4.a)

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -60 \\ -4 \\ -30 \\ 36 \end{bmatrix}$$

$$A = \begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix}, \quad b = \begin{bmatrix} -60 \\ -4 \\ -30 \\ 36 \end{bmatrix}$$

b)

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix} \rightarrow \begin{matrix} (2) - (4/12)(1) \\ (3) - (6/12)(1) \\ (4) - (-4/12)(1) \end{matrix} \rightarrow \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -4 & 10 & 24 \\ 0 & -2 & 1 & 8 \\ 0 & -8 & 12 & 24 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 1/2 & & 1 & 0 \\ -1/3 & & & 1 \end{bmatrix}$$

Pivoting -> swap (4) & (2)

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -4 & 10 & 24 \\ 0 & -2 & 1 & 8 \\ 0 & -8 & 12 & 24 \end{bmatrix} \rightarrow \begin{matrix} (4) \\ \\ \\ (2) \end{matrix} \rightarrow \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & -2 & 1 & 8 \\ 0 & -4 & 10 & 24 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/2 & & 1 & 0 \\ 1/3 & & & 1 \end{bmatrix} \quad \mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & -2 & 1 & 8 \\ 0 & -4 & 10 & 24 \end{bmatrix} \rightarrow \begin{matrix} (3) - (-2/-8)(2) \\ (4) - (-4/-8)(2) \end{matrix} \rightarrow \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 12 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/2 & 1/4 & 1 & 0 \\ 1/3 & 1/2 & & 1 \end{bmatrix}$$

Pivoting -> swap (4) & (3)

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 12 \end{bmatrix} \rightarrow \begin{matrix} \\ \\ (4) \\ (3) \end{matrix} \rightarrow \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & -2 & 2 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/4 & & 1 \end{bmatrix} \quad \mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & -2 & 2 \end{bmatrix} \rightarrow \begin{matrix} \\ \\ \\ (4) - (-2/4)(3) \end{matrix} \rightarrow \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/4 & -1/2 & 1 \end{bmatrix}$$

Where (i) is the ith row of A

Thus, using LU factorization, PA=LU, we find that

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/4 & -1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

c)

To solve Ax=b,

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -60 \\ -4 \\ -30 \\ 36 \end{bmatrix}$$

Since we have PA=LU where

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 12 & 12 & -24 & -48 \\ 4 & 0 & 2 & 8 \\ 6 & 4 & -11 & -16 \\ -4 & -12 & 20 & 40 \end{bmatrix} \quad \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/4 & -1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

We first solve Lz = Pb,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/4 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -60 \\ -4 \\ -30 \\ 36 \end{bmatrix} = \begin{bmatrix} -60 \\ 36 \\ -4 \\ -30 \end{bmatrix}$$

Using forward substitution,
we get that $z_1 = -60$,
Since $\left(-\frac{1}{3}\right) z_1 + (1) z_2 = 36$, we get that $z_2 = 16$
Since $\left(\frac{1}{3}\right) z_1 + \left(\frac{1}{2}\right) z_2 + (1) z_3 = -4$, we get that $z_3 = 8$
Since $\left(\frac{1}{2}\right) z_1 + \left(\frac{1}{4}\right) z_2 + \left(-\frac{1}{2}\right) z_3 + (1) z_4 = -30$, we get that $z_4 = 0$

Then we solve Ux=z,

$$\begin{bmatrix} 12 & 12 & -24 & -48 \\ 0 & -8 & 12 & 24 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -60 \\ 16 \\ 8 \\ 0 \end{bmatrix}$$

Using backward substitution,
we get that $x_4 = 0$,
Since $(4) x_3 + (12) x_4 = 8$, we get that $x_3 = 2$
Since $(-8) x_2 + (12) x_3 + (24) x_4 = 16$, we get that $x_2 = 1$
Since $(12) x_1 + (12) x_2 + (-24) x_3 + (-48) x_4 = -60$, we get that $x_1 = -2$

Therefore, using the LU factorization, we get that $\mathrm{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$