14.1 Functions of Several Variables

New Definitions

Level Curves $f(x, y) = k$ where k is a constant.

Level Surfaces $f(x, y, z) = k$

14.2 Limits and Continuity

Definition of Limits

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{1}$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon \tag{2}$$

When the Limit Does Not Exist

If we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Limit Laws Approaching along axes and a constant

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c \tag{3}$$

The **Squeeze Theorem** also holds.

Continuity

The direct substitution property: A function f is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \tag{4}$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

14.3 Partial Derivatives

Partial Derivatives

$$f_x(a,b) = g'(a) \quad \text{where} \quad gx = f(x,b) \tag{1}$$

Definition of Derivatives

The partial derivatives are defined by

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned} \tag{2}$$

Notation

If $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Finding Partial Derivatives

To find f_x, treat y as a constant and differentiate $f(x,y)$ with respect to x.
To find f_y, treat x as a constant and differentiate $f(x,y)$ with respect to y.

Interpretations

The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.
Partial derivatives can also be interpreted as *rates of change*. If $z = f(x,y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

More Than Two Variables

Treated similarly to two variable functions. If $w = f(x,y,z)$, then $f_x = \partial w/\partial x$ with respect to x when y and z are held constant. Similar notation follows.

Higher Derivatives

Second Partial Derivatives The partial derivatives of the partial derivatives, $(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$. The notation is similar to the notation of higher derivatives of functions of one variable.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{3}$$

14.4 Tangent Planes and Linear Approximations

Tangent Planes

Suppose a surface S had equation $z = f(x, y)$ with continuous first derivatives. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P.

Tangent Plane The plane to S at P that contains tangent lines T_1 and T_2.

At $P(x_0, y_0, z_0)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

Linear Approximations

At point $(a, b, f(a, b))$, the **linearlization** is

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{2}$$

The approximation of $f(x, y)$ is called the **linear approximation**.

Differentiable Functions

Theorem for differentiability If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Even if *directional derivatives* exist at a point in every direction, the function still may not be differentiable at that point. But if a function is differentiable at a point, then *all* directional derivatives will exist at that point.

Three or More Variables

Defined similarly.