VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 5

Jiang Yicheng
515370910224

November 14, 2016

Exercise 5.1

i)

If $f(x+iy) = u(x,y) + v(x,y)i$ is harmonic, then

$$\begin{aligned} 0 = \Delta f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \\ &= \frac{\partial^2}{\partial x^2}(u+vi) + \frac{\partial^2}{\partial y^2}(u+vi) \\ &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2} \\ &= \Delta u + i(\Delta v) \end{aligned}$$

Since $u, v : \Omega \to \mathbb{R}$, $\Delta u, \Delta v \in \mathbb{R}, u, v \in C^2(\mathbb{R})$. So $\Delta u + i(\Delta v) = 0 \Leftrightarrow \Delta u = 0 \wedge \Delta v = 0$. So u, v are harmonic.

ii)

For a function v which satisfies the Cauchy-Riemann differential equations with u

$$\begin{aligned} \Delta f &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} + i\frac{\partial}{\partial y}\frac{\partial u}{\partial x} \end{aligned}$$

Since u, v are potential functions, $\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}, \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x}$. So $f(x+yi) = u(x,y) + iv(x,y) \in C^2(\Omega), \Delta f = 0$, i.e. f is harmonic.

According to Cauchy-Riemann differential equations, the harmonic conjugate of u satisfies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $u(x,y) = x^3 - 3xy^2$,

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2, \frac{\partial v}{\partial x} = -(-6xy) = 6xy$$

so

$$v = \int 6xy dx = 3x^2 y + C(y)$$

where $C(y)$ is a real function of y only. So

$$3x^2 - 3y^2 = \frac{\partial v}{\partial y} = 3x^2 + \frac{\partial C(y)}{\partial y}$$

So

$$\frac{\partial C(y)}{\partial y} = -3y^2 \Leftrightarrow C(y) = \int -3y^2 dy = -y^3 + C$$

where $C \in \mathbb{R}$ is a constant.

To sum up, a harmonic conjugate of u is

$$v(x,y) = 3x^2 y - y^3$$

Exercise 5.2

Since $|a| < r < |b|$, series $\sum_{i=0}^{\infty}\left(\frac{a}{z}\right)^i$ and $\sum_{j=0}^{\infty}\left(\frac{z}{b}\right)^j$ are convergent and therefore

$$\begin{aligned}
\oint_\gamma \frac{1}{(z-a)(z-b)}dz &= \oint_\gamma \frac{1}{bz(1-\frac{a}{z})(\frac{z}{b}-1)}dz \\
&= -\oint_\gamma \frac{1}{bz}\sum_{i=0}^{\infty}\left(\frac{a}{z}\right)^i \cdot \sum_{j=0}^{\infty}\left(\frac{z}{b}\right)^j dz \\
&= -\oint_\gamma \frac{1}{bz}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left(\frac{a}{z}\right)^j \cdot \left(\frac{z}{b}\right)^{i-j} dz \\
&= -\sum_{i=0}^{\infty}\sum_{j=0}^{i} a^j b^{j-i-1}\oint_\gamma z^{i-2j-1}dz
\end{aligned}$$

Choose parametrization as $\gamma : [0, 2\pi) \to (t), \gamma(t) = re^{it}, r > 0$, then

$$\oint_\gamma \frac{1}{z}dz = \int_0^{2\pi}\frac{ire^{it}}{re^{it}}dt = 2\pi i$$

$$\forall n \in \mathbb{Z}, n \neq -1, \oint_\gamma z^n dz = \int_0^{2\pi} ire^{i(n+1)t}dt = \frac{1}{n+1}re^{i(n+1)t}|_0^{2\pi} = 0$$

So only for $i-2j-1 = -1$, i.e. $i = 2j$, the integral will not vanish, and therefore

$$\begin{aligned}
\oint_\gamma \frac{1}{(z-a)(z-b)}dz &= -\sum_{i=0}^{\infty}\sum_{j=0}^{i} a^j b^{j-i-1}\oint_\gamma z^{i-2j-1}dz \\
&= -2\pi i\sum_{j=0}^{\infty} a^j b^{-j-1} = -\frac{2\pi i}{b}\sum_{j=0}^{\infty}\left(\frac{a}{b}\right)^j \\
&= -\frac{2\pi i}{b}\cdot\frac{1}{1-\frac{a}{b}} \\
&= \frac{2\pi i}{a-b}
\end{aligned}$$

So $\oint_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}$.

Exercise 5.3

Since e^{ix^2} is holomorphic in an open set containing the Γ_R, Cauchy's theorem gives

$$\int_0^R e^{ix^2}\ dx + \int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2} iRe^{it}dt + \int_R^0 e^{i(re^{i\frac{\pi}{4}})^2} r\ dr = 0$$

and we have

$$\int_0^R e^{i(re^{i\frac{\pi}{4}})^2} e^{i\frac{\pi}{4}}\ dr = e^{i\frac{\pi}{4}}\int_0^R e^{-r^2}\ dr \xlongequal{t=\sqrt{2}r} \frac{1}{2}(1+i)\cdot\frac{1}{2}\int_{-\sqrt{2}R}^{\sqrt{2}R} e^{-t^2/2}\ dt$$

then let $R \to \infty$ and we get that

$$\int_0^\infty e^{i(re^{i\frac{\pi}{4}})^2} e^{i\frac{\pi}{4}}\ dr = \frac{1}{4}(1+i) \cdot \sqrt{2\pi}$$

On the other hand,

$$\begin{aligned}\Big|\int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2} iRe^{it}dt\Big| &\leqslant R\int_0^{\frac{\pi}{4}} \left|e^{iR^2(cos(2t)+isin(2t))}\right|dt = R\int_0^{\frac{\pi}{4}} e^{-R^2 sin(2t)}dt \\ &\leqslant R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{2}{\pi}\cdot 2t}dt = -\frac{\pi R}{4R^2}e^{-R^2\frac{2}{\pi}\cdot 2t}\Big|_0^{\frac{\pi}{4}} = -\frac{\pi R}{4R^2}(e^{-R^2}-1) \\ &= \frac{(1-e^{-R^2})\pi}{4R} \leqslant \frac{\pi}{4R} \xrightarrow{R\to\infty} 0\end{aligned}$$

So

$$\int_0^\infty e^{ix^2}\ dx = -\lim_{R\to\infty}\int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2} iRe^{it}dt + \int_0^\infty e^{i(re^{i\frac{\pi}{4}})^2} r\ dr = \frac{\sqrt{2\pi}}{4} + \frac{\sqrt{2\pi}}{4}i$$

So $\int_0^\infty sin(x^2)\ dx = Im(\int_0^\infty e^{ix^2}\ dx) = \dfrac{\sqrt{2\pi}}{4}, \int_0^\infty cos(x^2)\ dx = Re(\int_0^\infty e^{ix^2}\ dx) = \dfrac{\sqrt{2\pi}}{4}$, i.e.

$$\int_0^\infty sin(x^2)\ dx = \int_0^\infty cos(x^2)\ dx = \frac{\sqrt{2\pi}}{4}$$

Exercise 5.4

Since $\dfrac{e^{iz}-1}{2iz}$ is holomorphic in an open set containing the semi circle, Cauchy's theorem gives

$$\int_{-R}^{-\varepsilon} \frac{e^{iz}-1}{2iz}\ dz + \oint_{-C_\varepsilon} \frac{e^{iz}-1}{2iz}\ dz + \int_\varepsilon^R \frac{e^{iz}-1}{2iz}\ dz + \oint_{C_R} \frac{e^{iz}-1}{2iz}\ dz = 0$$

Since,

$$\oint_{C_R} \frac{e^{iz}-1}{2iz}\ dz = \int_0^\pi \frac{e^{iRe^{it}}-1}{2iRe^{it}} iRe^{it}dt = \frac{1}{2}\int_0^\pi e^{iRcost-Rsint}dt - \frac{\pi}{2}$$

$$\begin{aligned}\Big|\int_0^\pi e^{iRcost-Rsint}dt\Big| &\leqslant \int_0^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\cdot t}dt + \int_{\frac{\pi}{2}}^\pi e^{R\frac{2}{\pi}\cdot(t-\pi)}dt \\ &= -\frac{\pi}{2R}e^{-R\frac{2}{\pi}\cdot t}\Big|_0^{\frac{\pi}{2}} + \frac{\pi}{2Re^{2R}}e^{R\frac{2}{\pi}\cdot t}\Big|_{\frac{\pi}{2}}^\pi \\ &= -\frac{\pi}{2R}(e^{-R}-1) + \frac{\pi}{2Re^{2R}}(e^{2R}-e^R) \\ &= \frac{\pi}{2R}(1-e^{-R}+1-e^{-R}) \xrightarrow{R\to\infty} 0\end{aligned}$$

So let $R \to \infty$ and we will obtain

$$\int_{-\infty}^{-\varepsilon} \frac{e^{iz}-1}{2iz}\ dz + \oint_{-C_\varepsilon} \frac{e^{iz}-1}{2iz}\ dz + \int_\varepsilon^\infty \frac{e^{iz}-1}{2iz}\ dz = \frac{\pi}{2}$$

Since $\dfrac{e^{iz}-1}{2iz}=\dfrac{1}{2i}\sum_{j=0}^{\infty}\dfrac{1}{(j+1)!}(iz)^j$,

$$\begin{aligned}&\Big|\oint_{-C_\varepsilon}\frac{e^{iz}-1}{2iz}\ dz\Big|\\ =&\Big|\int_{\pi}^{0}\frac{\sum_{j=1}^{\infty}\frac{1}{j!}(i\varepsilon e^{it})^j}{2i\varepsilon e^{it}}\cdot i\varepsilon e^{it}dt\Big|\leqslant\frac{1}{2}\int_0^{\pi}\sum_{j=1}^{\infty}\frac{1}{j!}|\varepsilon|^j dt\\ =&\frac{1}{2}\int_0^{\pi}e^{|\varepsilon|}-1dt=\frac{e^{|\varepsilon|}-1}{2}\pi\overset{|\varepsilon|\to 0}{\longrightarrow}0\end{aligned}$$

So let $\varepsilon\to 0$ and we will obtain

$$\frac{\pi}{2}=\int_{-\infty}^{0}\frac{e^{iz}-1}{2iz}\ dz+\int_0^{\infty}\frac{e^{iz}-1}{2iz}\ dz=\frac{1}{2i}\int_{-\infty}^{\infty}\frac{cosz}{z}dz+\frac{1}{2}\int_{-\infty}^{\infty}\frac{sinz}{z}dz-\int_{-\infty}^{\infty}\frac{1}{2iz}dz$$

Since $\int_{-\infty}^{\infty}\frac{1}{2iz}dz=0$, $\int_{-\infty}^{\infty}\frac{sinz}{z}dz=\pi$. Since $\frac{sinz}{z}$ is odd, $\int_{-\infty}^{0}\frac{sinz}{z}dz=\int_0^{\infty}\frac{sinz}{z}dz$. So

$$\int_0^{\infty}\frac{sinx}{x}dx=\frac{\pi}{2}$$

Exercise 5.5

i)

Since $\left|\dfrac{R^2}{z}\right|>R$, $\dfrac{f(\zeta)}{\zeta-R^2/\overline{z}}$ is holomorphic function on the disc D_0 centered at the origin and of radius R, so the integral of $\dfrac{f(\zeta)}{\zeta-R^2/\overline{z}}$ around the circle of radius R

$$\oint_{C_R}\frac{f(\zeta)}{\zeta-R^2/\overline{z}}dz=0$$

i.e.

$$\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-R^2/\overline{z}}\cdot iRe^{i\varphi}d\varphi=0$$

According to Cauchy's Integral Formula, $\forall|z|<R$

$$\begin{aligned}f(z)&=\frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{\zeta-z}dz=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}\cdot iRe^{i\varphi}d\varphi-\frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})Re^{i\varphi}}{Re^{i\varphi}-R^2/\overline{z}}d\varphi\\ &=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\varphi})\Big(\frac{Re^{i\varphi}}{Re^{i\varphi}-z}+\frac{\overline{z}}{Re^{-i\varphi}-\overline{z}}\Big)d\varphi\\ &=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\varphi})\Big(\frac{R^2-|z^2|}{(Re^{i\varphi}-z)(Re^{-i\varphi}-\overline{z})}\Big)d\varphi\\ &=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\varphi})\frac{1}{2}\Big(\frac{R^2-\overline{z}Re^{i\varphi}+zRe^{i\varphi}-|z|^2+R^2+\overline{z}Re^{i\varphi}-zRe^{i\varphi}-|z|^2|}{(Re^{i\varphi}-z)(Re^{-i\varphi}-\overline{z})}\Big)d\varphi\\ &=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\varphi})\frac{1}{2}\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}+\frac{Re^{-i\varphi}+\overline{z}}{Re^{-i\varphi}-\overline{z}}\Big)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\varphi})Re\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\Big)d\varphi\end{aligned}$$

To sum up,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})Re\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\Big)d\varphi$$

ii)

For $z = re^{i\theta}$,

$$\begin{aligned}
&Re\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\Big)\\
=&Re\Big(\frac{Re^{i\varphi}+re^{i\theta}}{Re^{i\varphi}-re^{i\theta}}\Big)\\
=&Re\Big(\frac{(Rcos\varphi+rcos\theta)+i(Rsin\varphi+rsin\theta)}{(Rcos\varphi-rcos\theta)+i(Rsin\varphi-rsin\theta)}\Big)\\
=&Re\Big(\frac{((Rcos\varphi+rcos\theta)+i(Rsin\varphi+rsin\theta))((Rcos\varphi-rcos\theta)-i(Rsin\varphi-rsin\theta))}{((Rcos\varphi-rcos\theta)+i(Rsin\varphi-rsin\theta))((Rcos\varphi-rcos\theta)-i(Rsin\varphi-rsin\theta))}\Big)\\
=&\frac{(Rcos\varphi+rcos\theta)(Rcos\varphi-rcos\theta)+(Rsin\varphi+rsin\theta)(Rsin\varphi-rsin\theta)}{(Rcos\varphi-rcos\theta)^2+(Rsin\varphi-rsin\theta)^2}\\
=&\frac{R^2cos^2\varphi-r^2cos^2\theta+R^2sin^2\varphi-r^2sin^2\theta}{(R^2+r^2-2Rr(cos\varphi cos\theta+sin\varphi sin\theta)}\\
=&\frac{R^2-r^2}{R^2-2Rrcos(\theta-\varphi)+r^2}
\end{aligned}$$

So

$$Re\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\Big) = \frac{R^2-r^2}{R^2-2Rrcos(\theta-\varphi)+r^2}$$

for $z = re^{i\theta}$.

Exercise 5.6

i)

Set $f(x+yi) := u(x,y)+iv(x,y)$ where $v(x,y)$ is a harmonic conjugate to the function $u(x,y)$. Then f is holomorphic on the disc centered at the origin and of radius 1. Then for any $0 \leqslant r < 1, \theta \in \mathbb{R}$

$$\begin{aligned}
&f(rcos\theta+irsin\theta)\\
=&\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})Re\Big(\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\Big)d\varphi\\
=&\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi+isin\varphi)\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi\\
=&\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(cos\varphi,sin\varphi)d\varphi+\frac{i}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)v(cos\varphi,sin\varphi)d\varphi
\end{aligned}$$

where $P_r(\theta-\varphi) = \dfrac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}$. Since u, v are real valued functions,

$$u(rcos\theta, rsin\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(cos\varphi,sin\varphi)d\varphi$$

ii)

For $r = 1$, it's the solution to the *Dirichlet problem*. Also from *i*) we can know that $u(rcos\theta, rsin\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(cos\varphi, sin\varphi)d\varphi = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)f(cos\varphi, sin\varphi)d\varphi$ for $r < 1$.

$$
\begin{aligned}
&\Delta u(x,y)\\
=&\Delta_{r,\theta}u(r,\theta) = \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\\
=&\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi\\
&+\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi\\
&+\frac{\partial^2}{\partial r^2}\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi\\
=&\frac{1}{r^2}\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)2r(1-r^2)\frac{4r(1+r^2)-(1+10r^2+r^4)cos(\theta-\varphi)+4r^2cos^3(\theta-\varphi)}{(1-2rcos(\theta-\varphi)+r^2)^4}d\varphi\\
&+\frac{1}{r}\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)\frac{-4r+2(1+r^2)cos(\theta-\varphi)}{(1-2rcos(\theta-\varphi)+r^2)^2}d\varphi\\
&+\frac{1}{2\pi}\int_0^{2\pi} f(cos\varphi + isin\varphi)\\
&\frac{(12r^4+8r^2-4)+(-4r^5-40r^3-4r)cos(\theta-\varphi)+(8r^4+32r^2+8)cos^2(\theta-\varphi)-16rcos^3(\theta-\varphi)}{(1-2rcos(\theta-\varphi)+r^2)^4}d\varphi\\
&=0
\end{aligned}
$$

So u is harmonic and therefore it's a solution to the Dirichlet problem for the unit disc.

Exercise 5.7

For $r < 1$,

$$
\begin{aligned}
&u(r,\theta)\\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\varphi)u(1,\varphi)d\varphi\\
=&-\frac{1}{2\pi}\int_{-\pi}^{0}\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi + \frac{1}{2\pi}\int_0^{\pi}\frac{1-r^2}{1-2rcos(\theta-\varphi)+r^2}d\varphi\\
=&-\frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}tan\frac{\varphi-\theta}{2}\Big)\Big|_{-\pi}^{0} + \frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}tan\frac{\varphi-\theta}{2}\Big)\Big|_{0}^{\pi}\\
=&\frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}tan\frac{\theta}{2}\Big) - \frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}cot\frac{\theta}{2}\Big) + \frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}tan\frac{\theta}{2}\Big)\\
&+\frac{1}{\pi}arctan\Big(\frac{1+r}{1-r}cot\frac{\theta}{2}\Big)\\
=&\frac{2}{\pi}arctan\Big(\frac{1+r}{1-r}tan\frac{\theta}{2}\Big)
\end{aligned}
$$

So

$$u(r,\theta) = \begin{cases} \frac{2}{\pi} arctan\left(\frac{1+r}{1-r} tan\frac{\theta}{2}\right), 0 \leqslant r < 1 \\ -1 \qquad , r = 1 \wedge -\pi \leqslant \theta < 0 \\ 1 \qquad , r = 1 \wedge 0 \leqslant \theta < \pi \end{cases}$$

```
a = RevolutionPlot3D[
  {2 / Pi * ArcTan[(1 + x) / (1 - x) * Tan[y / 2]]}, {x, 0, 1},
  {y, -Pi, Pi}]
```

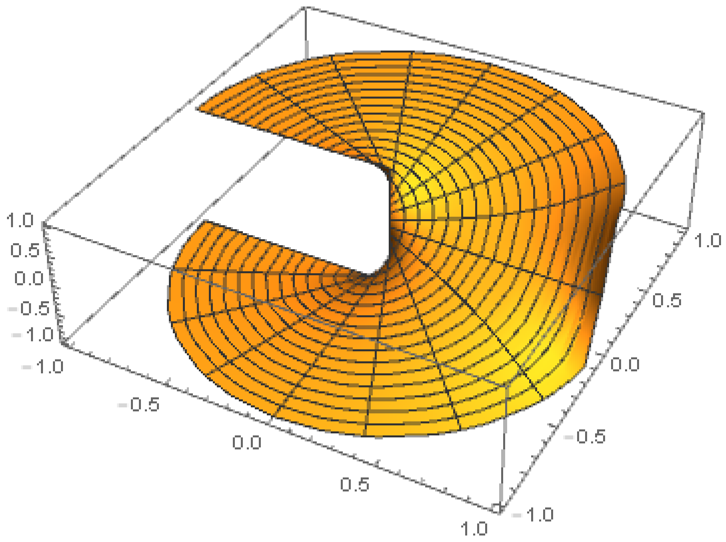

Figure 1: The figure of the solution to the Dirichlet problem

N