Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 10

Date Due: 10:00 AM, Thursday, the 8th of December 2016

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Exercise 10.1.

i) Show that

$$\mathcal{B} := \left\{ \frac{1}{\sqrt{2}}, \cos(\pi n x), \sin(\pi n x) \right\}_{n=1}^{\infty}$$

is an orthonormal system in $L^2([-1,1])$.

ii) Show that if $\{e_n\}$ is an orthonormal system in $L^2([-1,1])$, then $\{\widetilde{e}_n\}$ defined by

$$\widetilde{e}_n(x) = \sqrt{\frac{2}{b-a}} \cdot e_n\left(\frac{2}{b-a}\left(x - \frac{b+a}{2}\right)\right)$$

is an orthonormal system in $L^2([a,b])$.

iii) Use ii) to construct orthonormal systems from $\mathcal{B}$ in i) for the spaces $L^2([-\pi,\pi])$, and $L^2([0,L])$ for any $L > 0$.

(2 + 2 + 2 Marks)

Exercise 10.2. Calculate the Fourier series of the function f defined on $[-1,1]$ and given by $f(x) = x^2$. Evaluate the series at a suitable point to find the value of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By evaluating the series at a different point, find the value of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$$

(4 Marks)

Exercise 10.3. Consider the equation for a vibrating beam of length $l > 0$,

$$u_{tt} + c^2 u_{xxxx} = 0, \qquad (x,t) \in (0,l) \times \mathbb{R}_+.$$

Find the solution to the initial-boundary value problem

$$u(0,t) = u(l,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(l,t) = 0, \qquad t \in \mathbb{R}_+,$$
$$u(x,0) = x(l-x), \qquad u_t(x,0) = 0, \qquad x \in (0,l).$$

(4 Marks)

Exercise 10.4. Use a separation-of-variables approach to solve the damped wave equation

$$c^2 u_{xx} - u_{tt} - \mu u_t = 0, \qquad (x,t) \in (0,L) \times \mathbb{R}_+,\ L > 0,$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0$$

and initial conditions

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right), \qquad u_t(x,0) = 0, \qquad x \in [0,L].$$

(4 Marks)

Exercise 10.5. Solve the equation

$$u_{xx} + u_{yy} = u, \qquad (x, y) \in (0, \pi) \times (0, a), \quad a > 0,$$

with boundary conditions

$$u(0, y) = u(\pi, y) = u(x, 0) = 0, \qquad u(x, a) = 1, \qquad (x, y) \in [0, \pi] \times [0, a].$$

(4 Marks)

Exercise 10.6. Show that the telegraph equation with $\alpha, \beta > 0$,

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}, \qquad (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with the condition

$$\sup_{x \in \mathbb{R}_+} |u(x, t)| < \infty, \qquad t \in \mathbb{R}_+$$

and initial signal

$$u(0, t) = U_0 \cos(\omega t), \qquad \omega, U_0 > 0,$$

does not have a solution of the form $u(x, t) = X(x) \cdot T(t)$. Next, show that there exists a solution of the from

$$u(x, t) = U_0 e^{-Ax} \cos(\omega t + Bx)$$

for certain constants A and B. How are A and B determined from α, β, ω? (Thus, not every problem has a solution that can be found from a separation-of-variables approach.)
(2 + 3 Marks)

Exercise 10.7. From the recurrence relations obtained in Exercise 9.3, deduce[1]

$$J_\nu'(x) = -J_{\nu+1}(x) + \frac{\nu J_\nu(x)}{x}, \qquad J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu J_\nu(x)}{x}$$

for $\nu \in \mathbb{R}$. Use the first of these relations and l'Hôpital's rule to evaluate

$$\lim_{\beta \to \alpha} \frac{\alpha J_\nu(\beta) J_\nu'(\alpha) - \beta J_\nu'(\beta) J_\nu(\alpha)}{\alpha^2 - \beta^2} = \frac{1}{2} J_\nu'(\alpha)^2$$

where $\alpha \in \mathbb{R}$ is arbitrary. This proves that

$$\|J_\nu(\alpha \sqrt{\cdot})\|^2_{L^2([0,1])} = J_\nu'(\alpha)^2.$$

(3 Marks)

Exercise 10.8. The functions

$$I_\nu(x) := e^{-\nu\pi i/2} J_\nu(ix), \qquad \nu \in \mathbb{R},$$

are called the *modified Bessel functions of the first kind.*

i) Show that

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

Deduce that $I_\nu(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, $I_\nu(x) \neq 0$ for $x \neq 0$ and $I_{-n}(x) = I_n(x)$ for $n \in \mathbb{N}$.

ii) For $\nu \in \mathbb{R}$ we define the *modified Bessel functions of the second kind*[2] by

$$K_\nu(x) := \frac{\pi}{2} e^{\nu\pi i/2} (i J_\nu(ix) - Y_\nu(ix)),$$

where Y_ν is the Bessel function of the second kind introduced in Exercise 9.8 of Assignment 11. Use the results of this exercise to find the series expansion of $K_0(x)$ and verify that K_0 diverges at $x = 0$.

iii) Show that I_ν and K_ν both satisfy the differential equation

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0.$$

(4 + 3 + 3 Marks)

[1] This exercise follows Korenev's book, pages 96-7.

[2] Sometimes also called *Macdonald functions* (e.g., in Korenev) or *modified Bessel functions of the third kind.*