Assignment 7

1. State (t, W_t)

   Action $\{\pi_t, C_t\}$

   Reward $U(C_t) = \log(C_t)$

The return at time t

$$\int_t^T e^{-\rho(s-t)} \log(C_s)\, ds$$

$$V^*(t, W_t) = \max_{\pi, c} E_t \{ \int_t^T e^{-\rho(s-t)} \log(C_s)\, ds$$

$$= \max_{\pi, c} E_t \{ \int_t^{t_1} e^{-\rho(s-t)} \log(C_s)\, ds$$

$$+ e^{-\rho(t_1 - t)} V^*(t_1, W_{t_1})$$

$$\Rightarrow e^{-\rho t} V^*(t, W_t) = \max_{\pi, c} E_t \{ \int_t^{t_1} e^{-\rho s} \log(C_s)\, ds$$

$$+ e^{-\rho t_1} V^*(t_1, W_{t_1})$$

$$\Rightarrow \max_{\pi_t, C_t} E_t \{ d(e^{-\rho t} V^*(t, W_t)) + e^{-\rho t} \log(C_t)\, dt] = 0$$

$$\Rightarrow \max_{\pi_t, C_t} E_t \{ dV^*(t, W_t) + \log(C_t)\, dt] = \rho V^*(t, W_t)\, dt$$

$$\max_{\pi_t, c_t} \left\{ \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W}((\pi_t(\mu - r) + r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W^2} \cdot \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \log(c_t) \right\} = \rho V^*(t, W_t)$$

Let $\phi(t, W_t; c_t, \pi_t)$

$$= \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W}((\pi_t(\mu - r) + r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W^2} \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \log(c_t)$$

$$\frac{\partial \phi}{\partial \pi_t} = (\mu - r)\frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \pi_t \sigma^2 W_t = 0$$

$$\pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t}(\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \sigma^2 W_t}$$

$$\frac{\partial \phi}{\partial c_t} = -\frac{\partial V^*}{\partial W} + \frac{1}{c_t} = 0$$

$$c_t = \left(\frac{\partial V^*}{\partial W}\right)^{-1}$$

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W}\left(\left(\frac{-\frac{\partial V^*}{\partial W_t}(\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2}\sigma^2 \cancel{W_t}} + r\right)W_t - \left(\frac{\partial V^*}{\partial W}\right)^{-1}\right)$$

$$+ \frac{\partial^2 V}{\partial W^2}\frac{\sigma^2 W_t^2}{2}\frac{\left(\frac{\partial V}{\partial W_t}\right)^2(\mu - r)^2}{\left(\frac{\partial^2 V}{\partial W^2}\sigma^2 W_t\right)^2} - \log\frac{\partial V}{\partial W} = \rho V$$

$$\Rightarrow \frac{\partial V^*}{\partial t} + \underbrace{\frac{-\left(\frac{\partial V^*}{\partial W}\right)^2(\mu - r)^2}{\frac{\partial^2 V}{\partial W^2}\sigma^2}}_{} + r\frac{\partial V}{\partial W}\cdot W_t - 1$$

$$+ \underbrace{\frac{(\mu - r)^2}{2\sigma^2}\frac{\left(\frac{\partial V}{\partial W_t}\right)^2}{\frac{\partial^2 V}{\partial W^2}}}_{} - \log\frac{\partial V}{\partial W} = \rho V$$

$$\Rightarrow \frac{\partial V}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2}\frac{\left(\frac{\partial V}{\partial W}\right)^2}{\frac{\partial^2 V}{\partial W^2}} + \frac{\partial V}{\partial W}r\cdot W_t - 1 - \log\frac{\partial V}{\partial W}$$

$$= \rho V \qquad (1)$$

Boundry condition:

$$V(T, W_T) = \varepsilon \log W_T$$

guess solution

$$V^{\gamma}(t, W_t) = f(t) \log W_t$$

$$\varepsilon^{f(t)} \log W_t$$

$$\frac{\partial V^{\gamma}}{\partial t} = f'(t) \log W_t$$

$$\log f(t) \log W(t)$$

$$\frac{\partial V^{\gamma}}{\partial W_t} = \frac{f(t)}{W_t}$$

$$\frac{f'(t)}{f(t)} \log (W(t))$$

$$\frac{\partial^2 V^{\gamma}}{\partial W_t^2} = -\frac{f(t)}{W_t^2}$$

$$\frac{\log f(t)}{W(t)}$$

Substitute into ①, we get that

$$\underbrace{f'(t) \log W}_{} - \frac{(\mu-r)^2}{2\sigma^2} f(t)$$

$$+ \quad r f(t) \quad -1 - \log \frac{f(t)}{\underbrace{W_t}_{}} = \rho f(t) \log W_t$$

$$\therefore \; f'(t) \log W + \log W - \rho f(t) \log W$$

$$= + \frac{(\mu-r)^2}{2\sigma^2} f(t) - r f(t) + 1 + \log (f(t))$$

It seems that since W and $f(t)$ are independent here, we should have that

$$f'(t) + 1 - \rho f(t) = 0$$

$$\Rightarrow \quad f(t) = \frac{1 + (\rho\varepsilon - 1)\, e^{-\rho(T-t)}}{\rho}$$