VV286 Review 1

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Separable Equations

$$\frac{dy}{dx} = f(x)g(y), \qquad y(\xi) = \eta$$

1. $g(\eta) \neq 0$

$$\int_{\eta}^{y} \frac{ds}{g(s)} = \int_{\xi}^{x} f(t)dt \qquad \text{(Unique solution)}$$

2. $g(\eta) = 0$
   2.1 Obvious solution
   $$y(x) = \eta$$
   2.2 Check $\int_{\eta}^{y} \frac{ds}{g(s)}$ in a small neighbourhood of η

Linear Equations

$$a_1(x)y' + a_0(x)y = f(x)$$

1. Solve $a_1(x)y' + a_0(x)y = 0$ to find y_{hom}.
2. Use y_{hom} to find y_{part}. Set $y_{\text{part}}(x) = c(x)y_{\text{hom}}(x)$, then solve $c(x)$.

$$y = y_{\text{part}} + C \cdot y_{\text{hom}}$$

Transformable Equations

$y' = f(ax + by + c); b \neq 0$

$$u(x) = ax + by(x) + c$$

$y' = f(y/x)$

$$u(x) = \frac{y(x)}{x}$$

$y' + gy + hy^{\alpha} = 0, \alpha \neq 1$ (Bernoulli's equation)

$$u(x) = (y(x))^{1-\alpha}$$

$$\begin{aligned}
&y' + gy + hy^{\alpha} = 0\\
\Rightarrow &(1-\alpha)y^{-\alpha}y' + (1-\alpha)gy^{1-\alpha} + (1-\alpha)h = 0\\
\Rightarrow &(y^{1-\alpha})' + (1-\alpha)gy^{1-\alpha} + (1-\alpha)h = 0\\
\Rightarrow &u' + (1-\alpha)gu + (1-\alpha)h = 0
\end{aligned}$$

Note

1. $\alpha > 0$, $y = 0$
2. α is odd, $y_- = -y_+$

$y' + gy + hy^2 = k$ (Ricatti's equation)

1. Guess or given a solution ϕ
2. For other solution y, set $u = y - \phi$, then

$$\begin{cases} y' + gy + hy^2 = k \\ \phi' + g\phi + h\phi^2 = k \end{cases}$$

$$\Rightarrow (y' - \phi') + g(y - \phi) + h(y - \phi)(y + \phi) = 0$$

$$\Rightarrow u' + gu + hu(u + 2\phi) = 0$$

$$\Rightarrow u' + (g + 2\phi h)u + hu^2 = 0$$

$h(x, y)y' + g(x, y) = 0$

$$h(x, y)y' + g(x, y) = 0 \Rightarrow \left\langle \begin{pmatrix} 1 \\ y' \end{pmatrix}, \begin{pmatrix} g(x, y) \\ h(x, y) \end{pmatrix} \right\rangle = 0$$

Find a potential function $U(x, y)$ whose gradient at each point is parallel to the vector $\begin{pmatrix} g(x, y) \\ h(x, y) \end{pmatrix}$ i.e.

$$\nabla U(x, y) = M(x, y) \cdot \begin{pmatrix} g(x, y) \\ h(x, y) \end{pmatrix}$$

Then for any constant C, $U(x, y) = C$ is a solution.

Requirement

$$\frac{\partial M(x,y)g(x,y)}{\partial y} = \frac{\partial M(x,y)h(x,y)}{\partial x} \quad \text{(Rotation is zero)}$$

i.e.

$$\frac{\partial M}{\partial y} g + M \frac{\partial g}{\partial y} = \frac{\partial M}{\partial x} h + M \frac{\partial h}{\partial x}$$

Assumption

1. M depends only on x or only on y
2. M depends only on $x \cdot y$

Exercise 1.9

$$\left(\frac{x^2}{y} + 3\frac{y}{x}\right) y' + \left(3x + \frac{6}{y}\right) = 0$$

Implict Differential Equations

Slope parametrization

Given y'' exists and $y'' \neq 0$, y' is monotonic function of x. We can use slope to parametrize the solution curve.

$$p = y'(x) = y'(x(p))$$

$$\frac{dy(p)}{dp} = \frac{d}{dp} y(x(p)) = \frac{dy}{dx}\bigg|_{x=x(p)} \cdot \frac{dx(p)}{dp} = p \cdot \frac{dx(p)}{dp}$$

$F(y, y'; x) = 0$

1. Try to use slope parametrization. Solve

$$F(y(p), p; x(p)) = 0, y'(p) = px'(p)$$

2. Straight line solution.

General Implicit Differential Equation

Use slope parametrization,

$$
\begin{aligned}
&F(y, y'; x) = 0 \\
\Rightarrow &F(y(p), p; x(p)) = 0 \\
\xrightarrow{\partial/\partial p} &F_x \dot{x} + F_y \dot{y} + F_p = 0 \\
\xrightarrow{y'(p)=px'(p)} &\dot{x} = -\frac{F_p}{F_x + pF_y}, \quad \dot{y} = -\frac{pF_p}{F_x + pF_y}
\end{aligned}
$$

$y = xy' + g(y')$ (Clairaut's equation)

Assume $g \in C^1(I)$ for some interval I.

1. Use slope parametrization, $y(p) = x(p) \cdot p + g(p)$, then

$$y'(p) = px'(p) + x(p) + g'(p)$$

   Since $y'(p) = px'(p)$,

$$x(p) = -g'(p), \quad y(p) = -pg'(p) + g(p)$$

2. Straight line solution: $y = cx + g(c), c \in I$

$y = xf(y') + g(y')$ (d'Alembert's equation)

Assume $f, g \in C^1(I)$ for some interval I.

1. Use slope parametrization, $y(p) = x(p) \cdot f(p) + g(p)$, then

   $$y'(p) = f(p)x'(p) + f'(p)x(p) + g'(p)$$

   Since $y'(p) = px'(p)$, $x'(p) = \dfrac{f'(p)x(p) + g'(p)}{p - f(p)}$

2. Straight line $y = cx + d$ is solution if and only if $c = f(c), d = g(c)$

Concept

Equilibrium solution

$$x_{equi}(t) = constant$$

Steady-state solution

$$x_{ss}(t) = \lim_{t\to\infty} x(t)$$

Transient solution

$$x(t) - x_{ss}$$

Steady-state solution is often (but not always) equal to the equilibrium solution

$$
\begin{aligned}
y &= \frac{1}{1-e^{-x}} \\
\Rightarrow y' &= -\frac{e^{-x}}{(1-e^{-x})^2} = \frac{1-e^{-x}-1}{(1-e^{-x})^2} = y - y^2 \\
\Rightarrow y' &= y(1-y)
\end{aligned}
$$

Equilibrium solution: $y_1 = 1$, $y_2 = 0$

Steady-state solution: $y_{ss}(x) = \lim\limits_{x\to\infty} \dfrac{1}{1-e^{-x}} = 1$

$$a_1(x)y' + a_0(x)y = f(x), \quad y(\xi) = \eta$$

Differential Operator

$$L = a_1 \frac{d}{dx} + a_0$$

Homogeneous/Inhomogeneous

$f(x) = 0$ for all $x \in I$; $\exists x \in I$, $f(x) \neq 0$.

Initial Condition

η is called initial condition for y.

$\eta = 0$ homogeneous initial condition

Data

The pair $\{f, \eta\}$

Singular Point

If $a_1(x_0) = 0$, we say that x_0 is a singular point for L.

Implicit Equation

If $F(y_0, p_0; x_0) = 0$ and $F(y, p; x) = 0$ can be solved for p as a function of x and y in some neighborhood U of (x_0, y_0, p_0), then (x_0, y_0, p_0) is said to be a ***regular line element***, otherwise a ***singular line element***.

A solution y of the implicit ODE $F(y, y'; x) = 0$ is said to be ***regular*** on an interval $I \subset \mathbb{R}$ if for all $x \in I$ the line elements $(x, y(x), y'(x))$ are regular.

A point (x, y) is said to be a ***singular point*** of the ODE if there exists a singular line element (x, y, p).

Envelope

A one-parameter family of smooth curves in $\mathbb{R}^2$ is a set

$$\mathcal{F} = \{\mathcal{C}_s, s \in I\}$$

where $I \subset \mathbb{R}$ is some interval and each $\mathcal{C}_s$ is a smooth curve. An ***envelope*** of $\mathcal{F}$ is a curve $\mathcal{E}$ such that every point of $\mathcal{E}$ is tangent to a curve in $\mathcal{F}$.