# VV286
# Honors Mathematics IV
# Ordinary Differential Equations
# Assignment 3

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# 1

## 1.1

**Proof:** $\forall \varphi(x) = e^{-x^2/2}p(x) \in V$,

$$\begin{aligned}
H\varphi =& -\frac{d^2}{dx^2}e^{-x^2/2}p(x) + x^2e^{-x^2/2}p(x) \\
=& -\frac{d}{dx}(e^{-x^2/2}\cdot(-x)p(x) + e^{-x^2/2}p'(x)) + x^2e^{-x^2/2}p(x) \\
=& e^{-x^2/2}\cdot(-x)xp(x) + e^{-x^2/2}(p(x) + xp'(x)) \\
& - (e^{-x^2/2}\cdot(-x)p'(x) + e^{-x^2/2}p''(x)) + x^2e^{-x^2/2}p(x) \\
=& e^{-x^2/2}(p(x) + 2xp'(x) - p''(x))
\end{aligned}$$

Since $p(x) \in \mathcal{P}(\mathbb{R})$, $p(x) + 2xp'(x) - p''(x) \in \mathcal{P}(\mathbb{R})$. So $H\varphi \in V$.
So $H\varphi \in V$ if $\varphi \in V$.

## 1.2

$\forall \varphi(x) = e^{-x^2/2}p_1(x) \in V, \psi(x) = e^{-x^2/2}p_2(x) \in V$,

$$\begin{aligned}
& \langle H\psi, \varphi\rangle - \langle \psi, H\varphi\rangle \\
=& \int_{-\infty}^{\infty} H\psi \cdot \varphi dx - \int_{-\infty}^{\infty} \psi \cdot H\varphi dx \\
=& \int_{-\infty}^{\infty} e^{-x^2/2}(p_2(x) + 2xp_2'(x) - p_2''(x)) \cdot e^{-x^2/2}p_1(x)dx \\
& - \int_{-\infty}^{\infty} e^{-x^2/2}p_2(x) \cdot e^{-x^2/2}(p_1(x) + 2xp_1'(x) - p_1''(x))dx \\
=& \int_{-\infty}^{\infty} e^{-x^2}(2xp_1(x)p_2'(x) - p_1(x)p_2''(x) - 2xp_1'(x)p_2(x) + p_2(x)p_1''(x))dx \\
=& e^{-x^2}(p_1'(x)p_2(x) - p_1(x)p_2'(x))\Big|_{-\infty}^{\infty}
\end{aligned}$$

Since $p_1(x), p_2(x) \in \mathcal{P}(\mathbb{R})$, then use L'Hopital's rule we can get that

$$\lim_{x\to\pm\infty} e^{-x^2}p_1'(x)p_2(x) = \lim_{x\to\pm\infty} e^{-x^2}p_1(x)p_2'(x) = 0$$

so $\langle H\psi, \varphi\rangle = \langle \psi, H\varphi\rangle$, i.e. $H$ is symmetric.

## 1.3

$\forall \varphi(x) = e^{-x^2/2}p(x) \in V,,$

$$\begin{aligned}
& (HA - AH)\varphi - 2A\varphi \\
=& ((-\frac{d^2}{dx^2}) + x^2)(-\frac{d}{dx} + x)e^{-x^2/2}p(x) - (-\frac{d}{dx} + x)((-\frac{d^2}{dx^2}) + x^2)e^{-x^2/2}p(x) \\
& - 2(-\frac{d}{dx} + x)e^{-x^2/2}p(x)
\end{aligned}$$

$$
\begin{aligned}
=&((-\frac{d^2}{dx^2})+x^2)(-(e^{-x^2/2}(-x)p(x)+e^{-x^2/2}p'(x))+xe^{-x^2/2}p(x))\\
&-(-\frac{d}{dx}+x)(e^{-x^2/2}(p(x)+2xp'(x)-p''(x)))\\
&-2(-(e^{-x^2/2}(-x)p(x)+e^{-x^2/2}p'(x))+xe^{-x^2/2}p(x))\\
=&e^{-x^2/2}(2xp(x)-p'(x)+2x(2xp(x)-p'(x))'-(2xp(x)-p'(x))'')\\
&+e^{-x^2/2}(-xp(x)-2x^2p'(x)+xp''(x)+p'(x)+2p'(x)+2xp''(x)-p'''(x))\\
&-e^{-x^2/2}(xp(x)+2x^2p'(x)-xp''(x))-2e^{-x^2/2}(2xp(x)-p'(x))\\
=&e^{-x^2/2}(6xp(x)+(4x^2-5)p'(x)-4xp''(x)+p'''(x))\\
&+e^{-x^2/2}(-xp(x)+(-2x^2+3)p'(x)+3xp''(x)-p'''(x))\\
&+e^{-x^2/2}(-5xp(x)+(-2x^2+2)p'(x)+xp''(x))\\
=&0
\end{aligned}
$$

So $(HA-AH)\varphi = 2A\varphi$ for all $\varphi \in V$. So $[H,A] = HA - AH = 2A$.

## 1.4

Since $\psi \in V$ is an eigenfunction of $H$ for the eigenvalue $\lambda \in \mathbb{R}$,

$$H\psi = \lambda\psi$$

then

$$H(A\psi) = 2A\psi + AH\psi = 2A\psi + A(\lambda\psi) = 2A\psi + \lambda A\psi = (\lambda+2)A\psi$$

So $A\psi$ is an eigenfunction of $H$ for the eigenvalue $\lambda + 2$.

## 1.5

$H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = 1$

$H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx} e^{-x^2} = -e^{x^2} \cdot e^{-x^2} \cdot (-2x) = 2x$

$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx}(e^{-x^2} \cdot (-2x)) = -2e^{x^2} \cdot e^{-x^2}((-2x)x+1) = 4x^2 - 2$

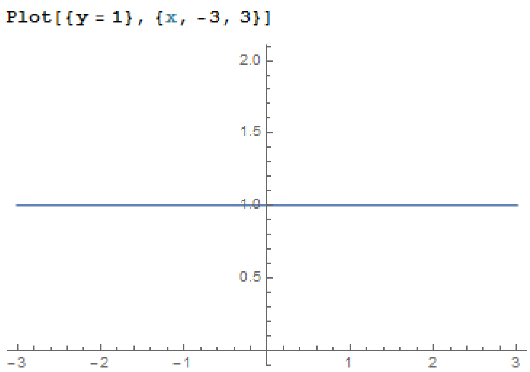


Figure 1: Figure for $H_0(x) = 1$

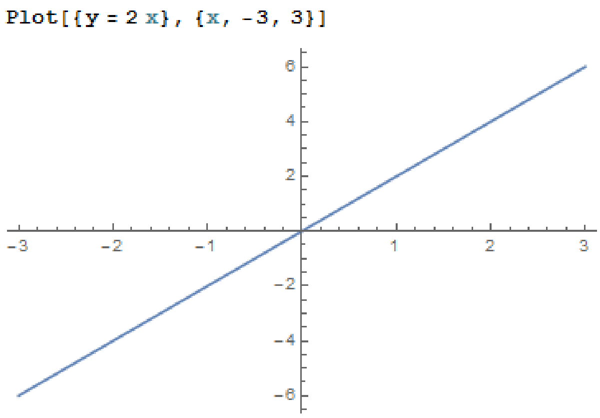


Figure 2: Figure for $H_1(x) = 2x$

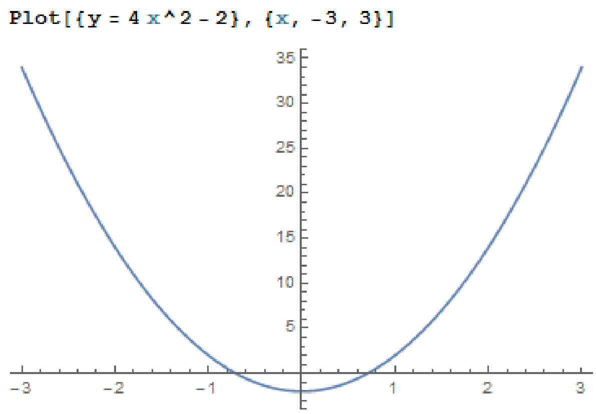


Figure 3: Figure for $H_2(x) = 4x^2 - 2$

## 1.6

$$H(e^{-x^2/2}) = e^{-x^2/2}(1 + 2x \cdot \frac{d1}{dx} - \frac{d^2 1}{dx^2}) = e^{-x^2/2}$$

$$\begin{aligned} e^{x^2/2}\Big(-\frac{d}{dx}\Big)(e^{-x^2/2}f(x)) &= -e^{x^2/2} \cdot (e^{-x^2/2}(-x)f(x) + e^{-x^2/2}f'(x)) \\ &= xf(x) + \frac{d}{dx}f(x) \\ &= Af(x) \end{aligned}$$

So $H(e^{-x^2/2}) = e^{-x^2/2}, Af(x) = e^{x^2/2}\Big(-\frac{d}{dx}\Big)(e^{-x^2/2}f(x))$.

Use induction to prove that $\psi_n(x) = e^{-x^2/2}H_n(x)$ is eigenfunctions of $H$ to eigenvalues $\lambda_n = 2n + 1, n \in \mathbb{N}$

1. When $n = 0$, $\psi_0(x) = e^{-x^2/2}$ and $H\psi_0(x) = (2 \cdot 0 + 1)\psi_0(x)$. So the statement holds when $n = 0$.

2. Assume that when $n = k$, $H\psi_k = (2k + 1)\psi_k$. Then according to former questions, $A\psi_k$ is an eigenfunction of $H$ for the eigenvalue $2k + 1 + 2 = 2(k + 1) + 1$.

$$
\begin{aligned}
A\psi_k &= -e^{x^2/2}\Big(\frac{d}{dx}\Big)(e^{-x^2/2}\psi_k(x)) \\
&= -e^{x^2/2}\Big(\frac{d}{dx}\Big)(e^{-x^2/2}\cdot e^{-x^2/2}(-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}) \\
&= (-1)^{k+1}e^{x^2/2}\Big(\frac{d}{dx}\Big)(\frac{d^k}{dx^k}e^{-x^2}) \\
&= (-1)^{k+1}e^{-x^2/2}\cdot e^{x^2}(\frac{d^{k+1}}{dx^{k+1}}e^{-x^2}) \\
&= e^{-x^2/2}H_{k+1}(x) \\
&= \psi_{k+1}
\end{aligned}
$$

so $\psi_{k+1}$ is an eigenfunction of $H$ for the eigenvalue $2(k+1)+1$. So the statement also holds when $n = k+1$.

To sum up, the eigenfunctions of $H$ to eigenvalues $\lambda_n = 2n+1, n \in \mathbb{N}$, may be written in the form $\psi_n(x) = e^{-x^2/2}H_n(x)$.

## 1.7

$$
\begin{aligned}
&H_{n+1}(x) - 2xH_n(x) - H_n'(x) \\
=&(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) - 2x(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) - \frac{d}{dx}((-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})) \\
=&(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) - 2x(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \\
&+ (-1)^{n+1}(e^{x^2}\cdot 2x\frac{d^n}{dx^n}(e^{-x^2}) + e^{x^2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2})) \\
=&2(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) - 4x(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \\
=&2(-1)^{n+1}e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}\cdot(-2x)) - 4x(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \\
=&0
\end{aligned}
$$

So $H_{n+1}(x) = 2xH_n(x) + H_n'(x)$. Now we use induction to prove that $H_n' = 2nH_{n-1}$.

1. When $n = 1$, $H_1' = (2x)' = 2 = 2\cdot 1\cdot H_0$, so the statement holds when $n = 1$.
2. Assume that when $n = k$, $H_k' = 2kH_{k-1}$, then

$$
\begin{aligned}
H_{k+1}' &= (2xH_k(x) + H_k'(x))' \\
&= 2H_k(x) + 2xH_k'(x) + H_k''(x) \\
&= 2H_k(x) + 2x(2kH_{k-1}(x)) + (2kH_{k-1}(x))' \\
&= 2H_k(x) + 4kxH_{k-1}(x) + 2k(H_k(x) - 2xH_{k-1}(x)) \\
&= 2(k+1)H_k(x)
\end{aligned}
$$

So when $n = k+1$ the statement also holds.

To sum up, $H_n' = 2nH_{n-1}, n \in \mathbb{N}^*$.

## 1.8

$\forall n \in \mathbb{N}^*$,

$$H_n(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2}) = H_n(x)\frac{H_{n-1}(x)}{(-1)^{n-1}e^{x^2}}$$

since $\psi_n(x) = e^{-x^2}H_n(x) \in V$, $H_n(x) \in \mathcal{P}(\mathbb{R})$. So $H_n(x), H_{n-1}(x) \in \mathcal{P}(\mathbb{R})$. Then

$$\underset{x\to\pm\infty}{lim} H_n(x)\frac{H_{n-1}(x)}{(-1)^{n-1}e^{x^2}} = 0$$

So $\forall n \in \mathbb{N}^*$

$$\begin{aligned}
||\psi_n(x)||^2 =& \langle\psi_n, \psi_n\rangle = \int_{-\infty}^{\infty} \psi_n^2(x)dx = \int_{-\infty}^{\infty} (e^{-x^2/2}H_n(x))^2 dx \\
=& \int_{-\infty}^{\infty} (e^{-x^2}(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})H_n(x))dx \\
=& (-1)^n \int_{-\infty}^{\infty} H_n(x)\cdot\frac{d^n}{dx^n}(e^{-x^2})dx \\
=& (-1)^n(H_n(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_n'(x)\cdot\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})dx) \\
=& (-1)^{n+1}\cdot 2n\int_{-\infty}^{\infty} H_{n-1}(x)\cdot\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})dx \\
=& 2n||\psi_{n-1}(x)||^2
\end{aligned}$$

For $n = 0$,

$$||\psi_0(x)||^2 = \int_{-\infty}^{\infty}(e^{-x^2/2})^2dx \overset{t=\sqrt{2}x}{=\!=\!=} \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}dx = \frac{\sqrt{2\pi}}{\sqrt{2}} = \sqrt{\pi}$$

So $\forall n \in \mathbb{N}^*$

$$\begin{aligned}
||\psi_n(x)||^2 &= 2n||\psi_{n-1}(x)||^2 = (2n)(2(n-1))2n||\psi_{n-2}(x)|| \\
&= \cdots \\
&= (2n)(2(n-1))\cdots(2\cdot 1)||\psi_0(x)||^2 \\
&= 2^n n!\sqrt{\pi}
\end{aligned}$$

To sum up, $\forall n \in \mathbb{N}$, $||\psi_n||^2 = \sqrt{\pi}2^n n!$

# 2

## 2.1

$$\begin{aligned}
det(A - \lambda\mathbb{1}) = 0 \Leftrightarrow det\begin{pmatrix}-2-\lambda & -2\\ -5 & 1-\lambda\end{pmatrix} = 0 \Leftrightarrow (-2-\lambda)(1-\lambda) - (-2)\cdot(-5) = 0 \\
\Leftrightarrow \lambda^2 + \lambda - 12 = 0 \Leftrightarrow \lambda = -4 \vee \lambda = 3
\end{aligned}$$

For $\lambda_1 = -4$,

$$(A - \lambda_1 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 = v_2$$

Hence any vector of the form $v = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_1 = -4$ and the corresponding eigenspace is $V_{\lambda_1} = V_{-4} = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$

For $\lambda_2 = 3$,

$$(A - \lambda_2 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 = -\frac{2}{5} v_2$$

Hence any vector of the form $v = \alpha \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ , $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_2 = 3$ and the corresponding eigenspace is $V_{\lambda_2} = V_3 = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \alpha \in \mathbb{R}\}$

To sum up,

1. One eigenvalue of $A$ is $\lambda_1 = -4$, eigenvector is $v = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\backslash\{0\}$ and the corresponding eigenspace is $V_{\lambda_1} = V_{-4} = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$.

2. Another eigenvalue of $A$ is $\lambda_2 = 3$, eigenvector is $v = \alpha \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \alpha \in \mathbb{R}\backslash\{0\}$ and the corresponding eigenspace is $V_{\lambda_2} = V_3 = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \alpha \in \mathbb{R}\}$.

## 2.2

$$\begin{aligned} det(B - \lambda \mathbb{1}) = 0 \Leftrightarrow det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow (2-\lambda)(1-\lambda)^2 - (1-\lambda) - (1-\lambda) &= 0 \\ \Leftrightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3 \end{aligned}$$

For $\lambda_1 = 0$,

$$(B - \lambda_1 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 = v_2 = v_3$$

Hence any vector of the form $v = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_1 = 0$ and the corresponding eigenspace is $V_{\lambda_1} = V_0 = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$

For $\lambda_2 = 1$,

$$(B - \lambda_2 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 = -v_3 \wedge v_2 = 0$$

Hence any vector of the form $v = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_2 = 1$ and the corresponding eigenspace is $V_{\lambda_2} = V_1 = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha \in \mathbb{R}\}$

For $\lambda_3 = 3$,

$$(B - \lambda_3 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow -2v_1 = v_2 = -2v_3$$

Hence any vector of the form $v = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_3 = 3$ and the corresponding eigenspace is $V_{\lambda_3} = V_3 = \{v \in \mathbb{R}^2 : v = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$

To sum up,

1. One eigenvalue of $B$ is $\lambda_1 = 0$, eigenvector is $v = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\backslash\{0\}$ and the corresponding eigenspace is $V_{\lambda_1} = V_0 = \{v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$.
2. One eigenvalue of $B$ is $\lambda_2 = 1$, eigenvector is $v = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha \in \mathbb{R}\backslash\{0\}$ and the corresponding eigenspace is $V_{\lambda_2} = V_1 = \{v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha \in \mathbb{R}\}$.
3. Another eigenvalue of $A$ is $\lambda_3 = 3$, eigenvector is $v = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\backslash\{0\}$ and the corresponding eigenspace is $V_{\lambda_3} = V_3 = \{v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}\}$.

## 3

Set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Set

$$\begin{aligned}
&F(x_1, \cdots, x_n, \lambda) \\
=&\langle x, Ax\rangle + \lambda(1 - |x|^2) \\
=&\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix} \rangle + \lambda(1 - (x_1^2 + x_2^2 + \cdots + x_n^2)) \\
=&\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j + \lambda(1 - (x_1^2 + x_2^2 + \cdots + x_n^2))
\end{aligned}$$

Since $A = A^T$, $\forall i, j \in [1, n] \cap \mathbb{N}, a_{ij} = a_{ji}$. Then $\forall i \in [1, n] \cap \mathbb{N}$

$$\frac{d}{dx_i}F(x_1, \cdots, x_n, \lambda) = \sum_{j=1}^{n}(a_{ij} + a_{ji})x_j - 2\lambda x_i = 2(a_{j1}x_1 + \cdots + (a_{ii} - \lambda)x_i + \cdots + a_{jn}x_n)$$

Because $D_{1-|x|^2} = (-2x_1, -2x_2, \cdots, -2x_n)$always has a $1 \times 1$ submatrix with determinant different from zero, so we can apply the Lagrange multiplier rule. Then when $Q_A(x) = \langle x, Ax\rangle$ reaches maximum or minimum,

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases} \Leftrightarrow \begin{cases} (A - \lambda\mathbb{1})x = 0 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases}$$

So when $Q_A(x) = \langle x, Ax\rangle$ reaches maximum or minimum, the Lagrange multiplier is an eigenvalue of $A$ (This is because the equations are the same and $x_1^2 + \cdots + x_n^2 = 1$ ensures that the $\lambda$ has corresponding non-trival eigenvector so that $\lambda$ can be an eigenvalue). And therefore

$$Q_A(x) = \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda\langle x, x\rangle = \lambda$$

So the maximum (resp. minimum) value of $Q_A$ when restricted to the unit sphere is given by the largest (resp. smallest) eigenvalue of A.

## 4

### 4.1

Since $r = h = 30cm = 0.3m, M = 1kg, m = 0.1kg$,

$$\frac{M}{12}(3r^2 + 4h^2) + mh^2 = \frac{1}{12}(3+4)\cdot 0.3^2 + 0.1\cdot 0.3^2 = 0.0615, -mrh = -0.1\cdot 0.3\cdot 0.3 = -0.009$$

$$\frac{M}{12}(3r^2+4h^2)+m(h^2+r^2)=0.0615+0.1\cdot 0.3^2=0.0705$$
$$\frac{M}{2}r^2+mr^2=\frac{1}{2}\cdot 0.3^2+0.1\cdot 0.3^2=0.054$$

So

$$I=\begin{pmatrix}0.0615 & 0 & -0.009\\ 0 & 0.0705 & 0\\ -0.009 & 0 & 0.054\end{pmatrix}$$

$$\overrightarrow{L}=I\overrightarrow{\omega}=\begin{pmatrix}0.0615 & 0 & -0.009\\ 0 & 0.0705 & 0\\ -0.009 & 0 & 0.054\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-0.009\\0\\0.054\end{pmatrix}$$

$$T=\frac{1}{2}\langle\overrightarrow{\omega},I\overrightarrow{\omega}\rangle=\frac{1}{2}\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}-0.009\\0\\0.054\end{pmatrix}\rangle=0.027$$

To sum up, $I=\begin{pmatrix}0.0615 & 0 & -0.009\\ 0 & 0.0705 & 0\\ -0.009 & 0 & 0.054\end{pmatrix}, \overrightarrow{L}=\begin{pmatrix}-0.009\\0\\0.054\end{pmatrix}, T=0.027.$

## 4.2

$$det(I-\lambda\mathbb{1})=0\Leftrightarrow det\begin{pmatrix}0.0615-\lambda & 0 & -0.009\\ 0 & 0.0705-\lambda & 0\\ -0.009 & 0 & 0.054-\lambda\end{pmatrix}=0$$
$$\Leftrightarrow(0.0705-\lambda)((0.0615-\lambda)(0.054-\lambda)-0.009^2)=0$$
$$\Leftrightarrow\lambda_1=0.0705,\lambda_2=0.048,\lambda_3=0.0675$$

For $\lambda_1=0.0705$,

$$(I-\lambda_1\mathbb{1})v=0\Leftrightarrow\begin{pmatrix}-0.009 & 0 & -0.009\\ 0 & 0 & 0\\ -0.009 & 0 & -0.0165\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow v_1=v_3=0$$

Hence any vector of the form $v=\alpha\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\alpha\in\mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_1=0.0705$ .

For $\lambda_2=0.048$,

$$(I-\lambda_2\mathbb{1})v=0\Leftrightarrow\begin{pmatrix}0.0135 & 0 & -0.009\\ 0 & 0.0225 & 0\\ -0.009 & 0 & 0.006\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow 3v_1=2v_3\wedge v_2=0$$

Hence any vector of the form $v=\alpha\begin{pmatrix}2\\0\\3\end{pmatrix}$, $\alpha\in\mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_2=0.048$ .

For $\lambda_3=0.0675$,

$$(I-\lambda_3\mathbb{1})v=0\Leftrightarrow\begin{pmatrix}-0.006 & 0 & -0.009\\ 0 & 0.003 & 0\\ -0.009 & 0 & -0.0135\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow 2v_1=-3v_3\wedge v_2=0$$

Hence any vector of the form $v = \alpha \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$, $\alpha \in \mathbb{R}\backslash\{0\}$ is an eigenvector $\lambda_3 = 0.0675$.

**So principal moments of inertia are** $\lambda_1 = 0.0705, \lambda_2 = 0.048, \lambda_3 = 0.0675$**, and the principal axes of inertia are** $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$**,** $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ **and** $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$**.**

According to former exercises, $T$ is maximal (minimal) if and only if $\lambda$ is maximal. So for axis $\frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $T$ is minimal; for axis $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $T$ is maximal.

For $\overrightarrow{\omega}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $\overrightarrow{L} = I\overrightarrow{\omega}_1 = 0.048 \cdot \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, for $\overrightarrow{\omega}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{L} = I\overrightarrow{\omega}_2 = 0.0705 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. So the nutation is parallel to the corresponding axis.