Q1

$$S(x) = \begin{cases} p_1(x) \\ p_2(x) \\ p_3(x) \end{cases}$$

$$S(x) = \begin{cases} 20 + 25x + 18x^2 + a_1x^3 & -3 \le x < -1 \\ 26 + a_2x + a_3x^2 + a_4x^3 & -1 \le x < 0 \\ 26 + 19x + a_5x^2 + x^3 & 0 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} 25 + 36x + 3a_1x^2 & -3 \le x < -1 \\ a_2 + 2a_3x + 3a_4x^2 & -1 \le x < 0 \\ 19 + 2a_5x + 3x^2 & 0 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} 36 + 6a_1x & -3 \le x < -1 \\ 2a_3 + 6a_4x & -1 \le x < 0 \\ 2a_5 + 6x & 0 \le x \le 3 \end{cases}$$

For natural spline:

$$S''(x_1) = S''(x_n) = 0$$

From $S(x)$, we can see that $x_1 = -3,\ \ x_2 = -1,\ \ x_3 = 0,\ \ x_4 = 3$

Thus, the constraints are

<u>Interpolant constraints:</u>

$$\begin{gathered} p_1(x_1) = y_1 \\ p_1(x_2) = y_2 \\ p_2(x_2) = y_2 \\ p_2(x_3) = y_3 \\ p_3(x_3) = y_3 \\ p_3(x_4) = y_4 \end{gathered}$$

We can see that

$$\begin{gathered} p_1(x_2) = p_1(-1) = y_2 = p_2(x_2) = p_2(-1) \\ p_2(x_3) = p_2(0) = y_3 = p_3(x_3) = p_3(0) \end{gathered}$$

Or

$$\begin{gathered} 20 + 25(-1) + 18(-1)^2 + a_1(-1)^3 = 26 + a_2(-1) + a_3(-1)^2 + a_4(-1)^3 \\ \rightarrow\ 20 - 25 + 18 - a_1 = 26 - a_2 + a_3 - a_4 \\ \rightarrow\ 20 - 25 + 18 - 26 = a_1 - a_2 + a_3 - a_4 \\ \rightarrow\ a_1 - a_2 + a_3 - a_4 = -13 \end{gathered}$$

$$\begin{gathered} 26 + a_2(0) + a_3(0)^2 + a_4(0)^3 = 26 + 19(0) + a_5(0)^2 + (0)^3 \\ 26 = 26 \end{gathered}$$

<u>Differentiability constraints:</u>

$$\begin{gathered} {p_1}'(x_2) = {p_2}'(x_2) \quad \text{or} \quad {p_1}'(-1) = {p_2}'(-1) \\ {p_2}'(x_3) = {p_3}'(x_3) \quad \text{or} \quad {p_2}'(0) = {p_3}'(0) \end{gathered}$$

We can see that

$$25 + 36(-1) + 3a_1(-1)^2 = a_2 + 2a_3(-1) + 3a_4(-1)^2$$
$$\rightarrow 25 - 36 + 3a_1 = a_2 - 2a_3 + 3a_4$$
$$\rightarrow -3a_1 + a_2 - 2a_3 + 3a_4 = -11$$
$$\rightarrow 3a_1 - a_2 + 2a_3 - 3a_4 = 11$$

$$a_2 + 2a_3(0) + 3a_4(0)^2 = 19 + 2a_5(0) + 3(0)^2$$
$$\rightarrow a_2 = 19$$

<u>Twice differentiability constraints:</u>

$$p_1''(x_2) = p_2''(x_2) \quad \text{or} \quad p_1''(-1) = p_2''(-1)$$
$$p_2''(x_3) = p_3''(x_3) \quad \text{or} \quad p_2''(0) = p_3''(0)$$

We can see that

$$36 + 6a_1(-1) = 2a_3 + 6a_4(-1)$$
$$\rightarrow 36 - 6a_1 = 2a_3 - 6a_4$$
$$\rightarrow 3a_1 + a_3 - 3a_4 = 18$$

$$2a_3 + 6a_4(0) = 2a_5 + 6(0)$$
$$\rightarrow a_3 - a_5 = 0$$

<u>Natural cubic spine constraints</u>:

$$S''(x_1) = S''(x_n) = 0$$

Or

$$36 + 6a_1(-3) = 0 \quad and \quad 2a_5 + 6(3) = 0$$
$$\rightarrow a_1 = 2 \quad and \quad a_5 = -9$$

Put together all the constraints, we have

<u>Interpolant constraints:</u>

$$a_1 - a_2 + a_3 - a_4 = -13$$

<u>Differentiability constraints:</u>

$$3a_1 - a_2 + 2a_3 - 3a_4 = 11$$
$$a_2 = 19$$

<u>Twice differentiability constraints:</u>

$$3a_1 + a_3 - 3a_4 = 18$$
$$a_3 - a_5 = 0$$

<u>Natural cubic spine constraints</u>:

$$a_1 = 2$$
$$a_5 = -9$$

Now we solve the system of equations using the last 5 constrains:

$$a_1 = 2$$
$$a_2 = 19$$
$$a_5 = -9$$
$$a_3 - (-9) = 0$$
$$\rightarrow a_3 = -9$$
$$3(2) + (-9) - 3a_4 = 18$$
$$\rightarrow a_4 = -7$$

We have $\{a_1,\ a_2,\ a_3,\ a_4,\ a_5\} = \{2,\ 19, -9, -7, -9\}$
To check that the other two constraints:

$$a_1 - a_2 + a_3 - a_4 = -13$$
$$\rightarrow 2 - 19 - 9 + 7 = -19 \neq -13$$

$$3a_1 - a_2 + 2a_3 - 3a_4 = 11$$
$$3(2) - 19 + 2(-9) - 3(-7) = -10 \neq 11$$

Thus, there is no values for $\{a_1,\ a_2,\ a_3,\ a_4,\ a_5\}$ that satisfies all the constraints for S(x) and the natural cubic spline, meaning that it is NOT possible to find values for $\{a_1,\ a_2,\ a_3,\ a_4,\ a_5\}$ so that S(x) is a natural cubic spline.