VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 8

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Exercise 8.2

i)

Set $x(t) = \sum\limits_{k=0}^{\infty} a_k t^k$, then

$$tx'(t) = t\sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}t^{k+1}$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k$$

Since

$$x'' - 2tx' + \lambda x = 0$$

then we can obtain that

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k - 2\sum_{k=0}^{\infty}(k+1)a_{k+1}t^{k+1} + \lambda\sum_{k=0}^{\infty} a_k t^k = 0$$

i.e.

$$(2a_2 + \lambda a_0) + \sum_{k=0}^{\infty}((k+3)(k+2)a_{k+3} - 2(k+1)a_{k+1} + \lambda a_{k+1})t^{k+1} = 0$$

So

$$\begin{cases} 2a_2 + \lambda a_0 = 0 \\ (k+3)(k+2)a_{k+3} + (\lambda - 2(k+1))a_{k+1} = 0, k \in \mathbb{N} \end{cases}$$

So for $k = 2j+1, j \geqslant 1$

$$a_{2j+1} = \frac{4j-2-\lambda}{(2j+1)(2j)}a_{2j-1} = \frac{\prod\limits_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!}a_1$$

for $k = 2j+2, j \geqslant 0$

$$a_2 = \frac{-\lambda}{2}a_0 = \frac{0-\lambda}{2}a_0$$

$$a_{2j+2} = \frac{4j-\lambda}{(2j+2)(2j+1)}a_{2j} = \frac{\prod\limits_{i=1}^{j}(4i-\lambda)}{(2j+2)!/2}a_2 = \frac{\prod\limits_{i=0}^{j}(4i-\lambda)}{(2j+2)!}a_0 \quad (j \geqslant 1)$$

Set $a_0 = c_1, a_1 = 0$ and we get

$$x_1(t) = c_1 + \sum_{j=0}^{\infty}\frac{\prod\limits_{i=0}^{j}(4i-\lambda)}{(2j+2)!}c_1 t^{2j}$$

Set $a_0 = 0, a_1 = c_2$ and we get

$$x_2(t) = c_2 t + \sum_{j=1}^{\infty}\frac{\prod\limits_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!}c_2 t^{2j+1}$$

ii)

If $n = 2m+1, m \in \mathbb{N}$, then $\forall j \geqslant m+1$, since $\lambda = 2n = 4m+2$

$$a_{2j+1} = \frac{\prod_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!} a_1 = 0$$

and $\forall j \leqslant m$

$$a_{2j+1} = \frac{\prod_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!} a_1 \neq 0$$

and therefore if $m > 0$

$$x_2(t) = c_2 t + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!} c_2 t^{2j+1} = c_2 t + \sum_{j=1}^{m} \frac{\prod_{i=1}^{j}(4i-2-\lambda)}{(2j+1)!} c_2 t^{2j+1}$$

if $m = 0$, $x_2(t) = c_2 t$

This is a polynomial of degree n and it is a solution to Hermite equation.

If $n = 2m, m \in \mathbb{N}$, then $\forall j \geqslant m+1$, since $\lambda = 2n = 4m$

$$a_{2j+2} = \frac{\prod_{i=0}^{j}(4i-\lambda)}{(2j+2)!} a_0 = 0$$

and $\forall j \leqslant m$

$$a_{2j+2} = \frac{\prod_{i=0}^{j}(4i-\lambda)}{(2j+2)!} a_0 \neq 0$$

and therefore if $m > 0$

$$x_1(t) = c_1 + \sum_{j=0}^{\infty} \frac{\prod_{i=0}^{j}(4i-\lambda)}{(2j+2)!} c_1 t^{2j} = c_1 + \sum_{j=0}^{m} \frac{\prod_{i=0}^{j}(4i-\lambda)}{(2j+2)!} c_1 t^{2j}$$

if $m = 0$, $x_1(t) = c_1$ This is a polynomial of degree n and it is a solution to Hermite equation.

To sum up, the Hermite equation has a polynomial solution of degree n if $\lambda = 2n$.

Exercise 8.3

i)

Set $y(t) = t^r \sum_{k=0}^{\infty} a_k t^k$, then

$$y'(t) = \sum_{k=0}^{\infty} (k+r) a_k t^{k+r-1}$$

$$y''(t) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r-2}$$

Since

$$2ty'' + (1-2t)y' - y = 0$$

then we can obtain that

$$\begin{aligned}
0 =&2t\sum_{k=0}^{\infty}(k+r)(k+r-1)a_kt^{k+r-2} + (1-2t)\sum_{k=0}^{\infty}(k+r)a_kt^{k+r-1} - \sum_{k=0}^{\infty}a_kt^{k+r}\\
=&\sum_{k=0}^{\infty}2(k+r)(k+r-1)a_kt^{k+r-1} + \sum_{k=0}^{\infty}(k+r)a_kt^{k+r-1} - \sum_{k=0}^{\infty}2(k+r)a_kt^{k+r} - \sum_{k=0}^{\infty}a_kt^{k+r}\\
=&(2r(r-1)a_0 + ra_0)t^{r-1}\\
&+\sum_{k=0}^{\infty}(2(k+r+1)(k+r)a_{k+1} + (k+r+1)a_{k+1} - 2(k+r)a_k - a_k)t^{k+r}
\end{aligned}$$

So

$$\begin{cases} 2r(r-1)a_0 + ra_0 = 0 \\ (k+r+1)(2k+2r+1)a_{k+1} = (2k+2r+1)a_k, k\in\mathbb{N}^* \end{cases} \Leftrightarrow \begin{cases} r = 0 \vee r = \dfrac{1}{2} \\ a_{k+1} = \dfrac{1}{k+r+1}a_k, k\in\mathbb{N} \end{cases}$$

then for $r = 0$ we obtain that $\forall k\in\mathbb{N}, a_k = \dfrac{1}{k!}a_0$

$$x_1(t) = \sum_{k=0}^{\infty}a_kt^k = a_0\sum_{k=0}^{\infty}\frac{1}{k!}t^k = a_0e^t$$

for $r = \dfrac{1}{2}$ we obtain that $\forall k\in\mathbb{N}^*, a_k = a_0\prod\limits_{i=1}^{k}\dfrac{1}{i+\frac{1}{2}} = a_0\dfrac{2^k}{\prod\limits_{i=1}^{k}(2i+1)} = a_0\dfrac{2^k}{\prod\limits_{i=0}^{k}(2i+1)}$

$$x_2(t) = t^{\frac{1}{2}}\sum_{k=0}^{\infty}a_kt^k = a_0t^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{2^k}{\prod\limits_{i=0}^{k}(2i+1)}t^k$$

So two independent solutions of $2ty'' + (1-2t)y' - y = 0$ are

$$y_1(t) = c_1e^t, y_2(t) = c_2t^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{2^k}{\prod\limits_{i=0}^{k}(2i+1)}t^k, c_1, c_2\in\mathbb{R}$$

ii)

Set $y(t) = t^r\sum\limits_{k=0}^{\infty}a_kt^k$, then

$$y'(t) = \sum_{k=0}^{\infty}(k+r)a_kt^{k+r-1}$$

$$y''(t) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_kt^{k+r-2}$$

Since

$$t^2y'' + (t - t^2)y' - y = 0$$

then we can obtain that

$$\begin{aligned}
0 =& t^2\sum_{k=0}^{\infty}(k+r)(k+r-1)a_kt^{k+r-2} + (t-t^2)\sum_{k=0}^{\infty}(k+r)a_kt^{k+r-1} - \sum_{k=0}^{\infty}a_kt^{k+r}\\
=& \sum_{k=0}^{\infty}(k+r)(k+r-1)a_kt^{k+r} + \sum_{k=0}^{\infty}(k+r)a_kt^{k+r} - \sum_{k=0}^{\infty}(k+r)a_kt^{k+r+1} - \sum_{k=0}^{\infty}a_kt^{k+r}\\
=& (r(r-1)a_0 + ra_0 - a_0)t^r\\
&+ \sum_{k=1}^{\infty}((k+r)(k+r-1)a_k + (k+r)a_k - (k+r-1)a_{k-1} - a_k)t^{k+r}
\end{aligned}$$

So

$$\begin{cases} r(r-1)a_0 + ra_0 - a_0 = 0 \\ (k+r+1)(k+r-1)a_k = (k+r-1)a_{k-1}, k \in \mathbb{N}^* \end{cases}$$

then for $r = 1$ we obtain that $\forall k \in \mathbb{N}, a_k = \dfrac{2}{(k+2)!}a_0$

$$x_1(t) = \sum_{k=0}^{\infty}a_kt^{k+1} = a_0\sum_{k=0}^{\infty}\frac{2}{(k+2)!}t^{k+1} = -a_0(\frac{1}{t}+1) + a_0\frac{e^t}{t}$$

for $r = -1$ we obtain that $\forall k \in \mathbb{N}^*, k \geqslant 3, a_k = a_2\prod_{i=3}^{k}\dfrac{1}{i} = \dfrac{2a_2}{k!}, a_1 = a_0,$

$$x_2(t) = \sum_{k=0}^{\infty}a_kt^{k-1} = \frac{a_0}{t} + a_0 + 2a_2\sum_{k=0}^{\infty}\frac{1}{k!}t^{k-1} - 2a_2(\frac{1}{t}+1) = (a_0 - 2a_2)(\frac{1}{t}+1) + 2a_2\frac{e^t}{t}$$

So two independent solutions for $t^2y'' + (t-t^2)y' - y = 0$ is

$$y_1(t) = c_1(\frac{1}{t}+1), y_2(t) = c_2e^t, c_1, c_2 \in \mathbb{R}$$

iii)

Set $y(t) = t^r\sum\limits_{k=0}^{\infty}a_kt^k$, then

$$y'(t) = \sum_{k=0}^{\infty}(k+r)a_kt^{k+r-1}$$

$$y''(t) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_kt^{k+r-2}$$

Since

$$t^2y'' - t(1+t)y' + y = 0$$

then we can obtain that

$$
\begin{aligned}
0 =& t^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r-2} - t(1+t) \sum_{k=0}^{\infty} (k+r) a_k t^{k+r-1} + \sum_{k=0}^{\infty} a_k t^{k+r} \\
=& \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r} - \sum_{k=0}^{\infty} (k+r) a_k t^{k+r} - \sum_{k=0}^{\infty} (k+r) a_k t^{k+r+1} + \sum_{k=0}^{\infty} a_k t^{k+r} \\
=& (r(r-1)a_0 - r a_0 + a_0) t^r \\
&+ \sum_{k=1}^{\infty} ((k+r)(k+r-1) a_k - (k+r) a_k - (k+r-1) a_{k-1} + a_k) t^{k+r}
\end{aligned}
$$

So

$$
\begin{cases}
r(r-1)a_0 - r a_0 + a_0 = 0 \\
(k+r-1)^2 a_k = (k+r-1) a_{k-1}, k \in \mathbb{N}^*
\end{cases}
$$

then $r = 1$ and we obtain that $\forall k \in \mathbb{N}, a_k = \dfrac{1}{k!} a_0$

$$
x_1(t) = t \sum_{k=0}^{\infty} a_k t^k = a_0 t \sum_{k=0}^{\infty} \frac{1}{k!} t^k = a_0 t e^t
$$

and

$$
\begin{aligned}
x_2(t) =& \frac{\partial}{\partial r} \Big(t^r \sum_{k=0}^{\infty} a_k(r) t^k \Big)_{r=1} = \Big(t^r \ln t \sum_{k=0}^{\infty} a_k(r) t^k + t^r \sum_{k=0}^{\infty} a_k'(r) t^k \Big)_{r=1} \\
=& a_0 t e^t \ln t + t \Big(\sum_{k=0}^{\infty} a_k'(r) t^k \Big)_{r=1}
\end{aligned}
$$

Since for $k \geqslant 1$,

$$
\begin{aligned}
\frac{a_k'(1)}{a_k(1)} =& \Big(\frac{d}{dr} \ln a_k(r) \Big)_{r=1} = \frac{d}{dr} \Big(\ln a_0(r) - \sum_{i=1}^{k} \ln(i+r-1) \Big)_{r=1} \\
=& \frac{a_0'(1)}{a_0(1)} - \sum_{i=1}^{k} \frac{1}{i}
\end{aligned}
$$

then for $k \geqslant 1$,

$$
a_k' = \Big(\frac{a_0'}{a_0} - \sum_{i=1}^{k} \frac{1}{i} \Big) \frac{1}{k!} a_0
$$

so

$$
\begin{aligned}
x_2(t) =& a_0 t e^t \ln t + t \sum_{k=0}^{\infty} a_0' \frac{1}{k!} t^k - t \big(a_0 \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{k} \frac{1}{i} \Big) \frac{1}{k!} t^k \big) \\
=& a_0' t e^t + a_0 \big(t e^t \ln t - \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{k} \frac{1}{i} \Big) \frac{1}{k!} t^{k+1} \big)
\end{aligned}
$$

So two independent solutions for $t^2 y'' - t(1+t) y' + y = 0$ is

$$
y_1(t) = c_1 t e^t, y_2(t) = c_2 \big(t e^t \ln t - \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{k} \frac{1}{i} \Big) \frac{1}{k!} t^{k+1} \big)
$$

Exercise 8.4

i)

Set $y(x) = x^r \sum\limits_{k=0}^{\infty} a_k x^k$, then

$$y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}$$

$$y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Since

$$x^2y'' + xy' + (x^2-\nu^2)y = 0$$

then we can obtain that

$$\begin{aligned}
0 =& x^2\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} + x\sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} + (x^2-\nu^2)x^r\sum_{k=0}^{\infty}a_k x^k \\
=& \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + \sum_{k=0}^{\infty}a_k x^{k+r+2} - \nu^2\sum_{k=0}^{\infty}a_k x^{k+r} \\
=& r(r-1)a_0x^r + (1+r)ra_1x^{1+r} + ra_0x^r + (1+r)a_1x^{1+r} - \nu^2a_0x^r - \nu^2a_1x^{1+r} \\
&+ \sum_{k=0}^{\infty}(k+r+2)(k+r+1)a_{k+2} + (k+r+2)a_{k+2} + a_k - \nu^2a_{k+2})x^{x+r+2}
\end{aligned}$$

So

$$\begin{cases} (r^2-\nu^2)a_0 = 0 \\ ((r+1)^2-\nu^2)a_1 = 0 \\ ((k+r+2)^2-\nu^2)a_{k+2} = -a_k, k \geqslant 2 \end{cases} \Leftrightarrow \begin{cases} r = \nu \vee r = -\nu \\ a_1 = 0 \\ ((k+r+2)^2-\nu^2)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

For $r = \nu$,

$$\begin{aligned}
a_{2k+1} =& 0, k \in \mathbb{N} \\
a_{2k} =& -\frac{1}{(2k)(2k+2\nu)}a_{2k-2} = -\Big(\frac{1}{2}\Big)^2\frac{1}{k(k+\nu)}a_{2k-2} \\
=& \Big(\frac{1}{2}\Big)^{2k}\frac{(-1)^k}{k!\prod\limits_{i=1}^{k}(i+\nu)}a_0, k \in \mathbb{N}^*
\end{aligned}$$

Set $a_0 = \dfrac{1}{2^\nu\Gamma(\nu+1)}$, then $a_{2k} = \Big(\dfrac{1}{2}\Big)^{2k}\dfrac{(-1)^k}{k!\prod\limits_{i=1}^{k}(i+\nu)}\dfrac{1}{2^\nu\Gamma(\nu+1)} = \Big(\dfrac{1}{2}\Big)^{2k+\nu}\dfrac{(-1)^k}{k!\Gamma(1+k+\nu)}$

So one solution is

$$J_\nu(x) = x^r\sum_{k=0}^{\infty}a_kx^k = x^\nu\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^{2n+\nu}\frac{(-1)^n}{n!\Gamma(1+n+\nu)}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+n+\nu)}\Big(\frac{x}{2}\Big)^{2n+\nu}$$

ii)

For $r=-\nu$, if 2ν is not an integer

$$\begin{cases} r=\nu \vee r=-\nu \\ a_1=0 \\ a_{k+2}=-\dfrac{1}{(k+2)(k+2-2\nu)}a_k, k\geqslant 0 \end{cases}$$

So

$$\begin{aligned} a_{2k+1}&=0, k\in\mathbb{N} \\ a_{2k}&=-\frac{1}{(2k)(2k-2\nu)}a_{2k-2}=-\left(\frac{1}{2}\right)^2\frac{1}{k(k-\nu)}a_{2k-2} \\ &=\left(\frac{1}{2}\right)^{2k}\frac{(-1)^k}{k!\prod\limits_{i=1}^{k}(i-\nu)}a_0, k\in\mathbb{N}^* \end{aligned}$$

If $0<\nu<1$, we can set $a_0=\dfrac{2^\nu}{\Gamma(1-\nu)}$, then

$$a_{2k}=\left(\frac{1}{2}\right)^{2k}\frac{(-1)^k}{k!\prod\limits_{i=1}^{k}(i-\nu)}\frac{2^\nu}{\Gamma(1-\nu)}=\left(\frac{1}{2}\right)^{2k-\nu}\frac{(-1)^k}{k!\Gamma(1+k-\nu)}$$

this is also hold for $k=0$. So one solution is

$$J_{-\nu}(x)=x^r\sum_{k=0}^{\infty}a_kx^k=x^{-\nu}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n-\nu}\frac{(-1)^n}{n!\Gamma(1+n-\nu)}x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+n-\nu)}\left(\frac{x}{2}\right)^{2n-\nu}$$

If $\lceil\nu\rceil=m>1$, we can set $a_0=\dfrac{2^\nu}{\Gamma(-\nu)}=\dfrac{2^\nu\prod\limits_{i=1}^{m}(i-\nu)}{\Gamma(m-\nu)}$, then

$$a_{2k}=\left(\frac{1}{2}\right)^{2k}\frac{(-1)^k}{k!\prod\limits_{i=1}^{k}(i-\nu)}\frac{2^\nu\prod\limits_{i=1}^{m}(i-\nu)}{\Gamma(m-\nu)}=\left(\frac{1}{2}\right)^{2k-\nu}\frac{(-1)^k}{k!\Gamma(1+k-\nu)}$$

this is also hold for $k=0$. So one solution is

$$J_{-\nu}(x)=x^r\sum_{k=0}^{\infty}a_kx^k=x^{-\nu}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n-\nu}\frac{(-1)^n}{n!\Gamma(1+n-\nu)}x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+n-\nu)}\left(\frac{x}{2}\right)^{2n-\nu}$$

that is for $\nu<0, 2\nu\notin\mathbb{N}$,

$$J_{-\nu}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+n-\nu)}\left(\frac{x}{2}\right)^{2n-\nu}$$

iii)

Since $a_{2n-2} = -(2n-2+2)(2n-2+2-2n)a_{2n} = 0$, $\forall 2k \leqslant 2n-2, k \in \mathbb{N}, a_{2k} = 0$. And $\forall k \in \mathbb{N}, 2k \geqslant 2n+2$

$$\begin{aligned}
a_{2k} =& -\frac{1}{(2k)(2k-2n)}a_{2k-2} = -\Big(\frac{1}{2}\Big)^2\frac{1}{k(k-n)}a_{2k-2} \\
=&\Big(\frac{1}{2}\Big)^{2(k-n)}\frac{(-1)^{(k-n)}}{\Big(\prod\limits_{i=n+1}^{k} i\Big)\Big(\prod\limits_{i=n+1}^{k}(i-n)\Big)}a_{2n} \\
=&\Big(\frac{1}{2}\Big)^{2(k-n)}\frac{(-1)^{(k-n)}n!}{(k-n)!k!}a_{2n}
\end{aligned}$$

This is also true for $2k = 2n$. Set $a_{2n} = \dfrac{(-1)^n}{2^n n!\Gamma(n)}$, then

$$\begin{aligned}
J_{-n}(x) =& x^r\sum_{k=0}^{\infty} a_k x^k = x^{-n}\sum_{k=n}^{\infty}\Big(\frac{1}{2}\Big)^{2(k-n)}\frac{(-1)^{(k-n)}n!}{(k-n)!k!}a_{2n}x^{2k} \\
=&\sum_{t=0}^{\infty}\Big(\frac{1}{2}\Big)^{2t}\frac{(-1)^t n!}{t!(t+n)!}a_{2n}x^{2t+n} \\
=&(-1)^n\sum_{t=0}^{\infty}\Big(\frac{1}{2}\Big)^{2t+n}\frac{(-1)^t}{t!\Gamma(t+n+1)}x^{2t+n} \\
=&(-1)^n J_n(x)
\end{aligned}$$

So $J_{-n} = (-1)^n J_n$ for $n \in \mathbb{N}$, and J_{-n} does not yield a second independent solution.

iv)

Set $y_2(x) = J_\nu c(x)$, since $x^2y'' + xy' + (x^2-\nu^2)y = 0$,

$$\begin{aligned}
0 =& x^2(J_\nu''(x)c(x) + 2J_\nu'(x)c'(x) + J_\nu(x)c''(x)) + x(J'(x)c(x) + J_\nu(x)c'(x)) + (x^2-\nu^2)J_\nu(x)c(x) \\
=& c(x)(x^2J_\nu''(x) + xJ_\nu(x) + (x^2-\nu^2)J_\nu(x)c(x)) + (2x^2J_\nu'(x) + xJ_\nu(x))c'(x) + x^2J_\nu(x)c''(x) \\
=& (2x^2J_\nu'(x) + xJ_\nu(x))c'(x) + x^2J_\nu(x)c''(x)
\end{aligned}$$

So

$$\begin{aligned}
&\frac{dc'(x)}{dx} = -2\frac{J_\nu'(x)}{J_\nu(x)} - \frac{1}{x}\frac{dc(x)}{dx} \\
\Rightarrow&\frac{dc'(x)}{dx} = (-2\frac{d\ln|J_\nu(x)|}{dx} - \frac{1}{x})c'(x) \\
\Rightarrow&\ln|c'(x)| = (-2\ln|J_\nu(x)| - \ln|x|) \\
\Rightarrow&c'(x) = \frac{1}{x\cdot J_\nu^2(x)} \\
\Rightarrow&c(x) = \int\frac{dx}{x\cdot J_\nu^2(x)}
\end{aligned}$$

So a second solution can be formally represented as

$$y_2(x) = J_\nu(x)\int\frac{dx}{x\cdot J_\nu^2(x)}$$

Exercise 8.5

i)

Set $u(t) = x^{-1/2}y(x)$ and $t = \dfrac{2}{3}x^{3/2}$, then

$$\begin{aligned}\frac{du(t)}{dt} &= (\frac{d}{dx}x^{-1/2}y(x))\frac{dx}{dt} = (-\frac{1}{2}x^{-3/2}y(x) + x^{-1/2}y'(x))(3/2)^{2/3}\frac{2}{3}t^{-1/3} \\ &= (-\frac{1}{2}x^{-3/2}y(x) + x^{-1/2}y'(x))x^{-1/2} \\ &= -\frac{1}{2x^2}y(x) + \frac{y'(x)}{x}\end{aligned}$$

$$\begin{aligned}\frac{d^2u(t)}{dt^2} &= \frac{d}{dx}(-\frac{1}{2x^2}y(x) + \frac{y'(x)}{x})\frac{dx}{dt} \\ &= (\frac{1}{x^3}y(x) - \frac{1}{2x^2}y'(x) - \frac{y'(x)}{x^2} + \frac{1}{x}y''(x))x^{-1/2} \\ &= x^{-7/2}y(x) - \frac{3}{2}x^{-5/2}y'(x) + y''(x)x^{-3/2}\end{aligned}$$

So

$$\begin{aligned}&t^2u'' + tu' + (t^2 - (1/3)^2)u \\ =&\frac{4}{9}x^3(x^{-7/2}y(x) - \frac{3}{2}x^{-5/2}y'(x) + y''(x)x^{-3/2}) + \frac{2}{3}x^{3/2}(-\frac{1}{2x^2}y(x) + \frac{y'(x)}{x}) \\ &+ (\frac{4}{9}x^3 - (1/3)^2)x^{-1/2}y(x) \\ =&\frac{4}{9}x^{-1/2}y(x) - \frac{2}{3}x^{1/2}y'(x) - \frac{4}{9}x^{5/2}y(x) - \frac{1}{3}x^{-1/2}y(x) + \frac{2}{3}x^{1/2}y'(x) + \frac{4}{9}x^{5/2}y(x) - \frac{1}{9}x^{-1/2}y(x) \\ =&0\end{aligned}$$

So Airy's equation is Bessel's equation of order $\nu = 1/3$.

ii)

The general solution of $t^2u'' + tu' + (t^2 - (1/3)^2)u = 0$ is

$$u(t) = c_1J_{1/3}(t) + c_2J_{-1/3}(t)$$

So

$$\begin{aligned}y(x) &= x^{1/2}u(t) = x^{1/2}(c_1J_{1/3}(t) + c_2J_{-1/3}(t)) \\ &= x^{1/2}(c_1J_{1/3}(\frac{2}{3}x^{3/2}) + c_2J_{-1/3}(\frac{2}{3}x^{3/2}))\end{aligned}$$

To sum up, the general solution to Airy's equation is

$$y(x) = x^{1/2}\Big(c_1J_{1/3}(\frac{2}{3}x^{3/2}) + c_2J_{-1/3}(\frac{2}{3}x^{3/2})\Big)$$