Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 6

Date Due: 10:00 AM, Thursday, the 10th of November 2016

Exercise 6.1. Suppose that $f\colon \Omega \to \mathbb{C}$ be holomorphic on a domain $\Omega \subset \mathbb{C}$. Let $z_0 \in \Omega$ be such that $B_r(z_0) \subset \Omega$ and suppose that for some $M > 0$,

$$|f(z)| \le M \qquad\qquad \text{for all } z \in B_r(z_0).$$

If $f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ is the power series representation of f, show that

$$|c_n| \le \frac{M}{r^n}$$

(2 Marks)

Exercise 6.2.

i) Use Exercise 6.1 to prove *Liouville's Theorem*: Any bounded, entire function must be constant.

ii) Deduce the *Fundamental Theorem of Algebra*: Every polynomial of degree $n \ge 1$ has at least one zero. (*Instructions*: If f is a polynomial of degree $n \ge 1$, show that f is unbounded. Consider then $g = 1/f$ and show that g must be entire and bounded if f has no zero, contradicting Liouville's Theorem.)

(2 + 2 Marks)

Exercise 6.3. Let $\Omega \subset \mathbb{C}$ be open and $g, h\colon \Omega \to \mathbb{C}$ be holomorphic at a point $z_0 \in \Omega$. Assume that h has a simple zero at z_0. Prove that

$$\operatorname{res}_{z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

(2 Marks)

Exercise 6.4. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,dx = \pi \frac{e^{-a}}{a}, \qquad \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx = \pi e^{-a}, \qquad a > 0.$$

(3 + 3 Marks)

Exercise 6.5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

(3 Marks)

Exercise 6.6. Show that

$$\int_0^{\infty} \frac{x \sin x}{(x^2 + 4)^2}\,dx = \frac{\pi}{8e^2}.$$

(3 Marks)

Exercise 6.7. Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi, \qquad n \in \mathbb{N}.$$

(3 Marks)

Exercise 6.8. Evaluate the following integrals using residue calculus:

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + a^2}\,dx, \qquad \int_0^{\infty} \frac{\ln x}{x^2 + a^2}\,dx, \qquad a \in \mathbb{R}.$$

(3 + 3 Marks)

Exercise 6.9. If possible, use the Heaviside operator method to solve $y'' + y = f(x)$ for

i) $f(x) = 3x + 5x^4$,

ii) $f(x) = e^{\mu x}$, $\mu \in \mathbb{R}$.

(3 + 3 Marks)

Exercise 6.10. Find the Laplace transform of the following functions:

i) $\sinh(bt)$, $b \in \mathbb{R}$,

ii) $\cos(bt)$, $b \in \mathbb{R}$,

iii) $t\sin(at)$, $a \in \mathbb{R}$,

iv) $t^2 \sinh(bt)$, $b \in \mathbb{R}$,

v) $\sqrt{t}$ (use the Euler gamma function).

vi) $1/\sqrt{t}$ (use the Euler gamma function).

(6 × 1 Mark)

Exercise 6.11. Use the Laplace transform to solve the following initial-value problems:

$$y''' - 6y'' + 11y' - 6y = e^{4t}, \qquad y(0) = y'(0) = y''(0) = 0$$

$$y'' + y' + y = H(t-\pi) - H(t-2\pi), \qquad y(0) = 1, \qquad y'(0) = 0$$

$$y'' + y = \begin{cases} \cos t & 0 \le t < \pi/2 \\ 0 & \pi/2 \le t < \infty \end{cases}, \qquad y(0) = 3, \qquad y'(0) = -1$$

You may use Laplace transform tables to look up the inverse transform.
(3 × 3 Marks)