Summary of Hofmann and Streicher's Groupoid Model

Overview

In their seminal 1994 paper, *The groupoid model refutes uniqueness of identity proofs*, Martin Hofmann and Thomas Streicher constructed a model of intensional Martin-Löf Type Theory (MLTT) in which identity types are interpreted as hom-sets in groupoids. This model provides a semantic justification for the fact that certain principles, such as function extensionality and uniqueness of identity proofs (UIP), are *not derivable* in MLTT.

The Groupoid Interpretation

In the groupoid model:

- A type A is interpreted as a *groupoid* $\mathcal{A}$.
- A term $a : A$ is interpreted as an *object* $a \in \mathcal{A}$.
- The identity type $x =_A y$ is interpreted as the set of *isomorphisms* (i.e., morphisms in the groupoid) from x to y, denoted $\mathrm{Hom}_{\mathcal{A}}(x, y)$.

This means that identity types can have multiple distinct inhabitants, corresponding to different isomorphisms between objects.

Failure of UIP

Because identity types are interpreted as hom-sets, and hom-sets in general groupoids can contain multiple distinct morphisms, the model validates the existence of multiple distinct proofs of equality. That is,

$$x = y \quad \text{can have} \quad p \neq q \in \mathrm{Hom}_{\mathcal{A}}(x, y),$$

which refutes UIP:

$$\text{UIP:} \quad \forall x, y : A,\ \forall p, q : x = y,\ p = q.$$

Failure of Function Extensionality

Functions between types are interpreted as functors between groupoids. That is:

- A function $f : A \to B$ is interpreted as a functor $F : \mathcal{A} \to \mathcal{B}$.
- An equality $f = g$ between functions corresponds to a *natural isomorphism* between functors.

Even if two functors $F, G : \mathcal{A} \to \mathcal{B}$ are pointwise isomorphic (i.e., $F(a) \cong G(a)$ for all a), they may not be *naturally* isomorphic, because naturality requires a coherent family of morphisms:

$$\forall f : a \to a', \quad G(f) \circ \eta_a = \eta_{a'} \circ F(f).$$

Therefore, the implication

$$\forall x : A,\ f(x) = g(x) \Rightarrow f = g$$

is *not valid* in the groupoid model. Thus, **function extensionality is not derivable** in MLTT.

Conclusion

Hofmann and Streicher's groupoid model provides a powerful demonstration that certain extensional principles, like function extensionality and UIP, are not provable from the core rules of intensional MLTT. This proves their *independence* from the theory and establishes that if one wants these principles, they must be *postulated* or derived from stronger assumptions such as the univalence axiom.