Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 4

Date Due: 10:00 AM, Thursday, the 20$^{\text{th}}$ of October 2016

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Exercise 4.1. The matrix

$$A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{pmatrix}$$

has the single eigenalue $\lambda = 3$. Find a basis of generalized eigenvectors and derive the Jordan normal form of A without explicitly multiplying $U^{-1}AU$.
(3 Marks)

Exercise 4.2. A particle of mass $m = 1$ travels in $\mathbb{R}^2$ under the influence of a constant linear force field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. Using Newton's second law to derive a differential equation for the position x and velocity v, verify that

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ F & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}, \qquad \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

where $F \in \operatorname{Mat}(2, \mathbb{R})$. Find the general solution for the system of equations when $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(4 Marks)

Exercise 4.3. We consider a linear chain of springs consisting of three mass points $m > 0$ that are linked by springs with spring constants $k > 0$. We denote the equilibirum positions of the mass points by r_1, r_2, r_3 and the positions at time by $x_1(t), x_2(t), x_3(t)$, respectively. Hence $d_j(t) := x_j(t) - r_j$ describes the displacement of the jth mass point at time t.

i) Use the laws of Newton and Hooke to derive the differential equation

$$\ddot{d} = Ad, \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \qquad A = \frac{k}{m} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{$**$}$$

ii) Transform $(**)$ into an equivalent first-order system $\binom{\dot{v}}{\dot{d}} = B\binom{v}{d}$ as in Exercise 4.3.

iii) For simplicity, set $k = m = 1$. Use Mathematica to find the eigenvalues, eigenvectors and the Jordan normal form J of B. Also obtain from Mathematica the matrix S such that $B = SJS^{-1}$.

iv) Obtain $\Phi(t) = e^{Bt} = Se^{Jt}S^{-1}$ from Mathematica. Verify that $\Phi(0) = \mathbb{1}$.

v) Extract the 6 linearly independent fundamental solutions $(d(t), v(t))$ und plot 6 pairs of graphs as follows: Plot the three curves $d_1(t), d_2(t), d_3(t)$ using different colors in the same graph. Then plot $v_1(t), v_2(t), v_3(t)$ together in a second graph. Do this for all six fundamental solutions.

vi) Let $r_1 = 4$, $r_2 = 8$ and $r_3 = 12$. Find the solution $(d(t), v(t))$ corresponding to the initial conditions $v(0) = (1, 0, -1)$ and $d(0) = (2, 0, 0)$ and plot the solution curves $x_1(t), x_2(t), x_3(t)$ together in a single graph, using different colors for each curve.

   Solution: The graph should look like the course title image on Canvas.

(2 + 1 + 3 + 2 + 4 + 3 Marks)

Exercise 4.4. Let $A \in \text{Mat}(n \times n, \mathbb{C})$ be any matrix and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A, counted with multiplicities. Show that

$$\det A = \prod_{i=1}^{n} \lambda_i, \qquad \operatorname{tr} A = \sum_{i=1}^{n} \lambda_i.$$

Furthermore, show that

$$\det(e^A) = e^{\operatorname{tr} A}.$$

(Hint: transform A to Jordan normal form and use the properties of the trace and determinant.)
(1 + 1 + 2 Marks)

Exercise 4.5. Find the solution to the initial value problem[1]

$$y''' + y' = \sec t \tan t, \qquad y''(0) = y'(0) = y(0) = 0.$$

(3 Marks)

Exercise 4.6. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -b^2/(4a^2) & -b/a \end{pmatrix}.$$

is not diagonalizable for any values of $a, b \in \mathbb{R}$.
(2 Marks)

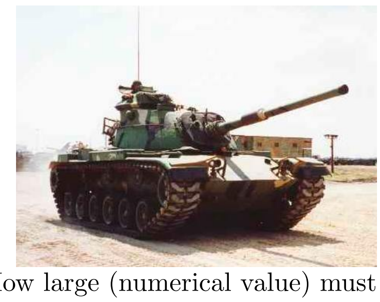
Exercise 4.7. Use the method of reduction of order to find the general solution to the following differential equations.[2]

$$y'' - \frac{2(t+1)}{t^2+2t-1}y' + \frac{2}{t^2+2t-1}y = 0, \qquad y_1(t) = t+1,$$
$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0, \qquad y_1(t) = \frac{\sin t}{\sqrt{t}}.$$

(2 × 2 Marks)

Exercise 4.8. A small object of mass 1 kg is attached to a spring with spring constant 1 N/m and is immersed in a viscous medium with damping constant 2 Ns/m. At time $t = 0$ the mass is lowered 1/4 m and given an initial velocity of 1 m/s in the upward direction. Show that the mass will overshoot its equilibrium position once, and then creep back to equilibrium.
(3 Marks)

Exercise 4.9. A small object of mass 4 kg is attached to an elastic spring with spring constant 64 N/m, and is acted upon by an external force $F(t) = A\cos^3(\omega t)$, $A > 0$. Find all values of ω for which resonance occurs.
(3 Marks)



Exercise 4.10. The gun of a U.S. M60 tank is attached to a spring-mass-dashpot system (i.e., a damped spring-mass system) with spring constant $100\alpha^2$ and damping constant 200α, in their appropriate units. The mass of the gun is 150 kg. Assume that the displacement $y(t)$ of the gun from its rest position after being fired at time $t = 0$ satisfies the initial value problem

$$150y'' + 200\alpha y' + 100\alpha^2 y = 0, \qquad y(0) = 0, \qquad y'(0) = 100\,\mathrm{m\,/\,s}.$$

It is desired that one second later, the quantity $y^2 + (y')^2$ be less than 0.01. How large (numerical value) must α be to guarantee that this is so? (The spring-mass-dashpot mechanism in the M60 tanks supplied by the U.S. to Israel are critically damped, for this situation is preferable in desert warfare were one has to fire again as quickly as possible.)[3]
(4 Marks)

[1] *Braun*, Section 3.12, Ex. 8
[2] *Braun*, Section 2.2, Ex. 10ff
[3] *Braun*, Section 2.6. Ex. 7; some numerical values have been changed.