14.1 Functions of Several Variables

New Definitions

Level Curves $f(x, y) = k$ where k is a constant.

Level Surfaces $f(x, y, z) = k$

14.2 Limits and Continuity

Definition of Limits

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{1}$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon \tag{2}$$

When the Limit Does Not Exist

If we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Limit Laws Approaching along axes and a constant

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c \tag{3}$$

The **Squeeze Theorem** also holds.

Continuity

The direct substitution property: A function f is called
continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \tag{4}$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

14.3 Partial Derivatives

Partial Derivatives

$$f_x(a,b) = g'(a) \quad \text{where} \quad gx = f(x,b) \tag{1}$$

Definition of Derivatives

The partial derivatives are defined by

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned} \tag{2}$$

Notation

If $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Finding Partial Derivatives

To find f_x, treat y as a constant and differentiate $f(x,y)$ with respect to x.
To find f_y, treat x as a constant and differentiate $f(x,y)$ with respect to y.

Interpretations

The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y=b$ and $x=a$.
Partial derivatives can also be interpreted as *rates of change*. If $z = f(x,y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

More Than Two Variables

Treated similarly to two variable functions. If $w = f(x,y,z)$, then $f_x = \partial w/\partial x$ with respect to x when y and z are held constant. Similar notation follows.

Higher Derivatives

Second Partial Derivatives The partial derivatives of the partial derivatives, $(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$. The notation is similar to the notation of higher derivatives of functions of one variable.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{3}$$

14.4 Tangent Planes and Linear Approximations

Tangent Planes

Suppose a surface S had equation $z = f(x, y)$ with continuous first derivatives. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P.

Tangent Plane The plane to S at P that contains tangent lines T_1 and T_2.

At $P(x_0, y_0, z_0)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

Linear Approximations

At point $(a, b, f(a, b))$, the **linearlization** is

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{2}$$

The approximation of $f(x, y)$ is called the **linear approximation**.

Differentiable Functions

Theorem for differentiability If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Even if *directional derivatives* exist at a point in every direction, the function still may not be differentiable at that point. But if a function is differentiable at a point, then *all* directional derivatives will exist at that point.

Three or More Variables

Defined similarly.

14.5 The Chain Rule

Chain Rule for Functions of Two Variables

Chain Rule Case 1 Suppose $z = f(x, y)$, $x = g(t)$, $y = h(t)$, and all are differentiable. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{1}$$

Chain Rule Case 2 Suppose $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$, and all are differentiable. Then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{2}$$

A tree diagram can help remember the Chain Rule (supposedly).

Implicit Differentiation

Writing $F(x, y) = 0$ which defines y implicitly in terms of x, that is $y = f(x)$ so $F(x, f(x)) = 0$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{3}$$

For z given implicitly as $z = f(x, y)$ by $F(x, y, z) = 0$, that is $F(x, y, f(x, y)) = 0$, we get similar formulas:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \tag{4}$$

Note in both cases the fraction appears inverted, but that is a consequence of solving for the implicit derivative(s).

14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives

For differentiable f, f has a directional derivative in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b \tag{1}$$

If $\vec{u}$ makes angle θ with the positive x-axis, we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$.

The Gradient Vector Since $D_u f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle$, we give the first vector a special name, **grad** f or ∇f ("del" or "nabla").

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \tag{2}$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u} \tag{3}$$

Functions of Three Variables

Defined similarly.

$$\nabla f = \langle f_x, f_y, f_z \rangle \tag{4}$$

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u} \tag{5}$$

Maximizing the Directional Derivatives

Suppose f is differentiable.

$$\max(D_u f(\vec{x})) = |\nabla f(\vec{x})| \tag{6}$$

This occurs when $\vec{u}$ has the same direction as $\nabla f(\vec{x})$, or in my own notation

$$\vec{u} = k \nabla f(\vec{x}) \tag{7}$$

Normalize $\nabla f(\vec{x})$ to get $\vec{u}$.

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq 0$, define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.
Using the standard equation of a plane, we can write

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{8}$$

The **normal line** is perpendicular to the tangent plane and given by $\nabla F(x_0, y_0, z_0)$. It has symmetric equations

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)} \tag{9}$$

14.7 Maximum and Minimum Values

Local maximum and minimum

If f has a local maximum or minimum at (a, b) and the partial derivatives exist, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0 \tag{1}$$

Critical Point More generally, a point (a, b) is called a critical point if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

2nd Derivative Test

If $f_x(a, b) = 0$ and $f_y(a, b) = 0$, let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 \tag{2}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a **saddle point** of f.

Absolute Maximum and Minimum Values

A **closed set** is one that contains all its boundary points.

Extreme Value Theorem There exists an absolute maximum and absolute minimum on a closed, bounded set D.

Closed Interval Method To find the absolute maximum and minimum values of f on D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum, the smallest is the absolute minimum.

14.8 Lagrange Multipliers

Geometrically, the maximum or minimum of $f(x, y)$ subject to $g(x, y) = k$ is when the two curves touch, that is when they have a common tangent line.

$$\nabla f(x_0, y_0, z_0) = \lambda \ \nabla g(x_0, y_0, z_0) \tag{1}$$

Lambda is called a **Lagrange multiplier**.
In component form,

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k \tag{2}$$

This is a system of four equations and four unknowns. For functions of two variables, a similar equation holds.

Two Constraints

For $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$,

$$\nabla f(x_0, y_0, z_0) = \lambda \ \nabla g(x_0, y_0, z_0) + \mu \ \nabla h(x_0, y_0, z_0) \tag{3}$$

In component form,

$$\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x, y, z) &= k \\ h(x, y, z) &= c \end{aligned} \tag{4}$$