VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 10

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Exercise 10.1

i)

$$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} dx = \frac{1}{2}(1-(-1)) = 1$$

$\forall n, k \in \mathbb{N}^*$,

$$\langle \cos(\pi n x), \cos(\pi n x) \rangle = \int_{-1}^{1} \cos(\pi n x) \cdot \cos(\pi n x) dx = \frac{x + \frac{1}{2n\pi}\sin(2\pi n x)}{2}|_{-1}^{1} = 1$$

$$\langle \sin(\pi n x), \sin(\pi n x) \rangle = \int_{-1}^{1} \sin(\pi n x) \cdot \sin(\pi n x) dx = \frac{x + \frac{1}{2n\pi}\cos(2\pi n x)}{2}|_{-1}^{1} = 1$$

$$\langle \frac{1}{\sqrt{2}}, \sin(\pi n x) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \sin(\pi n x) dx = \frac{-\cos(\pi n x)}{\sqrt{2}n\pi}|_{-1}^{1} = 0$$

$$\langle \frac{1}{\sqrt{2}}, \cos(\pi n x) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \cos(\pi n x) dx = \frac{\sin(\pi n x)}{\sqrt{2}n\pi}|_{-1}^{1} = 0$$

$$\begin{aligned}
\langle \sin(\pi n x), \cos(\pi k x) \rangle &= \int_{-1}^{1} \sin(\pi n x)\cos(\pi k x) dx \\
&= \int_{-1}^{0} \sin(\pi n x)\cos(\pi k x) dx + \int_{0}^{1} \sin(\pi n x)\cos(\pi k x) dx \\
&= \int_{1}^{0} \sin(\pi n x)\cos(\pi k x) dx + \int_{0}^{1} \sin(\pi n x)\cos(\pi k x) dx \\
&=0
\end{aligned}$$

So $\mathcal{B} = \{\frac{1}{\sqrt{2}}, \cos(\pi n x), \sin(\pi n x)\}_{n=1}^{\infty}$ is an orthonormal system in $L^2([-1,1])$.

ii)

Since $\{e_n\}$ is an orthonormal system in $L^2([-1,1])$, $\forall n, k \in \mathbb{N}, n \neq k$

$$\langle e_n, e_n \rangle = \int_{-1}^{1} e_n^2(x) dx = 1, \langle e_n, e_k \rangle = \int_{-1}^{1} e_n(x) e_k(x) dx = 0$$

so $\forall n, k \in \mathbb{N}, n \neq k$

$$\langle \tilde{e}_n, \tilde{e}_n \rangle = \frac{2}{b-a} \int_a^b e_n^2\Big(\frac{2}{b-a}\Big(x - \frac{b+a}{2}\Big)\Big) dx \xlongequal{t=\frac{2}{b-a}(x-\frac{b+a}{2})} \frac{2}{b-a}\int_{-1}^{1} e_n^2(t) \cdot \frac{b-a}{2} dt = 1$$

$$\begin{aligned}
\langle \tilde{e}_n, \tilde{e}_k \rangle =& \frac{2}{b-a} \int_a^b e_n\Big(\frac{2}{b-a}\Big(x - \frac{b+a}{2}\Big)\Big) e_k\Big(\frac{2}{b-a}\Big(x - \frac{b+a}{2}\Big)\Big) dx \\
\xlongequal{t=\frac{2}{b-a}(x-\frac{b+a}{2})}& \frac{2}{b-a}\int_{-1}^{1} e_n(t) e_k(t) \cdot \frac{b-a}{2} dt = 0
\end{aligned}$$

So $\{\tilde{e}_n\}$ is an orthonormal system in $L^2([a,b])$.

iii)

For the spaces $L^2([-\pi,\pi])$, an orthonormal systems is

$$\{\frac{1}{\sqrt{2}}\sqrt{\frac{2}{\pi-(-\pi)}}, \sqrt{\frac{2}{\pi-(-\pi)}}\cos(\frac{2}{\pi-(-\pi)}\pi nx), \sqrt{\frac{2}{\pi-(-\pi)}}\sin(\frac{2}{\pi-(-\pi)}\pi nx)\}_{n=1}^{\infty}$$

i.e.

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\}_{n=1}^{\infty}$$

For the spaces $L^2([0,L])$, an orthonormal systems is

$$\{\frac{1}{\sqrt{2}}\sqrt{\frac{2}{L-0}}, \sqrt{\frac{2}{L-0}}\cos(\pi n\frac{2}{L-0}(x-\frac{L+0}{2})), \sqrt{\frac{2}{L-0}}\sin(\pi n\frac{2}{L-0}(x-\frac{L+0}{2}))\}_{n=1}^{\infty}$$

i.e.

$$\{\frac{1}{\sqrt{L}}, (-1)^n\sqrt{\frac{2}{L}}\cos(\frac{2\pi n}{L}x), (-1)^n\sqrt{\frac{2}{L}}\sin(\frac{2\pi n}{L}x)\}_{n=1}^{\infty}$$

Exercise 10.2

$$\langle x^2, \frac{1}{\sqrt{2}}\rangle = \int_{-1}^{1}\frac{1}{\sqrt{2}}x^2dx = \frac{\sqrt{2}}{3}$$

$$\begin{aligned}
\langle x^2, \cos(\pi nx)\rangle &= \int_{-1}^{1}x^2\cos(\pi nx)dx \\
&= \frac{1}{\pi n}(x^2\sin(\pi nx)|_{-1}^{1} - 2\int_{-1}^{1}x\sin(\pi nx)dx) \\
&= \frac{2}{\pi n}(\frac{1}{\pi n}(x\cos(\pi nx)|_{-1}^{1} - \int_{-1}^{1}\cos(\pi nx)dx)) \\
&= \frac{2}{\pi^2n^2}(2\cdot(-1)^n) \\
&= \frac{4(-1)^n}{\pi^2n^2}
\end{aligned}$$

$$\begin{aligned}
\langle x^2, \sin(\pi nx)\rangle &= \int_{-1}^{1}x^2\sin(\pi nx)dx \\
&= -\frac{1}{\pi n}(x^2\cos(\pi nx)|_{-1}^{1} - 2\int_{-1}^{1}x\cos(\pi nx)dx) \\
&= \frac{2}{\pi n}(\frac{1}{\pi n}(x\sin(\pi nx)|_{-1}^{1} - \int_{-1}^{1}\sin(\pi nx)dx)) \\
&= 0
\end{aligned}$$

So the Fourier series of the function $f(x)=x^2, x\in[-1,1]$ is

$$f(x) = \frac{\sqrt{2}}{3}\cdot\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi^2n^2}\cos(\pi nx) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi^2n^2}\cos(\pi nx)$$

Since $x \in [-1, 1]$, then

$$1 = f(1) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(\pi n) \Rightarrow \frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$0 = f(0) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(0) \Rightarrow 0 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

To sum up,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Exercise 10.3

Set $u(x,t) = X(x)T(t)$ and we obtain that

$$X(x)T''(t) + c^2 X''''(x)T(t) = 0$$

Since the left hand side depends only on t and the right-hand side depends only on x, they must both be constant. Set $\frac{1}{c^2 T} T_{tt} = -\frac{1}{X} X_{xxxx} = \lambda \in \mathbb{R}$, then

$$X'''' = -\lambda X, T'' = c^2 \lambda T$$

and Dirichlet boundary conditions and initial conditions become

$$X(0)T(t) = X(l)T(t) = 0,\ X''(0)T(t) = X''(l)T(t) = 0, t \in \mathbb{R}^+$$

$$X(x)T(0) = x(l-x),\ X(x)T'(0) = 0, x \in (0, l)$$

So either $\forall t > 0, T = 0$ or $X(0) = X(l) = X''(0) = X''(l) = 0$, and $T'(0) = 0$

1. $X(0) = X(l) = X''(0) = X''(l) = 0$

   Since $X'''' = -\lambda X$, we can make an ansatz of the form $X_\lambda(x) = e^{\rho(\lambda)x}$ and get that

   $$(\rho(\lambda))^4 = -\lambda$$

   (a) $\lambda > 0$

   Then $\rho(\lambda) = \pm\frac{\sqrt[4]{4\lambda}}{2}(1+i), \pm\frac{\sqrt[4]{4\lambda}}{2}(1-i)$ So the general solution is given by

$$\begin{aligned}
X_\lambda(x) =& c_1 e^{\frac{\sqrt[4]{4\lambda}}{2}(1+i)x} + c_2 e^{-\frac{\sqrt[4]{4\lambda}}{2}(1+i)x} + c_3 e^{\frac{\sqrt[4]{4\lambda}}{2}(1-i)x} + c_4 e^{-\frac{\sqrt[4]{4\lambda}}{2}(1-i)x} \\
=& e^{\frac{\sqrt[4]{4\lambda}}{2}x}(C_1 \cos(\frac{\sqrt[4]{4\lambda}}{2}x) + C_2 \sin(\frac{\sqrt[4]{4\lambda}}{2}x)) \\
&+ e^{-\frac{\sqrt[4]{4\lambda}}{2}x}(C_3 \cos(\frac{\sqrt[4]{4\lambda}}{2}x) + C_4 \sin(\frac{\sqrt[4]{4\lambda}}{2}x)) \\
=& (e^{\frac{\sqrt[4]{4\lambda}}{2}x} C_1 + e^{-\frac{\sqrt[4]{4\lambda}}{2}x} C_3) \cos(\frac{\sqrt[4]{4\lambda}}{2}x) + (e^{\frac{\sqrt[4]{4\lambda}}{2}x} C_2 + e^{-\frac{\sqrt[4]{4\lambda}}{2}x} C_4) \sin(\frac{\sqrt[4]{4\lambda}}{2}x)
\end{aligned}$$

Since $X(0) = X(l) = X''(0) = X''(l) = 0$

$$\begin{cases} C_1 + C_3 = 0 \\ C_2 - C_4 = 0 \\ (e^{\frac{\sqrt[4]{4\lambda}}{2}l}C_1 + e^{-\frac{\sqrt[4]{4\lambda}}{2}l}C_3)\cos(\frac{\sqrt[4]{4\lambda}}{2}l) + (e^{\frac{\sqrt[4]{4\lambda}}{2}l}C_2 + e^{-\frac{\sqrt[4]{4\lambda}}{2}l}C_4)\sin(\frac{\sqrt[4]{4\lambda}}{2}l) = 0 \\ (e^{\frac{\sqrt[4]{4\lambda}}{2}l}C_2 - e^{-\frac{\sqrt[4]{4\lambda}}{2}l}C_4)\cos(\frac{\sqrt[4]{4\lambda}}{2}l) + (-e^{\frac{\sqrt[4]{4\lambda}}{2}l}C_1 + e^{-\frac{\sqrt[4]{4\lambda}}{2}l}C_3)\sin(\frac{\sqrt[4]{4\lambda}}{2}l) = 0 \end{cases}$$

There is no $\lambda > 0$ such that $C_1, C_2, C_3, C_4 \neq 0$.

(b) $\lambda = 0$

So $X'''' = 0, T'' = 0 \Rightarrow X(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3, T(t) = d_1 + d_2 t$. Since $X(0) = X(l) = X''(0) = X''(l) = 0, T'(0) = 0$, e obtain that

$$c_1 = 0, c_1 + c_2 l + c_3 l^2 + c_4 l^3 = 0, 2c_3 = 0, 2c_3 + 6c_4 l = 0, d_2 = 0$$

i.e. $c_1 = c_2 = c_3 = c_4 = 0, d_2 = 0$. So

$$X(x) = 0, T(t) = d_1$$

which doesn't satisfy the initial condition.

(c) $\lambda < 0$

Then $\rho(\lambda) = \pm\sqrt[4]{|\lambda|}, \pm\sqrt[4]{|\lambda|}i$ So the general solution is given by

$$X_\lambda(x) = c_1 e^{\sqrt[4]{|\lambda|}x} + c_2 e^{-\sqrt[4]{|\lambda|}x} + c_3 e^{i\sqrt[4]{|\lambda|}x} + c_4 e^{-i\sqrt[4]{|\lambda|}x}$$

Since $X(0) = X(l) = X''(0) = X''(l) = 0$

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0 \\ c_1 + c_2 - c_3 - c_4 = 0 \\ c_1 e^{\sqrt[4]{|\lambda|}l} + c_2 e^{-\sqrt[4]{|\lambda|}l} + c_3 e^{i\sqrt[4]{|\lambda|}l} + c_4 e^{-i\sqrt[4]{|\lambda|}l} = 0 \\ c_1 e^{\sqrt[4]{|\lambda|}l} + c_2 e^{-\sqrt[4]{|\lambda|}l} - c_3 e^{i\sqrt[4]{|\lambda|}l} - c_4 e^{-i\sqrt[4]{|\lambda|}l} = 0 \end{cases}$$

To have non-trival solution, we obtain that

$$c_1 = c_2 = 0, c_3 = -c_4, \sqrt[4]{|\lambda|}l = k\pi \Rightarrow \lambda_k = -\left(\frac{k\pi}{l}\right)^4, k = 1, 2, \cdots$$

So

$$X_k(x) = c_3(e^{i\frac{k\pi}{l}x} - e^{-i\frac{k\pi}{l}x}) = A_k \sin\left(\frac{k\pi}{l}x\right)$$

and solve the equation $T'' = c^2\lambda T$ we obtain that

$$T_k(t) = B_k e^{i\frac{ck^2\pi^2}{l^2}t} + C_k e^{-i\frac{ck^2\pi^2}{l^2}t}$$

So

$$u(x,t) = \sum_{k=1}^{\infty}(B_k e^{i\frac{ck^2\pi^2}{l^2}t} + C_k e^{-i\frac{ck^2\pi^2}{l^2}t})(A_k \sin\left(\frac{k\pi}{l}x\right))$$

Considering the initial condition $u(x,0) = x(l-x), u_t(x,0) = 0$

$$\sum_{k=1}^{\infty}(D_k + E_k)(\sin\left(\frac{k\pi}{l}x\right)) = x(l-x), \sum_{k=1}^{\infty}(D_k - E_k)(\sin\left(\frac{k\pi}{l}x\right)) = 0$$

So $D_k = E_k$ and

$$u(x,t) = \sum_{k=1}^{\infty} D_k(e^{i\frac{ck^2\pi^2}{l^2}t} + e^{-i\frac{ck^2\pi^2}{l^2}t})(\sin\left(\frac{k\pi}{l}x\right)) = \sum_{k=1}^{\infty} F_k(\cos(\frac{ck^2\pi^2}{l^2}t))(\sin\left(\frac{k\pi}{l}x\right))$$

Expanding the function $u(x,0) = x(l-x)$ into a Fourier-sine series, we see that

$$\begin{aligned} x(l-x) &= \sum_{k=1}^{\infty} \frac{2}{l}\int_0^l x(l-x)(\sin\left(\frac{k\pi}{l}x\right))dx \cdot (\sin\left(\frac{k\pi}{l}x\right)) \\ &= \sum_{k=1}^{\infty} \frac{4l^2(1-(-1)^k)}{(k\pi)^3} \cdot (\sin\left(\frac{k\pi}{l}x\right)) \end{aligned}$$

So $F_k = \dfrac{4l^2(1-(-1)^k)}{(k\pi)^3}$ and

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4l^2(1-(-1)^k)}{(k\pi)^3}(\cos(\frac{ck^2\pi^2}{l^2}t))(\sin\left(\frac{k\pi}{l}x\right))$$

2. $\forall t > 0, T = 0$

   Then $u(x,t) = 0$

To sum up, the solution of the equation for a vibrating beam of length $l > 0$ is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4l^2(1-(-1)^k)}{(k\pi)^3}(\cos(\frac{ck^2\pi^2}{l^2}t))(\sin\left(\frac{k\pi}{l}x\right))$$

or

$$u(x,t) = 0$$

Exercise 10.4

Set $u(x,t) = X(x)T(t)$ and we obtain that

$$c^2X''(x)T(t) - X(x)T''(t) - \mu X(x)T'(t) = 0 \Rightarrow \frac{1}{X}X'' = \frac{1}{c^2T}(T'' + \mu T')$$

Since the left hand side depends only on x and the right-hand side depends only on t, they must both be constant. Set $\frac{1}{X}X'' = \frac{1}{c^2T}(T'' + \mu T') = \lambda \in \mathbb{R}$, then

$$X'' = \lambda X, T'' + \mu T' = c^2\lambda T$$

and Dirichlet boundary conditions and initial conditions become

$$X(0)T(t) = X(L)T(t) = 0, t \in \mathbb{R}^+$$

$$X(x)T(0) = \sin\left(\frac{\pi x}{L}\right),\ X(x)T'(0) = 0, x \in [0,L]$$

So either $\forall t > 0, T = 0$ or $X(0) = X(L) = 0$

1. $X(0) = X(L) = 0$

   We obtain eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$$

   and eigenfunctions

$$X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$$

   We next need to solve

$$T'' + \mu T' + \left(\frac{cn\pi}{L}\right)^2 T = 0$$

   (a) $\mu^2 < 4\left(\frac{c\pi}{L}\right)^2$

$$T_n(t) = B_n e^{\frac{-\mu L + i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}t} + C_n e^{\frac{-\mu L - i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}t}$$

   So

$$u(x,t) = \sum_{n=1}^{\infty}(B_n e^{\frac{-\mu L + i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}t} + C_n e^{\frac{-\mu L - i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}t}) \cdot (A_n \sin\left(\frac{n\pi}{L}x\right))$$

   Considering the initial condition $u(x,0) = \sin\left(\frac{\pi x}{L}\right), u_t(x,0) = 0$ and let's set $a_n = \frac{-\mu L + i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}, b_n = \frac{-\mu L - i\sqrt{4(cn\pi)^2 - \mu^2 L^2}}{2L}$

$$\sum_{n=1}^{\infty}(D_n + E_n)(\sin\left(\frac{n\pi}{L}x\right)) = \sin\left(\frac{\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty}(a_n D_n + b_n E_n)(\sin\left(\frac{n\pi x}{L}\right)) = 0$$

   then $\forall n > 1, D_n = E_n = 0, D_1 + E_1 = 1.$

$$u(x,t) = u(x,t) = (\frac{b_1}{b_1 - a_1}e^{a_1 t} + \frac{a_1}{a_1 - b_1}e^{b_1 t}) \cdot \sin\left(\frac{\pi x}{L}\right)$$

   (b) $\mu^2 = 4\left(\frac{c\pi}{L}\right)^2$

$$\forall n > 1, T(t) = B_n e^{\frac{-\mu L + \sqrt{\mu^2 L^2 - 4(cn\pi)^2}}{2L}t} + C_n e^{\frac{-\mu L - \sqrt{\mu^2 L^2 - 4(cn\pi)^2}}{2L}t}$$

$$n = 1, T(t) = (B_n + C_n t)e^{-\frac{\mu}{2}t}$$

   set $c_n = \frac{-\mu L + \sqrt{\mu^2 L^2 - 4(cn\pi)^2}}{2L}, d_n = \frac{-\mu L - \sqrt{\mu^2 L^2 - 4(cn\pi)^2}}{2L}$ for $n > 1$

   Considering the initial condition $u(x,0) = \sin\left(\frac{\pi x}{L}\right), u_t(x,0) = 0$

$$\sum_{n \neq 1}(D_n + E_n)(\sin\left(\frac{n\pi}{L}x\right)) + D_1 \sin\left(\frac{\pi}{L}x\right) = \sin\left(\frac{\pi x}{L}\right)$$

$$\sum_{n=2}^{\infty}(c_nD_n+d_nE_n)(\sin\left(\frac{n\pi x}{L}\right))+(E_1-\frac{\mu}{2}D_1)\sin\left(\frac{\pi}{L}x\right)=0$$

So $\forall n>1, D_n=E_n=0, D_1=1, E_1=\frac{\mu}{2}$.

$$u(x,t)=(1+\frac{\mu}{2}e^{-\frac{\mu}{2}t})\cdot\sin\left(\frac{\pi x}{L}\right)$$

(c) $\exists k\in\mathbb{N}^*, k>1, \mu^2=4\left(\frac{ck\pi}{L}\right)^2$

$$\forall n>k, T_n(t)=B_ne^{\frac{-\mu L+i\sqrt{4(cn\pi)^2-\mu^2L^2}}{2L}t}+C_ne^{\frac{-\mu L-i\sqrt{4(cn\pi)^2-\mu^2L^2}}{2L}t}$$

$$\forall n<k, T_n(t)=B_ne^{\frac{-\mu L+\sqrt{\mu^2L^2-4(cn\pi)^2}}{2L}t}+C_ne^{\frac{-\mu L-\sqrt{\mu^2L^2-4(cn\pi)^2}}{2L}t}$$

$$n=k, T_n(t)=(B_n+C_nt)e^{-\frac{\mu}{2}t}$$

set $a_n=\frac{-\mu L+i\sqrt{4(cn\pi)^2-\mu^2L^2}}{2L}, b_n=\frac{-\mu L-i\sqrt{4(cn\pi)^2-\mu^2L^2}}{2L}$ for $n>k$,
$c_n=\frac{-\mu L+\sqrt{\mu^2L^2-4(cn\pi)^2}}{2L}, d_n=\frac{-\mu L-\sqrt{\mu^2L^2-4(cn\pi)^2}}{2L}$ for $n<k$
Considering the initial condition $u(x,0)=\sin\left(\frac{\pi x}{L}\right), u_t(x,0)=0$

$$\sum_{n\neq k}(D_n+E_n)(\sin\left(\frac{n\pi}{L}x\right))+D_k\sin\left(\frac{k\pi}{L}x\right)=\sin\left(\frac{\pi x}{L}\right)$$

$$\sum_{n=k+1}^{\infty}(a_nD_n+b_nE_n)(\sin\left(\frac{n\pi x}{L}\right))+\sum_{n=1}^{k-1}(c_nD_n+d_nE_n)(\sin\left(\frac{n\pi x}{L}\right))+(E_k-\frac{\mu}{2}D_k)\sin\left(\frac{k\pi}{L}x\right)=0$$

So $\forall n>1, D_n=E_n=0, D_1+E_1=1$.

$$u(x,t)=(\frac{d_1}{d_1-c_1}e^{c_1t}+\frac{c_1}{c_1-d_1}e^{d_1t})\cdot\sin\left(\frac{\pi x}{L}\right)$$

(d) $\mu^2>4\left(\frac{c\pi}{L}\right)^2$ while there doesn't exist $k\in\mathbb{N}, \mu^2=4\left(\frac{ck\pi}{L}\right)^2$
Similarly, we can obtain that

$$u(x,t)=(\frac{d_1}{d_1-c_1}e^{c_1t}+\frac{c_1}{c_1-d_1}e^{d_1t})\cdot\sin\left(\frac{\pi x}{L}\right)$$

2. $\forall t>0, T=0$
   Then $u(x,t)=0$

To sum up, the solution of the equation for damped wave equation is

1. $\mu^2>4\left(\frac{c\pi}{L}\right)^2$, set $c_1=\frac{-\mu L+\sqrt{\mu^2L^2-4(c\pi)^2}}{2L}, d_1=\frac{-\mu L-\sqrt{\mu^2L^2-4(c\pi)^2}}{2L}$

$$u(x,t)=(\frac{d_1}{d_1-c_1}e^{c_1t}+\frac{c_1}{c_1-d_1}e^{d_1t})\cdot\sin\left(\frac{\pi x}{L}\right)$$

2. $\mu^2 = 4\left(\frac{c\pi}{L}\right)^2$

$$u(x,t) = (1 + \frac{\mu}{2}e^{-\frac{\mu}{2}t}) \cdot \sin\left(\frac{\pi x}{L}\right)$$

3. $\mu^2 < 4\left(\frac{cn\pi}{L}\right)^2$, set$a_1 = \frac{-\mu L + i\sqrt{4(c\pi)^2 - \mu^2 L^2}}{2L}, b_1 = \frac{-\mu L - i\sqrt{4(c\pi)^2 - \mu^2 L^2}}{2L}$

$$u(x,t) = u(x,t) = (\frac{b_1}{b_1 - a_1}e^{a_1 t} + \frac{a_1}{a_1 - b_1}e^{b_1 t}) \cdot \sin\left(\frac{\pi x}{L}\right)$$

or

$$u(x,t) = 0$$

Exercise 10.5

Set $u(x,y) = X(x)Y(y)$ and we obtain that

$$X''(x)Y(y) + X(x)Y''(y) = X(x)Y(y) \Rightarrow \frac{1}{X}X'' = \frac{1}{Y}(Y - Y'')$$

Since the left hand side depends only on x and the right-hand side depends only on y, they must both be constant. Set$\frac{1}{X}X'' = \frac{1}{Y}(-Y'' + Y) = \lambda \in \mathbb{R}$, then

$$X'' = \lambda X, Y'' = (1 - \lambda)Y$$

and Dirichlet boundary conditions become

$$X(0)Y(y) = X(\pi)Y(y) = X(x)Y(0) = 0, X(x)Y(a) = 1, (x,y) \in [0,\pi] \times [0,a]$$

So $Y(0) = 0, X(x)Y(a) = 1$ and either $\forall 0 < y < a, Y = 0$ or $X(0) = X(\pi) = 0$

1. $X(0) = X(\pi) = 0$

   We obtain eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{\pi}\right)^2 = -n^2, n = 1, 2, 3, \cdots$$

   and eigenfunctions

$$X_n(x) = A_n \sin\left(\frac{n\pi}{\pi}x\right) = A_n \sin(nx)$$

   We next need to solve

$$Y'' = (1 + n^2)Y$$

   we obtain that

$$Y_n(y) = B_n e^{\sqrt{1+n^2}y} + C_n e^{-\sqrt{1+n^2}y}$$

   Since $Y(0) = 0$, $B_n + C_n = 0$,

$$Y_n(y) = B_n(e^{\sqrt{1+n^2}y} - e^{-\sqrt{1+n^2}y})$$

So

$$u(x,y) = \sum_{n=1}^{\infty} D_n(e^{\sqrt{1+n^2}y} - e^{-\sqrt{1+n^2}y})(\sin(nx))$$

Expanding the function $u(x,a) = 1$ into a Fourier-sine series, we see that

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} \frac{2}{\pi}\int_0^{\pi} 1 \cdot (\sin(nx))dx \cdot (\sin(nx)) \\ &= \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \cdot (\sin(nx)) \end{aligned}$$

So $D_n = \dfrac{1-(-1)^n}{n(e^{\sqrt{1+n^2}a} - e^{-\sqrt{1+n^2}a})}$ and

$$u(x,y) = \sum_{n=1}^{\infty} \frac{(1-(-1)^n)(e^{\sqrt{1+n^2}y} - e^{-\sqrt{1+n^2}y})}{n(e^{\sqrt{1+n^2}a} - e^{-\sqrt{1+n^2}a})} \sin(nx)$$

2. $\forall 0 < y < a, Y = 0$ then $u(x,y) = 0$

So the solution is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{(1-(-1)^n)(e^{\sqrt{1+n^2}y} - e^{-\sqrt{1+n^2}y})}{n(e^{\sqrt{1+n^2}a} - e^{-\sqrt{1+n^2}a})} \sin(nx)$$

or

$$u(x,y) = 0$$

Exercise 10.6

Set $u(x,t) = X(x)T(t)$ and we obtain that

$$X(x)T''(t)+(\alpha+\beta)X(x)T'(t)+\alpha\beta X(x)T(t) = c^2X''(x)T(t) \Rightarrow \frac{1}{X}X'' = \frac{1}{c^2T}(T''+(\alpha+\beta)T'+\alpha\beta T)$$

Since the left hand side depends only on x and the right-hand side depends only on t, they must both be constant. Set $\frac{1}{X}X'' = \frac{1}{c^2T}(T'' + (\alpha+\beta)T' + \alpha\beta T) = \lambda \in \mathbb{R}$, then

$$X'' = \lambda X, T'' + (\alpha+\beta)T' + \alpha\beta T = c^2\lambda T$$

Since $\alpha, \beta > 0$,

$$T(t) = e^{-\frac{\alpha+\beta}{2}t}G(t)$$

then $u(x,t) = e^{-\frac{\alpha+\beta}{2}t}v(x,t)$ and therefore $u(0,t) = e^{-\frac{\alpha+\beta}{2}t}v(0,t)$ which can not be turned into

$$u(0,t) = U_0\cos(\omega t)$$

So the telegraph equation does not have a solution of the form $u(x,t) = X(x) \cdot T(t)$.

Set $u(x,t) = U_0 e^{-Ax}\cos(\omega t + Bx)$, then we can see that it satisfies the condition and initial signal and

$$\begin{aligned}
&-U_0\omega^2 e^{-Ax}\cos(\omega t + Bx) - (\alpha+\beta)\omega U_0 e^{-Ax}\sin(\omega t + Bx) + \alpha\beta U_0 e^{-Ax}\cos(\omega t + Bx)\\
=&c^2 U_0 e^{-Ax}((A^2 - B^2)\cos(\omega t + Bx) + 2AB\sin(\omega t + Bx))\\
\Rightarrow&(c^2(A^2 - B^2) + \omega^2 - \alpha\beta)\cos(\omega t + Bx) + (2c^2AB + (\alpha+\beta)\omega)\sin(\omega t + Bx) = 0\\
\Rightarrow&\begin{cases} c^2(A^2 - B^2) + \omega^2 - \alpha\beta = 0\\ 2c^2AB + (\alpha+\beta)\omega = 0\end{cases}\\
\Rightarrow&(c^2(A^2 - B^2) + \omega^2 - \alpha\beta)\cos(\omega t + Bx) + (2c^2AB + (\alpha+\beta)\omega)\sin(\omega t + Bx) = 0\\
\Rightarrow&\begin{cases} A^4 - \dfrac{\alpha\beta - \omega^2}{c^2}A^2 - \Big(\dfrac{\alpha+\beta}{2c^2}\omega\Big)^2 = 0\\ AB = -(\alpha+\beta)\omega/2c^2\end{cases}
\end{aligned}$$

Set $f(y) = y^2 - \dfrac{\alpha\beta - \omega^2}{c^2}y - \Big(\dfrac{\alpha+\beta}{2c^2}\omega\Big)^2$, then $f(y) < 0$. So $f(y) = 0$ always have positive solution and therefore the equation always has solution for A and B.

So there exists a solution of the form

$$u(x,t) = U_0 e^{-Ax}\cos(\omega t + Bx)$$

where A and B is determined by $\begin{cases} c^2(A^2 - B^2) + \omega^2 - \alpha\beta = 0\\ 2c^2AB + (\alpha+\beta)\omega = 0\end{cases}$.

Exercise 10.7

Since we know that $J'_\nu(x) = \dfrac{1}{2}(J_{\nu-1}(x) - J_{\nu+1}(x))$ and $2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x)$

$$J'_\nu(x) =\frac{1}{2}(J_{\nu-1}(x) - J_{\nu+1}(x)) = \frac{1}{2}(\frac{2\nu}{x}J_\nu(x) - J_{\nu+1}(x) - J_{\nu+1}(x)) = -J_{\nu+1}(x) + \frac{\nu J_\nu(x)}{x}$$

$$J'_\nu(x) =\frac{1}{2}(J_{\nu-1}(x) - J_{\nu+1}(x)) = \frac{1}{2}(J_{\nu-1}(x) - \frac{2\nu}{x}J_\nu(x) + J_{\nu-1}(x)) = J_{\nu-1}(x) - \frac{\nu J_\nu(x)}{x}$$

i.e. $J'_\nu(x) = -J_{\nu+1}(x) + \dfrac{\nu J_\nu(x)}{x}, J'_\nu(x) = J_{\nu-1}(x) - \dfrac{\nu J_\nu(x)}{x}$.

When $\beta \to \alpha$, $\alpha J_\nu(\beta)J'_\nu(\alpha) - \beta J'_\nu(\beta)J_\nu(\alpha) \to 0, \alpha^2 - \beta^2 \to 0$, then we can try to use l'Hôpital's rule and obtain that

$$
\begin{aligned}
&\lim_{\beta\to\alpha}\frac{\alpha J_\nu(\beta)J_\nu'(\alpha)-\beta J_\nu'(\beta)J_\nu(\alpha)}{\alpha^2-\beta^2}\\
=&\frac{\alpha J_\nu(\beta)(-J_{\nu+1}(\alpha)+\frac{\nu J_\nu(\alpha)}{\alpha})-\beta(-J_{\nu+1}(\beta)+\frac{\nu J_\nu(\beta)}{\beta})J_\nu(\alpha)}{\alpha^2-\beta^2}\\
=&\frac{J_{\nu+1}(\beta)J_\nu(\alpha)+\beta J_\nu(\alpha)(J_\nu(\beta)-\frac{(\nu+1)(J_{\nu+1}(\beta))}{\beta})-\alpha J_{\nu+1}(\alpha)(-J_{\nu+1}(\beta)+\frac{\nu J_\nu(\beta)}{\beta})}{-2\beta}\\
=&\frac{-J_\nu'(-J_{\nu+1}+\frac{\nu J_\nu(\alpha)}{\alpha})}{-2\alpha}\\
=&\frac{1}{2}J_\nu'(\alpha)^2
\end{aligned}
$$

So

$$||J_\nu(\alpha\sqrt{\cdot})||^2_{L^2([0,1])}=2\int_0^1 xJ_\nu^2(\alpha x)dx=2\lim_{\beta\to\alpha}\frac{\alpha J_\nu(\beta)J_\nu'(\alpha)-\beta J_\nu'(\beta)J_\nu(\alpha)}{\alpha^2-\beta^2}=J_\nu'(\alpha)^2$$

Exercise 10.8

i)

$$
\begin{aligned}
I_\nu(x)=&e^{-\nu\pi i/2}J_\nu(ix)\\
=&e^{-\nu\pi i/2}\sum_{n=0}^\infty\frac{(-1)^n}{n!\Gamma(1+n+\nu)}\Big(\frac{ix}{2}\Big)^{2n+\nu}=e^{-\nu\pi i/2}\sum_{n=0}^\infty\frac{(-1)^n i^{2n+\nu}}{n!\Gamma(1+n+\nu)}\Big(\frac{x}{2}\Big)^{2n+\nu}\\
=&e^{-\nu\pi i/2}e^{\nu\pi i/2}\sum_{n=0}^\infty\frac{(-1)^n\cdot(-1)^n}{n!\Gamma(1+n+\nu)}\Big(\frac{x}{2}\Big)^{2n+\nu}\\
=&\sum_{n=0}^\infty\frac{1}{n!\Gamma(1+n+\nu)}\Big(\frac{x}{2}\Big)^{2n+\nu}
\end{aligned}
$$

So $I_\nu(x)=\sum_{m=0}^\infty\frac{1}{m!\Gamma(1+m+\nu)}\Big(\frac{x}{2}\Big)^{2m+\nu}$, and therefore $I_\nu(x)\in\mathbb{R}$ for all $x\in\mathbb{R}, I_\nu(x)\neq 0$ for $x\neq 0$. Moreover,

$$
\begin{aligned}
J_{-n}(x)=(-1)^nJ_n(x)\Rightarrow&J_{-n}(x)=(e^{i\pi})^nJ_n(x)\\
\Rightarrow&e^{n\pi i/2}J_{-n}(x)=e^{-n\pi i/2}J_n(x)\\
\Rightarrow&I_{-n}(x)=I_n(x)
\end{aligned}
$$

So $I_{-n}(x)=I_n(x)$ for all $n\in\mathbb{N}$

ii)

$$
\begin{aligned}
\lim_{\nu\to 0} K_\nu(x) =& \lim_{\nu\to 0}\frac{\pi}{2}e^{\nu\pi i/2}(iJ_\nu(ix)-Y_\nu(ix))\\
=&\frac{\pi}{2}(\lim_{\nu\to 0} iJ_\nu(ix)-\lim_{\nu\to 0} Y_\nu(ix))\\
=&\frac{\pi}{2}(i\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(1+m)}\left(\frac{x}{2}\right)^{2m}-\frac{2}{\pi}J_0(ix)\Big(\ln\left(\frac{ix}{2}\right)+\gamma\Big)+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m\sum_{k=1}^{m}\frac{1}{k}}{(m!)^2}\left(\frac{ix}{2}\right)^{2k})\\
=&\frac{\pi}{2}i\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}-\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\Big(\ln\left(\frac{x}{2}\right)+\frac{\pi}{2}i+\gamma\Big)+\sum_{m=1}^{\infty}\frac{\sum_{k=1}^{m}\frac{1}{k}}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\\
=&\sum_{m=1}^{\infty}\frac{\sum_{k=1}^{m}\frac{1}{k}-\ln\left(\frac{x}{2}\right)-\gamma}{(m!)^2}\left(\frac{x}{2}\right)^{2m}-(\ln\left(\frac{x}{2}\right)+\gamma)
\end{aligned}
$$

Since $\ln\left(\frac{x}{2}\right)$ diverges at $x=0$, $K_0(x)$ diverges at $x=0$.

iii)

First we know that

$$
\begin{aligned}
&x^2(I_\nu(x))''+x(I_\nu(x))'-(x^2+\nu^2)I_\nu(x)\\
=&x^2e^{-\nu\pi i/2}J''_\nu(ix)(i)^2+xe^{-\nu\pi i/2}J'_\nu(ix)i-(x^2+\nu^2)e^{-\nu\pi i/2}J_\nu(ix)\\
=&e^{-\nu\pi i/2}((ix)^2J''_\nu(ix)+(ix)J_\nu(ix)+((ix)^2-\nu^2)J_\nu(ix))\\
=&0
\end{aligned}
$$

Also since

$$
\begin{aligned}
K_\nu(x) =&\frac{\pi}{2}e^{\nu\pi i/2}(iJ_\nu(ix)-Y_\nu(ix))\\
=&\frac{\pi}{2}e^{\nu\pi i/2}(iJ_\nu(ix)-\frac{J_\nu(ix)\cos(\nu\pi)-J_{-\nu}(ix)}{\sin(\nu\pi)})\\
=&\frac{\pi}{2}e^{\nu\pi i/2}(\frac{i\sin(\nu\pi)J_\nu(ix)-J_\nu(ix)\cos(\nu\pi)+J_{-\nu}(ix)}{\sin(\nu\pi)})\\
=&\frac{\pi}{2}e^{\nu\pi i/2}\frac{-e^{-\nu\pi i}J_\nu(ix)+J_{-\nu}(ix)}{\sin(\nu\pi)}\\
=&\frac{\pi}{2}\frac{-e^{-\nu\pi i/2}J_\nu(ix)+e^{\nu\pi i/2}J_{-\nu}(ix)}{\sin(\nu\pi)}\\
=&\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)}
\end{aligned}
$$

then

$$
\begin{aligned}
&x^2(K_\nu(x))'' + x(K_\nu(x))' - (x^2+\nu^2)K_\nu(x) \\
=&\frac{\pi}{2\sin(\nu\pi)}((x^2(I_\nu(x))'' + x(I_\nu(x))' - (x^2+\nu^2)I_\nu(x)) \\
&- (x^2(I_{-\nu}(x))'' + x(I_{-\nu}(x))' - (x^2+(-\nu)^2)I_\nu(x))) \\
=&0
\end{aligned}
$$

So I_ν and K_ν both satisfy the differential equation

$$x^2y'' + xy' - (x^2+\nu^2)y = 0$$