Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 5

Date Due: 10:00 AM, Thursday, the 3rd of November 2016

Exercise 5.1. Let $\Omega \subset \mathbb{R}^2$ be a connected, open set. A function $u \in C^2(\Omega)$ such that $\Delta u = 0$ (where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator) is called *harmonic*.

i) Show that if f is a holomorphic function given by $f(x+iy) = u(x,y) + v(x,y)i$, where $u, v \colon \Omega \to \mathbb{R}$ are real functions, then u and v are harmonic.

ii) If u is harmonic, the Cauchy-Riemann differential equations define a harmonic function v such that $f(x+iy) := u(x,y) + v(x,y)i$ is holomorphic. This function v is called the *harmonic conjugate* of u. Find a harmonic conjugate to the function $u(x,y) = x^3 - 3xy^2$.

(2 + 2 Marks)

Exercise 5.2. Show by direct integration that if $|a| < r < |b|$, then

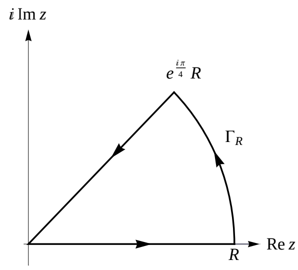
$$\oint_\gamma \frac{1}{(z-a)(z-b)} = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.
(1 Mark)

Exercise 5.3. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

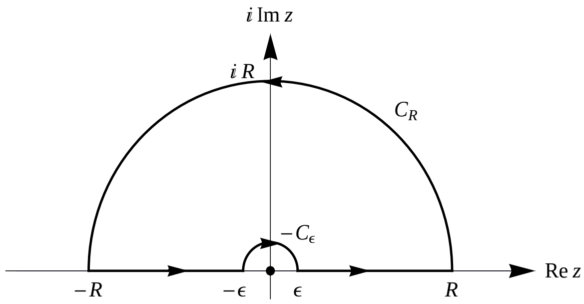
by integrating along the toy contour Γ_R (a *sector*) shown at right. These integrals are called the *Fresnel integrals*. They play an important role in optical scattering.
(2 Marks)



Exercise 5.4. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

by integrating the function $f(z) = (e^{iz} - 1)/(2iz)$ along the indented semi-circle shown at right.
(2 Marks)



Exercise 5.5. Let f be a holomorphic function on the disc D_0 centered at the origin and of radius R_0.

i) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi.$$

[Hint: Note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

ii) Show that

$$\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\varphi) + r^2} \qquad \text{for } z = re^{i\theta}.$$

(3 + 2 Marks)

Exercise 5.6. Let u be a real-valued function defined on the unit disc $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x,y) = 0 \qquad \text{for all } (x,y) \in D.$$

i) Deduce the *Poisson integral representation formula* from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(e^{i\varphi})\, d\varphi$$

where $P_r(\gamma)$ is the *Poison kernel* for the unit disc given by

$$P_r(\gamma) = \frac{1-r^2}{1 - 2r\cos\varphi + r^2}.$$

ii) The *Dirichlet problem for the unit disc* is a boundary value problem for the Laplace equation, viz.

$$\begin{aligned} \triangle u(x,y) &= 0, && (x,y) \in D, \\ u(x,y) &= f(x,y), && (x,y) \in \partial D = S^1. \end{aligned}$$

where $f \colon S^1 \to \mathbb{R}$ is a continuous function. Prove that a solution to the Dirichlet problem for the unit disc is given by

$$u(x,y) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) f(\cos\varphi, \sin\varphi)\, d\varphi & r < 1 \\ f(\cos\theta, \sin\theta) & r = 1 \end{cases} \qquad (*)$$

whenever $x = r\cos\theta$, $y = r\sin\theta$. (You need to show the converse of your previous arguments, i.e., that u defined by $(*)$ is harmonic in D. It is clear from the definition that this u satisfies the boundary condition $u|_{\partial D} = f$ and is then the unique solution to the Dirichlet problem.)

(2 + 2 Marks)

Exercise 5.7. Use the Poisson formula for the Dirichlet problem in the unit disk to find the solution of the Dirichlet problem

$$\Delta_{r,\theta} u(r,\theta) = 0, \qquad (r,\theta) \in (0,1) \times [-\pi, \pi), \qquad u(1,\varphi) = \begin{cases} -1 & -\pi \le \theta < 0, \\ 1 & 0 \le \theta < \pi, \end{cases}$$

in terms of elementary functions. Plot the graph of the solution using Mathematica. You may use that

$$\frac{1}{2\pi} \int \frac{1-r^2}{1 + r^2 - 2r\cos(t-\varphi)}\, dt = \frac{1}{\pi} \arctan\Big(\frac{1+r}{1-r} \tan\frac{t-\varphi}{2}\Big).$$

(Pay careful attention the branches of the arctangent. The solution will be a continuous function of φ.)
(4 Marks)