VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 6

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November 10, 2016

Exercise 6.1

According to theorem 2.2.20 and 2.2.18,

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{\frac{n!}{2\pi i}\oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}{n!} = \frac{1}{2\pi i}\oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

So

$$|c_n| = \frac{1}{2\pi}\left|\oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right| \leqslant \frac{2\pi r}{2\pi}\frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} \leqslant r\frac{M}{r^{n+1}} = \frac{M}{r^n}$$

i.e. $|c_n| \leqslant \dfrac{M}{r^n}$

Exercise 6.2

i)

If $f : \mathbb{C} \to \mathbb{C}$ is a bounded, entire function, then for some $M > 0$

$$|f(z)| \leqslant M$$

for all $z \in \mathbb{C}$. Then if we set that $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, $\forall r > 0$ we have

$$|c_n| \leqslant \frac{M}{r^n}$$

So $c_0 \leqslant M, \forall n > 0, c_n = 0$. So f is constant.

To sum up, any bounded, entire function must be constant.

ii)

If f is a polynomial of degree $n \geqslant 1$, then f is not a constant. And therefore, according to Liouville's Theorem, f is unbounded or not entire. While any polynomial is entire, then f is unbounded.

Assume that f has no zero, then define $g := \dfrac{1}{f}$. Since f is unbounded and entire polynomial, g is bounded and entire. And therefore, according to Liouville's Theorem, g is constant. So $f = \dfrac{1}{g}$ is constant. This leads to contradiction.

So f at least has one zero.

Exercise 6.3

Since h has a simple zero at z_0, in a small disc centered at z_0, h has the expansion

$$h(z) = \sum_{i=1}^{\infty} a_i (z - z_0)^i$$

so $\dfrac{h(z)-h(z_0)}{z-z_0} = a_1 + \sum_{i=2}^{\infty} a_i(z-z_0)^{i-1}$. Since h has a simple zero at z_0, $a_1 \neq 0$. Then

$$h'(z_0) = \lim_{z\to z_0} \frac{h(z)-h(z_0)}{z-z_0} = a_1 \neq 0$$

Moreover, since $\dfrac{g}{h}$ has a simple pole at z_0,

$$res_{z_0}\frac{g(z)}{h(z)} = \lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)} = \lim_{z\to z_0}\frac{g(z)}{\dfrac{h(z)-h(z_0)}{(z-z_0)}} = \frac{g(z_0)}{h'(z_0)}$$

So $res_{z_0}\dfrac{g(z)}{h(z)} = \dfrac{g(z_0)}{h'(z_0)}$.

Exercise 6.4

i) $\int_{-\infty}^{\infty}\dfrac{\cos x}{x^2+a^2}dx$

Set $f(z) = \dfrac{e^{iz}}{z^2+a^2}$. Set $C_R = \{z \in \mathbb{C} : z = R\cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$ be a semi-circle segment in the upper half-plane and $R > a$. Then

$$\sup_{0\leqslant\theta\leqslant\pi}\left|\frac{1}{R^2e^{2i\theta}+a^2}\right| \xrightarrow{R\to\infty} 0$$

According to Jordan's Lemma,

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

Since f has two poles $z_1 = ai, z_2 = -ai$.

$$rez_{ai}f = \lim_{z\to ai}(z-ai)\frac{e^{iz}}{z^2+a^2} = \frac{e^{-a}}{2ai}, rez_{-ai}f = \lim_{z\to -ai}(z+ai)\frac{e^{iz}}{z^2+a^2} = -\frac{e^{a}}{2ai}$$

Use toy contour $\ell = \{z \in \mathbb{C} : z = R\cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi \vee -R \leqslant z \leqslant R\}$, $R > a$, then according to The Residue Theorem,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = \int_{\ell} f(z) = 2\pi i rez_{ai}f = \pi\frac{e^{-a}}{a}$$

So

$$\int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{x^2+a^2}dx = \int_{-\infty}^{\infty} f(z)dz = \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz\right) = \pi\frac{e^{-a}}{a}$$

So

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx = \pi\frac{e^{-a}}{a}$$

ii) $\int_{-\infty}^{\infty} \dfrac{x\sin x}{x^2+a^2}dx$

Set $f(z) = \dfrac{ze^{iz}}{z^2+a^2}$. Set $C_R = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$ be a semi-circle segment in the upper half-plane and $R > a$. Then

$$\sup_{0\leqslant\theta\leqslant\pi} \left|\frac{Re^{i\theta}}{R^2e^{2i\theta}+a^2}\right| \xrightarrow{R\to\infty} 0$$

According to Jordan's Lemma,

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$$

Since f has two poles $z_1 = ai, z_2 = -ai$.

$$rez_{ai}f = \lim_{z\to ai}(z-ai)\frac{ze^{iz}}{z^2+a^2} = \frac{aie^{-a}}{2ai} = e^{-a}/2$$

Use toy contour $\ell = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi \vee -R \leqslant z \leqslant R\}$, $R > a$, then according to The Residue Theorem,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = \int_{\ell} f(z) = 2\pi i rez_{ai}f = \pi e^{-a}i$$

So

$$\int_{-\infty}^{\infty} \frac{x\cos x + ix\sin x}{x^2+a^2}dx = \int_{-\infty}^{\infty} f(z)dz = \lim_{R\to\infty}(\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz) = \pi e^{-a}i$$

So

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}dx = \pi e^{-a}$$

Exercise 6.5

Set $f(z) = \dfrac{1}{1+z^4}$. Set $C_R = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$ be a semi-circle segment in the upper half-plane and $R > 1$. Then

$$\left|\int_{C_R} f(z)dz\right| = \int_0^{\pi} \left|\frac{iRe^{i\theta}}{1+R^4e^{i4\theta}}\right| d\theta \leqslant \pi \frac{R}{|1+R^4e^{i4\theta}|} \xrightarrow{R\to\infty} 0$$

Since f has two poles $z_1 = e^{i\frac{\pi}{4}}, z_2 = e^{i\frac{3\pi}{4}}$ in this region

$$rez_{e^{i\frac{\pi}{4}}}f = \lim_{z\to e^{i\frac{\pi}{4}}}(z-e^{i\frac{\pi}{4}})\frac{1}{1+z^4} = \frac{1}{4}e^{-i\frac{3\pi}{4}}, rez_{e^{i\frac{3\pi}{4}}}f = \lim_{z\to e^{i\frac{3\pi}{4}}}(z-e^{i\frac{3\pi}{4}})\frac{1}{1+z^4} = \frac{1}{4}e^{-i\frac{\pi}{4}}$$

Use toy contour $\ell = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi \vee -R \leqslant z \leqslant R\}$, $R > 1$, then according to The Residue Theorem,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = \int_{\ell} f(z) = 2\pi i(rez_{e^{i\frac{\pi}{4}}}f + rez_{e^{i\frac{3\pi}{4}}}f) = \frac{\pi}{\sqrt{2}}$$

So

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = \int_{-\infty}^{\infty} f(z)dz = \lim_{R\to\infty}(\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz) = \frac{\pi}{\sqrt{2}}$$

So

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$$

Exercise 6.6

Set $f(z)=\dfrac{ze^{iz}}{(z^2+4)^2}$. Set $C_R=\{z\in\mathbb{C}:z=R\cdot e^{i\theta},0\leqslant\theta\leqslant\pi\}$ be a semi-circle segment in the upper half-plane and $R>2$. Then

$$\sup_{0\leqslant\theta\leqslant\pi}\left|\frac{Re^{i\theta}}{(R^2e^{2i\theta}+4)^2}\right|\xrightarrow{R\to\infty}0$$

According to Jordan's Lemma,

$$\lim_{R\to\infty}\int_{C_R}f(z)dz=0$$

Since f has one pole in order two $z_1=2i$ in this region

$$rez_{2i}f=\lim_{z\to 2i}\frac{d}{dz}(z-2i)^2\frac{ze^{iz}}{(z^2+4)^2}=\frac{1}{8e^2}$$

Use toy contour $\ell=\{z\in\mathbb{C}:z=R\cdot e^{i\theta},0\leqslant\theta\leqslant\pi\vee -R\leqslant z\leqslant R\}$, $R>2$, then according to The Residue Theorem,

$$\int_{-R}^{R}f(z)dz+\int_{C_R}f(z)dz=\int_{\ell}f(z)=2\pi i(rez_{2i}f)=\frac{\pi i}{4e^2}$$

So

$$\int_{-\infty}^{\infty}\frac{x\cos x+ix\sin x}{(x^2+4)^2}dx=\int_{-\infty}^{\infty}f(z)dz=\lim_{R\to\infty}(\int_{-R}^{R}f(z)dz+\int_{C_R}f(z)dz)=\frac{\pi i}{4e^2}$$

Since $\dfrac{x\sin x}{(x^2+4)^2}$ is even,

$$\int_0^{\infty}\frac{x\sin x}{(x^2+4)^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+4)^2}dx=\frac{\pi}{8e^2}$$

Exercise 6.7

Set $f(z)=\dfrac{1}{(1+z^2)^{n+1}}$. Set $C_R=\{z\in\mathbb{C}:z=R\cdot e^{i\theta},0\leqslant\theta\leqslant\pi\}$ be a semi-circle segment in the upper half-plane and $R>1$. Then

$$\left|\int_{C_R}f(z)dz\right|=\int_0^{\pi}\left|\frac{iRe^{i\theta}}{(1+R^2e^{i2\theta})^{n+1}}\right|d\theta\leqslant\pi\left|\frac{1}{(1+R^2e^{i2\theta})^{n+1}}\right|\xrightarrow{R\to\infty}0$$

Since f has one pole in order $n+1$ $z_1=i$ in this region

$$\begin{aligned}rez_{i}f&=\frac{1}{n!}\lim_{z\to i}\frac{d^n}{dz^n}(z-i)^{n+1}\frac{1}{(1+z^2)^{n+1}}\\&=\frac{(-1)^n(n+1)(n+2)\cdots(n+n))}{n!}\frac{1}{2^{2n+1}\cdot(-1)^n i}\\&=-\frac{(2n)!}{(n!)^2\cdot 2^{2n+1}}i\\&=-\frac{1}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}i\end{aligned}$$

Use toy contour $\ell = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi \vee -R \leqslant z \leqslant R\}$, $R > 1$, then according to The Residue Theorem,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = \int_{\ell} f(z) = 2\pi i(rez_i f) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\pi$$

So

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}dx = \int_{-\infty}^{\infty} f(z)dz = \lim_{R\to\infty}(\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\pi$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\pi$$

Exercise 6.8

i)

Set $f(z) = \dfrac{\sqrt{z}}{z^2+a^2}$. Set $C_R = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant 3\pi/4\}$ be a 3/4-circle segment in the upper half-plane and $R > |a|$. Then

$$\Big|\int_{C_R} f(z)dz\Big| = \int_0^{3\pi/4} \Big|\frac{iRe^{i\theta}\sqrt{R}e^{i\frac{\theta}{2}}}{(a^2+R^2e^{i2\theta})}\Big|d\theta \leqslant \pi\Big|\frac{R^{3/2}}{R^2e^{i2\theta}+a^2}\Big| \xrightarrow{R\to\infty} 0$$

Set $C_0 = \{z \in \mathbb{C} : z = re^{i\frac{3\pi}{4}}, 0 \leqslant r \leqslant R\}$. Then

$$\int_{C_0} f(z)dz = \int_0^R \frac{e^{i\frac{3\pi}{4}}\sqrt{r}e^{i\frac{3\pi}{8}}}{(a^2-r^2)}dr - \int_0^R \frac{e^{i\frac{3\pi}{4}}\sqrt{r}e^{i\frac{3\pi}{8}}}{(a^2-r^2)}dr = 0$$

Since f has one pole in order 1 $z_1 = |a|i$ in this region

$$rez_{|a|i}f = \lim_{z\to|a|i}(z-|a|i)\frac{\sqrt{z}}{z^2+a^2} = -\frac{e^{i\frac{\pi}{4}}}{2\sqrt{|a|}}i$$

Use toy contour $\ell = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant 3\pi/4 \vee 0 \leqslant z \leqslant R \vee z = re^{i\frac{3\pi}{4}}, 0 \leqslant r \leqslant R\}$, $R > |a|$, then according to The Residue Theorem,

$$\int_0^R f(z)dz + \int_{-C_0} f(z)dz + \int_{C_R} f(z)dz = \int_{\ell} f(z) = 2\pi i(rez_{|a|i}f) = \frac{\pi e^{i\frac{\pi}{4}}}{\sqrt{|a|}}$$

So

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2+a^2}dx = \int_0^{\infty} f(z)dz = Re\lim_{R\to\infty}(\int_0^R f(z)dz + \int_{C_R} f(z)dz) = \frac{\pi}{\sqrt{2|a|}}$$

So,

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2+a^2}dx = \frac{\pi}{\sqrt{2|a|}}$$

ii)

Set $f(z)=\dfrac{\ln z}{z^2+a^2}$. Set $C_R=\{z\in\mathbb{C}:z=R\cdot e^{i\theta},0\leqslant\theta\leqslant 3\pi/4\}$ be a 3/4-circle segment in the upper half-plane and $R>|a|$. Then

$$\Big|\int_{C_R}f(z)dz\Big|=\int_0^{3\pi/4}\Big|\frac{iRe^{i\theta}(\ln R+i\theta)}{R^2e^{i2\theta}+a^2}\Big|d\theta\leqslant 3\pi/4\Big|\frac{R(\ln R+i\theta)}{R^2e^{i2\theta}+a^2}\Big|\xrightarrow{R\to\infty}0$$

Set $C_0=\{z\in\mathbb{C}:z=re^{i\frac{3\pi}{4}},0\leqslant r\leqslant R\}$. Then

$$\int_{C_0}f(z)dz=\int_0^R\frac{e^{i\frac{3\pi}{4}}(\ln r+i\frac{3\pi}{4})}{(a^2-r^2)}dr=e^{i\frac{3\pi}{4}}(\frac{3\pi i}{8}\ln\Big|\frac{1+R}{1-R}\Big|+\int_0^R\frac{\ln r}{a^2-r^2})\xlongequal{R\to\infty}0$$

Since f has one pole in order 1 $z_1=|a|i$ in this region

$$rez_{|a|i}f=\lim_{z\to|a|i}(z-|a|i)\frac{\ln z}{z^2+a^2}=\frac{1}{4}\frac{2\ln|a|+\pi i}{|a|i}$$

Use toy contour $\ell=\{z\in\mathbb{C}:z=R\cdot e^{i\theta},0\leqslant\theta\leqslant 3\pi/4\vee 0\leqslant z\leqslant R\vee z=re^{i\frac{3\pi}{4}},0\leqslant r\leqslant R\}$, $R>|a|$, then according to The Residue Theorem,

$$\int_0^R f(z)dz+\int_{-C_0}f(z)dz+\int_{C_R}f(z)dz=\int_\ell f(z)=2\pi i(rez_{|a|i}f)=\frac{\ln|a|}{|a|}+\frac{\pi i}{2|a|}$$

So

$$\int_0^\infty\frac{\ln x}{x^2+a^2}dx=\int_0^\infty f(z)dz=\lim_{R\to\infty}(\int_{-R}^R f(z)dz+\int_{C_R}f(z)dz)=\frac{\ln|a|}{|a|}$$

So,

$$\int_0^\infty\frac{\ln x}{x^2+a^2}dx=\frac{\ln|a|}{|a|}$$

Exercise 6.9

i) $y''+y=3x+5x^4$

Use Heaviside Operator Method,

$$\begin{aligned}D^2y+y=3x+5x^4\Rightarrow y&=\frac{1}{1+D^2}(3x+5x^4)\\
\Rightarrow y&=\sum_{n=0}^\infty(-1)^nD^{2n}(3x+5x^4)\\
\Rightarrow y&=3x+5x^4-60x^2+120+\sum_{n=3}^\infty(-1)^nD^{2n-6}(D^6(3x+5x^4))\\
\Rightarrow y&=5x^4-60x^2+3x+120\end{aligned}$$

So the solution is $y=5x^4-60x^2+3x+120$.

ii)$y'' + y = e^{\mu x}$

Use Heaviside Operator Method,

$$\begin{aligned}
D^2 y + y = e^{\mu x} \Rightarrow y &= \frac{1}{1+D^2} e^{\mu x} \\
\Rightarrow y &= \sum_{n=0}^{\infty} (-1)^n D^{2n} e^{\mu x} \\
\Rightarrow y &= \sum_{n=0}^{\infty} (-1)^n \mu^{2n} e^{\mu x} \\
\Rightarrow y &= \frac{e^{\mu x}}{1+\mu^2}
\end{aligned}$$

So the solution is $y = \dfrac{e^{\mu x}}{1+\mu^2}$.

Exercise 6.10

i)$\mathscr{L}(\sinh(bt))$

$\forall p > |b|$,

$$\begin{aligned}
\mathscr{L}(\sinh(bt))(p) &= \int_0^{\infty} \frac{e^{bt} - e^{-bt}}{2} e^{-pt} dt \\
&= \frac{1}{2} \int_0^{\infty} e^{(b-p)t} - e^{-(b+p)t} dt \\
&= \frac{1}{2} \big(\frac{1}{b-p} e^{(b-p)t} \big|_0^{\infty} + \frac{1}{b+p} e^{-(b+p)t} dt \big|_0^{\infty} \big) \\
&= \frac{1}{2} \big(\frac{1}{b-p} (0-1) + \frac{1}{b+p} (0-1) \big) \\
&= \frac{b}{p^2 - b^2}
\end{aligned}$$

And for $p \leqslant |b|$, the integral doesn't converge.

So $\mathscr{L}(\sinh(bt))(p) = \dfrac{b}{p^2 - b^2}, p > |b|$

ii)$\mathscr{L}(\cos(bt))$

$\forall p > 0$,

$$\begin{aligned}
\mathscr{L}(\cos(bt))(p) &= \int_0^{\infty} \cos(bt) e^{-pt} dt = \frac{1}{b} \int_0^{\infty} e^{-pt} d(\sin(bt)) \\
&= \frac{1}{b} (\sin(bt) e^{-pt} \big|_0^{\infty} - \int_0^{\infty} \sin(bt)(-pe^{-pt}) dt) \\
&= -\frac{p}{b^2} \big(\cos(bt) e^{-pt} \big|_0^{\infty} - \int_0^{\infty} \cos(bt)(-pe^{-pt}) dt \big) \\
&= -\frac{p}{b^2} \big(0 - 1 + p \mathscr{L}(\cos(bt))(p) \big)
\end{aligned}$$

So

$$(b^2+p^2)\mathscr{L}(\cos(bt))(p) = p \Rightarrow \mathscr{L}(\cos(bt))(p) = \frac{p}{p^2+b^2}$$

And for $p \leqslant 0$, the integral doesn't converge.

So $\mathscr{L}(\cos(bt))(p) = \dfrac{p}{p^2+b^2}, p > 0$

iii)$\mathscr{L}(t\sin(at))$

$\forall p > 0$,

$$\frac{d}{dp}\mathscr{L}(\sin(at))(p) = \frac{d}{dp}\int_0^\infty \sin(at)e^{-pt}dt = \int_0^\infty \sin(at)(-te^{-pt})dt = -\mathscr{L}(t\sin(at))$$

So

$$\mathscr{L}(t\sin(at)) = -\frac{d}{dp}\mathscr{L}(\sin(at))(p) = -\frac{d}{dp}\frac{a}{p^2+a^2} = \frac{2ap}{(p^2+a^2)^2}$$

And for $p \leqslant 0$, the integral doesn't converge.

So $\mathscr{L}(t\sin(at)) = \dfrac{2ap}{(p^2+a^2)^2}, p > 0$

iv)$\mathscr{L}(t^2\sinh(bt))$

$\forall p > |b|$,

$$\frac{d^2}{dp^2}\mathscr{L}(\sin(bt))(p) = \frac{d}{dp}\int_0^\infty \sinh(bt)(-te^{-pt})dt\int_0^\infty \sinh(bt)t^2e^{-pt}dt = \mathscr{L}(t^2\sinh(bt))$$

So

$$\mathscr{L}(t^2\sinh(bt)) = \frac{d^2}{dp^2}\mathscr{L}(\sinh(bt))(p) = \frac{d^2}{dp^2}\frac{b}{p^2-b^2} = \frac{d}{dp}\frac{-2bp}{(p^2-b^2)^2} = \frac{6bp^2+2b^3}{(p^2-b^2)^3}$$

And for $p \leqslant |b|$, the integral doesn't converge.

So $\mathscr{L}(t\sin(bt)) = \dfrac{6bp^2+2b^3}{(p^2-b^2)^3}, p > |b|$

v)$\mathscr{L}(\sqrt{t})$

$\forall p > 0$,

$$\begin{aligned}\mathscr{L}(\sqrt{t})(p) &= \int_0^\infty \sqrt{t}e^{-pt}dt = p^{-\frac{1}{2}}\int_0^\infty (pt)^{\frac{1}{2}}e^{-pt}dt = p^{-\frac{3}{2}}\int_0^\infty (z)^{\frac{1}{2}}e^{-z}dz \\ &= p^{-\frac{3}{2}}\Gamma(\frac{1}{2})\end{aligned}$$

And for $p \leqslant 0$, the integral doesn't converge.

So $\mathscr{L}(\sqrt{t})(p) = p^{-\frac{3}{2}}\Gamma(\dfrac{1}{2}), p > 0$

vi)$\mathscr{L}(1/\sqrt{t})$

$\forall p > 0$,

$$\mathscr{L}(1/\sqrt{t})(p) = \int_0^{\infty} 1/\sqrt{t}e^{-pt}dt = p^{\frac{1}{2}}\int_0^{\infty}(pt)^{-\frac{1}{2}}e^{-pt}dt = p^{-\frac{1}{2}}\int_0^{\infty}(z)^{-\frac{1}{2}}e^{-z}dz$$
$$= p^{-\frac{1}{2}}\Gamma(-\frac{1}{2})$$

And for $p \leqslant 0$, the integral doesn't converge.

So $\mathscr{L}(\sqrt{t})(p) = p^{-\frac{1}{2}}\Gamma(-\frac{1}{2}), p > 0$

Exercise 6.11

i)$y''' - 6y'' + 11y' - 6y = e^{4t}, y(0) = y'(0) = y''(0) = 0$

Set $Y(p) = (\mathscr{L}y)(p)$, then

$$(\mathscr{L}y')(p) = p\cdot(\mathscr{L}y)(p) - y(0) = pY(p)$$
$$(\mathscr{L}y'')(p) = p\cdot(\mathscr{L}y')(p) - y'(0) = p^2Y(p)$$
$$(\mathscr{L}y''')(p) = p\cdot(\mathscr{L}y'')(p) - y''(0) = p^3Y(p)$$

and

$$(\mathscr{L}e^{4t})(p) = \frac{1}{p-4}, p > 4$$

So apply Laplace transform to the equation and we get

$$(p^3 - 6p^2 + 11p - 6)Y(p) = \frac{1}{p-4} \Rightarrow Y(p) = \frac{1}{(p-1)(p-2)(p-3)(p-4)}$$
$$\Rightarrow Y(p) = -\frac{1}{6}\frac{1}{p-1} + \frac{1}{2}\frac{1}{p-2} - \frac{1}{2}\frac{1}{p-3} + \frac{1}{6}\frac{1}{p-4}$$

Since $\mathscr{L}^{-1}\frac{1}{p-a} = e^{at}$ for $p > a$, and $p > 4$ in this question,

$$y(t) = \mathscr{L}^{-1}Y(p) = -\frac{1}{6}e^t + \frac{1}{2}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{4t}$$

So the solution is $y(t) = -\frac{1}{6}e^t + \frac{1}{2}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{4t}$

ii)$y'' + y' + y = H(t-\pi) - H(t-2\pi), y(0) = 1, y'(0) = 0$

Set $Y(p) = (\mathscr{L}y)(p)$, then

$$(\mathscr{L}y')(p) = p\cdot(\mathscr{L}y)(p) - y(0) = pY(p) - 1$$
$$(\mathscr{L}y'')(p) = p\cdot(\mathscr{L}y')(p) - y'(0) = p^2Y(p) - p$$

and

$$\Big(\mathscr{L}(H(t-\pi) - H(t-2\pi))\Big)(p) = \frac{e^{-\pi p}}{p} - \frac{e^{-2\pi p}}{p}, p > 0$$

So apply Laplace transform to the equation and we get

$$(p^2+p+1)Y(p) = \frac{e^{-\pi p}}{p} - \frac{e^{-2\pi p}}{p} + p + 1$$

$$\Rightarrow Y(p) = \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} (\frac{e^{-\pi p}}{p} - \frac{e^{-2\pi p}}{p} + \frac{1}{2}) + \frac{p+\frac{1}{2}}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow Y(p) = \frac{2}{\sqrt{3}} \mathscr{L}(e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t))\mathscr{L}(H(t-\pi)-H(t-2\pi)) + \frac{1}{\sqrt{3}}\mathscr{L}(e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t))$$
$$+\mathscr{L}(e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t))$$

For $t \geqslant 2\pi$

$$\begin{aligned}
& e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t) * ((H(t-\pi)-H(t-2\pi)) \\
=& \int_0^t e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot ((H(s-\pi)-H(s-2\pi))ds \\
=& \int_0^t e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot ((H(s-\pi)-H(s-2\pi))ds \\
=& \int_0^\pi e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 0ds + \int_\pi^{2\pi} e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 1ds \\
&+ \int_{2\pi}^t e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 0ds \\
=& e^{\frac{1}{2}(s-t)}(\sin(\frac{\sqrt{3}}{2}(s-t)) + \sqrt{3}\cos\frac{\sqrt{3}}{2}(s-t))\Big|_\pi^{2\pi} \\
=& 2e^{\frac{1}{2}(2\pi-t)}\sin(\frac{\sqrt{3}}{2}(2\pi-t)+\frac{\pi}{3}) - 2e^{\frac{1}{2}(\pi-t)}\sin(\frac{\sqrt{3}}{2}(\pi-t)+\frac{\pi}{3})
\end{aligned}$$

For $\pi \leqslant t < 2\pi$

$$\begin{aligned}
& e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t) * ((H(t-\pi)-H(t-2\pi)) \\
=& \int_0^\pi e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 0ds + \int_\pi^t e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 1ds \\
=& e^{\frac{1}{2}(s-t)}(\sin(\frac{\sqrt{3}}{2}(s-t)) + \sqrt{3}\cos\frac{\sqrt{3}}{2}(s-t))\Big|_\pi^t \\
=& \sqrt{3} - e^{\frac{1}{2}(\pi-t)}\sin(\frac{\sqrt{3}}{2}(\pi-t)+\frac{\pi}{3})
\end{aligned}$$

For $t < \pi$

$$\begin{aligned}
& e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t) * ((H(t-\pi)-H(t-2\pi)) \\
=& \int_0^\pi e^{-\frac{1}{2}(t-s)}\sin(\frac{\sqrt{3}}{2}(t-s)) \cdot 0ds \\
=& 0
\end{aligned}$$

So

$$y(t)=\begin{cases}\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}), t<\pi\\ \sqrt{3}-e^{\frac{1}{2}(\pi-t)}\sin(\frac{\sqrt{3}}{2}(\pi-t)+\frac{\pi}{3})+\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}), \pi\leqslant t<2\pi\\ 2e^{\frac{1}{2}(2\pi-t)}\sin(\frac{\sqrt{3}}{2}(2\pi-t)+\frac{\pi}{3})-2e^{\frac{1}{2}(\pi-t)}\sin(\frac{\sqrt{3}}{2}(\pi-t)+\frac{\pi}{3}))+\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}), 2\pi\leqslant t\end{cases}$$

iii)$y''+y=\begin{cases}\cos t, 0\leqslant t\leqslant \pi/2\\ 0, \pi/2\leqslant t<\infty\end{cases}=\cos t\cdot H(\frac{\pi}{2}-t), y(0)=3, y'(0)=-1$

Set $Y(p)=(\mathscr{L}y)(p)$, then

$$(\mathscr{L}y')(p)=p\cdot(\mathscr{L}y)(p)-y(0)=pY(p)-3$$
$$(\mathscr{L}y'')(p)=p\cdot(\mathscr{L}y')(p)-y'(0)=p^2Y(p)-3p+1$$

So apply Laplace transform to the equation and we get

$$\begin{aligned}&(p^2+1)Y(p)=\mathscr{L}(\cos t\cdot H(\frac{\pi}{2}-t))+3p-1\\ \Rightarrow&Y(p)=\frac{1}{p^2+1}\cdot\mathscr{L}(\cos t\cdot H(\frac{\pi}{2}-t))+3\frac{p}{p^2+1}-\frac{1}{p^2+1}\\ \Rightarrow&Y(p)=\mathscr{L}(\sin t)\cdot\mathscr{L}(\cos t\cdot H(\frac{\pi}{2}-t))+3\mathscr{L}(\cos t)-\mathscr{L}(\sin t)\end{aligned}$$

For $t\geqslant\frac{\pi}{2}$

$$\begin{aligned}&(\sin t)*(\cos t\cdot H(\frac{\pi}{2}-t))\\ =&\int_0^t(\sin(t-s)(\cos s\cdot H(\frac{\pi}{2}-s))ds=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin t+\sin(t-2s)ds+0\\ =&\frac{1}{2}(s\sin t\big|_0^{\frac{\pi}{2}}+\frac{1}{2}\cos(t-2s)\big|_0^{\frac{\pi}{2}})\\ =&\frac{\pi}{4}\sin t-\frac{1}{2}\cos t\end{aligned}$$

For $0\leqslant t<\frac{\pi}{2}$

$$\begin{aligned}&(\sin t)*(\cos t\cdot H(\frac{\pi}{2}-t))\\ =&\int_0^t(\sin(t-s)(\cos s\cdot H(\frac{\pi}{2}-s))ds=\frac{1}{2}\int_0^{t}\sin t+\sin(t-2s)ds\\ =&\frac{1}{2}(s\sin t\big|_0^{t}+\frac{1}{2}\cos(t-2s)\big|_0^{t})\\ =&\frac{t}{2}\sin t\end{aligned}$$

So

$$y(t)=\begin{cases}(\frac{\pi}{4}-1)\sin t+\frac{5}{2}\cos t, t\geqslant\frac{\pi}{2}\\ (\frac{t}{2}-1)\sin t+3\cos t, 0\leqslant t<\frac{\pi}{2}\end{cases}$$