Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 9

Date Due: 10:00 AM, Thursday, the 1st of December 2016

UNIVERSITY OF MICHIGAN

JOINT INSTITUTE

交大密西根学院

Exercise 9.1. Find two independent solutions of the Bessel equation of order 3/2,

$$x^2y'' + xy' + (x^2 - 9/4)y = 0.$$

(3 Marks)

Exercise 9.2. The goal of this exercise is to prove the *Poisson integral formula* for the Bessel functions,

$$J_n(x) = \frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi \cos\bigl(x\cos\theta\bigr)\sin^{2n}\theta\,d\theta, \qquad n \in \mathbb{N}. \tag{1}$$

The formula is actually valid for all $n \in \mathbb{R}$ with $n > -1/2$, but we restrict ourselves to $n \in \mathbb{N}$ here.

i) Prove (e.g., using integration by parts) that

$$\int_0^\pi \cos^{2m}\theta\sin^{2n}\theta\,d\theta = \frac{(2m)!}{2^{2m}m!}\frac{(2n)!}{2^{2n}n!}\frac{\pi}{(n+m)!}, \qquad n, m \in \mathbb{N}. \tag{2}$$

ii) Insert the series expansion $\cos(x\cos\theta) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}(x\cos\theta)^{2k}$ in the right-hand side of (1), exchange integration and summation, and use (2) to obtain the series expansion (1) of Exercise 8.4.

iii) Show that for $n \in \mathbb{N}$,

$$\int_0^\pi \sin\bigl(x\cos\theta\bigr)\sin^{2n}\theta\,d\theta = 0$$

and deduce the alternative form of Poisson's integral,

$$J_n(x) = \frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi e^{i(x\cos\theta)}\sin^{2n}\theta\,d\theta, \qquad n \in \mathbb{N}.$$

iv) Substitute $\xi = \cos\theta$ to write the Poisson integral in yet another form, as

$$J_n(x) = \frac{(2x)^n n!}{\pi(2n)!}\int_{-1}^1 e^{ix\xi}(1-\xi^2)^{n-1/2}\,d\xi, \qquad n \in \mathbb{N}.$$

(2 + 2 + 2 + 1 Marks)

Exercise 9.3. Use the series expansions developed in Exercise 8.4 to prove the relations

$$\frac{d}{dx}\bigl(x^\nu J_\nu(x)\bigr) = x^\nu J_{\nu-1}(x), \qquad \frac{d}{dx}\bigl(x^{-\nu} J_\nu(x)\bigr) = -x^{-\nu} J_{\nu+1}(x),$$

$$2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x), \qquad J_\nu'(x) = \frac{1}{2}\bigl(J_{\nu-1}(x) - J_{\nu+1}(x)\bigr).$$

for $\nu \in \mathbb{R}$.
(2 + 2 + 1 + 1 Marks)

Exercise 9.4. In the lecture we have studied the oscillations of a suspended chain of constant density ϱ. Show that if $\varrho = \varrho(x) = \varrho_0 \cdot x^\mu$, $\mu \geq 0$, where x is the vertical coordinate as in the lecture, the fundamental frequencies are given by

$$\omega = \frac{1}{2\sqrt{\mu+1}}\sqrt{g/l}\cdot\alpha_{\mu,n}$$

where $\alpha_{\mu,n}$, $n = 1, 2, \ldots$ is the nth zero of J_μ, the Bessel function of the first kind of order μ. (*Literature*: see Korenev's book, page 217.)
(3 Marks)

Exercise 9.5. The wave equation for a vibrating string without any influence of external forces is

$$c^2 u_{xx} = u_{tt}, \qquad c^2 = \frac{T}{\varrho}, \tag{3}$$

where T is the tension and ϱ the density of the string. The *normal modes* of the string are given by

$$u(x,t) = X(x)e^{i\omega t}$$

for suitable frequencies $\omega > 0$. Suppose that the string has length l and the ends are fixed, so $u(0) = u(l) = 0$.

i) Let $\varrho(x) = \varrho_0$ be constant. Find the equation that $X(x)$ satisfies and solve it. What are the frequencies of the normal modes?

ii) Suppose that $\varrho(x) = \varrho_0(1 + kx/l) =: \varrho_0\xi$, where $k > 0$. Show that X satisfies the Airy equation

$$X'' + \kappa^2 \xi X = 0$$

where $\kappa^2 = \varrho_0\omega^2 l^2/(k^2 T)$. Deduce the normal frequencies

$$\omega^2 = \frac{9\mu^2 k^2 T}{4\varrho_0 l^2}$$

where μ is a solution of

$$J_{1/3}(\mu)J_{-1/3}(\lambda\mu) = J_{-1/3}(\mu)J_{1/3}(\lambda\mu), \qquad \lambda = (1+k)^{3/2}.$$

(2 + 4 Marks)

Exercise 9.6. You are asked to construct a steel flag pole with a circular cross-section 20 cm in diameter. It is to be made exclusively of steel but may be hollow or solid. What is the maximum height to which you can build the flagpole?
(3 Marks)

Exercise 9.7. The goal of this exercise is to establish some basic properties of the logarithmic derivative of the Euler gamma function. We define

$$\psi(x) := \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

where $\Gamma(x) := \int_0^\infty e^{-y}y^{x-1}\,dy$ denotes the Euler gamma function.

i) Differentiate the relation $\Gamma(x+1) = x\Gamma(x)$ to show that

$$\psi(x+1) = \frac{1}{x} + \psi(x).$$

for $x > 0$.

ii) The *Euler number* $\gamma \approx 0.5772\ldots$ is defined as $\gamma := -\Gamma'(1)$. Show that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

for $n \in \mathbb{N}$.[1]

iii) By applying Stirling's formula to the gamma function and (formally!) differentiating it, show that

$$\psi(x+1) = \ln(x) + \frac{1}{2x} + O\left(\frac{1}{x^2}\right)$$

and deduce

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln(n) + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

(2 + 2 + 2 Marks)

[1] It is not known whether γ is rational or irrational.

Exercise 9.8. For $\nu \in \mathbb{R} \setminus \mathbb{Z}$ the *Bessel function of the second kind* is defined as

$$Y_\nu(x) := \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

yielding a second independent solution to Bessel's differential equation of order ν. For $n \in \mathbb{Z}$ we define

$$Y_n(x) := \lim_{\nu \to n} Y_\nu(x),$$

where the limit on the right is evaluated using l'Hospital's rule.

i) Show that

$$Y_0(x) = \lim_{\nu \to 0} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} = \frac{2}{\pi} \left. \frac{dJ_\nu(x)}{d\nu} \right|_{\nu=0}$$

ii) Use the results of Exercise 9.7 to deduce the series expansion

$$Y_0(x) = \frac{2}{\pi} J_0(x) \left(\ln\left(\frac{x}{2}\right) + \gamma \right) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} H_n$$

where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$.

(It is then possible to find the series expansion for Y_n, $n \in \mathbb{Z}$, by using the recurrence formulas for J_n. The result of this tedious procedure can be found, e.g., at the very beginning of Koronev's book.)
(3 + 3 Marks)