VV286 RC8

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Fourier Series

The space $L^2([a,b])$ is defined as the completion of $C([a,b])$ with respect to the norm $||\cdot||_2$

$$||f||_2 = \sqrt{\int_a^b |f(x)|^2 dx}$$

$$L^2([a,b]) := \left\{ f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 dx < \infty \right\}$$

with inner product

$$\langle f, g \rangle := \int_a^b \overline{f(x)} g(x) dx \qquad f, g \in L^2([a,b])$$

The orthonormal system basis in $L^2([-1,1])$

$$\mathcal{B}_{[-1,1]} = \left\{\frac{1}{\sqrt{2}}, \cos(\pi n x), \sin(\pi n x)\right\}_{n=1}^{\infty}$$

The orthonormal system basis in $L^2([a,b])$

$$\mathcal{B}_{[a,b]} = \left\{\tilde{e}_n(x) : \tilde{e}_n(x) = \sqrt{\frac{2}{b-a}} \cdot e_n\left(\frac{2}{b-a}\left(x - \frac{b+a}{2}\right)\right)\right\}_{n=1}^{\infty}$$

where $e_n(x) \in \mathcal{B}_{[-1,1]}$

The basis in $L^2([0, L])$

1. The Fourier-Euler Basis:

$$\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

2. The Fourier-Cosine Basis:

$$\mathcal{B}_2 = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

3. The Fourier-Sine Basis:

$$\mathcal{B}_3 = \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

The basis in $L^2([-\pi,\pi])$

The Fourier-Euler Basis:

$$\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \right\}_{n=1}^{\infty}$$

Complex Fourier-Euler Basis

Complex Fourier-Euler basis in $L^2([-\pi,\pi])$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty}$$

Complex Fourier-Euler basis in $L^2([-L,L])$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$$

The Fourier Expansion

For a function $f \in L^2([a, b])$,

$$\lim_{N\to\infty} S_N(x) = \sum_{e_n \in \mathcal{B}_{[a,b]}} \langle e_n(x), f(x) \rangle e_n(x)$$

$$= \sum_{e_n \in \mathcal{B}_{[a,b]}} \int_a^b f(x)\overline{e_n(x)} dx \cdot e_n(x)$$

Complex Fourier series for periodic function

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}x}$$

$$a_k = \frac{1}{T}\int_0^T f(x)e^{-jk\frac{2\pi}{T}x}dx$$

Dirichlet's rule

Let $f \in L^2([a,b])$ be piecewise continuously differentiable. Then

1. On any subinterval $[a', b'] \subset [a,b]$ with $a' > a, b' < b$ on which f is continuous the Fourier series converges uniformly towards f.
2. At any point $x \in [a,b]$, we have the pointwise limit

$$S_N(x) \xrightarrow{N\to\infty} \frac{\lim\limits_{y\nearrow x} f(y) + \lim\limits_{y\searrow x} f(y)}{2}$$

Using Fourier series to evaluate series

Example

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + 2\pi) = f(x)$ and

$$f(x) = e^x \qquad \text{for } -\pi < x < \pi$$

Find the Fourier series of f and use it to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2}$$

Solution

$$\int_{-\pi}^{\pi} f(x)\cdot\frac{1}{\sqrt{2\pi}}e^{-inx}\;dx=\int_{-\pi}^{\pi} e^{x}\cdot\frac{1}{\sqrt{2\pi}}e^{-inx}\;dx$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{1-ni}e^{x}e^{-inx}\Big|_{-\pi}^{\pi}$$

$$=\frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{1-ni}(e^{\pi}-e^{-\pi})$$

Let $x=\pi$, then

$$\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{(-1)^n}{1-ni}(e^{\pi}-e^{-\pi})\cdot\frac{1}{\sqrt{2\pi}}e^{inx}\Big|_{x=\pi}$$

$$=\frac{e^{\pi}-e^{-\pi}}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-ni}=\frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1+2\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right)$$

According to Dirichlet's rule,

$$\frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1+2\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right)=\frac{\lim\limits_{y\nearrow\pi}f(y)+\lim\limits_{y\searrow\pi}f(y)}{2}$$

$$=\frac{e^{\pi}+e^{-\pi}}{2}$$

So

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2}=\frac{1}{2}\left(\frac{\pi(e^{\pi}+e^{-\pi})}{e^{\pi}-e^{-\pi}}-1\right)$$

Partial Differential Equation

1. Use separation of variables

$$u(x_1, \cdots, x_n) = u_1(x_1) \cdot u_2(x_2) \cdots u_n(x_n)$$

2. Change the boundary condition
3. Solve one equation $Lu = \lambda u$ with initial consition and boundary condition (find eigenvalues together with eigenfunction)
4. Solve other equations
5. Let the whole solution satisfy boundary condition
   5.1 Usually expand boundary conditions to Fourier series and determine the coefficients

Example

Find the general solution of the Laplace equation $\triangle u = 0$ on the rectangle $\Omega = [0, a] \times [0, b]$ with boundary conditions

$$u(0, y) = 0, u(a, y) = 0, 0 < y < b$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, u(x, b) = g(x), 0 < x < a$$

where $g : [0, a] \to \mathbb{R}$ is a continuous function. Then find the solution if

$$g(x) = \begin{cases} x, 0 < x < a/2 \\ a - x, a/2 < x < a \end{cases}$$

Solution

We make a separation-of-variables ansatz $u(x,y) = X(x)Y(y)$, yielding

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow X''Y + XY'' = 0$$

and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

So

$$X'' = \lambda X, \, Y'' = -\lambda Y$$

with initial conditions and boundary conditions

$$X(0)Y(y) = 0, X(a)Y(y) = 0, 0 < y < b$$

$$X(x)Y'(0) = 0, X(x)Y(b) = g(x), 0 < x < a$$

To find non-trival solution, the condition can be changed to

$$X(0) = X(a) = Y'(0) = 0, X(x)Y(b) = g(x), 0 < x < a$$

For $X'' = \lambda X$, we use ansatz $X(x) = e^{\rho(\lambda)x}$, and therefore

$$\rho(\lambda)^2 = \lambda$$

1. $\lambda = 0$

$X(x) = c_0 + c_1 x$ to satisfy the initial conditions and boundary conditions, $X(x) = 0$

2. $\lambda > 0$

$$\rho(\lambda) = \pm\sqrt{\lambda} \Rightarrow X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Since $X(0) = X(a) = 0$,

$$c_1 + c_2 = 0, c_1 e^{\sqrt{\lambda}a} + c_2 e^{-\sqrt{\lambda}a} = 0 \Rightarrow c_1 = c_2 = 0$$

3. $\lambda < 0$

$$\rho(\lambda) = \pm\sqrt{-\lambda}i \Rightarrow X(x) = c_1 e^{i\sqrt{-\lambda}x} + c_2 e^{-i\sqrt{-\lambda}x}$$
$$= c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$$

Since $X(0) = X(a) = 0$,

$$c_2 = 0, c_1 \sin(\sqrt{-\lambda}a) + c_2 \cos(\sqrt{-\lambda}a) = 0 \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{a}$$

So

$$X_n(x) = c_n \sin(\frac{n\pi x}{a}), Y'' = \frac{n^2\pi^2}{a^2}Y$$

Then

$$Y_n(y) = d_n e^{n\pi y/a} + f_n e^{-n\pi y/a}$$

Considering $Y'(0) = 0$

$$d_n \cdot n\pi/a - f_n \cdot n\pi/a = 0 \Rightarrow d_n = f_n$$

So

$$u(x, y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} c_n d_n (e^{n\pi y/a} + e^{-n\pi y/a}) \sin(\frac{n\pi x}{a})$$

Expand $g(x)$ into Fourier sine series

$$\begin{aligned} &\int_0^a g(x) \sin(n\pi x/a) dx \\ =&\int_0^{a/2} x \sin(n\pi x/a) dx + \int_{a/2}^{a} (a - x) \sin(n\pi x/a) dx \\ =&\frac{2a^2}{n^2\pi^2} \sin(n\pi/2) \end{aligned}$$

So

$$g(x) = \sum_{n=1}^{\infty} \frac{2}{a} \left(\frac{2a^2}{n^2\pi^2} \sin(n\pi/2) \right) \sin(\frac{n\pi x}{a})$$

Since $u(x, b) = g(x)$

$$c_n d_n(e^{n\pi b/a} + e^{-n\pi b/a}) = \frac{4a}{n^2\pi^2} \sin(n\pi/2)$$

So

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4a \sin(n\pi/2)}{e^{n\pi b/a} + e^{-n\pi b/a}} (e^{n\pi y/a} + e^{-n\pi y/a}) \sin(\frac{n\pi x}{a})$$

Summary

- Use ansatz to find power series solution of non-constant coefficient 2nd-order ODE
  1. Normal one: $x(t) = \sum_{n=0}^{\infty} a_n t^n$
  2. Frobenius: $x(t) = t^r \sum_{n=0}^{\infty} a_n t^n$
     - 2.1 $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{N}$
     - 2.2 $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \in \mathbb{N}$:
       1)try;
       2)$\left.\dfrac{dx_1(t)}{dr}\right|_{r=r_2} \Rightarrow a_n'(r)$
     - 2.3 $r_1, r_2 \in \mathbb{C}$

- Change equation to Bessel Equation and find solution
  1. Given ~~(or find)~~ substitution (see *bessel.pdf*)

$$z = g(x) \xlongequal{\text{usually}} \beta x^{\gamma}, u(g(x)) = v(x, y(x)) \xlongequal{\text{usually}} x^{\alpha} y(x)$$

  2. If no information of order, calaulate

$$y'(x) = -\alpha x^{-\alpha-1} u(g(x)) + x^{-\alpha} \frac{du(z)}{dz}\bigg|_{z=g(x)} \frac{d(g(x))}{dx}$$
$$= -\alpha x^{-1} y(x) + x^{-\alpha} u'(z)|_{z=g(x)} g'(x)$$

$$y''(x) = \frac{d}{dx}(-\alpha x^{-1} y(x) + x^{-\alpha} u'(z)|_{z=g(x)} g'(x))$$

  3. Insert and find solution of corresponding Bessel equation $J_{\nu}(z), J_{-\nu}(z)$, then the soltuion is

$$y(x) = c_1 x^{-\alpha} J_{\nu}(g(x)) + c_2 x^{-\alpha} J_{-\nu}(g(x))$$

- Evaluate series using Fouries series
  1. According to domain of given function, use proper basis for Fourier series
     1.1 $[0, L]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi nx}{L}\right) \right\}_{n=1}^{\infty}$
     1.2 $[-\pi, \pi]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}$
     1.3 $[-L, L]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$
  2. Calculate the coefficients (Pay attention to constant!)
  3. Choose appropriate value for x and use ***Dirichlet's rule***

- Solve PDE (usually only two variables)
  1. Use separation of variables
     $$u(x_1, \cdots, x_m) = u_1(x_1) \cdot u_2(x_2) \cdots u_m(x_m)$$
  2. Change the boundary condition
  3. According to boundary condition, choose one equation $Lu_1 = \lambda u_1$ to solve. (find proper λ)
  4. Solve other equations for each λ
     $$(u_i)_n(x_i) = (c_i)_n \cdot f(\rho(\lambda)_n x_i)$$
  5. Let the whole solution satisfy boundary condition
     $$u(x_1, \cdots, x_m) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{m} (u_i)_n(x_i) \right)$$
     5.1 Usually expand boundary conditions to Fourier series and determine the coefficients