The Sturm-Liouville Problem

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Introduction

- Mathematicians Sturm and Liouville were interested in differential equations derived from the theory of heat
- Had an idea, known as the Sturm-Liouville problem which has many applications in the world of physics
- Want to explore the theory and some applications

The Sturm-Liouville Problem

The Sturm-Liouville problem is a boundary value problem for a real second order differential equation of the form,

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \tag{1}$$

on the interval $0 < x < 1$, where the boundary conditions are

$$a_1 y(0) + a_2 y'(0) = 0 \ , \ b_1 y(1) + b_2 y'(1) = 0. \tag{2}$$

Regular Sturm-Liouville equation in a closed finite interval, $[a, b]$, when the functions $p(x), r(x)$ are positive $\forall x \in [a, b]$, where $p(x), q(x)$ and $r(x)$ are continuous and bounded in the interval.

Lagrange's Identity

Let $L[y] = \lambda r(x)y$. Then we can write the Sturm-Liouville problem as

$$L[y] = -[p(x)y']' + q(x)y. \tag{3}$$

For two functions $u(x), v(x)$ on the interval $[0, 1]$, the **inner product** of $u(x)$ and $v(x)$ is defined as

$$(u, v) = \int_0^1 u(x)v(\bar{x})dx, \tag{4}$$

The **adjoint of an operator**, say Λ, is denoted by Λ^* such that it satisfies

$$(\Lambda[u], v) = (u, \Lambda^*[v]) \tag{5}$$

for all $u(x), v(x)$.
We call the operator **self-adjoint** if $\Lambda = \Lambda^*$.

Lagrange's Identity

Computing $\int_0^1 L[u]v dx$ and $\int_0^1 uL[v]dx$, we see

$$\int_0^1 L[u]v - uL[v]dx = \Big[-p(x)(u'(x)v(x) - u(x)v'(x)) \Big]_0^1. \tag{6}$$

Then if u, v satisfy the boundary conditions, we have

$$\int_0^1 L[u]v - uL[v]dx = 0. \tag{7}$$

and so **Lagrange's Identity is self-adjoint**.

Eigenvalues and Eigenfunctions

An Example:

$$y'' + \lambda y = 0, \text{ where } y(0) = 0 \text{ and } y(1) = 0. \tag{8}$$

We want to find $y(t)$. Take $y(t) = e^{\mu t}$ for some constant $\mu \in \mathbb{C}$. Then $y''(t) = \mu^2 e^{\mu t}$ and substituting into the problem yields,

$$e^{\mu t}(\mu^2 + \lambda) = 0. \tag{9}$$

Then $\mu = \pm i\sqrt{\lambda}$ and so $y = a_1 e^{i\sqrt{\lambda}t} + a_2 e^{-i\sqrt{\lambda}t}$ for constants $a_1, a_2 \in \mathbb{C}$. Since $y(0) = 0$, we have that $a_1 = -a_2$.

Eigenvalues and Eigenfunctions

Using the boundary condition $y(1) = 0$,

$$a_1 e^{i\sqrt{\lambda}} - a_1 e^{-i\sqrt{\lambda}} = 0. \tag{10}$$

Then

$$a_1(2i \sin\sqrt{\lambda}) = 0. \tag{11}$$

a_1 is nonzero so $\sqrt{\lambda} = n\pi$, then $\lambda_n = n^2\pi^2$, and we have n solutions of the form $y_n(t) = 2ia_1 \sin n\pi t$.
Each $y_n(t)$ is called an **eigenfunction** corresponding to each **eigenvalue**, λ_n.

Eigenvalues and Eigenfunctions

Theorem

(Spectral theorem) Any regular Sturm-Liouville problem has an infinite sequence of real eigenvalues, and can be ordered according to increasing magnitude such that $\lambda_0 < \lambda_1 < \ldots$ *with* $\lim_{n\to\infty} \lambda_n = \infty$*. Furthermore, the eigenfunctions corresponding to* λ_n *have exactly n zeros in the interval* $a < x < b$*.*

Theorem

All eigenvalues and eigenfunctions of the Sturm-Liouville problem are real.

Eigenvalues and Eigenfunctions

Theorem

If ϕ_1, ϕ_2 are two eigenfunctions of the Sturm-Liouville problem with corresponding eigenvalues λ_1, λ_2 respectively where $\lambda_1 \neq \lambda_2$ then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)dx = 0 \tag{12}$$

where $r(x)$ is the weight function.

This theorem illustrates the notion of **orthogonality** of eigenfunctions with respect to $r(x)$.

Eigenvalues and Eigenfunctions

Definition

Given an eigenfunction ϕ_n for some $n \in \mathbb{N}$, the eigenfunction is said to be **normalised** if it satisfies

$$\int_0^1 r(x)\phi_n^2(x)dx = 1, \tag{13}$$

This is known as the **normalisation condition**.

Fourier Sine Expansion

Theorem

The function $f(x)$ can be expanded into the series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{14}$$

where $\phi_1, \phi_2, ..., \phi_n$ are the normalised eigenfunctions of the Sturm-Liouville problem. The series converges to $\frac{f(x+)+f(x-)}{2}$ at every point along $0 < x < 1$.

The Nonhomogeneous Boundary Value Problem

When does the nonhomogeneous boundary value problem have a solution? Consider a nonhomogeneous boundary value problem

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x) \tag{15}$$

$$a_1 y(0) + a_2 y'(0) = 0 \ , \ b_1 y(1) + b_2 y'(1) = 0. \tag{16}$$

Now assume an unknown solution $y = \phi(x)$ of the nonhomogeneous boundary value problem can be expressed in a series

$$\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{17}$$

where $c_n = \int_0^1 r(x)\phi(x)\phi_n(x)dx$ for $n = 1, 2, \ldots.$

The Nonhomogeneous Boundary Value Problem

1. For $\mu = \lambda_n$ and $d_n \neq 0$, there is no solution.
2. For $\mu = \lambda_n$ and $d_n = 0$, c_n has an arbitrary multiple of the eigenfunction ϕ_n and the boundary value problem has a solution but it is not unique.
3. For $\mu \neq \lambda_n$, there is a solution

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{d_n}{\lambda_n - \mu} \phi_n(x). \tag{18}$$

Note

$$d_n = \int_0^1 r(x) \frac{f(x)}{r(x)} \phi_n(x) dx = \int_0^1 f(x) \phi_n(x) dx, \quad n = 1, 2, \ldots. \tag{19}$$

Example of the Nonhomogeneous Boundary Value Problem

Consider

$$y'' + 4\pi^2 y = x, \quad y(0) = 0, y(1) = 0. \tag{20}$$

Since $\mu = 4\pi^2 = \lambda_2$ and $d_2 = \frac{1}{\pi}$, there is no solution.
Now consider

$$y'' + 4\pi^2 y = 1, \quad y(0) = 0, y(1) = 0. \tag{21}$$

In this case $d_2 = 0$. Thus, there is a solution.

Mean Convergence

Theorem

The function $f(x)$ can be expanded into the series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{22}$$

where $\phi_1, \phi_2, ..., \phi_n$ are the normalised eigenfunctions of the Sturm-Liouville problem. The series converges to $\frac{f(x+)+f(x-)}{2}$ at every point along $0 < x < 1$.

- Pointwise convergence
- Mean convergence : other type of convergence, which is useful for series of orthogonal functions (eigenfunctions).

Mean Convergence

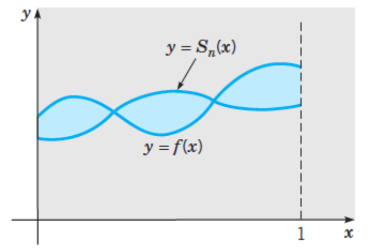


Figure: Approximation of $f(x)$ by $S_n(x)$.

Mean square error R_n of the approximation to S_n to f,

$$R_n(a_1, ..., a_n) = \int_0^1 r(x)[f(x) - S_n(x)]^2 dx, \tag{23}$$

where

$$a_i = \int_0^1 r(x) f(x) \phi_i(x) dx, \quad i = 1, ..., n. \tag{24}$$

Mean Convergence

Definition

We say that the set $\phi_1, ..., \phi_n, ...$ is **complete** with respect to mean square convergence for a set of functions if the series $f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x)$ converges in the mean for each function f.

Definition

A function is **square integrable** on the interval $0 \leq x \leq 1$ if both f and f^2 are integrable on that interval.

Theorem

The eigenfunctions ϕ_i of the Sturm-Liouville problem are complete with respect to mean convergence for the square integrable set of functions on $0 \leq x \leq 1$.

Singular Sturm-Liouville Problem

Our Sturm-Liouville problem,

$$-[p(x)y']' + q(x)y = \lambda r(x)y, \tag{25}$$

is **singular** when p is differentiable, q and r are continuous, and $p(x), r(x) > 0$ at all points in the open interval $0 < x < 1$, but at least one of these functions does not satisfy these conditions at one or both boundary points.

If this occurs, this boundary point is referred to as a **singular point**.

Example - Legendre's Equation

Legendre's equation is of the form

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \quad -1 < x < 1, \tag{26}$$

but can be written as a Sturm-Liouville problem in the form

$$-[(1 - x^2)y']' = \lambda y, \quad -1 < x < 1. \tag{27}$$

Here we have $p(x) = 1 - x^2$, $q(x) = 0$ and $r(x) = 1$.
The eigenvalues of this problem are given by $\lambda = n(n+1)$.
This problem is clearly singular as $p(1) = 0$.

Solving Bessel's Equation - Method of Frobenius

Bessel's Equation of order zero:

$$x^2y'' + xy' + x^2y = 0, \tag{28}$$

To solve, we use the substitution

$$y = \phi(r, x) = a_0x^r + \sum_{n=1}^{\infty} a_nx^{r+n}. \tag{29}$$

This gives us our first solution,

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \quad x > 0. \tag{30}$$

The function in brackets gives us J_0, the Bessel function of the first kind of order zero.

Solving Bessel's Equation

In order to find our second solution, a similar derivation is used. The second solution is

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \quad x > 0, \tag{31}$$

where $H_m = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{m}$.
We want to find the Bessel function of the second kind of order zero, Y_0. It is defined as

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2)J_0(x)], \tag{32}$$

where γ is the Euler-Máscheroni constant. Substituting in,

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right], \quad x > 0. \tag{33}$$

Solving Bessel's Equation

The general solution of Bessel's function of order zero is given by

$$y = c_1 J_0(x) + c_2 Y_0(x), \tag{34}$$

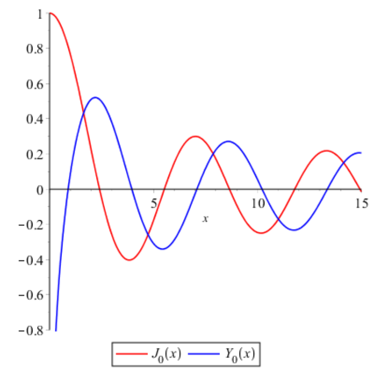


Figure: Graph showing Bessel's functions of order zero of the first and second kind.

Bessel's Equation as a Sturm-Liouville Problem

The Sturm-Liouville problem

$$xy'' + y' + \lambda xy = 0, \quad 0 < x < 1, \quad \lambda > 0, \tag{35}$$

can be reduced to Bessel's equation of order zero by substituting in the new independent variable $t = \sqrt{\lambda}x$, and then by multiplying through by $\frac{t}{\sqrt{\lambda}}$:

$$t^2y'' + ty' + t^2y = 0. \tag{36}$$

Therefore we know the solution to our original problem is

$$y = c_1 J_0(\sqrt{\lambda}x) + c_2 Y_0(\sqrt{\lambda}x). \tag{37}$$

Bessel's Equation as a Sturm-Liouville Problem

The boundary conditions:

$$y(0) = 0, \tag{38}$$
$$y(1) = 0. \tag{39}$$

can only be satisfied for the trivial solution, $c_1 = c_2 = 0$. We must consider the modified boundary condition:

$$y,\ y' \text{ bounded as } x \to 0. \tag{40}$$

This is satisfied for $c_2 = 0$ and our second boundary condition gives us:

$$J_0(\sqrt{\lambda}) = 0. \tag{41}$$

The eigenfunctions are given by:

$$\phi_n(x) = J_0(\sqrt{\lambda_n} x). \tag{42}$$

Bessel's Equation as a Sturm-Liouville Problem

The eigenfunctions satisfy the orthogonality relation

$$\int_0^1 x\phi_m(x)\phi_n(x)\ dx = 0, \quad m \neq n \tag{43}$$

with respect to the weight function $r(x) = x$. We assume that

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_n}x). \tag{44}$$

Bessel's Equation as a Sturm-Liouville Problem

We multiply this by $xJ_0(\sqrt{\lambda_m}x)$ and integrate term-by-term from $x = 0$ to $x = 1$,

$$\int_0^1 xf(x)J_0(\sqrt{\lambda_m}x)\ dx = \sum_{n=1}^{\infty} c_n \int_0^1 xJ_0(\sqrt{\lambda_m}x)J_0(\sqrt{\lambda_n}x)\ dx. \tag{45}$$

Due to our orthogonality condition, the right hand side collapses into a single term. Therefore

$$c_m = \frac{\displaystyle\int_0^1 xf(x)J_0(\sqrt{\lambda_m}x)\ dx}{\displaystyle\int_0^1 xJ_0^2(\sqrt{\lambda_m}x)\ dx}, \tag{46}$$

which determines the coefficients in the series.

Vibrations of a Circular Elastic Membrane

We have equation

$$a^2(u_{rr} + \frac{1}{r}u_r) = u_{tt}, \quad 0 < r < 1, \quad t > 0. \tag{47}$$

with boundary condition

$$u(1, t) = 0, \quad t \geq 0, \tag{48}$$

and initial conditions

$$u(r, 0) = f(r), \quad 0 \leq r \leq 1, \tag{49}$$

$$u_t(r, 0) = 0, \qquad 0 \leq r \leq 1, \tag{50}$$

We have that $u(r, t)$ is bounded for $0 \leq r \leq 1$.

Vibrations of a Circular Elastic Membrane

We then assume that $u(r,t) = R(r)T(t)$ and substitute in to our equation:

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{1}{a^2}\frac{T''}{T} = -\lambda^2. \tag{51}$$

This gives us

$$r^2R'' + rR' + \lambda^2 r^2 R = 0, \tag{52}$$

$$T'' + \lambda^2 a^2 T = 0. \tag{53}$$

We can easily solve our T equation:

$$T(t) = k_1 \sin\lambda at + k_2 \cos\lambda at. \tag{54}$$

Our R equation can be reduced to Bessel's equation of order zero by a change of variables. Therefore we know:

$$R = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r). \tag{55}$$

Vibrations of a Circular Elastic Membrane

Our boundary condition, $u(1,t) = 0$, leaves us with the equation

$$J_0(\lambda) = 0. \tag{56}$$

Eigenfunctions are $J_0(\lambda_n r)$, and therefore can be used as a basis for a series expansion for our given function f.
The fundamental solution to this problem is given by the following:

$$u_n(r,t) = J_0(\lambda_n r)\sin\lambda_n at, \quad n = 1, 2, \ldots \tag{57}$$
$$v_n(r,t) = J_0(\lambda_n r)\cos\lambda_n at, \quad n = 1, 2, \ldots \tag{58}$$

Vibrations of a Circular Elastic Membrane

We assume our function $u(r,t)$ can be expressed as an infinite linear combination of the solutions:

$$\begin{aligned} u(r,t) &= \sum_{n=1}^{\infty} [k_n u_n(r,t) + c_n v_n(r,t)] \\ &= \sum_{n=1}^{\infty} [k_n J_0(\lambda_n r) \sin \lambda_n a t + c_n J_0(\lambda_n r) \cos \lambda_n a t]. \end{aligned} \tag{59}$$

Our initial conditions give us:

$$u(r,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r), \tag{60}$$

$$u_t(r,0) = \sum_{n=1}^{\infty} \lambda_n a k_n J_0(\lambda_n r) = 0. \tag{61}$$

Vibrations of a Circular Elastic Membrane

Using our equation from earlier we find

$$k_n = 0, \quad c_n = \frac{\displaystyle\int_0^1 rf(r)J_0(\sqrt{\lambda_n}r)\ dr}{\displaystyle\int_0^1 r[J_0(\sqrt{\lambda_n}r)]^2\ dr}; \quad n = 1, 2, ... \tag{62}$$

Therefore the solution to the differential equation is:

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\cos\lambda_n at, \tag{63}$$

where the coefficients c_n are as defined as above.

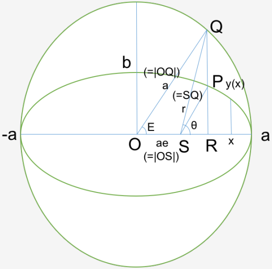
Limitations

However useful, this method is nevertheless quite restricted:

1. The problem must be linear,
2. Can be difficult to solve resulting ODEs,
3. Must be using a suitable coordinate system.

Kepler's Equation

We can find an area of a sector SPa using the ellipse function.



Kepler's equation is

$$M = E - e \sin E. \tag{64}$$

Kepler's Equation

- Joseph Louis Lagrange also contributed to Kepler's equation by using repeated differentiation.
- Bessel used integration and described a solution of Kepler's equation in the form

$$E = M + \sum_{n=1}^{\infty} b_n(e) \sin nM, \qquad (65)$$

  as a Fourier sine series.
- If $E = g(M)$ is the solution of Kepler's equation, then g has $M = 0$ and $M = \pi$ as its fixed points.

Kepler's Equation

It also can be expressed in Bessel function notation, $J_n(x)$

$$J_n(x) = \frac{1}{\pi}\int_0^\infty \cos(nE - x\sin E)dE \tag{66}$$

and Bessel's solution of Kepler's equation is

$$E = M + \sum_{n=1}^{\infty}\left(\frac{2}{n}J_n(ne)\right)\sin nM. \tag{67}$$

The Korteweg-de Vries Equation

Solitons are solitary waves with a single crest and the following properties

1. Stable; Speed and form does not change as it propagates
2. Does not change in speed and amplitude after it collides with another soliton
3. The greater the amplitude, the faster and skinnier the shape

Korteweg and de Vries formulated the $1+1$ dimensional soliton wave with the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{68}$$

where $u(x,t)$ is the vertical displacement(analogous to the potential in general wave equations) of the soliton.

Sturm-Liouville Equation's Fundamental Solution

For the sake of convenience, denote $q(x)$ as $u(x)$ for the potential and $\lambda = k^2$.

Now, we know that $y_{xx} + [k^2 - u(x)]y(x) = 0 \leftarrow$ our Sturm-Liouville/wave equation

For some random $u(x)$(unknown), how can we find solutions f_1, f_2?

Hint: $u(x)$ is the potential or "height" of a wave...if the wave propagates "forever," shouldn't it reach 0? Let's assume $u(x) \to 0$ as $|x| \to \infty$ then.

$\therefore y_{xx} + [k^2 - u(x)]y(x) = 0 \to y_{xx} + k^2 y(x) = 0$
$\Rightarrow$Solution can be expressed as a linear combination of $e^{\pm ikx}$ (at $x \to \pm\infty$).

Sturm-Liouville Equation's Fundamental Solution

We arbitrarily choose,

$$f_1(x,k) \to e^{ikx} \text{ as } x \to \infty \quad \text{and } f_2(x,k) = e^{-ikx} \text{ as } x \to -\infty$$

We can try to go further; use what we have learned for nonhonmogeneous linear ODEs.

$$y_{xx} + [k^2 - u(x)]y(x) = 0 \to y_{xx} + k^2 y(x) = u(x)$$

Variation of parameters can be used on our ansatz $y(x) = A(x)e^{ikx} + B(x)e^{-ikx}$ and we get,

$$A(x) = \frac{1}{2ik}\int_0^x u(x')y(x')e^{-ikx'}dx' + C_1$$

$$B(x) = -\frac{1}{2ik}\int_0^x u(x')y(x')e^{ikx'}dx' + C_2$$

Completing the Step; Finding C_1, C_2

Consider $y(x) = f_1(x, k)$ at $x \to \infty$. We get

1. exponents
2. integrals

$\to$ we can compare coefficients of e^{ikx}, e^{-ikx} and find

$$C_1 = 1 - \frac{1}{2ik} \int_0^\infty u(x') f_1(x', k) e^{-ikx'} dx'$$

$$C_2 = \frac{1}{2ik} \int_0^\infty u(x') f_1(x', k) e^{ikx'} dx'$$

As we set $y = f_1(x, k)$, substituting C_1, C_2 gives us $f_1(x, k)$. The exact same method applies for the case $y = f_2(x, k)$ and ...

Here are our f_1, f_2s

Using the identity $e^{ik(x-x')} - e^{-ik(x-x')} = 2i \sin k(x-x')$ we can write

$$f_1(x,k) = e^{ikx} - \frac{1}{k}\int_x^{\infty} \sin k(x-x')u(x')f_1(x',k)dx'$$

$$f_2(x,k) = e^{-ikx} + \frac{1}{k}\int_{-\infty}^{x} \sin k(x-x')u(x')f_2(x',k)dx'$$

Some notes about this; The integral equation of the form above are called **Volterra integral equation of the second kind**. Some facts...

1. The (improper) integrals in the equation will converge to a finite value at $\mathrm{Im}k > 0$
2. f_1, f_2 are analytic in the upper half of the complex plane where $\mathrm{Im}k > 0$.

Why Not Choose Some Other Solution?

We've simply chosen $f_1(x,k), f_2(x,k)$ for our discussion. But we can freely choose arbitrary solutions to express $y(x)$.

Alternatively we can choose functions such that

$$f_1(x,-k) \to e^{-ikx} \text{ as } x \to \infty \quad \text{and } f_2(x,-k) = e^{ikx} \text{ as } x \to -\infty$$

Why expand our choice of solutions? Because a useful consequence follows

Linear Independency

The Wronskian of functions $p(x), q(x)$ is

$$W(p, q) = pq_x - p_x q$$

We compute the Wronskians $\pm\infty$ for $W(f_1(x, k), f_1(x, -k))$ and $W(f_2(x, k), f_2(x, -k))$. $\rightarrow$ gives $-2ik, 2ik$ respectively

In fact, this is true for any value of x, thus

$$f_2(x, k) = c_{11}(k) f_1(x, k) + c_{12}(k) f_1(x, -k)$$

$$f_1(x, k) = c_{21}(k) f_2(x, -k) + c_{22}(k) f_2(x, k)$$

Some Physics...

The following are coefficients of reflection from right/left and the transmission coefficient

Definition

$$R_R(k) = \frac{c_{11}(k)}{c_{12}(k)},\ T_R(k) = \frac{1}{c_{12}(k)},\ R_L(k) = \frac{c_{22}(k)}{c_{21}(k)},\ T_L(k) = \frac{1}{c_{21}(k)}$$

Where does all of this come from? Observe the behaviour of $f_1(x,k), f_2(x,k)$ at $\pm\infty$

Waves to and from Infinity

$$f_1(x,k) = \begin{cases} c_{21}(k)e^{ikx} + c_{22}e^{-ikx} & (x \to -\infty), \\ e^{ikx} & (x \to \infty). \end{cases} \tag{69}$$

$$f_2(x,k) = \begin{cases} e^{-ikx} & (x \to -\infty), \\ c_{11}(k)e^{ikx} + c_{12}e^{-ikx} & (x \to \infty), \end{cases} \tag{70}$$

Scattering is a physical phenomenon of one-dimensional waves; In general, when a wave is incident upon an object there are waves that

1. reflect
2. transmit

Given information on the scattered waves, the method to find the potential $u(x)$ is called **inverse scattering**

An Image of a Wave Incident from Infinity

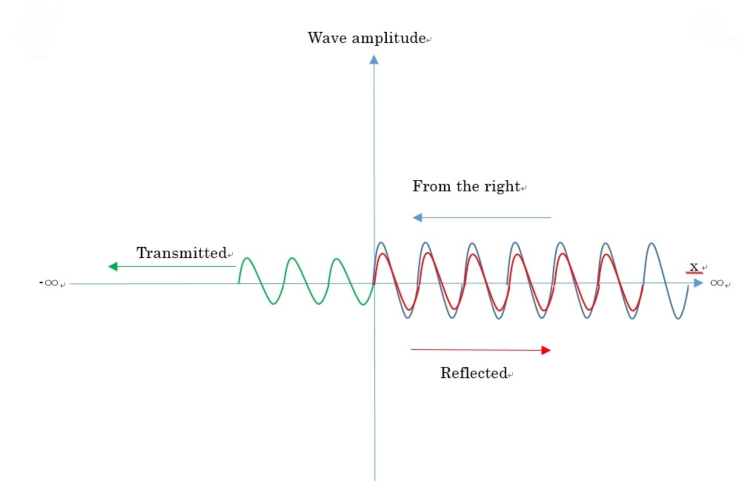


Figure: Scattering of a wave.

$A(x, x')$ to describe Scattering

Recall that

$$f_1(x,k) = e^{ikx} - \frac{1}{k}\int_x^\infty \sin k(x-x')u(x')f_1(x',k)dx'$$

$\rightarrow$ if $u(x) = 0$ then the integral will also be zero. This implies that the integral describes scattering of the waves.

We denote the function $A_R(x, x')$ to describe scattering of a wave from the right, $A_L(x, x')$ for a wave from the left. Then rewrite

$$f_1(x,k) = e^{ikx} + \int_x^\infty e^{ikx'}A_R(x,x')dx'$$

$$f_2(x,k) = e^{-ikx} + \int_{-\infty}^x e^{-ikx}A_L(x,x')dx'$$

Overall Flow

1. Consider the Sturm-Liouville equation as a one-dimensional wave equation and find solutions $f_1(x, \pm k), f_2(x, \pm k)$
2. The linear combination of solutions tell us about the behavior of the wave. The ratio of coefficients = ratio of amplitude gives reflection/transmission coefficients.
3. Observing the behavior of fundamental solutions at infinity, introduce functions A(x,x')to describe scattering.
4. Determine the time dependencies of coefficients using GLM(Gelfand-Levitan-Marchenko) equation.
5. Using the specified time dependent forms of the coefficients, solve the GLM to establish $u(x) = -2\frac{d}{dx}A_R(x, x') = 2\frac{d}{dx}A_L(x, x')$.
6. Use scattering information to obtain $u(x)$

$N = 1$ soliton

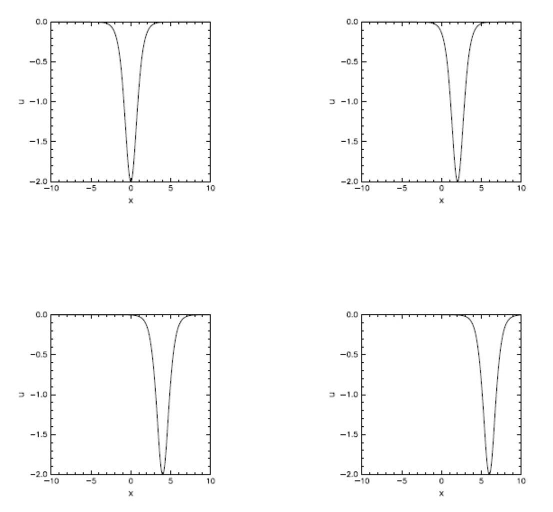


Figure: An $N = 1$ soliton example

$N = 2$ soliton

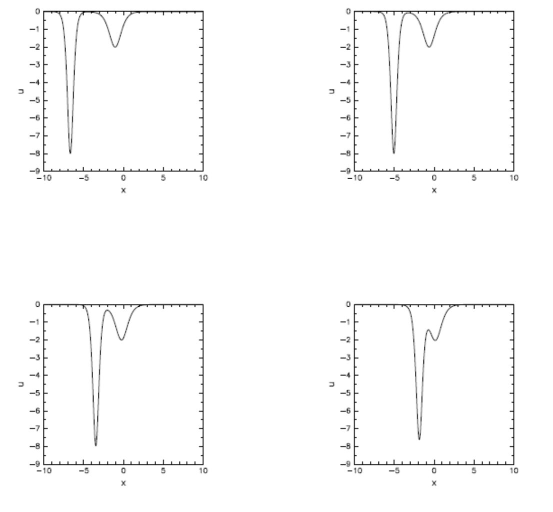


Figure: An $N = 2$ soliton example

Conclusion

We started with some basic ideas related to the Sturm-Liouville problem and considered

1. Homogeneous
2. Nonhomogeneous
3. Singular
4. Regular

cases and described methods to find solutions.

We further expanded on these facts to apply them to real life problems, demonstrating the significance of the Sturm-Liouville problem.