VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 1

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1 Exercise 1

1.1

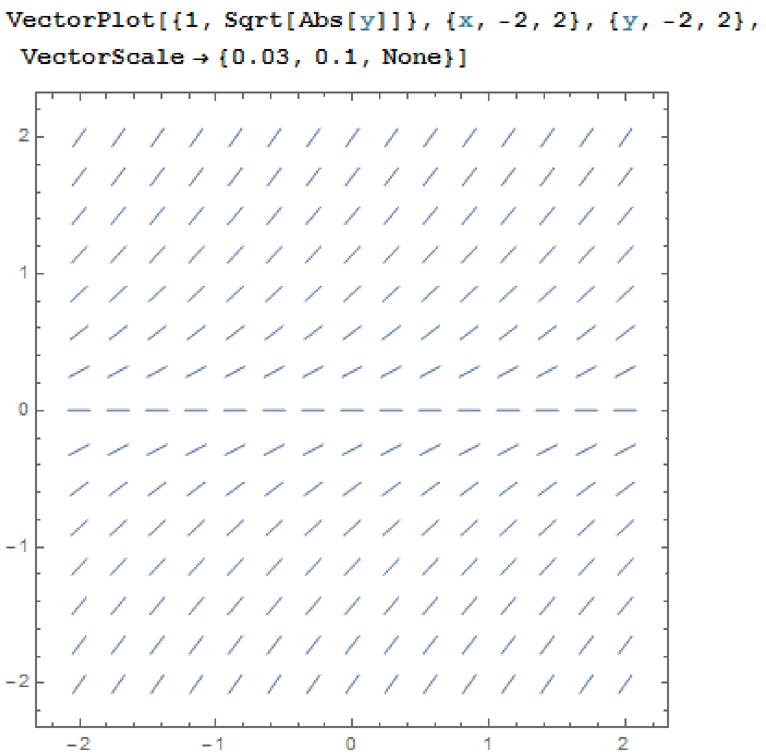

Figure 1: Direction field for $y' = \sqrt{|y|}$

1.2

If $\forall x \in R, y(x) = 0$, we can see that this is a solution to the differential equation $y' = \sqrt{|y|}$. For other possible solutions, they can be found through the equation

$$\int_0^y \frac{ds}{\sqrt{|s|}} = \int_1^x dt$$

we can see that the integral on the left-hand side left-hand side exists for y in a small neighborhood of 0 since

$$\int_0^y \frac{ds}{\sqrt{|s|}} = \begin{cases} \displaystyle\int_0^y \frac{ds}{\sqrt{s}}, s \geqslant 0 \\ \displaystyle\int_0^y \frac{ds}{\sqrt{-s}}, s < 0 \end{cases} = \begin{cases} 2\sqrt{y}, y \geqslant 0 \\ -2\sqrt{-y}, y < 0 \end{cases}$$

So the possible solution except $y = 0$ is given by $x - 1 = \begin{cases} 2\sqrt{y}, y \geqslant 0 \\ -2\sqrt{-y}, y < 0 \end{cases}$, i.e.$y = \begin{cases} \dfrac{1}{4}(x-1)^2, x \geqslant 1 \\ -\dfrac{1}{4}(x-1)^2, x < 1 \end{cases}$

For this function, first we can see that it's differentiable since $\underset{x\nearrow 1}{lim}\frac{dy}{dx} = 0 = \underset{x\searrow 1}{lim}\frac{dy}{dx}$. And for $x \geqslant 1$, $y \geqslant 0$ and $y' = \frac{1}{2}(x-1) = \sqrt{\frac{1}{4}(x-1)^2} = \sqrt{y}$; while for $x < 1$, $y < 0$ and $y' = -\frac{1}{2}(x-1) = \sqrt{\frac{1}{4}(x-1)^2} = \sqrt{-y}$. So $y' = \sqrt{|y|}$ always holds and therefore this is a solution for the differential equation.

To sum up, all solutions of this problem are

1. $y = 0(x \in \mathbb{R})$
2. $y = \begin{cases} \frac{1}{4}(x-1)^2, x \geqslant 1 \\ -\frac{1}{4}(x-1)^2, x < 1 \end{cases}$

2 Exercise 2

3 Exercise 3

Proof: Since $y_\xi(\xi) = 0$, then $\forall \xi \in \overline{I}$

$$y' = \frac{d}{dx}\int_{x_0}^{x} f(\xi)y_\xi(x)d\xi = y_x(x)f(x) + \int_{x_0}^{x} f(\xi)y_\xi'(x)d\xi = \int_{x_0}^{x} f(\xi)y_\xi'(x)d\xi$$

Since $y_\xi'(\xi) = 0$, then $\forall \xi \in \overline{I}$

$$y'' = \frac{d}{dx}\int_{x_0}^{x} f(\xi)y_\xi'(x)d\xi = y_x'(x)f(x) + \int_{x_0}^{x} f(\xi)y_\xi''(x)d\xi = \int_{x_0}^{x} f(\xi)y_\xi''(x)d\xi$$

So we can see that $\forall n \in \mathbb{N}, n \leqslant p-1$, $y^{(n)} = \int_{x_0}^{x} f(\xi)y_\xi^{(n)}(x)d\xi, (y^{(0)} = y)$. Since $y_\xi^{(p-1)}(\xi) = \frac{1}{a_p(\xi)}$, then

$$y^{(p)} = \frac{d}{dx}\int_{x_0}^{x} f(\xi)y_\xi^{(p-1)}(x)d\xi = y_x^{(p-1)}(x)f(x) + \int_{x_0}^{x} f(\xi)y_\xi^{(p)}(x)d\xi = \frac{f(x)}{a_p(x)} + \int_{x_0}^{x} f(\xi)y_\xi^{(p)}(x)d\xi$$

$\forall n \in \mathbb{N}, n \leqslant p-1$, $y^{(n)}(x_0) = \int_{x_0}^{x_0} f(\xi)y_\xi^{(n)}(x)d\xi = 0$, so $y(x) = \int_{x_0}^{x} f(\xi)y_\xi(x)d\xi$ satisfies the initial condition. Moreover, since $\sum_{n=0}^{p}(a_n(x)y_\xi^{(n)}(x)) = 0$

$$\begin{aligned}
&a_p(x)y^{(p)} + \cdots + a_1(x)y' + a_0(x)y \\
=&a_p(x)(\frac{f(x)}{a_p(x)} + \int_{x_0}^{x} f(\xi)y_\xi^{(p)}(x)d\xi) + \sum_{n=0}^{p-1}(a_n(x)\int_{x_0}^{x} f(\xi)y_\xi^{(n)}(x)d\xi) \\
=&f(x) + \sum_{n=0}^{p}(a_n(x)\int_{x_0}^{x} f(\xi)y_\xi^{(n)}(x)d\xi) \\
=&f(x) + \int_{x_0}^{x} f(\xi)\sum_{n=0}^{p}(a_n(x)y_\xi^{(n)}(x))d\xi \\
=&f(x)
\end{aligned}$$

So $y(x) = \int_{x_0}^{x} f(\xi)y_\xi(x)d\xi$ solves $a_p(x)y^{(p)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$.

4 Exercise 4

Proof: Since y_{hom} is a solution of $a_1(x)y' + a_0(x)y = 0$, $a_1(x)y'_{hom}(x) + a_0(x)y_{hom}(x) = 0$

$$
\begin{aligned}
&a_1(x)y' + a_0(x)y = f(x) \\
\Rightarrow &a_1(x)(\frac{d}{dx}(c(x)y_{hom}(x))) + a_0(x)c(x)y_{hom}(x) = f(x) \\
\Rightarrow &a_1(x)c'(x)y_{hom}(x) + c(x)(a_1(x)y'_{hom}(x) + a_0(x)y_{hom}(x)) = f(x) \\
\Rightarrow &a_1(x)c'(x)y_{hom}(x) = f(x) \\
\Rightarrow &c'(x) = \frac{f(x)}{a_1(x)y_{hom}(x)} \\
\Rightarrow &\int_{x_0}^{x} c'(\xi)d\xi = \int_{x_0}^{x} \frac{f(\xi)}{a_1(\xi)y_{hom}(\xi)} d\xi \\
\Rightarrow &c(x) = \int_{x_0}^{x} f(\xi)d\xi(Let \;\; c(x_0) = 0, and \;\; use \;\; \forall\xi \in \overline{I} a_1(\xi)y_{hom}(\xi) = 1) \\
\Rightarrow &y_{part}(x) = c(x)y_{hom}(x) = y_{hom}(x)\int_{x_0}^{x} f(\xi)d\xi = \int_{x_0}^{x} f(\xi)y_{hom}(x)d\xi
\end{aligned}
$$

So this differential equation yields the same solution formula as the Duhamel principle.

5 Exercise 5

For "Washing of Feet",

$$\lambda y_0 = 12.6 \times 2^{150 \div 11} - 0.26 \times (2^{150 \div 11} - 1) \approx 157134$$

For "Woman Reading Music",

$$\lambda y_0 = 10.3 \times 2^{150 \div 11} - 0.3 \times (2^{150 \div 11} - 1) \approx 127337$$

For "Woman Playing Mandolin",

$$\lambda y_0 = 8.2 \times 2^{150 \div 11} - 0.17 \times (2^{150 \div 11} - 1) \approx 102252$$

All these values are unacceptably large. Thus, "Washing of Feet", "Woman Reading Music" and "Woman playing Mandolin" are forgeries.

6 Exercise 6

6.1

According to the question, we have the following differentiable equation:

$$\frac{dX}{dt} = k(60 - X)(150 - X), X(5) = 10$$

where k is a constant.

The unique solution of this equation can be obtained from:

$$\int_{10}^{X} \frac{ds}{(60 - s)(150 - s)} = \int_{5}^{t} kdt$$

So $90k(t-5) = \int_{10}^{X} (\frac{1}{60-s} - \frac{1}{150-s})ds = ln\frac{150-X}{60-X} - ln2.8$. When $t=0, X=0$, so $-450k = ln(2.5/2.8)$. So $X = 60 - \frac{90}{2.8 \cdot (25/28)^{(5-t)/5} - 1}$

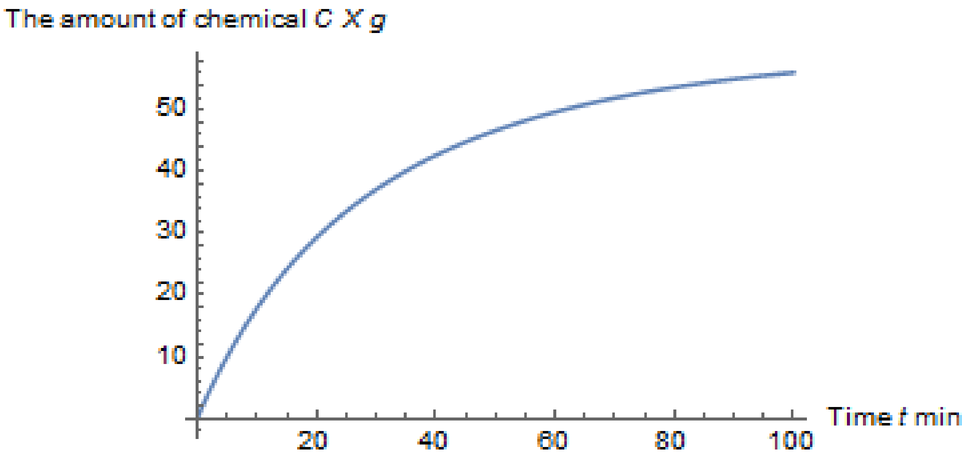

Figure 2: The amount of chemical C formed as a function of time

6.2

$$X(20) = 60 - \frac{90}{2.8 \cdot (25/28)^{(5-20)/5} - 1} \approx 29.323g$$

So there are $29.323g$ chemical C formed in 20 minutes.

6.3

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} 60 - \frac{90}{2.8 \cdot (25/28)^{(5-t)/5} - 1} = 60g$$

So the limiting amount of C as time $t \to \infty$ is $60g$.

6.4

$$M_A = 40 - 60 \times 2 \div 3 = 0g, M_B = 50 - 60 \times 1 \div 3 = 30g$$

So as time $t \to \infty$, chemicals A remains $0g$, chemicals B remains $30g$.

7 Exercise 7

7.1

The solution that pass through (2,1/4) is $y = \begin{cases} \frac{2}{2-3x}, x < 0 \\ \frac{2}{2+3x}, x > 0 \end{cases}$

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VectorPlot[{1, (y^2 - y) / x}, {x, -2, 2}, {y, -2, 2},
 VectorScale → {0.03, 0.1, None}]
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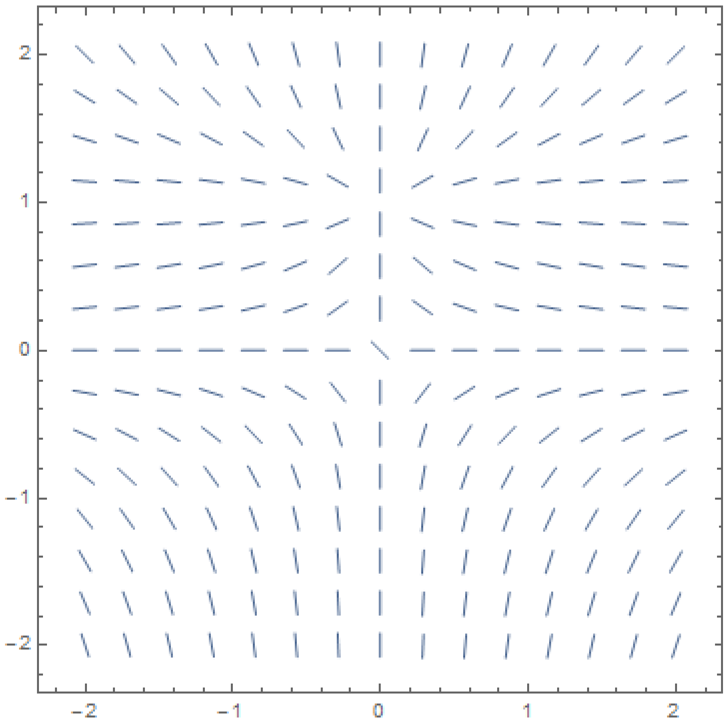

Figure 3: The direction field of the equation $xy' = y^2 - y$

7.2

The solution that pass through (1/2,1/2) is $y = \begin{cases} \dfrac{1}{1-2x}, x < 0 \\ \dfrac{1}{1+2x}, x > 0 \end{cases}$

7.3

Since (0,2) doesn't satisfy the differentiable equation, so there is no solution for this situation.

7.4

One solution that pass through (0,1) is $y = 1$

8 Exercise 8

8.1

1. $y = -1$ is a solution of the equation $y' = (1 + x)(1 + y)$
2. If there exists some x_0 such that $y(x_0) = y_0 \neq -1$, then the solution can be found

by

$$\int_{y_0}^{y} \frac{ds}{1+s} = \int_{x_0}^{x} 1 + tdt$$

So

$$t + \frac{1}{2}t^2|_{x_0}^{x} = ln(1+s)|_{y_0}^{y} \Rightarrow y = (1+y_0)e^{0.5x^2+x-x_0-0.5x_0^2} - 1$$

To sum up, the general solution to $y' = (1+x)(1+y)$ with initial value $y(x_0) = y_0$ is

$$y = (1+y_0)e^{0.5x^2+x-x_0-0.5x_0^2} - 1$$

.

8.2

The solution to $y' = e^{x+y+3}$ can be found by

$$\int_{y_0}^{y} \frac{ds}{e^s} = \int_{x_0}^{x} e^{t+3}dt$$

So

$$e^{t+3}|_{x_0}^{x} = -e^{-s}|_{y_0}^{y} \Rightarrow y = -ln(e^{-y_0} - e^{x+3} + e^{x_0+3})$$

To sum up, the general solution to $y' = e^{x+y+3}$ with initial value $y(x_0) = y_0$ is

$$y = -ln(e^{-y_0} + e^{x_0+3} - e^{x+3})$$

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