# Vv286 Honors Mathematics IV
# Ordinary Differential Equations

## Assignment 2

**Date Due: 10:00 AM, Thursday, the 29th of September 2016**

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**Exercise 2.1.** Show that the Ricatti differential equation

$$y' + g(x)y + h(x)y^2 = k(x) \qquad \text{on an open interval } I \subset \mathbb{R}$$

with $g, h \in C(I)$, $h \in C^1(I)$, $h \neq 0$ on $I$, can be transformed into the linear differential equation of second order,

$$u'' + \left(g - \frac{h'}{h}\right)u' - khu = 0,$$

using the transformation

$$u(x) = e^{\int h(x)y(x)\,dx}.$$

**(2 Marks)**

**Exercise 2.2.** Show that the equation

$$h(x,y)y' + g(x,y) = 0,$$

if the orthogonal vector field is not a potential field, has an integrating factor of the form $M(x,y) = M(x \cdot y)$ if and only if

$$\frac{h_x - g_y}{xg - hy}$$

is a function of $x \cdot y$ only. Check that this is the case for

$$\left(\frac{x^2}{y} + 3\frac{y}{x}\right)y' + \left(3x + \frac{6}{y}\right) = 0.$$

and then solve the equation.
**(2 + 2 Marks)**

**Exercise 2.3.** Find an integrating factor (Euler multiplier) for the equation

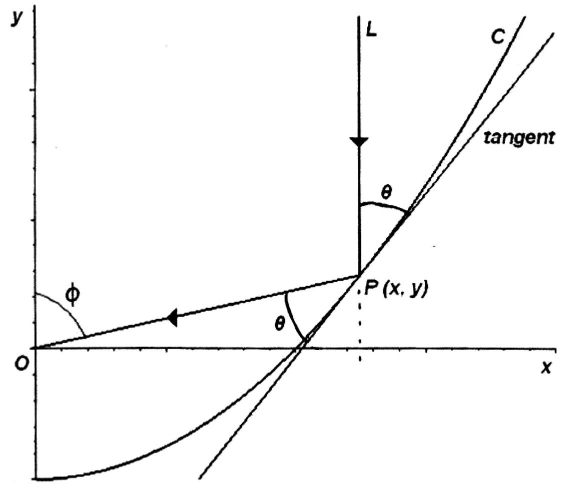
$$a_1(x)y' + a_0(x)y = f(x),$$

where $a_0, a_1, f$ are continuous functions on an interval $I \subset \mathbb{R}$. (Try guessing whther the factor might depend only on $x$ or only on $y$, for example.)

Then solve the equation in general using this factor. Compare with the formula for a particular solution of an inhomogeneous, linear, first-order ordinary differential equation obtained from Duhamel's principle.
**(3 Marks)**

**Exercise 2.4.** Show that the solution of Clairaut's equation obtained from a slope parametrization of the integral curve is always teh envelope of the straight-line solutions.
**(3 Marks)**

**Exercise 2.5.** Light strikes a plane curve $C$ in such a manner that all beams $L$ parallel to the $y$-axis are reflected to a single point $O$ (see the diagram below). The objective of this exercise is to determine the differential equation for the fucntion $y = f(x)$ describing the curve $C$.




i) Show geometrically that $\phi = 2\theta$, $\tan\phi = \frac{x}{y}$ and $\tan(\pi/2 - \theta) = \frac{dy}{dx}$.

ii) Use the identity $\tan(\pi/2 - x) = \frac{1}{\tan x}$ to show that $\tan\theta = \frac{dx}{dy}$.

iii) Use the identity $\tan(2x) = 2\tan x/(1 - \tan^2 x)$ to derive the ODE

$$x\left(\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} = x. \qquad (*)$$

iv) Substitute $w = x^2$ in $(*)$ to obtain a differential equation of Clairaut type. Solve this equation, and resubstitute $w = x^2$ to obtain a solution of $(*)$.

v) What do the above calculations tell you about the types of curves that can be used to focus rays into a single point?

**($3 \times \frac{1}{2} + 1 \times \frac{1}{2} + 2 + 2 + 2$ Marks)**

**Exercise 2.6.** Consider the initial value problem

$$y' = y^2 + x^2, \qquad y(0) = 0. \qquad (**)$$

i) Use Picard iteration to find a succession of approximate solutions $y_1, y_2, y_3, y_4$, starting from $y_0(x) = 0$. You may use Mathematica to help perform the integrations.

ii) Use Mathematica to obtain a numerical solution to $(**)$. Plot the numerical solution as well as $y_1, y_2, y_3, y_4$ in a single graph.

**($2 + 3$ Marks)**