1.

Denote K_1, K_2 as the number of junior and senior students who choose VE414, Y_1, Y_2 as the number of junior and senior students who successfully enroll in VE414. Denote P_1, P_2 as the probability of junior and senior students that would be admitted in.

We are now given that $K_1 = 71 + 16 = 87, K_2 = 10 + 34 = 44, Y_1 = 16, Y_2 = 34$. Assume that the instructor is not guilty, i.e. both junior and senior student share the same possibility p of being admitted into VE414. Then

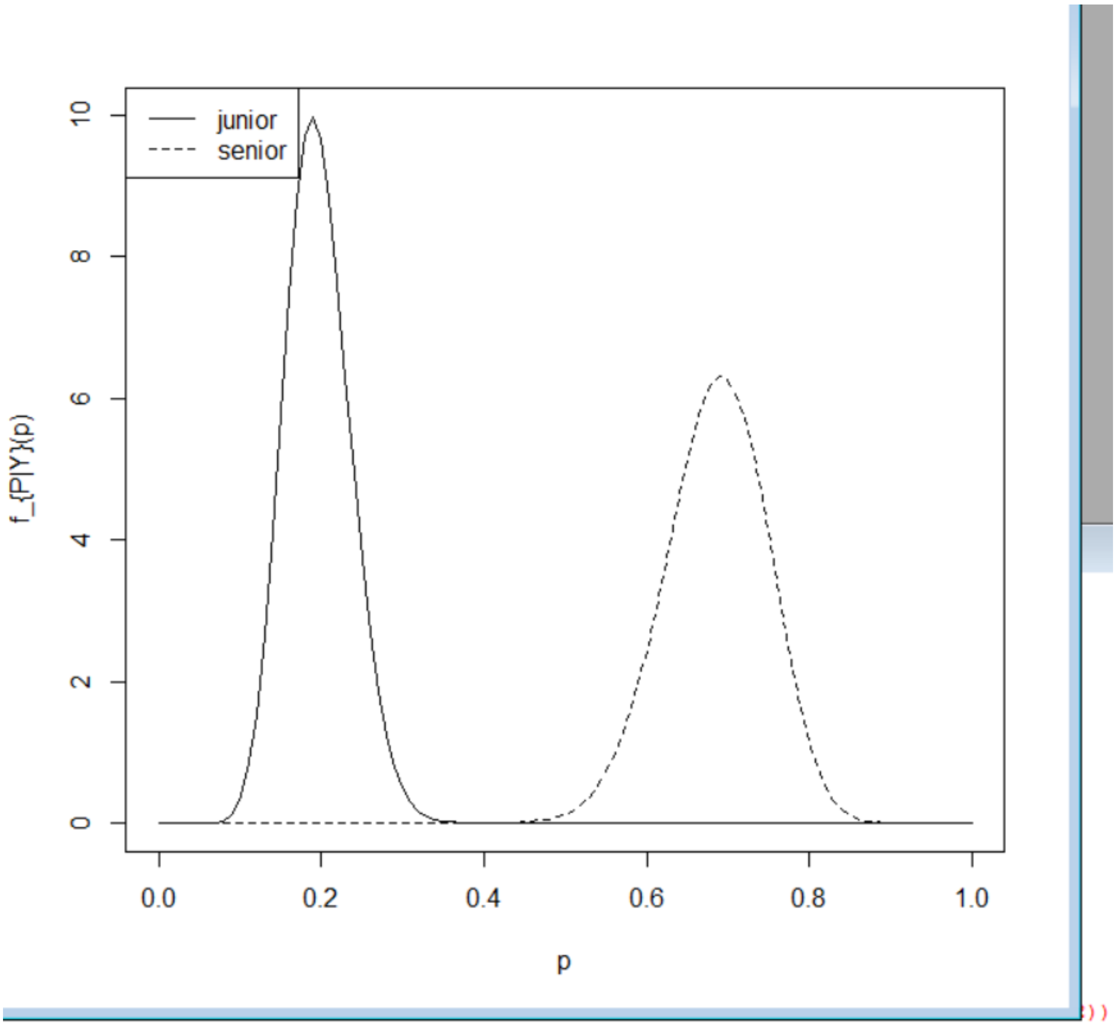
$$f_{Y_1|P} \sim \text{Binomial}(k_1, p), \quad f_{Y_2|P} \sim \text{Binomial}(k_2, p)$$

Known that the prior of p is

$$f_P(p) \sim \text{Beta(3,7)}$$

The posterior then is

$$f_{P|Y_1=16}(p) \sim \text{Beta}(3 + 16, 7 + 71), \quad f_{P|Y_2=34}(p) \sim \text{Beta}(3 + 34, 7 + 10)$$

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> curve(dbeta(x, 19, 78), xlim = c(0, 1), ylab = '', xlab = '')
> curve(dbeta(x, 37, 17), add = T, lty = 2, ylab = '', xlab = '')
> legend(par('usr')[1], par('usr')[4], xjust = 0, c('junior', 'senior'), lty = c(1, 2))
> title(ylab = 'f_{P|Y}(p)', xlab = 'p')
```

```
> l = qbeta(0.025, 19, 78)
> u = qbeta(0.975, 19, 78)
> cat(paste("(", l, sep = ""), paste(u, ")", sep = ""), "\n", sep = ",")
(0.123569256564545,0.280046247426744),
> l = qbeta(0.025, 37, 17)
> u = qbeta(0.975, 37, 17)
> cat(paste("(", l, sep = ""), paste(u, ")", sep = ""), "\n", sep = ",")
(0.556565984160423,0.800813353168366),
```

We can see that two creditable interval does not overlay with each other and therefore we can say that instructor is guilty.

2.

(a)

Given the Poisson distribution with PDF

$$f_{X_i|y}(x_i|y) = \frac{y^{x_i} e^{-y}}{x_i!}$$

the likelihood function for parameter y is

$$\mathcal{L}(y;x) = \prod_{i=1}^{n} f_{X_i|y}(x_i|y) = \frac{y^{\sum\limits_{i=1}^{n} x_i} e^{-ny}}{\prod\limits_{i=1}^{n} x_i!}$$

Given the gamma prior with mean of 15 and variance of 25,

$$f_Y(y) = \begin{cases} \dfrac{1}{\Gamma(9)\cdot(\frac{5}{3})^9} y^{9-1} e^{-\frac{3y}{5}} & , \quad y > 0 \\ 0 & , \quad y \leqslant 0 \end{cases}$$

the posterior of y is

$$f_{Y|\bar{x}}(y) = \begin{cases} A \cdot y^{n\bar{x}+8} e^{-(n+\frac{3}{5})y} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}$$

where A is a scale constant. Since

$$\begin{aligned} 1 &= A \cdot \int_{-\infty}^{\infty} f_{Y|\bar{x}}(y)\ dy = A \cdot \int_{0}^{+\infty} y^{n\bar{x}+8} e^{-(n+\frac{3}{5})y}\ dy \\ \Rightarrow A &= \frac{1}{\Gamma(n\bar{x}+9)\cdot(\frac{1}{n+\frac{3}{5}})^{n\bar{x}+9}} \end{aligned}$$

So the posterior is given by

$$f_{Y|\bar{x}}(y) = \begin{cases} \dfrac{1}{\Gamma(n\bar{x}+9)\cdot(\frac{1}{n+\frac{3}{5}})^{n\bar{x}+9}} \cdot y^{n\bar{x}+8} e^{-(n+\frac{3}{5})y} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}$$

(b)

We now have the prior of y

$$f_{Y|\bar{x}}(y) = \begin{cases} \dfrac{1}{\Gamma(n\bar{x}+9)\cdot(\frac{1}{n+\frac{3}{5}})^{n\bar{x}+9}} \cdot y^{n\bar{x}+8} e^{-(n+\frac{3}{5})y} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}$$

Given a new data point X^* with Poisson distribution

$$f_{X^*|y}(x^*|y) = \frac{y^{x^*} e^{-y}}{x^*!}$$

which is also the likelihood function for parameter y, then the posterior is

$$f_{Y|\bar{x}}(y) = \begin{cases} A \cdot y^{n\bar{x}+8+x^*} e^{-(n+\frac{3}{5}+1)y} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}$$

So it is still a Gamma distribution with PDF

$$f_{Y|\bar{x}}(y) = \begin{cases} \dfrac{1}{\Gamma(n\bar{x}+9+x^*)\cdot(\frac{1}{n+\frac{8}{5}})^{n\bar{x}+9+x^*}} \cdot y^{n\bar{x}+8+x^*} e^{-(n+\frac{8}{5})y} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}$$

3.

(a)

By using the uniform prior, we obtain the posterior

$$f_{P|X_3=2}(p) = \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} p^{3-1}(1-p)^{2-1} \sim \text{Beta}(3,2)$$

Use the Jeffreys Prior,

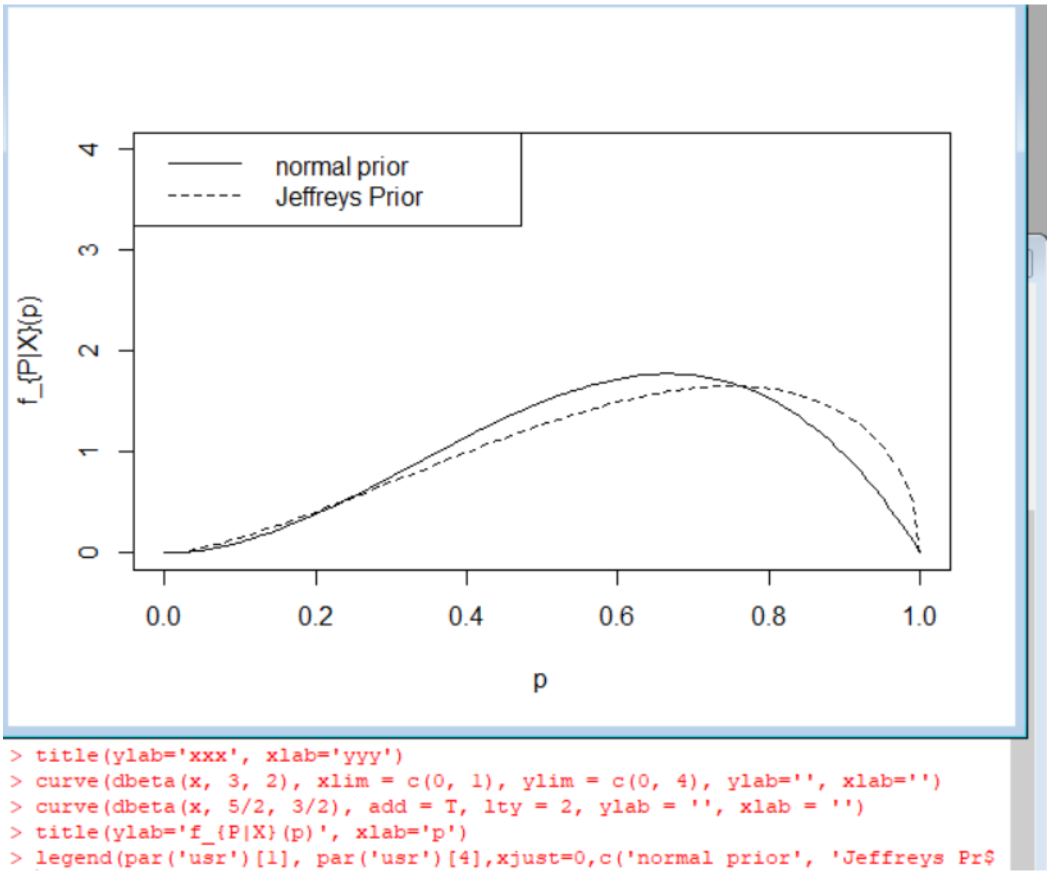
$$\begin{aligned} I(p) &= -\mathbb{E}\Big[\frac{\partial^2 \ln \mathcal{L}(p;X)}{\partial p^2}\Big] \\ &= -\mathbb{E}\Big[\frac{\partial^2}{\partial p^2}\ln\big(\frac{k!}{x!(k-x)!}p^x(1-p)^{k-x}\big)\Big] \\ &= -\mathbb{E}\Big[\frac{\partial^2}{\partial p^2}(x\ln p + (k-x)\ln(1-p))\Big] \\ &= -\mathbb{E}\Big[\frac{\partial}{\partial p}(\frac{x}{p} - \frac{k-x}{1-p})\Big] \\ &= -\mathbb{E}\Big[-\frac{x}{p^2} - \frac{k-x}{(1-p)^2}\Big] \\ &= \frac{1}{p^2}\mathbb{E}[X] + \frac{k}{(1-p)^2} - \frac{1}{(1-p)^2}\mathbb{E}[X] \\ &= \frac{kp(1-2p)}{p^2(1-p)^2} + \frac{k}{(1-p)^2} \\ &= \frac{2k}{p(1-p)} \end{aligned}$$

so

$$f_P(p) \propto \sqrt{\frac{2k}{p(1-p)}} \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$$

and therefore the posterior is

$$f_{P|X_3=2}(p) \sim \text{Beta}(\frac{1}{2}+2, \frac{1}{2}+1) \sim \text{Beta}(\frac{5}{2}, \frac{3}{2})$$



From the figure, we can see that two distribution are obviously different. Using Jefferys prior, the posterior seems to be shifted a little bit towards right compared with the one using normal prior.

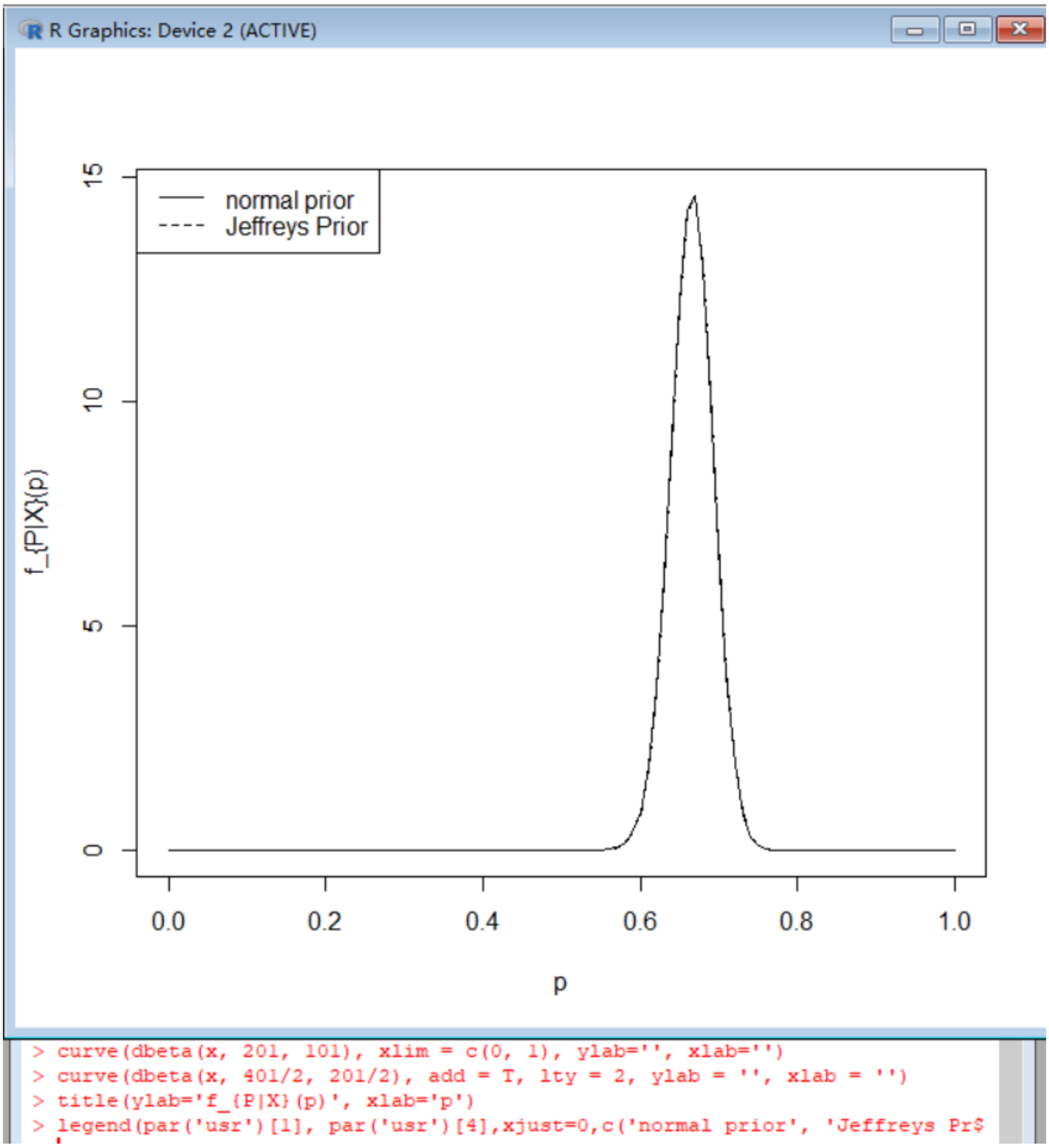
(b)

By using the uniform prior, we obtain the posterior

$$f_{P|X_{300}=200}(p) \sim \text{Beta}(201, 101)$$

Use the Jeffreys Prior,

$$f_{P|X_3=2}(p) \sim \text{Beta}(\frac{1}{2}+200, \frac{1}{2}+100) \sim \text{Beta}(\frac{401}{2}, \frac{201}{2})$$



From the figure, we can see that two distribution are almost the same and we cannot tell the difference.

(c)

Given $f_P(p) \propto p^{-1}(1-p)^{-1}$, the posterior is

$$f_{P|X_2=1} \propto f_{X2=1|P} \cdot f_P(p) \propto \frac{2!}{1!1!} p^1(1-p)^1 \cdot p^{-1}(1-p)^{-1} \propto 2$$

So $f_{P|X_2=1} = \begin{cases} 1, & p \in (0,1) \\ 0, & \text{otherwise} \end{cases}$

Although the prior is improper, through this example we see that it can help us lead to a proper posterior which is a distribution function. This posterior cannot be obtain through the given case $X_2 = 1$ directly. So we can say that using this improper prior we still can obtain some information.