane from a region R in the xy-plane,

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v))\, \mathbf{J}\, du\, dv \tag{2}$$