VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 4

Jiang Yicheng
515370910224

October 20, 2016

Exercise 4.1

$$(A-3\mathbb{1})v = \begin{pmatrix} 4-3 & -4 & -11 & 11 \\ 3 & -12-3 & -42 & 42 \\ -2 & 12 & 37-3 & -34 \\ -1 & 7 & 20 & -17-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -4 & -11 & 11 \\ 3 & -15 & -42 & 42 \\ -2 & 12 & 34 & -34 \\ -1 & 7 & 20 & -20 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_4 - v_3 \wedge v_2 = 3v_4 - 3v_3$$

$$\Rightarrow v = \begin{pmatrix} v_4 - v_3 \\ 3v_4 - 3v_3 \\ v_3 \\ v_4 \end{pmatrix} = v_3 \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

So we can find two independent eigenvectors by choosing

$$v_1 = \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Hence, $dimV_\lambda = 2$ and we need to find two generalized eigenvectors. $m = a_\lambda - dimV_\lambda + 1 = 4 - 2 + 1 = 3$. We can find that

$$(A-3\mathbb{1})^2 = \begin{pmatrix} 0 & 1 & 3 & -3 \\ 0 & 3 & 9 & -9 \\ 0 & -2 & -6 & 6 \\ 0 & -1 & -3 & 3 \end{pmatrix}, (A-3\mathbb{1})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that any vector solves $(A-3\mathbb{1})^3 v = 0$, and we can easily choose a vector such that $(A-3\mathbb{1})^2 v \neq 0$.

So we can set

$$v^{(3)} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v^{(2)} = (A-3\mathbb{1})v^{(3)} = \begin{pmatrix} 0 \\ -3 \\ 4 \\ 3 \end{pmatrix}, v^{(1)} = (A-3\mathbb{1})v^{(2)} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix} = -2v_1 - v_2$$

Then we set $U = (v_1, v^{(1)}, v^{(2)}, v^{(3)})$ and we can get that

$$U^{-1}AU = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and a basis of generalized eigenvectors is $\{\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$.

Exercise 4.2

Since $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a constant linear force field, then we can consider it as a matrix in 2×2, i.e.

$$F(x_1, x_2) = Fx$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. According to Newton's second law, sicne the mass of particle is $m = 1$,

$$F(x_1, x_2) = ma = v'$$

where $v' = a$ is the acceleration of the particle and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is the velocity of it. Moreover, according to the definition of velocity, we know that

$$v = x'$$

so

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ Fx \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ F & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

For $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, set $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then $det(A - \lambda\mathbb{1}) = 0 \Leftrightarrow \lambda = \pm 1, \pm i$.

1. $(A - \lambda)v = 0 \Leftrightarrow v_1 = v_2 = v_3 = v_4$, so we choose $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$
2. $(A + \lambda)v = 0 \Leftrightarrow v_1 = v_2 = -v_3 = -v_4$, so we choose $u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$
3. $(A - i\lambda)v = 0 \Leftrightarrow v_2 = -v_1 \wedge v_3 = iv_1 \wedge v_4 = -iv_1$, so we choose $u_3 = \begin{pmatrix} 1 \\ -1 \\ i \\ -i \end{pmatrix}$
4. $(A + i\lambda)v = 0 \Leftrightarrow v_2 = -v_1 \wedge v_3 = -iv_1 \wedge v_4 = iv_1$, so we choose $u_4 = \begin{pmatrix} 1 \\ -1 \\ -i \\ i \end{pmatrix}$

So A is diagonalizable and the fundamental system is

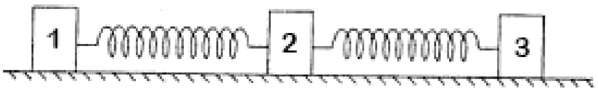
$$\mathscr{F} = \{e^t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, e^{it} \begin{pmatrix} 1 \\ -1 \\ i \\ -i \end{pmatrix}, e^{-it} \begin{pmatrix} 1 \\ -1 \\ -i \\ i \end{pmatrix}\}$$

So general solution for the system of equations is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ v_1(t) \\ v_2(t) \end{pmatrix} = Re \begin{pmatrix} c_1e^t + c_2e^{-t} + c_3e^{it} + c_4e^{-it} \\ c_1e^t + c_2e^{-t} - c_3e^{it} - c_4e^{-it} \\ c_1e^t - c_2e^{-t} + ic_3e^{it} - ic_4e^{-it} \\ c_1e^t - c_2e^{-t} - ic_3e^{it} + ic_4e^{-it} \end{pmatrix} = \begin{pmatrix} c_1e^t + c_2e^{-t} + c_5cost \\ c_1e^t + c_2e^{-t} - c_5cost \\ c_1e^t - c_2e^{-t} - c_5sint \\ c_1e^t - c_2e^{-t} + c_5sint \end{pmatrix}$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ are constant.

Exercise 4.3



i)

Set the acceleration of each mass is a_1, a_2, a_3, then $a_1 = \ddot{d}_1, a_2 = \ddot{d}_2, a_3 = \ddot{d}_3$. Then according to laws of Newton and Hooke, we get that:

$$ma_1 = F_1 = k(d_2 - d_1), ma_3 = F_2 = k(d_2 - d_3), ma_2 = -F_1 - F_2 = k(d_1 - d_2 - d_2 + d_3)$$

So

$$\ddot{d} = \begin{pmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{d}_3 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} d_2 - d_1 \\ d_1 - 2d_2 + d_3 \\ d_2 - d_3 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = Ad$$

So $\ddot{d} = Ad$

ii)

Set the velocity of mass is v_1, v_2, v_3, then $v_1 = \dot{d}_1, v_2 = \dot{d}_2, v_3 = \dot{d}_3$. Then

$$\begin{pmatrix} \dot{v} \\ \dot{d} \end{pmatrix} = \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{pmatrix} = \begin{pmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{d}_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{k}{m}(d_2 - d_1) \\ \frac{k}{m}(d_1 - 2d_2 + d_3) \\ \frac{k}{m}(d_2 - d_3) \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{k}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & 0 & 0 & 0 & \frac{k}{m} & -\frac{k}{m} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

So let $B = \begin{pmatrix} 0 & 0 & 0 & -\frac{k}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & 0 & 0 & 0 & \frac{k}{m} & -\frac{k}{m} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ and then $\begin{pmatrix} \dot{v} \\ \dot{d} \end{pmatrix} = B \begin{pmatrix} v \\ d \end{pmatrix}$.

iii)

When $k = m = 1$, $B = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

```
In[1]:= B = {{0, 0, 0, -1, 1, 0}, {0, 0, 0, 1, -2, 1}, {0, 0, 0, 0, 1, -1}, {1, 0, 0, 0, 0, 0},
          {0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}}

Out[1]= {{0, 0, 0, -1, 1, 0}, {0, 0, 0, 1, -2, 1}, {0, 0, 0, 0, 1, -1},
         {1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}}

In[2]:= Eigenvalues[B]
```

Out[2]= $\{i\sqrt{3}, -i\sqrt{3}, i, -i, 0, 0\}$

```
In[3]:= Solve[Thread[(B - I * Sqrt[3] * IdentityMatrix[6]).{a, b, c, d, e, f} ==
           {0, 0, 0, 0, 0, 0}], {a, b, c, d, e, f}]

        Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[3]= $\left\{\left\{b \to -2\,a,\ c \to a,\ d \to -\frac{i\,a}{\sqrt{3}},\ e \to \frac{2\,i\,a}{\sqrt{3}},\ f \to -\frac{i\,a}{\sqrt{3}}\right\}\right\}$

```
In[4]:= Solve[Thread[(B + I * Sqrt[3] * IdentityMatrix[6]).{a, b, c, d, e, f} ==
           {0, 0, 0, 0, 0, 0}], {a, b, c, d, e, f}]

        Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[4]= $\left\{\left\{b \to -2\,a,\ c \to a,\ d \to \frac{i\,a}{\sqrt{3}},\ e \to -\frac{2\,i\,a}{\sqrt{3}},\ f \to \frac{i\,a}{\sqrt{3}}\right\}\right\}$

```
In[5]:= Solve[Thread[(B - I * IdentityMatrix[6]).{a, b, c, d, e, f} == {0, 0, 0, 0, 0, 0}],
         {a, b, c, d, e, f}]

        Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[5]= {{b → 0, c → -a, d → -i a, e → 0, f → i a}}

In[6]:= Solve[Thread[(B + I * IdentityMatrix[6]).{a, b, c, d, e, f} == {0, 0, 0, 0, 0, 0}],
         {a, b, c, d, e, f}]

        Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[6]= {{b → 0, c → -a, d → i a, e → 0, f → -i a}}

In[7]:= Solve[Thread[(B - 0 * IdentityMatrix[6]).{a, b, c, d, e, f} == {0, 0, 0, 0, 0, 0}],
         {a, b, c, d, e, f}]

        Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[7]= {{a → 0, b → 0, c → 0, e → d, f → d}}
```

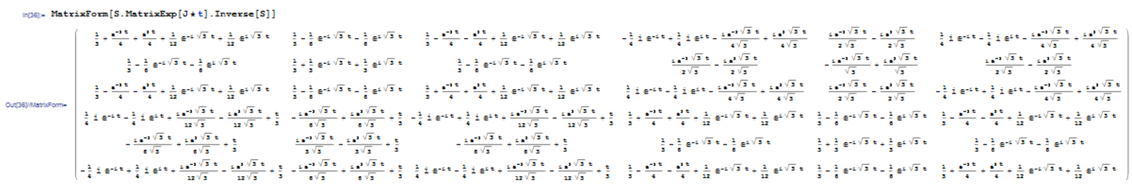
So eigenvalues of B are $\lambda_1 = \sqrt{3}i$, $\lambda_2 = -\sqrt{3}i$, $\lambda_3 = i$, $\lambda_4 = -i$, $\lambda_5 = 0$ and eigenvectors are $v_1 = a_1 \begin{pmatrix} \sqrt{3} \\ -2\sqrt{3} \\ \sqrt{3} \\ -i \\ 2i \\ -i \end{pmatrix}$, $v_2 = a_2 \begin{pmatrix} \sqrt{3} \\ -2\sqrt{3} \\ \sqrt{3} \\ i \\ -2i \\ i \end{pmatrix}$, $v_3 = a_3 \begin{pmatrix} 1 \\ 0 \\ -1 \\ -i \\ 0 \\ i \end{pmatrix}$, $v_4 = a_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ i \\ 0 \\ -i \end{pmatrix}$, $v_5 = a_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \setminus \{0\}$.

```
{S, J} = JordanDecomposition[B];
{MatrixForm[S], MatrixForm[J]}
```

$$\left\{\begin{pmatrix} 0 & 1 & i & -i & -i\sqrt{3} & i\sqrt{3} \\ 0 & 1 & 0 & 0 & 2i\sqrt{3} & -2i\sqrt{3} \\ 0 & 1 & -i & i & -i\sqrt{3} & i\sqrt{3} \\ 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -2 & -2 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{3} \end{pmatrix}\right\}$$

So the Jordan normal form J of B is $J = \begin{pmatrix} 0 & 1 & i & -i & -\sqrt{3}i & \sqrt{3}i \\ 0 & 1 & 0 & 0 & 2\sqrt{3}i & -2\sqrt{3}i \\ 0 & 1 & -i & i & -\sqrt{3}i & \sqrt{3}i \\ 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -2 & -2 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ and the matrix S such that $B = SJS^{-1}$ is $S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}i & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}i \end{pmatrix}$.

iv)



So $\Phi(t) = e^{Bt} = Se^{Jt}S^{-1} =$

$$\frac{1}{6}\begin{pmatrix} 2+3\cos t+\cos\sqrt{3}t & 2-2\cos\sqrt{3}t & 2-3\cos t+\cos\sqrt{3}t & -3\sin t-\sqrt{3}\sin\sqrt{3}t & 2\sqrt{3}\sin\sqrt{3}t & 3\sin t-\sqrt{3}\sin\sqrt{3}t \\ 2-2\cos\sqrt{3}t & 2+4\cos\sqrt{3}t & 2-2\cos\sqrt{3}t & 2\sqrt{3}\sin\sqrt{3}t & -4\sqrt{3}\sin\sqrt{3}t & 2\sqrt{3}\sin\sqrt{3}t \\ 2-3\cos t+\cos\sqrt{3}t & 2-2\cos\sqrt{3}t & 2+3\cos t+\cos\sqrt{3}t & 3\sin t-\sqrt{3}\sin\sqrt{3}t & 2\sqrt{3}\sin\sqrt{3}t & -3\sin t-\sqrt{3}\sin\sqrt{3}t \\ 3\sin t+\frac{1}{\sqrt{3}}\sin\sqrt{3}t+2t & -\frac{2}{\sqrt{3}}\sin\sqrt{3}t+2t & 3\sin t+\frac{1}{\sqrt{3}}\sin\sqrt{3}t+2t & 2+3\cos t+\cos\sqrt{3}t & 2-2\cos\sqrt{3}t & 2-3\cos t+\cos\sqrt{3}t \\ -\frac{2}{\sqrt{3}}\sin\sqrt{3}t+2t & \frac{4}{\sqrt{3}}\sin\sqrt{3}t+2t & -\frac{2}{\sqrt{3}}\sin\sqrt{3}t+2t & 2-2\cos\sqrt{3}t & 2+4\cos\sqrt{3}t & 2-2\cos\sqrt{3}t \\ -3\sin t-\frac{1}{\sqrt{3}}\sin\sqrt{3}t+2t & -\frac{2}{\sqrt{3}}\sin\sqrt{3}t+2t & 3\sin t+\frac{1}{\sqrt{3}}\sin\sqrt{3}t+2t & 2-3\cos t+\cos\sqrt{3}t & 2-2\cos\sqrt{3}t & 2+3\cos t+\cos\sqrt{3}t \end{pmatrix}$$

So $\Phi(0) = \frac{1}{6}\begin{pmatrix} 2+3+1 & 2-2 & 2-3+1 & 0 & 0 & 0 \\ 2-2 & 2+4 & 2-2 & 0 & 0 & 0 \\ 2-3+1 & 2-2 & 2+3+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+3+1 & 2-2 & 2-3+1 \\ 0 & 0 & 0 & 2-2 & 2+4 & 2-2 \\ 0 & 0 & 0 & 2-3+1 & 2-2 & 2+3+1 \end{pmatrix} = \mathbb{1}.$

v)

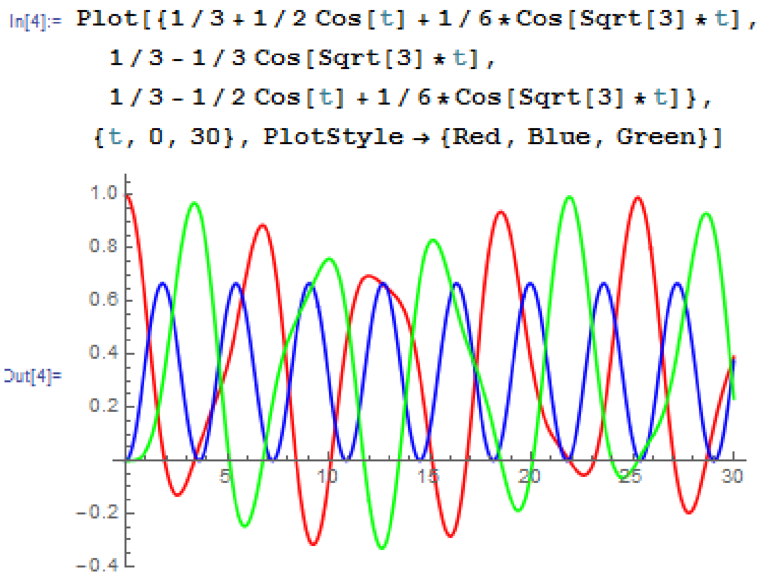


Figure 1: Figure for $v_1(t) = \frac{1}{3} + \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$, $v_2 = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$, $v_3(t) = \frac{1}{3} - \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$,red for $v_1(t)$, blue for v_2, green for v_3.

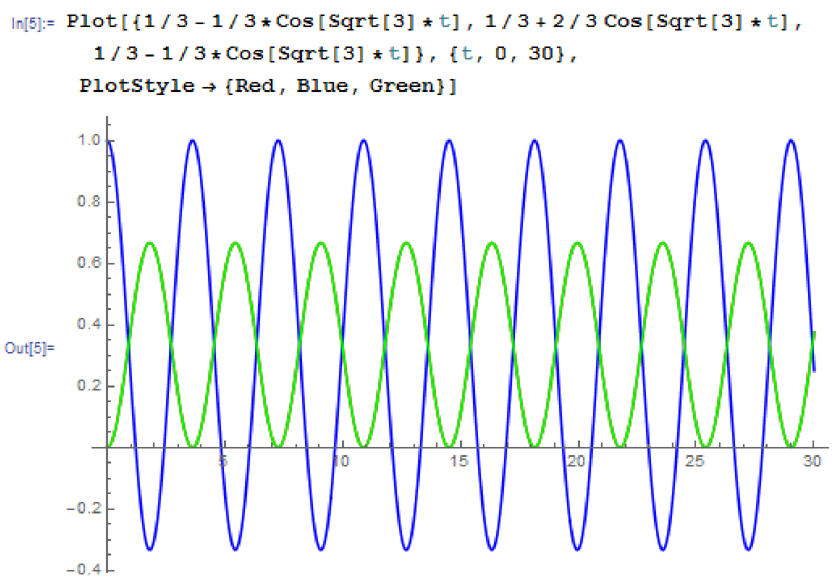


Figure 2: Figure for $v_1(t) = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$, $v_2 = \frac{1}{3} + \frac{1}{3}cos\sqrt{3}t$, $v_3(t) = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$,red for $v_1(t)$, blue for v_2, green for v_3.

```
In[11]:= Plot[{1 / 3 - 1 / 2 Cos[t] + 1 / 6 * Cos[Sqrt[3] * t],
          1 / 3 - 1 / 3 Cos[Sqrt[3] * t],
          1 / 3 + 1 / 2 Cos[t] + 1 / 6 * Cos[Sqrt[3] * t]},
         {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[11]=

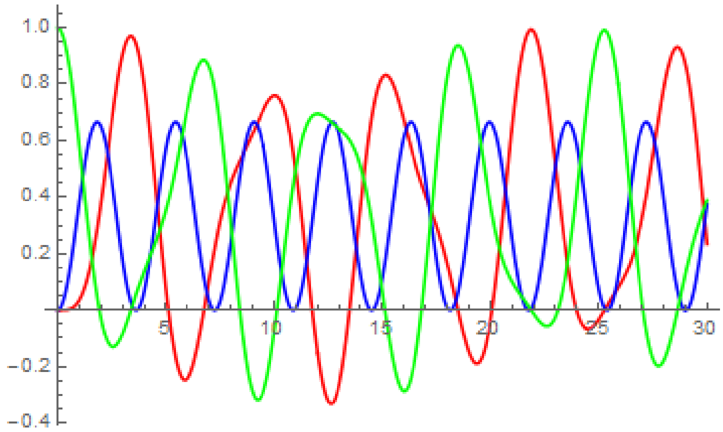


Figure 3: Figure for $v_3(t) = \frac{1}{3} + \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$, $v_2 = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$, $v_1(t) = \frac{1}{3} - \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$,red for $v_1(t)$, blue for v_2, green for v_3.

```
In[12]:= Plot[{-1 / 2 Sin[t] - 1 / 2 / Sqrt[3] * Sin[Sqrt[3] * t],
          1 / Sqrt[3] * Sin[Sqrt[3] * t],
          1 / 2 Sin[t] - 1 / 2 / Sqrt[3] * Sin[Sqrt[3] * t]},
         {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[12]=

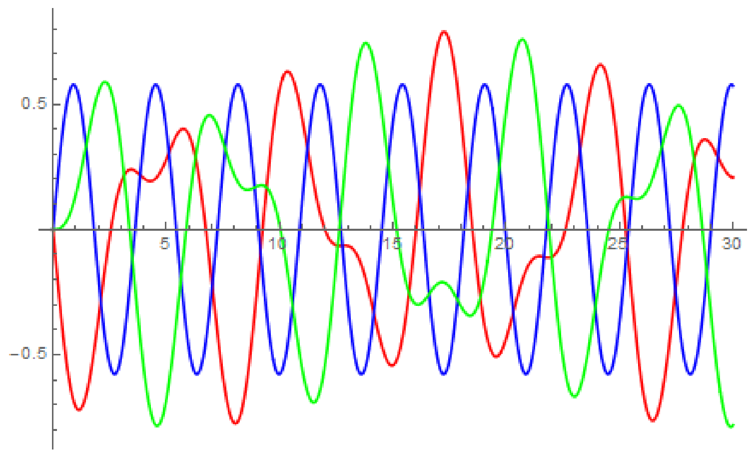


Figure 4: Figure for $v_1(t) = -\frac{1}{2}sint - \frac{1}{2\sqrt{3}}sin\sqrt{3}t$, $v_2 = \frac{1}{\sqrt{3}}sin\sqrt{3}t$, $v_3(t) = \frac{1}{2}sint - \frac{1}{2\sqrt{3}}sin\sqrt{3}t$,red for $v_1(t)$, blue for v_2, green for v_3.

```
In[13]:= Plot[{1 / Sqrt[3] * Sin[Sqrt[3] * t],
   -2 / Sqrt[3] * Sin[Sqrt[3] * t],
   1 / Sqrt[3] * Sin[Sqrt[3] * t]}, {t, 0, 30},
  PlotStyle → {Red, Blue, Green}]
```

Out[13]=

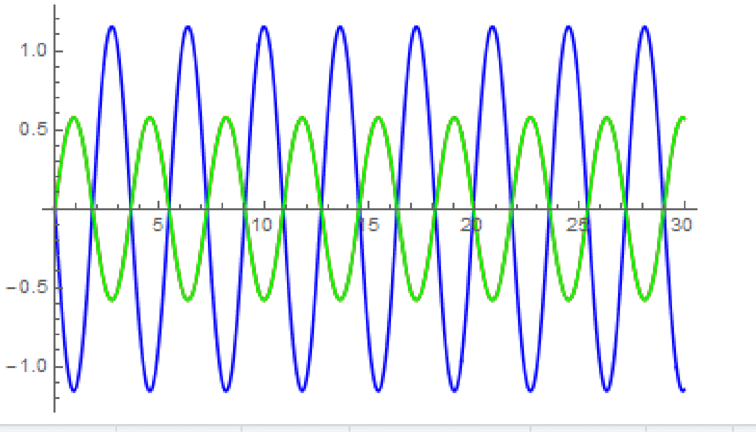


Figure 5: Figure for $v_1(t) = \frac{1}{\sqrt{3}} sin\sqrt{3}t, v_2 = -\frac{2}{\sqrt{3}} sin\sqrt{3}t, v_3(t) = \frac{1}{\sqrt{3}} sin\sqrt{3}t$,red for $v_1(t)$, blue for v_2, green for v_3.

```
In[14]:= Plot[{1 / 2 Sin[t] - 1 / 2 / Sqrt[3] * Sin[Sqrt[3] * t],
   1 / Sqrt[3] * Sin[Sqrt[3] * t],
   -1 / 2 Sin[t] - 1 / 2 / Sqrt[3] * Sin[Sqrt[3] * t]},
  {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[14]=

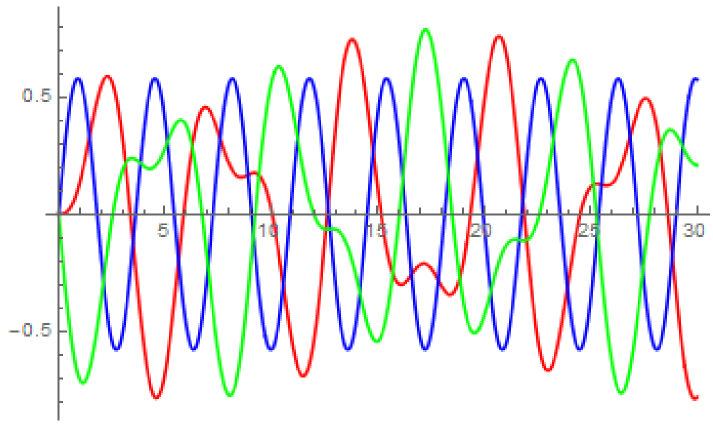


Figure 6: Figure for $v_3(t) = -\frac{1}{2} sint - \frac{1}{2\sqrt{3}} sin\sqrt{3}t, v_2 = \frac{1}{\sqrt{3}} sin\sqrt{3}t, v_1(t) = \frac{1}{2} sint - \frac{1}{2\sqrt{3}} sin\sqrt{3}t$, red for $v_1(t)$, blue for v_2, green for v_3.

```
In[17]:= Plot[{1/2*Sin[t] + 1/6/Sqrt[3]*Sin[Sqrt[3]*t] + t/3,
           -1/3/Sqrt[3]*Sin[Sqrt[3]*t] + t/3,
           -1/2*Sin[t] + 1/6/Sqrt[3]*Sin[Sqrt[3]*t] + t/3},
          {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[17]=

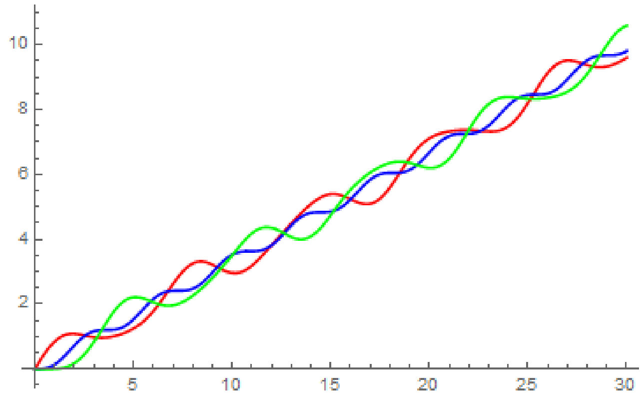


Figure 7: Figure for $d_1(t) = \frac{1}{2}sint + \frac{1}{6\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_2 = -\frac{1}{3\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_3(t) = -\frac{1}{2}sint + \frac{1}{6\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}$, red for $d_1(t)$, blue for d_2, green for d_3.

```
In[18]:= Plot[{-1/3/Sqrt[3]*Sin[Sqrt[3]*t] + t/3,
           2/3/Sqrt[3]*Sin[Sqrt[3]*t] + t/3,
           -1/3/Sqrt[3]*Sin[Sqrt[3]*t] + t/3}, {t, 0, 30},
          PlotStyle → {Red, Blue, Green}]
```

Out[18]=

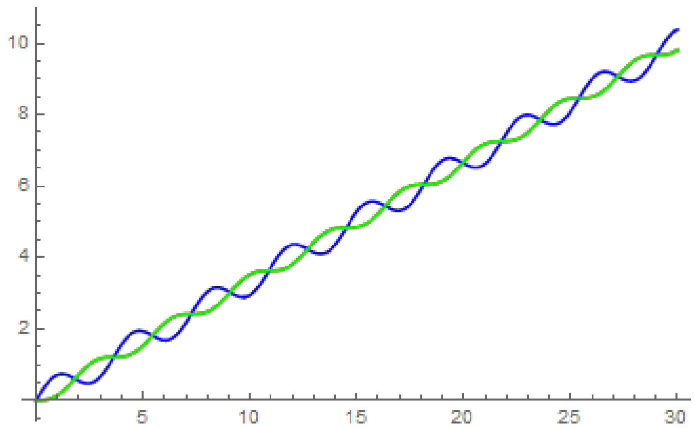


Figure 8: Figure for $d_1(t) = -\frac{1}{3\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_2 = \frac{2}{3\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_3(t) = -\frac{1}{3\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}$, red for $d_1(t)$, blue for d_2, green for d_3.

```
In[19]:= Plot[{-1 / 2 * Sin[t] + 1 / 6 / Sqrt[3] * Sin[Sqrt[3] * t] + t / 3,
    -1 / 3 / Sqrt[3] * Sin[Sqrt[3] * t] + t / 3,
    +1 / 2 * Sin[t] + 1 / 6 / Sqrt[3] * Sin[Sqrt[3] * t] + t / 3},
   {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[19]=

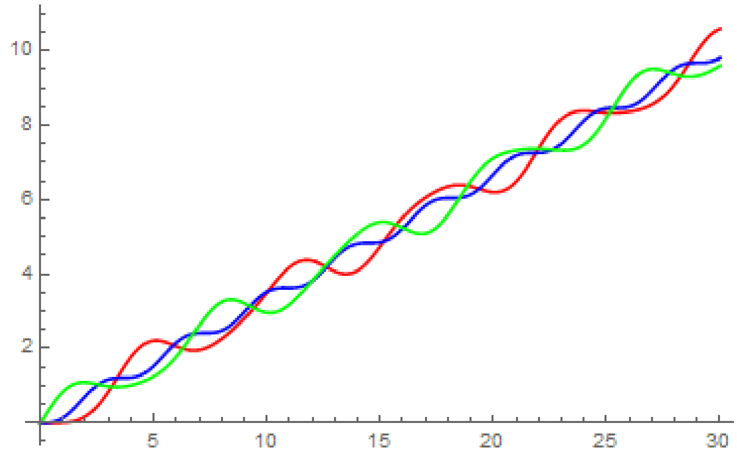


Figure 9: Figure for $d_3(t) = \frac{1}{2}sint + \frac{1}{6\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_2 = -\frac{1}{3\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}, d_1(t) = -\frac{1}{2}sint + \frac{1}{6\sqrt{3}}sin\sqrt{3}t + \frac{t}{3}$, red for $d_1(t)$, blue for d_2, green for d_3.

```
In[4]:= Plot[{1 / 3 + 1 / 2 Cos[t] + 1 / 6 * Cos[Sqrt[3] * t],
    1 / 3 - 1 / 3 Cos[Sqrt[3] * t],
    1 / 3 - 1 / 2 Cos[t] + 1 / 6 * Cos[Sqrt[3] * t]},
   {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

Out[4]=

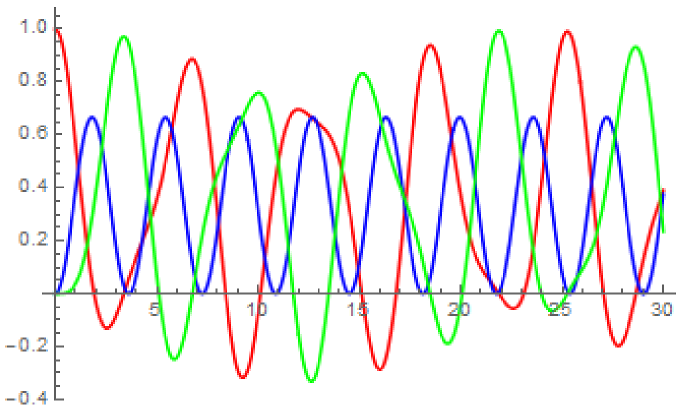


Figure 10: Figure for $d_1(t) = \frac{1}{3} + \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t, d_2 = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t, d_3(t) = \frac{1}{3} - \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$,red for $d_1(t)$, blue for d_2, green for d_3.

```
In[5]:= Plot[{1/3 - 1/3*Cos[Sqrt[3]*t], 1/3 + 2/3 Cos[Sqrt[3]*t],
  1/3 - 1/3*Cos[Sqrt[3]*t]}, {t, 0, 30},
 PlotStyle → {Red, Blue, Green}]
```

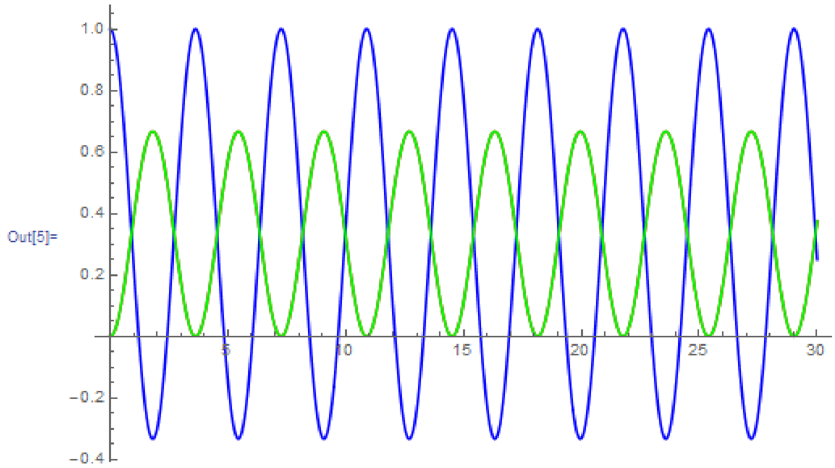


Figure 11: Figure for $d_1(t) = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$, $d_2 = \frac{1}{3} + \frac{1}{3}cos\sqrt{3}t$, $d_3(t) = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$,red for $d_1(t)$, blue for d_2, green for d_3.

```
In[11]:= Plot[{1/3 - 1/2 Cos[t] + 1/6*Cos[Sqrt[3]*t],
   1/3 - 1/3 Cos[Sqrt[3]*t],
   1/3 + 1/2 Cos[t] + 1/6*Cos[Sqrt[3]*t]},
  {t, 0, 30}, PlotStyle → {Red, Blue, Green}]
```

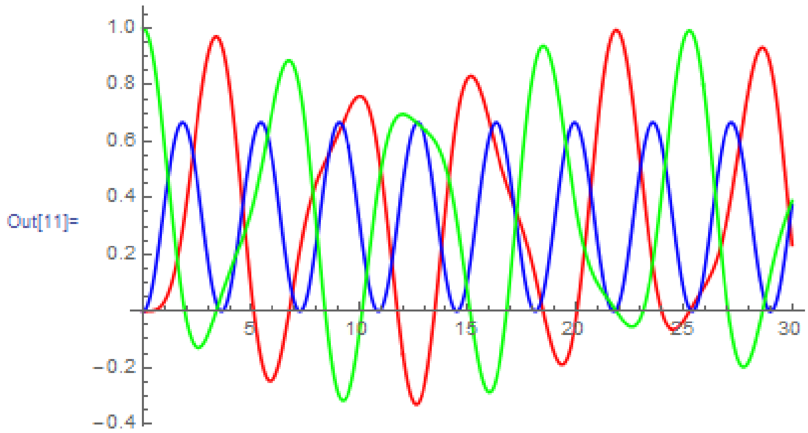


Figure 12: Figure for $d_3(t) = \frac{1}{3} + \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$, $d_2 = \frac{1}{3} - \frac{1}{3}cos\sqrt{3}t$, $d_1(t) = \frac{1}{3} - \frac{1}{2}cost + \frac{1}{6}cos\sqrt{3}t$,red for $d_1(t)$, blue for d_2, green for d_3.

vi)

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(0) \\ d(0) \end{pmatrix} = \Phi(0)C = \mathbb{1}C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

So

$$d = c_1 d^{(1)} + c_2 d^{(2)} + c_3 d^{(3)} + c_4 d^{(4)} + c_5 d^{(5)} + c_6 d^{(6)} = \begin{pmatrix} cost + sint + \frac{1}{3}cos\sqrt{3}t \\ \frac{2}{3} - \frac{2}{3}cos\sqrt{3}t \\ -cost - sint + \frac{1}{3}cos\sqrt{3}t \end{pmatrix}$$

So

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} cost + sint + \frac{1}{3}cos\sqrt{3}t + \frac{14}{3} \\ \frac{26}{3} - \frac{2}{3}cos\sqrt{3}t \\ -cost - sint + \frac{1}{3}cos\sqrt{3}t + \frac{38}{3} \end{pmatrix}$$

```
In[25]:= Plot[{Cos[t] + Sin[t] + 1 / 3 * Cos[Sqrt[3] * t] + 14 / 3,
    26 / 3 - 2 / 3 * Cos[Sqrt[3] * t],
    -Cos[t] - Sin[t] + 1 / 3 * Cos[Sqrt[3] * t] + 38 / 3},
   {t, 0, 60}, PlotStyle → {Red, Blue, Green},
   PlotRange → {{0, 60}, {0, 15}}]
```

Out[25]=

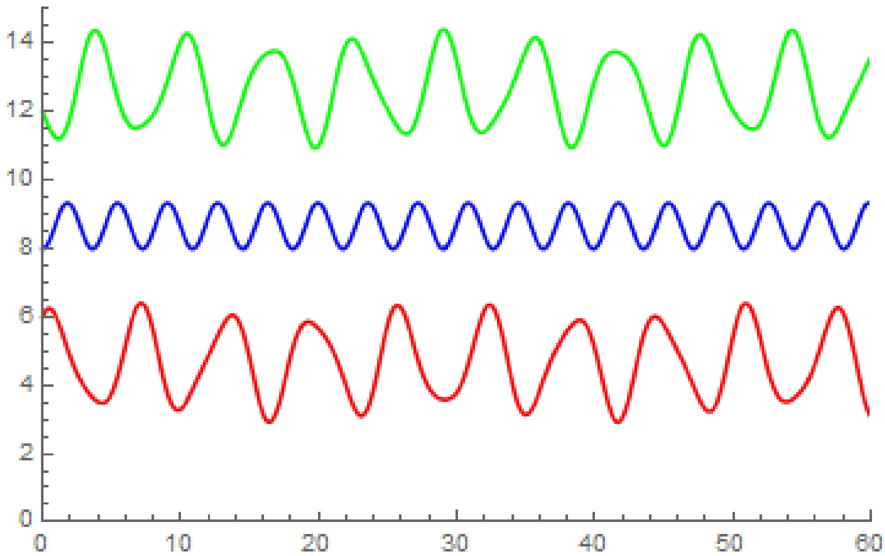


Figure 13: Figure for $x_1(t) = cost + sint + \frac{1}{3}cos\sqrt{3}t + \frac{14}{3}, x_2(t) = \frac{26}{3} - \frac{2}{3}cos\sqrt{3}t, x_3(t) = -cost - sint + \frac{1}{3}cos\sqrt{3}t + \frac{38}{3}$, red for $x_1(t)$, blue for $x_2(t)$, green for $x_3(t)$.

Exercise 4.4

Since $A \in Mat(n \times n, \mathbb{C})$ and A has n eigenvalues, then A is diagonlizable. Set $U \in Mat(n \times n, \mathbb{C})$ satifies that $U^{-1}AU = diag(\lambda_1, \cdots, \lambda_n) =: D$, then

$$detA = det\mathbb{1} \cdot detA = det(U^{-1}U) \cdot detA = det(U^{-1}AU) = \prod_{i=1}^{n} \lambda_i$$

$$trA = tr(UDU^{-1}) = tr(UU^{-1}D) = trD = \sum_{i=1}^{n} \lambda_i$$

So $det(e^A) = det(Ue^D U^{-1}) = det(e^D) = \prod_{i=1}^{n} e^{\lambda_i} = e^{\sum\limits_{i=1}^{n} \lambda_i} = e^{trA}$

To sum up, $detA = \prod_{i=1}^{n} \lambda_i, trA = \sum\limits_{i=1}^{n} \lambda_i, det(e^A) = e^{trA}$.

Exercise 4.5

Since $\int sec\,t\,tan\,tdt = -sec\,t + C$ where C is a constant and $y''' + y' = sec\,t\,tan\,t$, then

$$y'' + y = sec\,t + C$$

Since $y''(0) = y(0) = 0$, $0 + 0 = 1 + C$. So

$$y'' + y = sec\,t - 1$$

Set $x_1 = y, x_2 = \dot{x_1}$, then $x_1(0) = x_2(0) = 0$ and

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ sect - 1 - x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ sect - 1 \end{pmatrix}$$

Set $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $det(A - \lambda\mathbb{1}) = 0 \Leftrightarrow \lambda = \pm i$. For $\lambda_1 = i$, we find $u_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$, for $\lambda_2 = -i$, we find $u_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$. So the fundamental system is given by

$$\mathscr{F} = \{e^{\lambda_1 t}u_1, e^{\lambda_2 t}u_2\} = \{\begin{pmatrix} ie^{it} \\ -e^{it} \end{pmatrix}, \begin{pmatrix} ie^{-it} \\ e^{-it} \end{pmatrix}\}$$

Set $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1(t) \begin{pmatrix} ie^{it} \\ -e^{it} \end{pmatrix} + c_2(t) \begin{pmatrix} ie^{-it} \\ e^{-it} \end{pmatrix}$. Then

$$\begin{aligned} c_1(t) &= \int \frac{det \begin{pmatrix} 0 & ie^{-it} \\ sect - 1 & e^{-it} \end{pmatrix}}{det \begin{pmatrix} ie^{it} & ie^{-it} \\ -e^{it} & e^{-it} \end{pmatrix}} dt = \int 0.5(cost - isint - 1 + itant)dt \\ &= 0.5(sint + icost - t + iln|cost|) \end{aligned}$$

$$\begin{aligned} c_2(t) &= \int \frac{det \begin{pmatrix} ie^{it} & 0 \\ -e^{it} & sect - 1 \end{pmatrix}}{det \begin{pmatrix} ie^{it} & ie^{-it} \\ -e^{it} & e^{-it} \end{pmatrix}} dt = \int 0.5(1 - cost - isint + itant)dt \\ &= 0.5(-sint + icost + t - iln|cost|) \end{aligned}$$

So $y = (0.5(sint + icost - t - iln|cost|) + C_1)ie^{it} + (0.5(-sint + icost + t + iln|cost|) + C_2)ie^{-it}$.
Since $y'(0) = y(0) = 0$, $C_1 = C_2 = -0.5i$.

To sum up, the solution to the initial value problem is

$$y = -1 + cost + tsint + cost \cdot ln|cost|$$

Exercise 4.6

$$det(A - \lambda\mathbb{1}) = (-\lambda)(-b/a - \lambda) + b^2/(4a^2) = 0 \Leftrightarrow \lambda = -b/2a$$

$$(A + b/2a \cdot \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} b/(2a) & 1 \\ -b^2/(4a^2) & -b/(2a) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow v_2 = -\frac{b}{2a}v_1$$

So all the eigenvectors of A are $k\begin{pmatrix} -1 \\ \dfrac{b}{2a} \end{pmatrix}, k \neq 0$ and the eigenspace is

$$V = \{v : v = k\begin{pmatrix} -1 \\ \dfrac{b}{2a} \end{pmatrix}, k \in \mathbb{C}\}$$

Now I want to show that $V \neq \mathbb{R}^2$ by proving that $u = \begin{pmatrix} b \\ \dfrac{1}{2a} \end{pmatrix} \notin V$. This is because if $u \in V$, then $b = -k \wedge \dfrac{1}{2a} = \dfrac{bk}{2a}$, and therefore $b^2 + 1 = 0$. Since $b \in \mathbb{R}$, this is impossible. So $V \neq \mathbb{R}^2$.

To sum up, the matrix A is not diagonalizable for any values of $a, b \in \mathbb{R}$.

Exercise 4.7

i)

Since $y_1(t) = t + 1$ is a solution to the equation

$$y'' - \frac{2(t+1)}{t^2 + 2t - 1}y' + \frac{2}{t^2 + 2t - 1}y = 0$$

then we can set that $y_2(t) = v(t)(t + 1)$, then

$$\begin{aligned}
&y_2'' - \frac{2(t+1)}{t^2 + 2t - 1}y_2' + \frac{2}{t^2 + 2t - 1}y_2 = 0 \\
\Rightarrow &v''(t+1) + 2v' - \frac{2(t+1)}{t^2 + 2t - 1}(v'(t+1) + v) + \frac{2}{t^2 + 2t - 1}v(t+1) = 0 \\
\Rightarrow &v''(t+1) = \frac{4}{t^2 + 2t - 1}v' \\
\Rightarrow &\int \frac{1}{v'}d(v') = \int (\frac{1}{t + 1 - \sqrt{2}} - \frac{1}{t+1} + \frac{1}{t + \sqrt{2} + 1} - \frac{1}{t+1})dt \\
\Rightarrow &lnv' = ln|t + 1 - \sqrt{2}| + ln|t + \sqrt{2} + 1| - 2ln|t + 1| + C_1 \\
\Rightarrow &v' = C_1\Big|\frac{t^2 + 2t - 1}{t^2 + 2t + 1}\Big| = C_1\Big|1 - \frac{2}{(t+1)^2}\Big| \\
\Rightarrow &v(t) = \int C_1\Big|1 - \frac{2}{(t+1)^2}\Big|dt = C_1\Big|t + \frac{2}{t+1}\Big| + C_2 \\
\Rightarrow &y_2(t) = C_1|t^2 + t + 2| + C_2(t+1)
\end{aligned}$$

So the general solution is $y(t) = C_1|t^2 + t + 2| + C_2(t + 1)$, where $C_1, C_2 \in \mathbb{R}$ are constant.

ii)

Since $y_1(t) = \dfrac{sint}{\sqrt{t}}$ is a solution to the equation

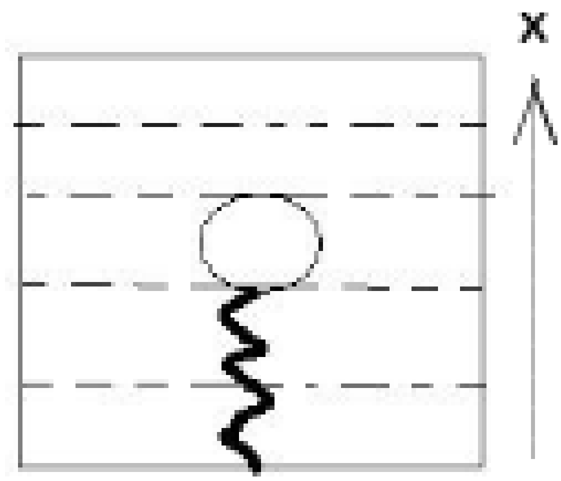
$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0$$

then we can set that $y_2(t) = v(t)\dfrac{sint}{\sqrt{t}}$, then

$$\begin{aligned}
&t^2y_2'' + ty_2' + (t^2 - \frac{1}{4})y_2 = 0 \\
\Rightarrow &t^2y_1v'' + (2t^2y_1' + ty_1)v' = 0 \\
\Rightarrow &v'' = -\frac{2t^{3/2}cost}{t^{3/2}sint}v' = -(2cot\,t)v' \\
\Rightarrow &\int \frac{1}{v'}d(v') = -\int (2cot\,t)dt \\
\Rightarrow &lnv' = -2ln|sint| + C_1 \\
\Rightarrow &v' = C_1\Big|\frac{1}{sin^2t}\Big| \\
\Rightarrow &v(t) = \int C_1\Big|\frac{1}{sin^2t}\Big|dt = C_1|cot\,t| + C_2 \\
\Rightarrow &y_2(t) = C_1\frac{|cost|}{\sqrt{t}} + C_2\frac{sint}{\sqrt{t}}
\end{aligned}$$

So the general solution is $y(t) = C_1\dfrac{|cost|}{\sqrt{t}} + C_2\dfrac{sint}{\sqrt{t}}$, where $C_1, C_2 \in \mathbb{R}$ are constant.

Exercise 4.8



Set the position of mass is $x(t)$, the velocity and acceleration are $\dot{x}(t) = v(t), \ddot{x}(t) = a(t)$. The equilibrium is at $k(0-x) = mg \Rightarrow x = -10$. Then according to laws of Newton and Hooke,

$$ma = k(x_0 - x) - mg - \beta v$$

where x_0 is the initial length of the spring and therefore we can set it as 0. Then according to the question,

$$x(0) = -10 - 1/4, v(0) = 1, k = 1, \beta = 2, m = 1$$

and use $g = 10N/kg$, we can get

$$\ddot{x} + 2\dot{x} + x + 10 = 0, x(0) = -1/4, \dot{x}(0) = 1$$

Set $x_1 = x, x_2 = \dot{x}$, then

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_2 - x_1 - 10 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

Set $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $det(A - \lambda \mathbb{1}) = 0 \Leftrightarrow \lambda = -1$. For $\lambda = -1$, we find $u_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. And from $(A + \mathbb{1})u_2 = u_1$, we find $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then set $U = (u_1, u_2)$ and we have

$$J = U^{-1}AU = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then

$$e^{Jt} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

So

$$Ue^{Jt} = \begin{pmatrix} 2e^{-t} & (2t+1)e^{-t} \\ -2e^{-t} & (1-2t)e^{-t} \end{pmatrix}$$

So the fundamental system is given by

$$\mathscr{F} = \{\begin{pmatrix} 2e^{-t} \\ -2e^{-t} \end{pmatrix}, \begin{pmatrix} (1+2t)e^{-t} \\ (1-2t)e^{-t} \end{pmatrix}\}$$

Set $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}(c_1(t) \begin{pmatrix} 2 \\ -2 \end{pmatrix} + c_2(t) \begin{pmatrix} 1+2t \\ 1-2t \end{pmatrix})$. Then

$$c_1(t) = \int \frac{det \begin{pmatrix} 0 & (1+2t)e^{-t} \\ -10 & (1-2t)e^{-t} \end{pmatrix}}{det \begin{pmatrix} 2e^{-t} & (1+2t)e^{-t} \\ -2e^{-t} & (1-2t)e^{-t} \end{pmatrix}} dt = \int 2.5(1+2t)e^t dt = 2.5(2t-1)e^t$$

$$c_2(t) = \int \frac{det \begin{pmatrix} 2e^{-t} & 0 \\ -2e^{-t} & -10 \end{pmatrix}}{det \begin{pmatrix} 2e^{-t} & (1+2t)e^{-t} \\ -2e^{-t} & (1-2t)e^{-t} \end{pmatrix}} dt = \int -5e^t dt = -5e^t$$

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}((2.5(2t-1)e^t + C_1) \begin{pmatrix} 2 \\ -2 \end{pmatrix} + (-5e^t + C_2) \begin{pmatrix} 1+2t \\ 1-2t \end{pmatrix})$

So $x(t) = x_1(t) = -10 + (2C_2t + 2C_1 + C_2)e^{-t}, x_2(t) = (-2C_2t - 2C_1 + C_2)e^{-t}$. Since $x_1(0) = -10 - 1/4, x_2(0) = 1$, $C_1 = -5/16, C_2 = 3/8$. So

$$x(t) = -10 + (\frac{3}{4}t - \frac{1}{4})e^{-t}$$

We can see that $x(t) \leqslant -10 \Leftrightarrow 3/4t - 1/4 \leqslant 0 \Leftrightarrow t \leqslant 1/3$. So at $t = 1/3$ the mass will over shoot its equilibrium and then it will never reach the equilibrium again. While since

$$\underset{t\to\infty}{lim} x(t) = \underset{t\to\infty}{lim} - 10 + (\frac{3}{4}t - \frac{1}{4})e^{-t} = -10$$

so mass will overshoot its equilibrium position only once and then creep back to equilibrium.

Exercise 4.9

Set the position of mass is $x(t)$, the velocity and acceleration are $\dot{x(t)} = v(t), \ddot{x(t)} = a(t)$. Set the initial position of one side of the spring is $x = 0$, then according to laws of Newton and Hooke,

$$ma(t) + kx(t) = F(t) = Acos^3(\omega t)$$

Then according to the question,

$$k = 64, m = 4$$

then we can get

$$4\ddot{x} + 64x = Acos^3(\omega t)$$

Set $x_1 = x, x_2 = \dot{x}$, then

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{A}{4}cos^3(\omega t) - 16x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A}{4}cos^3(\omega t) \end{pmatrix}$$

Set $A = \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix}$, $det(A - \lambda \mathbb{1}) = 0 \Leftrightarrow \lambda = \pm 4i$. For $\lambda_1 = -4i$, we find $u_1 = \begin{pmatrix} -1 \\ 4i \end{pmatrix}$, for $\lambda_2 = 4i$, we find $u_2 = \begin{pmatrix} 1 \\ 4i \end{pmatrix}$.

So the fundamental system is given by

$$\mathscr{F} = \{e^{\lambda_1 t}u_1, e^{\lambda_2 t}u_2\} = \{\begin{pmatrix} -e^{-4it} \\ 4ie^{-4it} \end{pmatrix}, \begin{pmatrix} e^{4it} \\ 4ie^{4it} \end{pmatrix}\}$$

Set $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1(t)\begin{pmatrix} -e^{-4it} \\ 4ie^{-4it} \end{pmatrix} + c_2(t)\begin{pmatrix} e^{4it} \\ 4ie^{4it} \end{pmatrix}$. Then

$$c_1(t) = \int \frac{det\begin{pmatrix} 0 & e^{4it} \\ \frac{A}{4}cos^3(\omega t) & 4ie^{4it} \end{pmatrix}}{det\begin{pmatrix} -e^{-4it} & e^{4it} \\ 4ie^{-4it} & 4ie^{4it} \end{pmatrix}} dt = \int \frac{A}{128i}e^{4it}(cos(3\omega t) + 3cos(\omega t))dt$$

$$= \frac{Ae^{4it}}{128i}(\frac{1}{9\omega^2 - 16}(4icos(3\omega t) + sin(3\omega t)) + \frac{3}{\omega^2 - 16}(4icos(\omega t) + sin(\omega t)))$$

$$c_2(t) = \int \frac{det\begin{pmatrix} -e^{-4it} & 0 \\ 4ie^{-4it} & \frac{A}{4}cos^3(\omega t) \end{pmatrix}}{det\begin{pmatrix} -e^{-4it} & e^{4it} \\ 4ie^{-4it} & 4ie^{4it} \end{pmatrix}} dt = \int \frac{A}{128i}e^{-4it}(cos(3\omega t) + 3cos(\omega t))dt$$

$$= \frac{Ae^{-4it}}{128i}(\frac{1}{9\omega^2 + 16}(4icos(3\omega t) + sin(3\omega t)) + \frac{3}{\omega^2 + 16}(4icos(\omega t) + sin(\omega t)))$$

So

$$x(t) = -\frac{Acos(3\omega t)}{81\omega^4 - 256} - \frac{Asin(3\omega t)}{4i(81\omega^4 - 256)} - \frac{3Acos(\omega t)}{\omega^4 - 256} - \frac{3Asin(\omega t)}{4i(\omega^4 - 256)} + C_2 e^{4it} - C_1 e^{-4it}$$

So when $\omega^4 = 256$ or $81\omega^4 - 256 = 0$, i.e.$\omega = 4$ or $\omega = 4/3$, resonance will occur.

Exercise 4.10

Set $x_1 = y, x_2 = \dot{x_1}$, then

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{4}{3}\alpha x_2 - \frac{2\alpha^2}{3} x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2\alpha^2}{3} & -\frac{4\alpha}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1(0) = 0, x_2(0) = 100$$

Set $A = \begin{pmatrix} 0 & 1 \\ -\frac{2\alpha^2}{3} & -\frac{4\alpha}{3} \end{pmatrix}$, $det(A - \lambda \mathbb{1}) = 0 \Leftrightarrow \lambda = (-\frac{2}{3} \pm \frac{\sqrt{2}}{3} i)\alpha$. For $\lambda_1 = (-\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha$, we find $u_1 = \begin{pmatrix} 1 \\ (-\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha \end{pmatrix}$, for $\lambda_2 = -(\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha$, we find $u_2 = \begin{pmatrix} 1 \\ -(\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha \end{pmatrix}$.

So the fundamental system is given by

$$\mathscr{F} = \{e^{(-\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha t} \begin{pmatrix} 1 \\ (-\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha \end{pmatrix}, e^{-(\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha t} \begin{pmatrix} 1 \\ -(\frac{2}{3} + \frac{\sqrt{2}}{3} i)\alpha \end{pmatrix}\}$$

Since $x_1(0) = 0, x_2(0) = 100$, then we can obtain that

$$x_1 = -\frac{75\sqrt{2}}{\alpha} i e^{\lambda_1 t} + \frac{75\sqrt{2}}{\alpha} i e^{\lambda_2 t}, x_2 = -\frac{75\sqrt{2}}{\alpha} i \lambda_1 e^{\lambda_1 t} + \frac{75\sqrt{2}}{\alpha} i \lambda_2 e^{\lambda_2 t}$$

So

$$\begin{aligned} y^2 + (y')^2|_{t=1} &= Re(-\frac{11250}{\alpha^2}((1 + \lambda_1^2)e^{2\lambda_1 t} + (1 + \lambda_2^2)e^{2\lambda_2 t}))|_{t=1} \\ &= Re(-\frac{11250}{\alpha^2}((1 + \lambda_1^2)e^{2\lambda_1} + (1 + \lambda_2^2)e^{2\lambda_2})) \\ &= -\frac{11250}{\alpha^2} e^{-\frac{4}{3}\alpha}((2 + \frac{4}{9}\alpha^2)cos\frac{2\sqrt{2}}{3}\alpha + \frac{8\sqrt{2}}{9}\alpha^2 sin\frac{2\sqrt{2}}{3}\alpha) \end{aligned}$$

So the α should satisfy that

$$-\frac{11250}{\alpha^2} e^{-\frac{4}{3}\alpha}((2 + \frac{4}{9}\alpha^2)cos\frac{2\sqrt{2}}{3}\alpha + \frac{8\sqrt{2}}{9}\alpha^2 sin\frac{2\sqrt{2}}{3}\alpha) \leqslant 0.01$$