1.

(a)

The upper bound of $q_Y(y) = A \cdot f_{Y|X=0.5}(y|0.5) = \exp\left(-\dfrac{(0.5-y)^2}{2}\right) \times \dfrac{1}{1+y^2}$ is given by

$$q_Y(y) \leqslant \exp\left(-\frac{(0.5-y)^2}{2}\right) = \sqrt{2\pi} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(0.5-y)^2}{2}\right)}_{\text{Normal(0.5,1)}}$$

then we can use rejection sampling. We will accept some $y \in Y \sim \text{Normal}(0.5, 1)$, if some $v \in V \sim \text{Uniform}(0, 1)$ satisfies that

$$v \leqslant \frac{q_Y(y)}{M \cdot g_Y(y)} = \frac{1}{1+y^2}$$

(b)

```julia
using Distributions
using Random

n = [50 250 750 1500 3000];
for i in n
    y_accept = Array{Float64, 1}(undef, i);
    j = 0;
    while j < i
        v = rand();
        y = rand(Normal(0.5, 1), 1);
        if v <= 1/(1 + y^2)
            j = j + 1;
            y_accept[j] = y;
        end
    end
    println("For n = " + i + ", mean is " + mean(y_accept));
end
```

```
For n = 50, mean is 0.24664324533864446
For n = 250, mean is 0.3002767543593847
For n = 750, mean is 0.2126923955648288
For n = 1500, mean is 0.26926520623385103
For n = 3000, mean is 0.26226569197702143
```

And we can complete the table

Grid size n	50	250	750	1500	3000
$\mathbb{E}[Y\|X = 0.5]$	0.24664	0.30028	0.21269	0.26927	0.26227

(c)

Choose proposal distribution as

$$g_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\Big(- \frac{(0.5 - y)^2}{2}\Big)$$

then the mean can be estimated by

$$\mathbb{E}[Y|X = 0.5] = \frac{\sum\limits_{i=1}^{n} y_i w_i}{\sum\limits_{i=1}^{n} w_i}$$

where $w_i = \dfrac{q_Y(y_i)}{g_Y(y_i)} = \dfrac{\sqrt{2\pi}}{1 + y_i^2} \in (0, \sqrt{2\pi}]$ which does not vary too much.

(d)

```julia
using Distributions
using Random

n = [50, 250, 750, 1500, 3000];
for i in n
    y = rand(Normal(0.5, 1), i);
    w = (2 * pi)^0.5./(1 .+ y.*y);
    println("For n = ", i, ", mean is ", sum(y .* w)/sum(w));
end
```

```
For n = 50, mean is 0.26921759842265336
For n = 250, mean is 0.2809231316011938
For n = 750, mean is 0.3399450184396458
For n = 1500, mean is 0.25364632578667273
For n = 3000, mean is 0.2752306045705318
```

And we can complete the table

Grid size n	50	250	750	1500	3000
$\mathbb{E}[Y\|X = 0.5]$	0.26922	0.28092	0.33995	0.25365	0.27523

2.

To use Gibbs sampling, we need to do sampling from

$$f_{Y_1|Y_2} \qquad \text{and} \qquad f_{Y_2|Y_1}$$

Given the joint distribution, f_{Y_1,Y_2}, we know that

$$f_{Y_1|Y_2} = \frac{f_{Y_1,Y_2}}{f_{Y_2}} \propto f_{Y_1,Y_2} \qquad \text{and} \qquad f_{Y_2|Y_1} = \frac{f_{Y_2,Y_1}}{f_{Y_1}} \propto f_{Y_1,Y_2}$$

so we can apply direct or indirect sampling. Since we can only do uniform sampling, while the two target distributions $f_{Y_1|Y_2}$ and $f_{Y_2|Y_1}$ are over $\mathbb{R}$, we need do random variable transformation first. We use

$$t = \arctan y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \Rightarrow \quad y = \tan t$$

then

$$f_{T_1|T_2=t_2}(t_1|t_2) = f_{Y_1|Y_2=\tan t_2}(\tan t_1|\tan t_2) \cdot \frac{1}{\cos^2 t_1} \propto f_{Y_1,Y_2}(\tan t_1, \tan t_2) \cdot \frac{1}{\cos^2 t_1} > 0$$

$$f_{T_2|T_1=t_1}(t_2|t_1) = f_{Y_2|Y_1=\tan t_1}(\tan t_2|\tan t_1) \cdot \frac{1}{\cos^2 t_2} \propto f_{Y_1,Y_2}(\tan t_2, \tan t_1) \cdot \frac{1}{\cos^2 t_2} > 0$$

and we will do Gibbs sampling for t_1 and t_2 first, then use inverse transformation to get random sample for y_1, y_2. Choose two initial values $t_1^{(0)} = t_2^{(0)} = 0$, for the k^{th} iteration, we draw $t_1^{(k)}$ from $f_{T_1|T_2=t_2^{(k-1)}}$ through rejection sampling by finding some $M \in (0, \infty)$ and proposal distribution

$$g_T(t) \sim \text{Uniform}(-\frac{\pi}{2}, \frac{\pi}{2})$$

This M exists since

$$\begin{aligned}
& f_{Y_1,Y_2}(\tan t_1, \tan t_2) \cdot \frac{1}{\cos^2 t_1} \\
=& A \cdot \exp\Big(-\frac{1}{2 \cdot 0.91}(\tan^2 t_1 - 0.6 \tan t_1 \tan t_2 + \tan^2 t_2)\Big) \cdot (1 + \tan^2 t_1) \\
<& A \cdot \exp\Big(-\frac{1}{2 \cdot 0.91}(\tan t_1 - 0.3 \tan t_2)^2\Big) \cdot (1 + \tan^2 t_1) \to 0 \qquad \text{as } t_1 \to \pm\frac{\pi}{2}
\end{aligned}$$

and it is continuous. After enough times iteration, say it is n, we get two random sample from $f_{T_1|T_2=t_2^{(k-1)}}$ and $f_{T_1|T_2=t_2^{(k-1)}}$

$$t_1^{(n)}, \quad t_2^{(n)}$$

then

$$y_1 = \tan t_1, \quad y_2 = \tan t_2$$

form a sample (y_1, y_2) from f_{Y_1,Y_2}.

3.

If we use the same $g_Y(y)$ for all iteration, then Metropolis-Hasting is just the Gibbs sampling. In other words, Gibbs sampling will accept all proposal while MH accept with some possibility.