I.1 Complex Charts and Complex Structures

Recall the classification of compact orientable 2-manifolds; each of these is a g-holed torus for some unique integer $g \geq 0$. This integer g is called the topological genus of the compact Riemann surface.

II.1 Functions on Riemann Surfaces

Let f be meromorphic at p, whose Laurent series in a local coordinate z is $\sum_n c_n(z - z_0)^n$. The order of f at p, denoted by $\mathrm{ord}_p(f)$, is the minimum exponent actually appearing (with nonzero coefficient in the Laurent series:

$$\mathrm{ord}_p(f) = \min\{n | c_n \neq 0\}$$

II.4 Global Properties of Holomorphic Maps

Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces.

The multiplicity of F at p, denoted $\mathrm{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \mapsto z^m$.

Proposition For each $y \in Y$, define $d_y(F)$ to be the sum of the multiplicities of F at the points of X mapping to y:

$$d_y(F) = \sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F)$$

Then $d_y(F)$ is constant, independent of y.

The degree of F, denoted $\deg(F)$, is the integer $d_y(F)$ for any $y \in Y$.

Theorem (Hurwitz's formula)

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} (\mathrm{mult}_p(F) - 1)$$

Problem Let X be the projective plane curve of degree d defined by the homogeneous polynomial $F(x, y, z) = x^d + y^d + z^d$. This curve is called the Fermat curve of degree d. Let $\pi : X \to \mathbb{P}^1$ be given by $\pi[x : y : z] = [x : y]$. Use Hurwitz's formula to show that the genus of the Fermat curve is

$$g(X) = \frac{(d-1)(d-2)}{2}$$