Convert to Bessel Equation

Given

$$a_1 x^{b_1} y'' + a_2 x^{b_2} y' + a_3 x^{b_3} y = 0$$

use substitution

$$z = g(x) = \beta x^{\gamma}, u(g(x)) = x^{\alpha} y(x)$$

Calculate

$$\begin{aligned}
y'(x) =& \frac{d}{dx}(x^{-\alpha} u(g(x))) \\
=& -\alpha x^{-\alpha-1} u(g(x)) + x^{-\alpha} \frac{du(z)}{dz}\Big|_{z=g(x)} \frac{dg(x)}{dx} \\
=& -\alpha x^{-\alpha-1} u(g(x)) + \beta\gamma x^{-\alpha+\gamma-1} u'(g(x)) \\
=& -\alpha x^{-1} y(x) + \beta\gamma x^{-\alpha+\gamma-1} u'(g(x))
\end{aligned}$$

$$\begin{aligned}
y''(x) =& \alpha x^{-2} y(x) - \alpha x^{-1} y'(x) + \beta\gamma(-\alpha+\gamma-1) x^{-\alpha+\gamma-2} u'(g(z)) \\
&+ \beta\gamma x^{-\alpha+\gamma-1} u''(g(x)) \cdot \frac{dg(x)}{dx} \\
=& (\alpha+\alpha^2) x^{-\alpha-2} u(g(x)) + \beta\gamma(-2\alpha+\gamma-1) x^{-\alpha+\gamma-2} u'(g(x)) \\
&+ \beta^2\gamma^2 x^{-\alpha+2\gamma-2} u''(g(x))
\end{aligned}$$

Insert into initial equation

$$\begin{aligned}
0 =& a_1 x^{b_1} y'' + a_2 x^{b_2} y' + a_3 x^{b_3} y \\
=& a_1(\alpha+\alpha^2) x^{b_1-\alpha-2} u(g(x)) + a_1\beta\gamma(-2\alpha+\gamma-1) x^{b_1-\alpha+\gamma-2} u'(g(x)) \\
&+ a_1\beta^2\gamma^2 x^{b_1-\alpha+2\gamma-2} u''(g(x)) \\
&+ (-a_2\alpha) x^{b_2-\alpha-1} u(g(x)) + a_2\beta\gamma x^{b_2-\alpha+\gamma-1} u'(g(x)) + a_3 x^{b_3-\alpha} u(g(x)) \\
=& a_1\beta^2\gamma^2 x^{b_1-\alpha+2\gamma-2} u''(g(x)) \\
&+ \Big(a_1\beta\gamma(-2\alpha+\gamma-1) x^{b_1-\alpha+\gamma-2} + a_2\beta\gamma x^{b_2-\alpha+\gamma-1}\Big) u'(g(x)) \\
&+ \Big(a_1(\alpha+\alpha^2) x^{b_1-\alpha-2} - a_2\alpha x^{b_2-\alpha-1} + a_3 x^{b_3-\alpha}\Big) u(g(x))
\end{aligned}$$

we first notice that

$$\frac{b_1-\alpha+2\gamma-2}{\gamma} = \frac{b_1-\alpha+\gamma-2}{\gamma} + 1 = \frac{b_1-\alpha-2}{\gamma} + 2$$

Compare with Bessel equation of order ν

$$z^2 u''(z) + z u'(z) + (z^2 - \nu^2) u(z) = 0$$

then we must have

$$\begin{cases} a_2 = 0 \vee b_1 - \alpha + \gamma - 2 = b_2 - \alpha + \gamma - 1 \\ (b_3 - \alpha)/\gamma = (b_1 - \alpha - 2)/\gamma + 2 \end{cases} \Rightarrow \begin{cases} a_2 = 0 \vee b_2 = b_1 - 1 \\ \gamma = \dfrac{b_3 - b_1 + 2}{2} \end{cases}$$

Consider the coefficients

$$a_1\beta^2\gamma^2 \cdot \beta^{-\frac{b_1-\alpha+2\gamma-2}{\gamma}} = \Big(a_1\beta\gamma(-2\alpha+\gamma-1)+a_2\beta\gamma\Big)\cdot\beta^{-\frac{b_1-\alpha+\gamma-2}{\gamma}} = a_3\cdot\beta^{-\frac{b_3-\alpha}{\gamma}}$$

i.e.

$$a_1\beta^2\gamma^2 = \beta\gamma(a_1(-2\alpha+\gamma-1)+a_2)\beta = a_3$$

so

$$\alpha = \frac{a_2}{2a_1}-\frac{1}{2}, \beta = \sqrt{\frac{a_3}{a_1}}\frac{1}{\gamma}$$

To sum up,

$$\alpha = \frac{a_2}{2a_1}-\frac{1}{2}, \beta = \sqrt{\frac{a_3}{a_1}}\frac{2}{b_3-b_1+2}, \gamma = \frac{b_3-b_1+2}{2}$$

Constraints: $a_2 = 0 \vee b_1 - b_2 = 1$

Note: Any other change to the initial equation may lead to failure by use this substitution. Only if there are some more constraints on a_i, b_i.