1.1 Plane Curves

An algebraic plane curve is a curve consisting of the points of the plane whose coordinates x, y satisfy an equation

$$f(x, y) = 0 \tag{1}$$

where f is a nonconstant polynomial. We fix a field k and assume that the coordinates x, y of points and the coefficients of f are elements of k.

Lemma Let k be an arbitrary field, $f \in k[x, y]$ an irreducible polynomial, and $g \in k[x, y]$ an arbitrary polynomial. If g is not divisible by f then the system of equations $f(x, y) = g(x, y) = 0$ has only a finite number of solutions.

An algebraically closed field k is infinite, and if f is not a constant, the curve with equation $f(x, y) = 0$ has infinitely many points. Because of this, it follows from the lemma that an irreducible polynomial $f(x, y)$ is uniquely determined, up to a constant multiple, by the curve $f(x, y) = 0$.

1.2 Rational Curves

We say that an irreducible algebraic curve X defined by $f(x, y) = 0$ is rational if there exist two rational functions $\varphi(t)$ and $\psi(t)$, at least one nonconstant, such that

$$f(\varphi(t), \psi(t)) \equiv 0 \tag{3}$$

as an identity in t.

1.3 Relation with Field Theory

Let X be the irreducible curve given by 1.1, (1). Consider rational functions $u(x, y) = p(x, y)/q(x, y)$, where p and q are polynomials in k such that the denominator $q(x, y)$ is not divisible by $f(x, y)$. We say that such a function $u(x, y)$ is a *rational function* defined on X; and two rational functions $p(x, y)/q(x, y)$ and $p_1(x, y)/q_1(x, y)$ defined on X are equal on X if the polynomial $p(x, y)q_1(x, y) - q(x, y)p_1(x, y)$ is divisible by $f(x, y)$. It is easy to check that rational functions on X, up to equality on X, form a field. This field is called the function field or field of rational functions of X, and denoted by $k(X)$.

Let X be a rational curve. By Lüroth's theorem, the field $k(X)$ is isomorphic to the field of rational functions $k(t)$. Suppose that this isomorphism takes x to $\varphi(t)$ and y to $\psi(t)$. This gives the parametrisation $x = \varphi(t)$, $y = \psi(t)$ of X.

Proposition The parametrisation $x = \varphi(t)$, $y = \psi(t)$ has the following properties:

(i) Except possibly for a finite number of points, any $(x_0, y_0) \in X$ has a representation $(x_0, y_0) = (\varphi(t_0), \psi(t_0))$ for some t_0.

(ii) Except possibly for a finite number of points, this representation is unique.

1.4 Rational Maps