1.

(a)

Since X_i are independent and identical distribution, Y_i are independent and identical distribution, assume α, β are independent, then the joint posterior can be found by

$$\begin{aligned}
&f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}} \\
=&A \cdot f_{\mathbf{X}|\{\mathbf{Y},\alpha,\beta\}} \cdot f_{\mathbf{Y}|\{\alpha,\beta\}} \cdot f_{\alpha,\beta} \\
=&\begin{cases} A \cdot \prod_{i=1}^{n} \frac{y_i^{x_i} e^{-y_i}}{x_i!} \cdot \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} y_i^{\alpha-1} e^{-\beta y_i} \cdot a e^{-a\alpha} \cdot \frac{c^b}{\Gamma(b)} \beta^{b-1} e^{-c\beta}, & y_i > 0, \alpha > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases} \\
=&\begin{cases} A \cdot \prod_{i=1}^{n} y_i^{x_i} e^{-y_i} \cdot \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} y_i^{\alpha-1} e^{-\beta y_i} \cdot e^{-a\alpha} \cdot \beta^{b-1} e^{-c\beta}, & y_i > 0, \alpha > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases}
\end{aligned}$$

where A is a constant.

(b)

X_i's independence and Y_i's independence has been given, and I assume α, β are independent.

(c)

The conditional posterior is

$$\begin{aligned}
&f_{\{\mathbf{Y},\beta\}|\{\mathbf{X},\alpha\}} = \frac{f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}}}{f_{\{\alpha|\mathbf{X}\}}} \\
=&B \cdot f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}} \\
=&\begin{cases} B \cdot \prod_{i=1}^{n} y_i^{x_i} e^{-y_i} \cdot \prod_{i=1}^{n} \beta^\alpha y_i^{\alpha-1} e^{-\beta y_i} \cdot \beta^{b-1} e^{-c\beta}, & y_i > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases} \\
=&\begin{cases} B \cdot \prod_{i=1}^{n} y_i^{x_i+\alpha-1} e^{-(1+\beta)y_i} \cdot \beta^{n\alpha+b-1} \cdot e^{-c\beta}, & y_i > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases}
\end{aligned}$$

Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\{\mathbf{Y},\beta\}|\{\mathbf{X},\alpha\}} d\mathbf{y} d\beta = 1$, and

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \prod_{i=1}^{n} y_i^{x_i+\alpha-1} e^{-(1+\beta)y_i} \cdot \beta^{n\alpha+b-1} \cdot e^{-c\beta} d\mathbf{y} d\beta \\
=&\int_0^\infty \beta^{n\alpha+b-1} \cdot e^{-c\beta} \cdot \prod_{i=1}^{n} \Gamma(x_i+\alpha) (\frac{1}{\beta+1})^{x_i+\alpha} d\beta
\end{aligned}$$

we obtain that

$$f_{\{\mathbf{Y},\beta\}|\{\mathbf{X},\alpha\}} = \begin{cases} \dfrac{\prod_{i=1}^{n} y_i^{x_i+\alpha-1} e^{-(1+\beta)y_i} \cdot \beta^{n\alpha+b-1} \cdot e^{-c\beta}}{\prod_{i=1}^{n} \Gamma(x_i+\alpha) \cdot \int_0^\infty \beta^{n\alpha+b-1} \cdot \prod_{i=1}^{n} (\frac{1}{\beta+1})^{x_i+\alpha} \cdot e^{-c\beta} d\beta}, & y_i > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases}$$

(d)

First, consider the conditional posterior,

$$\begin{aligned} f_{\{\mathbf{Y}\}|\{\mathbf{X},\alpha,\beta\}} &= \frac{f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}}}{f_{\{\alpha,\beta|\mathbf{X}\}}} \\ &= C \cdot f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}} \\ &= \begin{cases} C \cdot \prod_{i=1}^{n} y_i^{x_i+\alpha-1} e^{-(1+\beta)y_i} & , y_i > 0 \\ 0 & \text{,otherwise} \end{cases} \\ &= \begin{cases} \prod_{i=1}^{n} \dfrac{(\beta+1)^{x_i+\alpha}}{\Gamma(x_i+\alpha)} y_i^{x_i+\alpha-1} e^{-(\beta+1)y_i} & , y_i > 0 \\ 0 & \text{,otherwise} \end{cases} \end{aligned}$$

Then the marginal posterior is

$$\begin{aligned} f_{\{\alpha,\beta\}|\mathbf{X}} &= \frac{f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}}}{f_{\mathbf{Y}|\{\mathbf{X},\alpha,\beta\}}} \\ &= D \cdot \begin{cases} \prod_{i=1}^{n} \dfrac{\beta^\alpha \Gamma(x_i+\alpha)}{(\beta+1)^{x_i+\alpha}\Gamma(\alpha)} \cdot e^{-a\alpha} \cdot \beta^{b-1} e^{-c\beta} & , \alpha > 0, \beta > 0 \\ 0 & \text{,otherwise} \end{cases} \end{aligned}$$

where D is a constant.

2.

(a)

Denote $Y = 0$ and $Y = 1$ as the case of a grouper is a tiger grouper and a greasy grouper respectively, and

$$f_Y(y) = \begin{cases} \frac{2}{3}, & y = 0 \\ \frac{1}{3}, & y = 1 \end{cases}$$

Using $\mathbf{X}$ to denote the length and weight of a grouper. Since $f_{\mathbf{X}|Y} \sim$Normal(μ_y, Σ_y), the posterior is

$$
\begin{aligned}
f_{Y|\mathbf{X}} =& \frac{f_{\mathbf{X}|Y}\cdot f_Y}{f_{\mathbf{X}}} = A\cdot f_{\mathbf{X}|Y}\cdot f_Y \\
=& A\cdot\begin{cases}\frac{2}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_t}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})\Big), & y=0\\ \frac{1}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_g}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big), & y=1\end{cases}
\end{aligned}
$$

Denote $z=\sigma(x)$ as the optimal decision rule, and the loss function can be rewritten as

$$
C(y,z)=\begin{cases}0.1, & y=1, z=0\\ 0.9, & y=0,z=1\\ 0, & y=z\end{cases}=\begin{cases}0.1(1-z), & y=1\\ 0.9z, & y=0\end{cases}
$$

then the optimal decision rule is given by

$$
\begin{aligned}
z =& \sigma(x) = \arg\min_{z\in\{0,1\}}\sum_{y\in\{0,1\}} C(y,z)f_{Y|\mathbf{X}}\\
=& \arg\min_{z\in\{0,1\}}\Big\{0.9z\cdot\frac{2}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_t}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})\Big)\\
&+0.1(1-z)\cdot\frac{1}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_g}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big)\Big\}
\end{aligned}
$$

where $\mathbf{x}$ denote the pair of observed length and weight.

(b)

We can rewrite the decision rule as the following:
If

$$
\begin{aligned}
&0.9\cdot\frac{2}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_t}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})\Big)\\
<&0.1\cdot\frac{1}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_g}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big)
\end{aligned}
$$

then $z=\delta(x)=1$, else if

$$
\begin{aligned}
&0.9\cdot\frac{2}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_t}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})\Big)\\
\geqslant&0.1\cdot\frac{1}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_g}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big)
\end{aligned}
$$

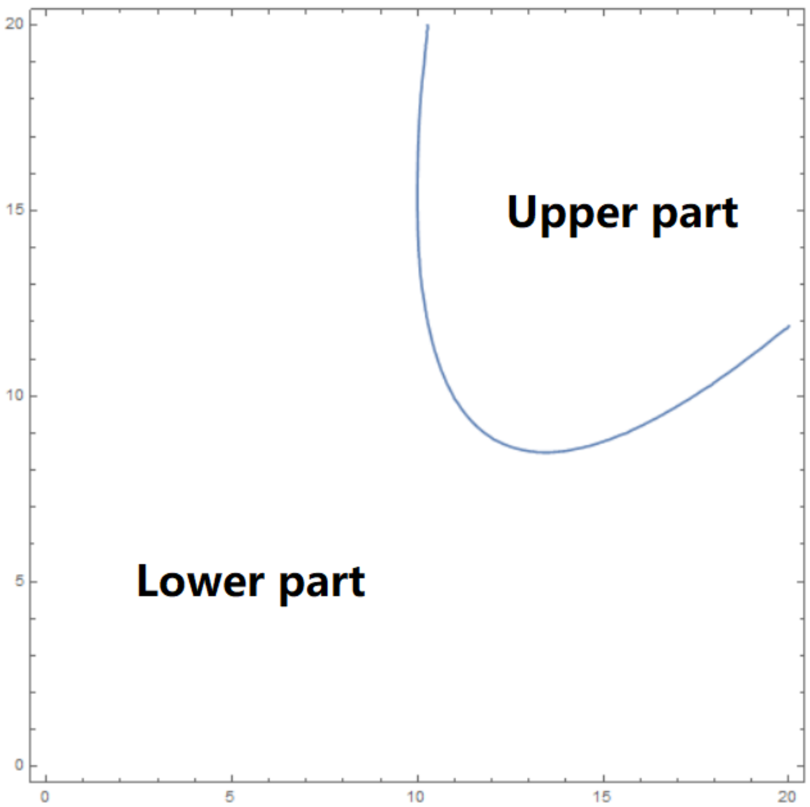
then $z=\delta(x)=0$. So we can obtain the boundary

$$
\begin{aligned}
&0.9\cdot\frac{2}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_t}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})\Big)\\
=&0.1\cdot\frac{1}{3}\cdot\frac{1}{2\pi\sqrt{\det\Sigma_g}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big)\\
\Rightarrow&\frac{\sqrt{\det\Sigma_t}}{18\sqrt{\det\Sigma_g}}\exp\Big(\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{t}})^T\Sigma_t^{-1}(\mathbf{x}-\mu_{\mathbf{t}})-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{g}})^T\Sigma_g^{-1}(\mathbf{x}-\mu_{\mathbf{g}})\Big)=1\\
\Rightarrow&\frac{\sqrt{23}}{18\sqrt{3}}\exp\Big(-\frac{1}{69}(20x_1^2-4(5x_2+92)x_1+5x_2^2+46x_2+2576)\Big)=1
\end{aligned}
$$

and plot it in Mathematica

```
In[1]:= ContourPlot[1/18*Sqrt[23/3]*Exp[-1/69 (2576 + 20 a^2 + 46 b + 5 b^2 - 4 a (92 + 5 b))] == 10,
          绘制等高线           平方根         指数形式
  {a, 0, 20}, {b, 0, 20}, ImageSize -> {500, 500}]
                          图像尺寸
```

Out[1]=



(c)

According to the given data, the mean weight and length of greasy groupers are larger than those of tiger groupers. So we can make decision that

1. If observed data point falls in The Upper part, we judge it as a greasy grouper
2. If observed data point falls in The Lower part, we judge it as a tiger grouper