SCP

$$Minimize \quad Z = \sum_{j=1}^{n} c_j X_j \qquad (1)$$

$$Subject\ to \quad \sum_{j=1}^{n} a_{ij} X_j \geq 1 \quad i = 1, 2, \dots, m \qquad (2)$$

$$X_j = \{0,1\} \ \ i = 1, 2, \dots, m \qquad (3)$$

Relax Constraints (2) with given non-negative Lagrangean multipliers λ_i $(i = 1, 2, \dots, m)$, resulting in the relaxed problem:

LR-SCP

$$Minimize \quad Z_D(\boldsymbol{\lambda}) = \sum_{j=1}^{n} (c_j - \sum_{i=1}^{m} \lambda_i a_{ij}) X_j + \sum_{i=1}^{m} \lambda_i \qquad (4)$$

$$Subject\ to \quad X_j = \{0,1\} \ \ i = 1, 2, \dots, m \qquad (5)$$

Ideally, $\boldsymbol{\lambda}$ should solve the dual problem $Z_D = minimize\ Z_D(\boldsymbol{\lambda})\ \ \forall \boldsymbol{\lambda}$

Easier to solve

Main ideas:

- An optimal objective of a given LR-SCP $Z_D^*(\boldsymbol{\lambda})$ gives a lower bound of the optimal objective of SCP Z^*, that is $Z_D^*(\boldsymbol{\lambda}) \leq Z^*$.

- A feasible solution to SCP with objective of Z^f gives an upper bound of Z^*, that is $Z_f \geq Z^*$

- Therefore, $Z_D^*(\boldsymbol{\lambda}) \leq Z^* \leq Z_f$.

1. How to solve LR-SCP (obtaining the lower bound)?

$$Minimize \quad Z_D(\boldsymbol{\lambda}) = \sum_{j=1}^{n}(c_j - \sum_{i=1}^{m} \lambda_i \, a_{ij})X_j + \sum_{i=1}^{m} \lambda_i \quad (4)$$

$$Subject\ to \quad X_j = \{0,1\} \;\; i = 1, 2, \dots, m \quad (5)$$

Define $C_j = (c_j - \sum_{i=1}^{m} \lambda_i a_{ij})$, if $C_j \leq 0$, $X_j = 1$, otherwise, $X_j = 0$.

2. How to get a feasible solution to SCP (obtaining an upper bound)?

Apply the heuristic of Balas and Ho (1980) using the lagrangean costs C_j.

Denoting by R the set of rows that are not yet covered, by S the set of j's that with $X_j = 1$, by p the iteration index

Step 1: Set $R = M, S = \emptyset, p = 1$

Step 2: If $R = \emptyset$, go to Step 3.

Otherwise, let $K_j = |R \cap M_j|$, choose $i_* \in R$ such that $\left|N_{i_*}\right| = \min_{i \in R} |N_i|$

choose $j(p) \notin S$ such that $f\left(C_{j(p)}, K_{j(p)}\right) = \min_{j \in N_{i_*}} f(C_j, K_j)$

set $R \leftarrow R \backslash M_{j(p)}, S \leftarrow S \cup \{j(p)\}, p \leftarrow p + 1$, go to Step 2.

Step 3: Consider the elements $j \in S$ in order, if $S \backslash \{j\}$ can cover all i, set $S = S \backslash \{j\}$. When all$j \in S$ are checked, a primal feasible is obtained.

Options for $f(C_j, K_j)$:

$C_j, C_j/K_j, C_j/log_2(K_j), C_j/K_j log_2(K_j), C_j/K_j ln(K_j)$

$$M = \{1, 2, \dots, m\}$$
$$N = \{1, 2, \dots, n\}$$
$$M_j = \left\{i | a_{ij} = 1, j \in N\right\}$$
$$N_i = \left\{j | a_{ij} = 1, i \in M\right\}$$
$$S = \left\{j \in N | X_j = 1\right\}$$

3. How to update λ?

Use subgradient procedures

$$\lambda_i^{q+1} = \max\left\{0, \lambda_i^q + t^q\left(1 - \sum_{j=1}^{n} a_{ij} X_j\right)\right\} \quad i = 1, 2, \ldots, m$$

$$t^q = \frac{\alpha^q (Z_{UB} - Z_D^*(\boldsymbol{\lambda}^q))}{\sum_{i=1}^{m} (1 - \sum_{j=1}^{n} a_{ij} X_j)^2}$$

$$\alpha^{q+1} = \beta \alpha^q \quad \beta \in (0, 1]$$

$q: iteration\ index$

The algorithm:

Step 1: Initialize $q = 0, \lambda_i^0$ and α^0.

Step 2: Compute $C_j = (c_j - \sum_{i=1}^{m} \lambda_i^q a_{ij})$, if $C_j \leq 0, X_j = 1$, otherwise, $X_j = 0$.

Step 3: Compute the objective value $Z_D^*(\boldsymbol{\lambda}^q)$ using X_j obtained in Step 2, denoted as Z_{LB}.

Step 4: Find a primal feasible solution by Balas and Ho heuristic, compute corresponding the objective value, this is an upper bound. Keep the minimum upper bound, denoted as Z_{UB}.

Step 5: if $Z_{LB} = Z_{UB}$, stop. Otherwise, update $\lambda_i^{q+1} = \max\left\{0, \lambda_i^q + t^q\left(1 - \sum_{j=1}^{n} a_{ij} X_j\right)\right\}$, where $t^q = \frac{\alpha^q (Z_{UB} - Z_D^*(\boldsymbol{\lambda}^q))}{\sum_{i=1}^{m}(1 - \sum_{j=1}^{m} a_{ij} X_j)^2}$ using X_j obtained in Step 3, $\alpha^{q+1} = \beta\alpha^q \;\; \beta \in (0, 1]$ and go to Step 2 unless some stopping criteria are met, e.g., $Z_{UB} - Z_{LB} < \epsilon$ or $\alpha^q < \epsilon$.