1.

(a)

Substitute $u=\dfrac{1}{1+y^2}$, then

$$
\begin{aligned}
&A|_{X=0.5}\\
=&\int_{-\infty}^{\infty}\exp\Big(-\frac{(0.5-y)^2}{2}\Big)\times\frac{1}{1+y^2}dy\\
=&\int_{-\infty}^{0}\exp\Big(-\frac{(0.5-y)^2}{2}\Big)\times\frac{1}{1+y^2}dy+\int_{0}^{\infty}\exp\Big(-\frac{(0.5-y)^2}{2}\Big)\times\frac{1}{1+y^2}dy\\
=&\int_{0}^{1}\exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot u\cdot\frac{1}{2\sqrt{u^3-u^4}}du\\
&+\int_{1}^{0}\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot u\cdot\Big(-\frac{1}{2\sqrt{u^3-u^4}}\Big)du\\
=&\int_{0}^{1}\exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot\frac{1}{2\sqrt{u-u^2}}du\\
&+\int_{0}^{1}\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot\frac{1}{2\sqrt{u-u^2}}du
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y|X=0.5]&=\int_{-\infty}^{\infty}y\cdot f_{Y|X=0.5}dy=\int_{-\infty}^{\infty}y\cdot\frac{1}{A}\exp\Big(-\frac{(0.5-y)^2}{2}\Big)\cdot\frac{1}{1+y^2}dy\\
&=\int_{0}^{1}\frac{1}{A}\exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot u\cdot\frac{-\sqrt{\frac{1}{u}-1}}{2\sqrt{u^3-u^4}}du\\
&\quad+\int_{1}^{0}\frac{1}{A}\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot u\cdot\Big(-\frac{\sqrt{\frac{1}{u}-1}}{2\sqrt{u^3-u^4}}\Big)du\\
&=-\int_{0}^{1}\frac{1}{A}\exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot\frac{1}{2u}du\\
&\quad+\int_{0}^{1}\frac{1}{A}\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big)\cdot\frac{1}{2u}du
\end{aligned}
$$

Using simple grid approximation,

$$
\begin{aligned}
&A|_{X=0.5}\\
\approx&\frac{1-0}{n}\sum_{i=1}^{n}\exp\Big(-\frac{(0.5+\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)\cdot\frac{1}{2\sqrt{\frac{2i-1}{2n}-\frac{(2i-1)^2}{4n^2}}}\\
&+\frac{1-0}{n}\sum_{i=1}^{n}\exp\Big(-\frac{(0.5-\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)\cdot\frac{1}{2\sqrt{\frac{2i-1}{2n}-\frac{(2i-1)^2}{4n^2}}}\\
=&\sum_{i=1}^{n}\frac{1}{\sqrt{(2n-2i+1)(2i-1)}}\Bigg(\exp\Big(-\frac{(0.5+\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)+\exp\Big(-\frac{(0.5-\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}[Y|X=0.5]\\
\approx&\frac{1-0}{nA}\sum_{i=1}^{n}\frac{1}{2(\frac{2i-1}{2n})}\Bigg(-\exp\Big(-\frac{(0.5+\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)+\exp\Big(-\frac{(0.5-\sqrt{\frac{2n}{2i-1}-1})^2}{2}\Big)\Bigg)
\end{aligned}
$$

(b)

```julia
n = Array{Int64, 1}(undef,5);
n[1] = 50; n[2] = 250; n[3] = 750; n[4] = 1500; n[5] = 3000;
for j in 1:1:5
    f = Array{Float64, 1}(undef, n[j]);
    ee = Array{Float64, 1}(undef, n[j]);
    for i in 1:1:n[j]
        f[i] = (exp(-(0.5+(2*n[j]/(2*i-1)-1)^0.5)^2/2) + exp(-(0.5-(2*n[j]/(2*i-1)-1)^0.5)^2/2))/((2
            *n[j]-2*i+1)*(2*i-1))^0.5;
        mean[i] = (-exp(-(0.5+(2*n[j]/(2*i-1)-1)^0.5)^2/2) + exp(-(0.5-(2*n[j]/(2*i-1)-1)^0.5)^2/2))
            /2/((2*i-1)/2/n[j]);
    end
    A = sum(f);
    E = sum(ee)/A/n[j];
    println("For n = ", n[j], ", Normallized constant is A = ", A);
    println("For n = ", n[j], ", E[Y|X=0.5] = ", E);
end
```

```
For n = 50, Normallized constant is A = 1.4740494468809806
For n = 50, E[Y|X=0.5] = 0.27985682751798313
For n = 250, Normallized constant is A = 1.5157641526863064
For n = 250, E[Y|X=0.5] = 0.2721090661427476
For n = 750, Normallized constant is A = 1.530032204520519
For n = 750, E[Y|X=0.5] = 0.26956796784954773
For n = 1500, Normallized constant is A = 1.535741183073878
For n = 1500, E[Y|X=0.5] = 0.2685653225100145
For n = 3000, Normallized constant is A = 1.5397781152124015
For n = 3000, E[Y|X=0.5] = 0.2678610138806316
julia>
```

And we can complete the table

Grid size n	50	250	750	1500	3000
$\mathbb{E}[Y\|X=0.5]$	0.27986	0.27211	0.26957	0.26857	0.26786

(c)

Substitute $u = \dfrac{1}{1+y^2}$, then

$$\begin{aligned}\mathbb{E}[Y|X=0.5] &= \int_{-\infty}^{\infty} y \cdot f_{Y|X=0.5} dy = \int_{-\infty}^{\infty} y \cdot \frac{1}{A} \exp\Big(-\frac{(0.5-y)^2}{2}\Big) \cdot \frac{1}{1+y^2} dy \\ &= \frac{1}{A} \int_0^1 \left(\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big) - \exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\right) \cdot \frac{u}{2u^2} du\end{aligned}$$

So our goal is to do direct sampling within $[0,1]$ with distribution

$$f_U(u) = \frac{1}{B}\left(\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big) - \exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\right) \cdot \frac{1}{2u^2}$$

the relation between the sample mean and the value we want is

$$\mathbb{E}[Y|X=0.5] = \frac{B}{A} \times \text{ sample mean}$$

where

$$A = \int_0^1 \left(\exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big) + \exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big)\right) \cdot \frac{1}{2\sqrt{u-u^2}} du$$

$$B = \int_0^1 \left(\exp\Big(-\frac{(0.5-\sqrt{\frac{1}{u}-1})^2}{2}\Big) - \exp\Big(-\frac{(0.5+\sqrt{\frac{1}{u}-1})^2}{2}\Big)\right) \cdot \frac{1}{2u^2} du$$

(d)

```julia
using Distributions
using Plotly
a = 0; b = 1;
n = 50;
fA = Array{Float64, 1}(undef, n);
fB = Array{Float64, 1}(undef, n);
u_grid = Array{Float64, 1}(undef, n);
for i in 1:1:n
    u_grid[i] = (b - a)/n * i-1/2/n;
    fB[i] = (-exp(-(0.5 + (1/u_grid[i]-1)^0.5)^2/2)+exp(-(0.5 - (1/u_grid[i]-1)^0.5)^2/2))/2/(u_grid[i]^2);
    fA[i] = (exp(-(0.5 + (1/u_grid[i]-1)^0.5)^2/2)+exp(-(0.5 - (1/u_grid[i]-1)^0.5)^2/2))/2/(u_grid[i]-u_grid[i]^2)^0.5;
end
posterior = fB/sum(fB);
A = (b - a)/n*sum(fA);
B = (b - a)/n*sum(fB);
m = 100;
samples = Array{Float64, 1}(undef, m);
distribution = Categorical(posterior);
for i in 1:1:m
    samples[i] = u_grid[rand(distribution)];
end
print(B/A*mean(samples))
```

Since sampling has some randomness, we do sampling with each pair of m, n for ten times, and take their meas as the final result. Here is the table recording all data

50		250		750		1500		3000	
100	1000	100	1000	100	1000	100	1000	100	1000
0.25407	0.28465	0.26989	0.27480	0.25340	0.28065	0.25783	0.27283	0.29350	0.26621
0.29710	0.27444	0.25934	0.27351	0.26038	0.26688	0.28019	0.26942	0.31311	0.27620
0.29251	0.28342	0.27405	0.26962	0.26226	0.27495	0.26341	0.26737	0.28783	0.26712
0.28655	0.27720	0.31464	0.27332	0.26826	0.27843	0.25576	0.26294	0.28089	0.27435
0.29914	0.27558	0.25808	0.26695	0.27315	0.27570	0.28655	0.26410	0.27893	0.26982
0.28196	0.27715	0.26168	0.26916	0.27414	0.27181	0.27249	0.26014	0.26563	0.26494
0.26002	0.28125	0.27068	0.27022	0.27933	0.26869	0.28216	0.26647	0.25266	0.26133
0.28162	0.27710	0.28735	0.27678	0.24637	0.27080	0.30826	0.26613	0.27100	0.25781
0.27074	0.28215	0.23380	0.26571	0.25735	0.26624	0.27769	0.26630	0.27078	0.27256
0.30203	0.27808	0.25576	0.26957	0.28215	0.27368	0.28267	0.26346	0.23202	0.26955
0.28257	0.27910	0.26853	0.27096	0.26568	0.27278	0.27670	0.26592	0.27463	0.26799

And therefore

Grid size n	50		250		750		1500		3000	
Sample size m	100	1000	100	1000	100	1000	100	1000	100	1000
$\mathbb{E}[Y\|X = 0.5]$	0.28257	0.27910	0.26853	0.27096	0.26568	0.27278	0.27670	0.26592	0.27463	0.26799

(e)

```julia
x = 0.5; a = -5; b = 5;
n = Array{Int64, 1}(undef,5);
n[1] = 50; n[2] = 250; n[3] = 750; n[4] = 1500; n[5] = 3000;
for j in 1:1:5
    if n[j] <= 1000
        y_grid = collect(
            range(a, length = n[j] , stop=b));
        newa = a ;
        l = (b - a)/n[j];
    elseif n[j] > 1000 && n[j] <= 2000
        nm = 1000;
        na = round(Int , (n[j] - nm) / 2) ;
        l = (b - a) / (nm - 1);
        newa = a - l * na ;
        y_grid = collect(
            range(newa, step=l, length=n[j]));
    else n[j] > 2000
        nm = round(Int, n[j] / 2) ;
        na = round(Int, (n[j] - nm) / 2) ;
        l = (b - a) / (nm - 1);
        newa = a - l * na ;
        y_grid = collect(
            range(newa, step=l, length=n[j])) ;
    end
    f = Array{Float64, 1}(undef, n[j]);
    for i in 1:1:n[j]
        f[i] = exp(-(0.5 - y_grid[i])^2/2)/(1 + y_grid[i]^2);
    end
    A = l * sum(f);
    for i in 1:1:n[j]
        f[i] = y_grid[i] * exp(-(0.5 - y_grid[i])^2/2)/(1 + y_grid[i]^2);
    end
    E = l * sum(f) / A;
    println("For n = ", n[j], ", E[Y|X=0.5] = ", E);
end
```

```
For n = 50, E[Y|X=0.5] = 0.26617556413726473
For n = 250, E[Y|X=0.5] = 0.26617525294322397
For n = 750, E[Y|X=0.5] = 0.26617519291542174
For n = 1500, E[Y|X=0.5] = 0.2661761233904551
For n = 3000, E[Y|X=0.5] = 0.2661761233906986
julia>
```

Here is the completed table

Grid size n	50	250	750	1500	3000
$\mathbb{E}[Y\|X = 0.5]$	0.26618	0.26618	0.26618	0.26618	0.26618

and we can see that by using this method, $\mathbb{E}[Y|X = 0.5]$ converges much more quickly than using variable transformation.