VV286 RC6

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November 15, 2017

The (Unilateral) Lapalace Transform

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$\sup_{t \in [0,\infty)} e^{-\beta t}|f(t)| < \infty \quad \text{for some } \beta \geqslant 0$$

Then the function $F : (\beta, \infty) \to \mathbb{R}$,

$$F(p) = (\mathscr{L} f)(p) = \int_0^\infty e^{-pt} f(t) dt$$

is called the Lapalace transform of f.

Table of Laplace Transforms

$g(t)$	$\mathcal{L}g(p)$	Comment
1	$\dfrac{1}{p}$	$p > 0$
$\delta(t)$	1	
t^n	$\dfrac{n!}{p^{n+1}}$	$n \in \mathbb{N},\ p > 0$
e^{at}	$\dfrac{1}{p-a}$	$p > a$
$\sin(bt)$	$\dfrac{b}{p^2+b^2}$	$b \in \mathbb{R},\ p > 0$
$\cos(bt)$	$\dfrac{p}{p^2+b^2}$	$b \in \mathbb{R},\ p > 0$
$H(t-a)$	$\dfrac{e^{-ap}}{p}$	$p > 0$
$f(t-a)H(t-a)$	$e^{-ap}\mathcal{L}f(p)$	
$e^{at}f(t)$	$\mathcal{L}f(p-a)$	
$f(at)$	$\dfrac{1}{a}\mathcal{L}f\left(\dfrac{p}{a}\right)$	
$f^{(n)}(t)$	$p^n\mathcal{L}f(p) - p^{n-1}f(0) - \dots f^{(n-1)}(0)$	$n \in \mathbb{N}$
$(-t)^n f(t)$	$\mathcal{L}f^{(n)}(p)$	$n \in \mathbb{N}$

Heaviside function

$$H : \mathbb{R} \mapsto \mathbb{R}, \qquad H(t) = \begin{cases} 1, t > 0 \\ 0, t \leqslant 0 \end{cases}$$

Delta function (not a function at all)

1. For $t \neq 0$,
$$\delta(t) = 0$$
2. $0 \in I \subset \mathbb{R}$
$$\int_I \delta(t) f(t) dt = f(0)$$

Inverting the Lapalace Transform

The Bromwich Integral

Let $\Omega \subset \mathbb{C}$ be an open set, $\beta \in \mathbb{R}$ and $F : \Omega \to \mathbb{C}$ analytic for all $z \in \mathbb{C}$ with $\mathrm{Re}z \geqslant \beta$. Then the Bromwich integral of F is

$$(\mathscr{M}F)(t) = \frac{1}{2\pi i}\int_{\mathcal{C}^*} e^{pt}F(p)dp = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}F(p)dp$$

1. For $t > 0$, the Bromwich integral $\mathscr{M}F(t)$ is usually calculated by closing the contour on the left and applying the residue theorem.
2. For $t < 0$ we close the contour on the right.

(To use Jordan's Lemma)

Convolution

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds$$

$$(\mathscr{L})(f * g) = (\mathscr{L} f) \cdot (\mathscr{L} g)$$

Solving an ODE with the Laplace Transform

To deal with discontinuous inhomogeneities and even inhomogeneities that are not functions at all.

$$ay'' + by' + cy = f(x), \quad y(0) = y_0, \quad y'(0) = y_1$$

1.

Apply the Laplace transform to both sides of the ODE/IVP;

$$\begin{aligned}(\mathscr{L}f')(p) &= \int_0^\infty e^{-pt} f'(t)dt = \int_0^\infty pe^{-pt} f(t)dt - f(0) \\ &= p \cdot (\mathscr{L}f)(p) - f(0)\end{aligned}$$

$$(\mathscr{L}f'')(p) = p^2(\mathscr{L}f)(p) - p \cdot f(0) - f'(0)$$

$$(ap^2 + bp + c)Y - (ap + b)y_0 - ay_1 = (\mathscr{L}f)(p)$$

2.

$$Y = (\mathscr{L} f)(p) \cdot \frac{1}{ap^2 + bp + c} + \frac{ay_0 p + by_0 + ay_1}{ap^2 + bp + c}$$

Find $g(x)$ such that $(\mathscr{L} g)(p) = \dfrac{1}{ap^2 + bp + c}$

The function g is called a Green's function for the differential equation.

3.

Use transform table and apply convolution to find inverse Laplace transform.

Example

$$y''+y=\begin{cases}\cos t, 0\leqslant t\leqslant \pi/2\\ 0, \pi/2\leqslant t<\infty\end{cases}=\cos t\cdot H(\frac{\pi}{2}-t), y(0)=3, y'(0)=-1$$

Solution

Set $Y(p) = (\mathcal{L}y)(p)$, then

$$(\mathcal{L}y')(p) = p \cdot (\mathcal{L}y)(p) - y(0) = pY(p) - 3$$

$$(\mathcal{L}y'')(p) = p \cdot (\mathcal{L}y')(p) - y'(0) = p^2 Y(p) - 3p + 1$$

So apply Laplace transform to the equation and we get

$$(p^2 + 1)Y(p) = \mathcal{L}(\cos t \cdot H(\frac{\pi}{2} - t)) + 3p - 1$$

$$\Rightarrow Y(p) = \frac{1}{p^2 + 1} \cdot \mathcal{L}(\cos t \cdot H(\frac{\pi}{2} - t)) + 3\frac{p}{p^2 + 1} - \frac{1}{p^2 + 1}$$

$$\Rightarrow Y(p) = \mathcal{L}(\sin t) \cdot \mathcal{L}(\cos t \cdot H(\frac{\pi}{2} - t)) + 3\mathcal{L}(\cos t) - \mathcal{L}(\sin t)$$

For $t \geqslant \frac{\pi}{2}$

$$
\begin{aligned}
&(\sin t) * (\cos t \cdot H(\frac{\pi}{2} - t)) \\
=&\int_0^t (\sin(t-s)(\cos s \cdot H(\frac{\pi}{2} - s))ds \\
=&\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin t + \sin(t-2s)ds \\
=&\frac{1}{2}(s\sin t\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\cos(t-2s)\Big|_0^{\frac{\pi}{2}}) \\
=&\frac{\pi}{4}\sin t - \frac{1}{2}\cos t
\end{aligned}
$$

For $0 \leqslant t < \frac{\pi}{2}$

$$
\begin{aligned}
&(\sin t) * (\cos t \cdot H(\frac{\pi}{2} - t)) \\
=&\int_0^t (\sin(t-s)(\cos s \cdot H(\frac{\pi}{2} - s))ds = \frac{1}{2}\int_0^t \sin t + \sin(t-2s)ds \\
=&\frac{1}{2}(s \sin t|_0^t + \frac{1}{2}\cos(t-2s)|_0^t) \\
=&\frac{t}{2}\sin t
\end{aligned}
$$

So

$$y(t) = \begin{cases} (\frac{\pi}{4} - 1)\sin t + \frac{5}{2}\cos t, t \geqslant \frac{\pi}{2} \\ (\frac{t}{2} - 1)\sin t + 3\cos t, 0 \leqslant t < \frac{\pi}{2} \end{cases}$$

The Fourier Transform

Let $f : \mathbb{R} \to \mathbb{C}$, if f is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

Then

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

exists for all $\xi \in \mathbb{R}$. $\widehat{f}(\xi)$ is called the Fourier Transform of f.

Properties

1. $$\widehat{(f')}(\xi) = i\xi \cdot \hat{f}(\xi)$$

2. $$\frac{d}{d\xi}\hat{f}(\xi) = \widehat{(-ix)f}(\xi)$$

3. $$\widehat{(f(ax))}(\xi) = \frac{1}{|a|}\widehat{f(x)}(\frac{\xi}{a})$$

4. $$\widehat{(e^{iat}f)}(\xi) = \hat{f}(\xi - a)$$

Decay

Let $\Omega \subset \mathbb{R}$ be bounded and $f : \mathbb{R} \setminus \Omega \to \mathbb{C}$.

1. If $f(x) = O(x^{-n})$ as $|x| \to \infty$ for some $n > 0$, then f is said to have polynomial decay at infinity.
2. If $f(x) = O(x^{-n})$ as $|x| \to \infty$ for all $n > 0$, then f is said to have faster-than-polynomial decay at infinity.
3. If $f(x) = O(e^{-b|x|})$ as $|x| \to \infty$ for some $b > 0$, then f is said to have exponential decay at infinity.

Decay Behavior of Fourier Transform

1. If $f \in C^\infty(\mathbb{R})$ and all derivatives of f are absolutely integrable, then $\widehat{f}$ has faster-than-polynomial decay at infinity.
2. If f is analytic on $S_a = \{z \in \mathbb{C} : |\mathrm{Im} z| < a\}$ for some $a > 0$ and there exists a constant $A > 0$ such that

$$f(z) = f(x + iy) \leqslant \frac{A}{1 + x^2}$$

for all $z = x + iy \in S_a$, (denote the set of all such kind of f as $\mathscr{F}_a$), then for any $0 \leqslant b < a$ there exists a constant $B > 0$ such that

$$|\widehat{f}(\xi)| \leqslant Be^{-b|\xi|}$$

for all $\xi \in \mathbb{R}$. ($\widehat{f}$ has exponential decay at infinity.)

Fourier Inversion Theorem

$\forall f \in \mathscr{F} = \bigcup_{\forall a \in \mathbb{R}^+} \mathscr{F}_a$, $\widehat{f}$ exists and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi$$

for all $x \in \mathbb{R}$

The Complex Fourier Transform

For $f : \mathbb{C} \to \mathbb{C}$ define the Fourier transform of f at $\xi + i\eta \in \mathbb{C}$ by

$$\widehat{f}(\xi + i\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\xi + i\eta)x} dx$$

Let $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) = O(e^{-b|x|})$ as $|x| \to \infty$ for some $b > 0$. Then $\widehat{f}$ exists and is analytic in the strip $S_b = \{z \in \mathbb{C} : |\mathrm{Im} z| < b\}$.

Fourier Inversion Formula

$f(x)$ is piecewise continuous, absolutely integrable, and continuously differentiable on each small interval (a_k, a_{k+1})

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi} d\xi$$

$$f(x^+) = \lim_{y \searrow x} f(y), \quad f(x^-) = \lim_{y \nearrow x} f(y)$$

Lapalace Transform

$$(\mathscr{L} f)(p) = \int_0^\infty e^{-pt} f(t) dt$$

$$(\mathscr{M} F)(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{pt} (\mathscr{L} f)(p) dp$$

Fourier Transform

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi$$

Series Methods for Second-Order Equations

Set $x(t) = \sum\limits_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k$$

Insert to find relation among coefficients a_k, a_{k+1}, a_{k+2}. If we can find two independent solutions (usually no t before x'', the easiest case), then we have done; else

Set $x(t) = t^r \sum_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=0}^{\infty} (k+r) a_k t^{k+r-1}$$

$$x''(t) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r-2}$$

Insert to solve r.

1. Two distinct real roots r_1 and r_2, $r_1 - r_2 \notin \mathbb{Z}$

$$x_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n, \quad x_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n$$

2. Two distinct real roots r_1 and r_2, $r_1 - r_2 \in \mathbb{Z}$

$$\begin{aligned}
& x_2(t) \\
=& \frac{\partial}{\partial r}\left(t^r \sum_{n=0}^{\infty} a_n(r) t^n\right)\Bigg|_{r=r_2} = t^r \ln t \sum_{n=0}^{\infty} a_n(r) t^n + t^r \sum_{n=0}^{\infty} a_n'(r) t^n \Bigg|_{r=r_2} \\
=& c \cdot x_1(t) \ln t + t^{r_2} \sum_{n=0}^{\infty} a_n'(r_2) t^n
\end{aligned}$$

Especially $c = 1$ for $r_1 = r_2$.

3. Complex root

$$x_1(t) = \mathrm{Re}\left(t^{r_1}\sum_{n=0}^{\infty} a_n t^n\right), \quad x_2(t) = \mathrm{Im}\left(t^{r_2}\sum_{n=0}^{\infty} b_n t^n\right)$$