3701_Assignment1

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2019/6/20

problem 1

Exercise1

(1)

$P(A\cap B) = P(A)(P(A^c\cap B) + P(A\cap B)) = P(A)P(A^c\cap B) + P(A)P(A\cap B)$

$P(A\cap B) - P(A)P(A\cap B) = P(A)P(A^c\cap B)$

$(1 - P(A))P(A\cap B) = P(A)P(A^c\cap B)$

$(1 - P(A))P(A)P(B) = P(A)P(A^c\cap B)$ So we get $P(A^c)P(B) = P(A^c\cap B)$

(2)

$P(A\cap B) = P(A)P(B) = [P(B\cap A) + P(B^c\cap A)]P(B)$

$P(A) = P(B\cap A) + P(B^c\cap A) = P(A)P(B) + P(B^c\cap A)$

$P(A) - P(A)P(B) = P(B^c\cap A)$

$P(A)(1 - P(B)) = P(B^c\cap A)$ So we get $P(A)P(B^c) = P(A\cap B^c)$

(3)

$P(A^c)P(B) = P(A^c\cap B) = P(B)[P(A^c\cap B) + P(A^c\cap B^c)]$ so $P(A^c) = P(A^c\cap B) + P(A^c\cap B^c) = P(A^c)P(B) + P(A^c\cap B^c)$

$P(A^c)(1 - P(B)) = P(A^c\cap B^c)$ So we get $P(A^c)P(A^c) = P(A^c\cap B^c)$

Exercise3

$P(A|B) = \frac{P(A|B)}{P(B)}$, $P(A|B^c) = \frac{P(A|B^c)}{P(B^c)}$ so $P(A\cap B) = P(A|B)P(B)$ $P(A\cap B^c) = P(A|B^c)P(B^c)$ $P(A) = P(A\cap B) + P(A\cap B^c)$ so we get $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

Exercise4

$Var(x) = E(x - E(x))^2 = E[x^2 - 2xE(x) + E(x)^2] = E(x^2) - E(2xE(x)) + E(x)^2 = E(x^2) - E(x)2E(x) + E(x)^2 = E(x^2) - 2E(x)^2 + E(x)^2 = E(x^2) - E(x)^2$

problem 2

Exercise3

$P(x, \theta) = (1 - \theta)^{x-1}\theta, \theta \in [0, 1]$

$E(x) = \sum_{x=1}^{n}(1 - \theta)^{x-1}\theta x = \theta\sum_{x=1}^{\infty}(1 - \theta)^{x-1}x$

because $\frac{d(1-\theta)^x}{d(1-\theta)} = x(1-\theta)^{x-1}$ so $E(x) = \theta\sum_{x=1}^{\infty}\frac{d(1-\theta)^x}{d(1-\theta)} = \theta\frac{d}{d(1-\theta)}\sum_{x=1}^{\infty}[(1-\theta)^x] = \theta\frac{d}{d(1-\theta)}\frac{1-\theta}{\theta}$ let $1-\theta = q$ $(1 - q)\frac{d}{dq}\frac{q}{1-q} = (1 - q)\frac{1}{(1-q)^2} = \frac{1}{1-q} = \frac{1}{\theta}$

Exercise5

$f(x, \theta_1, \theta_2) = \frac{1}{\theta_1-\theta_2}I_{\theta_1\leq x\leq\theta_2}$

For $x < \theta_1$ $F(x) = \int_{-\infty}^{\theta_1} f(x)dx = \int_{-\infty}^{\theta_1} = 0$

For $\theta_1 < x < \theta_2$ $F(x) = \int_{-\infty}^{\theta_1} f(x)dx + \int_{\theta_1}^{x} \frac{1}{\theta_2-\theta_1} = 0 + (\frac{x}{\theta_2-\theta_1})|_{\theta_1}^{x} = \frac{x-\theta_1}{\theta_2-\theta_1}$

For $x > \theta_2$ $F(x) = \int_{-\infty}^{\theta_1} f(x)dx + \int_{\theta_1}^{\theta_2} \frac{1}{\theta_2-\theta_1} + \int_{\theta_2}^{+\infty} f(x)dx = 0 + 1 + 0 = 1$

so $F(x) = 0, x < \theta_1$ $F(x) = \frac{x-\theta_1}{\theta_2-\theta_1}, \theta_1 \leq x < \theta_2$ $F(x) = 1, x > \theta_2$

Exercise6

From exercise5, we can use R to get the function: $f(x) = \frac{1}{b-a}, F(x) = \frac{x-a}{b-a}$ $F(x) = u$ so$x = (b-a)u + a = f^{-1}(u)$

```
uni.gen = function(n, a, b){
  x = c()
  for(i in 1:n){
  u = runif(n)
  x[i] = u[i] * (b - a) + a
}
  return(x)
}
```

problem 3

```
x_list = uni.gen(1000, 1, 3)
mean(x_list)
```

```
## [1] 2.024526
```

```
x_list1 = runif(1000, 1, 3)
mean(x_list1)
```

```
## [1] 2.023485
```

```
## the sample mean I generate is very close to the population mean (1+3)/2=2, the more sample we genera
```

problem 4

```
binom.gen = function(k,n,p){
  b = c()
  for(j in 1:k){
    b1 = c()
    u = runif(n)
    for(i in 1:n){
      b1[i] = ifelse(u[i]>p,0,1)
    }
    b[j] = sum(b1)

  }
  return(b)
}
set.seed(1729)
ber1 = binom.gen(100,1,0.5)
```

```r
ber2 = binom.gen(100,1,0.9)
sd(ber1)
```

```
## [1] 0.5025189
```

```r
sd(ber2)
```

```
## [1] 0.2386833
```

```r
##
```

the standard deviation of ber(0.5) is higher than ber(0.9), because$v(0.5) = 0.5(1-0.5) = 0.25, v(0.9) = 0.9(1-0.9) = 0.09$, this is because probability of 0.5 is more variable than the probability of 0.9 , $sd(0.5) = \sqrt{0.5(1-0.5)} = 0.5, sd(0.9) = \sqrt{0.9(1-0.9)} = 0.3$.

problem 5

```r
##(i)

1 - pnorm(32, 29, sqrt(5))
```

```
## [1] 0.08985625
```

```r
##(ii)

prob_fish = 1 - pnorm(32, 29, sqrt(5))
exptation = 1 / prob_fish
exptation
```

```
## [1] 11.12889
```

```r
##(iii)
set.seed(3701)
fish_weig = 0; pull_num = 1
fish_weig = rnorm(1,29,sqrt(5))
while(fish_weig <= 32){
  fish_weig = rnorm(1,29,sqrt(5))
  pull_num = pull_num + 1
}
pull_num
```

```
## [1] 6
```

problem 6

i.

Because X,Y are independent

$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$

$P(X \le x) = F_x(x), P(Y \le y = F_y(y)$ so $F_{x,y}(x, y) = P(X \le x, Y \le y) = P(X \le x)P(Y \le y) = F_x(x)F_y(y)$

ii.

$F_{x,y}(x, y) = F_x(x)F_y(y)$ $f_{x,y}(x, y) = \frac{d^2(F_x(x)F_y(y))}{dxdy} = \frac{dF_y(y)}{dy}\frac{dF_x(x)}{dx} = F^{'}(y)F^{'}(x) = f(y)f(x)$

iii.

$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} XYf_{x,y}(x,y)dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} XYf(x)f(y)dxdy = \int_{-\infty}^{+\infty} Xf(x)dx \int_{-\infty}^{+\infty} Yf(y)dy = E(X)E(Y)$

iv.

$cov(X,Y) = E[(X-E(X))(Y-E(Y))] = E[XY-XE(Y)-YE(X)+E(X)E(Y)] = E(XY)-E(XE(Y))-E(YE(X))+E(E(X)E(Y)) = E(XY)-E(Y)E(X)-E(X)E(Y)+E(X)E(Y)$ Because X,Y are independent, so $E(XY) = E(X)E(Y)$ so $cov(X,Y) = E(XY) - E(X)E(Y) = 0$

v.

```
set.seed(1729)
x.list = c();x2.list = c()
x.list = rbinom(1000,1,0.5)
x2.list = rnorm(1000,0,1)
cov(x.list,x2.list)
```

```
## [1] -0.01611453
```

The covariance is close to 0

problem 7

We need to simulate two uniform distribution over (0,1), then we get a square with the area of 1, then we draw a cricle in the square with the radius 1/2. We cast points in the area, meaning we siumlate big amount of points in the square. Then we calculate how many points are in the cycle. The porpotion of the points that are in the cycle times 4 is the estimation of pi. Mathematially, $S_{cycle} = \pi\frac{1}{4}$ $S_{square} = 1$ $S_{cycle}/S_{square} = \frac{1}{4}\pi$ so $S_{cycle}/S_{square} \times 4$ is almost π and the boundary is the function of cycle, we need to count how many points are in the cycle. the cycle function is $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$

```
## Monte Carlo Estimate
n = 100000
sum = 0
for(i in 1:n){
  x = runif(1,0,1)
  y = runif(1,0,1)
  if(x^2+y^2-x-y<=(-1/4))
    sum = sum+1
}
est.mc = (sum/n)*4
est.mc
```

```
## [1] 3.13668
```

problem 8

the pdf of the Cauchy distribution is $f(x;0,1) = \frac{1}{\pi(1+x^2)}$ and the cdf of the Cauchy distribution is $F(x;0,1) = \frac{1}{\pi}arctanx+\frac{1}{2}$ let $u = F(x;0,1)$ we get $F^{-1}(x;0,1) = \frac{1}{\pi}arctanx+\frac{1}{2}$ $arctanx = (u-\frac{1}{2})\pi$ so $x = tan[(u-\frac{1}{2})\pi] = F^{-1}(u)$ the code is

```
set.seed(1729)
cauchy.gen = function(n){
  x = c()
```

```
  u.list = runif(n)
  for(i in 1:n){
  x[i] = tan((u.list[i]-1/2)*pi)
  }
  return(x)
}
##example
cauchy.gen(100)
```

```
##   [1]   0.274378217  -0.199258616   0.233093421  -0.496872005   2.147329394
##   [6]  -0.287791252   0.035996994  -1.898970493   1.983408009  81.266615138
##  [11]   0.110932694  -0.209755621  -3.920214759   0.267539964   0.069285085
##  [16]   1.213901077  -0.089530496   1.455303503   0.474024825   1.213059365
##  [21]   0.239143196  -0.622762025  -0.780517664  -1.065626608   0.109075942
##  [26]  -1.660936445  -0.934627799  -0.038085312   0.477742589   6.865302452
##  [31]  -1.005953832   0.945801706   0.740654444   1.860665424  -0.425621270
##  [36]  -4.299108089   0.720827271  -8.012846462  -0.407348182   0.003194039
##  [41]  -0.646967992   0.107199279  -0.125834884  -0.513268418  12.779477121
##  [46]  -0.248356173   2.783173270   0.351746479   4.733677158 -13.412087231
##  [51] -25.741034125   1.279028988  -1.887456216  -0.465442521   1.810126447
##  [56]  -4.690533197  -3.414827164 -44.063191972   0.274838081  -5.169924460
##  [61]  -0.362735190  -3.896430490   2.463377563  -1.593712947  -3.301743342
##  [66]   1.015752732   0.156697158   1.966857195   0.655406587   5.741256264
##  [71]  -1.662669934  -0.452063067  -0.132147254   2.562667928   1.044352712
##  [76]   2.554855725 -10.847088837  -0.662176825  -1.573636876   0.916770958
##  [81]   0.091391663  -0.167664604  -0.475627111   0.016240008   0.952020884
##  [86]  -1.795024186   2.069468717   0.558217561  -0.005360418  -3.799073011
##  [91]   1.741354225  -1.984143446  -0.807282852  -0.777663426   0.022199366
##  [96]  -1.825721124   0.007727107   0.804202498   0.118121449  -5.593810815
```

problem 9

(1) Prove this is a proper density function

(i)$\frac{1}{\pi} > 0$ and $\frac{1}{\sqrt{1-x^2}} \geq 0$ so $f(x) \geq 0$

(ii)$\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx = \frac{1}{\pi} acrsinx|_0^1 = \frac{1}{\pi}(\frac{\pi}{2} - (-\frac{\pi}{2})) = \frac{1}{\pi}\pi = 1$ So this is a proper density

(2) use the conclusion to simulate the pi.

From the abrove, we can see that if we want to estimate π, we can get a series of point in the internal [-1,1], simulate a monte carlo simulation over that interval of the uniform interval.

```
## Monte Carlo Estimate
n = 800000
sum = 0
for(i in 1:n){
  u.gen = runif(1,-1,1)
  sum = sum + (1/(sqrt(1-(u.gen^2))))/(1/2)
}
est.mc = sum/n
est.mc
```

```
## [1] 3.140946
```