Problem III.4.7 Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Cech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1, \qquad \dim H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}$$

III.5 The Cohomology of Projective Space

Theorem 5.1 Let A be a noetherian ring, let $S = A[x_0, \ldots, x_r]$, and let $X = \operatorname{Proj} S$ be the projective space $\mathbb{P}^r_A$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (II, §5). For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{F}(n))$ (see II, §5). Then

(a) the natural map $S \to \Gamma_*(\mathcal{O}_X) = \oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$

(c) $H^r(X, \mathcal{O}_X(-r-1)) \simeq A$

(d) the natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \simeq A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.

Problem III.5.3 Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X, \mathcal{O}_X) \simeq k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

Hint: Use (I, 3.4)

(b) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational.