Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 1

Date Due: 10:00 AM, Thursday, the 22nd of September 2016

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Exercise 1.1. Consider the differential equation

$$y' = \sqrt{|y|}.$$

i) Use Mathematica to plot the direction field for this equation in some domain that incorporates bothe negative and positive x and y values.

ii) The solution of the initial value problem with $y(1) = 0$ are not unique. Find all solutions of this problem and verify that they actually are solutions (using the definition of a solution from the lecture slides).

(2 + 2 Marks)

Exercise 1.2. This Exercise was already posed in Vv285 (Ex. 6.1) and is repeated here for reference only:

i) Let $f, g\colon \mathbb{R} \to \mathbb{R}$, $F\colon \mathbb{R}^2 \to \mathbb{R}$. Show that

$$\frac{d}{dt}F(f(t), g(t)) = f'(t)\partial_1 F(f(t), g(t)) + g'(t)\partial_2 F(f(t), g(t))$$

where $\partial_1 F(x, y) = \frac{\partial}{\partial x}F(x, y)$ and $\partial_2 F(x, y) = \frac{\partial}{\partial y}F(x, y)$. *Hint:* Consider $F(f(t), g(t))$ as a composition of the map $G\colon \mathbb{R} \to \mathbb{R}^2$, $G(t) = (f(t), g(t))$ with $F\colon \mathbb{R}^2 \to \mathbb{R}$ and apply the chain rule.

ii) Let $f(x, y)$ be a differentiable function over $(a, b) \times (c, d)$ and let $\alpha, \beta\colon \mathbb{R} \ni [c, d] \to [a, b] \in \mathbb{R}$ be differentiable. Then the integral

$$I(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y)\, dx$$

is a differentiable function of y in $[c, d]$. Assuming that $\frac{\partial f}{\partial y}(x, y)$ is continuous and $\alpha'(y), \beta'(y)$ exist in $[a, b] \times [c, d]$, give a formula for $I'(y)$.

(0 Marks)

Exercise 1.3. Prove the following version of Duhamel's principle for ordinary differential equations of order n:

Let $I \subset \mathbb{R}$ be an open interval, $x_0 \in \overline{I}$, $p \in \mathbb{N} \setminus \{0, 1\}$ and $a_0, \dots, a_p, f$ continuous, real-valued functions on $\overline{I}$, where $a_p(x) \neq 0$ for all $x \in \overline{I}$. Let y_ξ solve the homogeneous equation

$$a_p(x)y_\xi^{(p)} + \cdots + a_1(x)y_\xi' + a_0(x)y_\xi = 0$$

with initial values

$$y_\xi^{(p-1)}(\xi) = \frac{1}{a_p(\xi)}, \qquad y_\xi^{(p-2)}(\xi) = 0, \qquad \dots, \qquad y_\xi'(\xi) = 0, \qquad y_\xi(\xi) = 0.$$

for $x \in \overline{I}$. Then

$$y(x) = \int_{x_0}^{x} f(\xi)y_\xi(x)\, d\xi.$$

solves

$$a_p(x)y^{(p)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

with initial values

$$y^{(p-1)}(x_0) = 0, \qquad y^{(p-2)}(x_0) = 0, \qquad \dots, \qquad y'(x_0) = 0, \qquad y(x_0) = 0.$$

(6 Marks)

Exercise 1.4. Another approach to obtaining a solution to an inhomogeneous, linear, first-order equation is as follows: If y_{hom} is a solution of

$$a_1(x)y' + a_0(x)y = 0,$$

then $c \cdot y_{\text{hom}}$ will of course solve the same equation for any constant $c \in \mathbb{R}$. The method of *variation of parameters* states that a solution to

$$a_1(x)y' + a_0(x)y = f(x) \tag{$*$}$$

can be found by setting $y_{\text{part}}(x) := c(x)y_{\text{hom}}(x)$ for some function $c(x)$, inserting into $(*)$ and obtaining a differential equation for c. Solving this differential equation then yields the same solution formula as the Duhamel principle. Verify this!
(3 Marks)

Exercise 1.5. This exercise refers to Section 1.3 of the textbook by Braun, which can be downloaded at **http://link.springer.com/chapter/10.1007/978-1-4612-4360-1_1** from within the SJTU network.

Read the discussion of the art forgeries in this chapter, and then demonstrate that the paintings "Washing of Feet," "Woman Reading Music" and "Woman playing Mandolin" are forgeries, as per Exercises 2-4 of the textbook section.
(3 Marks)

Suppose that a grams of chemical A are combined with b grams of chemical B to give chemical C. If there are M parts of A and N parts of B in the compund C (i.e., the chemical reaction equation is $M\,A + N\,B \to C$) and $X(t)$ is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t are, respectively,

$$a - \frac{M}{M+N}X \qquad \text{and} \qquad b - \frac{N}{M+N}X.$$

The law of mass action states that when no temperature change is involved, the rate at which the two substances react is proportional to the product of the amount of A and B that remain at time T:

$$\frac{dX}{dt} \sim \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right)$$

With a constant of proportionality $k > 0$ we obtain

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

where $\alpha = a(M+N)/M$ and $\beta = b(M+N)/N$. A reaction governed by this law is said to be a *second-order reation.*

Exercise 1.6. Two chemicals A and B are combind to form a chemical C in a second-order reaction. Initially ther are 40 grams of A and 50 grams of B and for each gram of B, 2 grams of A are used. It is observed that 10 grams of C are formed in 5 minutes.

i) Use Mathematica to plot the amount of chemical C formed as a function of time.

ii) How much is formed in 20 minutes?

iii) What is the limiting amount of C as time $t \to \infty$?

iv) How much of chemicals A and B remain as $t \to \infty$?

(2 + 1 + 1 + 2 Marks)

Exercise 1.7. Plot the direction field of the equation $xy' = y^2 - y$. Then find, if possible, at least one solution that passed through each of the indicated points:

a) $(2, 1/4)$, b) $(1/2, 1/2)$, c) $(0, 2)$, d) $(0, 1)$.

(Do not worry about proving uniqueness; stating one solution in each case is sufficient.)
(4 Marks)

Exercise 1.8. Find the general solution to the following equations:

a) $y' = (1+x)(1+y)$, b) $y' = e^{x+y+3}$.

(2+2 Marks)