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VV286 RC2

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Implict Differential Equations

Slope parametrization

Given y'' exists and $y'' \neq 0$, y' is monotonic function of x. We can use slope to parametrize the solution curve.

$$p = y'(x) = y'(x(p))$$

$$\frac{dy(p)}{dp} = \frac{d}{dp} y(x(p)) = \frac{dy}{dx}\Big|_{x=x(p)} \cdot \frac{dx(p)}{dp} = p \cdot \frac{dx(p)}{dp}$$

$F(y, y'; x) = 0$

1. Try to use slope parametrization. Solve

$$F(y(p), p; x(p)) = 0, y'(p) = px'(p)$$

2. Straight line solution.

General Implicit Differential Equation

Use slope parametrization,

$$
\begin{aligned}
&F(y, y'; x) = 0 \\
\Rightarrow &F(y(p), p; x(p)) = 0 \\
\Rightarrow &F_x \dot{x} + F_y \dot{y} + F_p = 0 \\
\xRightarrow{y'(p)=px'(p)} &\dot{x} = -\frac{F_p}{F_x + pF_y}, \quad \dot{y} = -\frac{pF_p}{F_x + pF_y}
\end{aligned}
$$

$y = xy' + g(y')$ (Clairaut's equation)

Assume $g \in C^1(I)$ for some interval I.

1. Use slope parametrization, $y(p) = x(p) \cdot p + g(p)$, then

$$y'(p) = px'(p) + x(p) + g'(p)$$

   Since $y'(p) = px'(p)$,

$$x(p) = -g'(p), \quad y(p) = -pg'(p) + g(p)$$

2. Straight line solution: $y = cx + g(c), c \in I$

The solution of Clairaut's equation obtained from a slope parametrization of the integral curve is always the envelope of the straight-line solutions.

Envelope equation

Given a one-parameter family of smooth curves in $\mathbb{R}^2$

$$F = \{\mathcal{C}_s, s \in I \subset \mathbb{R}\}$$

with each curve $\mathcal{C}_p$ parameterized by a function

$$\gamma(s, \cdot) : J \to \mathcal{C}_p, \quad t \mapsto \gamma(s, t)$$

Then the envelope $\mathcal{E}$ of F which is parametrized by $\gamma(s, \psi(s))$ can be found through

$$\frac{\partial \gamma_1}{\partial s}\frac{\partial \gamma_2}{\partial t} = \frac{\partial \gamma_1}{\partial t}\frac{\partial \gamma_2}{\partial s}, \quad t = \psi(s)$$

Another way to solve Clairaut's equation

1. Straight line solution: $y = cx + g(c), c \in I$
2. $\gamma(c, x) = \begin{pmatrix} x \\ cx + g(c) \end{pmatrix}$

$$\frac{\partial \gamma_1}{\partial c}\frac{\partial \gamma_2}{\partial x} = \frac{\partial \gamma_1}{\partial x}\frac{\partial \gamma_2}{\partial c} \Rightarrow 0 = x + g'(c)$$

We obtain the parametrization of $\mathcal{E}$ as $\gamma(c, -g'(c))$. So

$$y(c) = -cg'(c) + g(c)$$

$y = xf(y') + g(y')$ (d'Alembert's equation)

Assume $f, g \in C^1(I)$ for some interval I.

1. Use slope parametrization, $y(p) = x(p) \cdot f(p) + g(p)$, then
$$y'(p) = f(p)x'(p) + f'(p)x(p) + g'(p)$$
Since $y'(p) = px'(p)$, $x'(p) = \dfrac{f'(p)x(p) + g'(p)}{p - f(p)}$
2. Straight line $y = cx + d$ is solution if and only if $c = f(c), d = g(c)$

Example

$$y = (y \cdot y' + 2x) \cdot y'$$

Solution

Use slope parametrization,

$$
\begin{aligned}
& y(p) = p^2 y(p) + 2x(p)p, \quad y'(p) = px'(p) \\
\Rightarrow & y(p) = \frac{2px(p)}{1-p^2} \\
\Rightarrow & px'(p) = y'(p) = \frac{(2x(p) + 2px'(p))(1-p^2) - 2px(p)(-2p)}{(1-p^2)^2} \\
\Rightarrow & ((1-p^2)p - 2p)x'(p) = \frac{2(1+p^2)}{1-p^2}x(p) \\
\Rightarrow & x'(p) = -\frac{2}{p(1-p^2)}x(p) = -\frac{1}{p}\left(\frac{1}{1-p} + \frac{1}{1+p}\right)x(p) \\
\Rightarrow & x'(p) = -\left(\frac{1}{p} + \frac{1}{1-p} + \frac{1}{p} - \frac{1}{1+p}\right)x(p)
\end{aligned}
$$

Solution (continued)

So $\ln|x(p)| = -2\ln|p| + \ln|p-1| + \ln|p+1| + C$, i.e.

$$x(p) = C \cdot \left(1 - \frac{1}{p^2}\right)$$

and therefore $y(p) = \dfrac{2px(p)}{1-p^2} = -\dfrac{2C}{p}$ So

$$x = C \cdot \left(1 - \left(\frac{y}{2C}\right)^2\right) = C - \frac{y^2}{4C}$$

Straight line solution is given by $y = 0$

System of Equations

Higher Order Equations

Given an explicit ODE of order n,

$$x^{(n)}(t) = f(x, x', x'', \cdots, x^{(n-1)}, t)$$

set

$$x_1(t) = x(t), x_2(t) = x'(t), x_3(t) = x''(t), \cdots, x_n(t) = x^{(n-1)}(t)$$

and we obtain

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ f(x_1, x_2, \cdots, x_n, t) \end{pmatrix}$$

Initial Value Problem

$$\frac{dx}{dt} = F(x,t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

Picard Iteration

1. Guess a function $x^{(0)}(t) = x_0$ (constant)
2. Set $x^{(k+1)}(t) := x_0 + \int_{t_0}^{t} F(x^{(k)}(s), s)ds$, $k \in \mathbb{N}$

Under suitable conditions on the function F, the sequence of functions $(x^{(k)})$ will converge to a unique function $x(t)$ which satisfies

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)ds$$

Theorem of Picard-Lindelöf (Existence and Uniqueness)

Let $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is open and let $t_0 \in I$, where $I \subset \mathbb{R}$ is an interval. Suppose $F : \Omega \times I \to \mathbb{R}^n$ is a continuous function satisfying a Lipschitz estimate in x: there exists an $L > 0$ such that for all $x, y \in \Omega$ and all $t \in I$

$$||F(x,t) - F(y,t)|| \leqslant L||x - y||$$

then the initial value problem has a unique solution in some t-interval containing t_0.

Gronwall's Inequality (Stability)

Under same conditions, x, y satisfies

$$x'(t) = F(x,t), x(t_0) = x_0, y'(t) = F(y,t), y(t_0) = y_0$$

Then

$$||x(t) - y(t)|| \leqslant e^{L\cdot|t-t_0|}||x_0 - y_0||$$

Linear Systems of Equations (Explict Linear Higher Order ODE)

$$\frac{dx}{dt} = A(t)x + b(t), \quad t \in I \subset \mathbb{R}$$

where $A : I \to$Mat$(n \times n, \mathbb{R})$ is a matrix-valued function of t and $b : I \to \mathbb{R}^n$

Construction of Solutions

$$x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)$$

$$x_{\text{hom}}(t) = \sum_{k=1}^{n} \lambda_k x^{(k)}(t)$$

where $x^{(k)}(t)$, $k = 1, \cdots, n$ satisfy

$$\frac{dx^{(k)}}{dt} = A(t)x^{(k)}, \left(x^{(k)}(t_0) = b_k, x_0 = \sum_{k=1}^{n} \lambda_k b_k \right)$$

for some numbers $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and

$$\left(\underset{t \in I}{\forall} \sum_{k=1}^{n} \lambda'_k x^{(k)}(t) = 0 \right) \Rightarrow \lambda'_1 = \cdots = \lambda'_n = 0$$

Solution Space

A vector space contains all solution.

$$\mathrm{Span}\{x^{(1)}, \cdots, x^{(n)}\}$$

Fundamental System of Solutions

A set of functions giving a bassis of the solution space.

$$\{x^{(1)}, \cdots, x^{(n)}\}$$

Fundamental Matrix

$$X(t) = (x^{(1)}(t), \cdots, x^{(n)}(t))$$

Systems of Linear ODEs with Constant Coefficients

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0$$

Try

$$x(t) \overset{?}{=\!=} e^{At}x_0 \overset{\Delta}{=\!=} x_0\left(\mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}\right)$$

Well-defined

Use operator norm with euclidean norm in $\mathbb{R}^n$

$$||A|| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{|Ax|}{|x|}$$

Then

$$\sum_{k=1}^{\infty}\left|\left|\frac{A^k t^k}{k!}\right|\right| = \sum_{k=1}^{\infty}\frac{||A^k||\cdot|t^k|}{k!} \leqslant \sum_{k=1}^{\infty}\frac{||A||^k\cdot|t|^k}{k!} = e^{|t|\cdot||A||} - 1 < \infty$$

The series is absolutely convergent and therefore it's convergent to some matrix in $\mathrm{Mat}(n\times n,\mathbb{R})$

Satisfy the Differential Equation

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{d}{dt}\frac{A^k t^k}{k!}\\ &= \sum_{k=1}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\\ &= Ae^{At}\end{aligned}$$

And for $t = 0$

$$e^{At} = \mathbb{1} \Rightarrow x(0) = x_0$$

Eigenvalue Problem

Eigenvalue

Let V be a real or complex vector space and $L \in \mathcal{L}(V, V)$. Then a number $\lambda \in \mathbb{F}$ such that for some $x \neq 0$

$$Lx = \lambda x$$

Eigenvector

Any x such that $Lx = \lambda x$ holds is called an eigenvector for the eigenvalue λ.

Eigenspace

The subspace

$$V_\lambda = \{x \in V : Lx = \lambda x\}$$

The Eigenvalue Problem for Matrices

For a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{R})$

$$Ax = \lambda x \Leftrightarrow (A - \lambda \mathbb{1})x = 0 \Rightarrow det(A - \lambda \mathbb{1}) = 0$$

$p(\lambda) := det(A - \lambda \mathbb{1})$ is a polynomial of degree n. It has at most n distinct roots.

If A has n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, then it has precisely n independent eigenvectors $v_1, \cdots, v_n$ and $\mathbb{R}^n = \bigoplus_{j=1}^{n} V_{\lambda_j}$.

Diagonalizable Matrics

Set $U = (v_1, \cdots, v_n)$, $D = U^{-1}AU$

$$De_k = U^{-1}AUe_k = U^{-1}Av_k = U^{-1}(\lambda_k v_k) = \lambda_k e_k$$

So

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Matrix Powers

$$D^k = (U^{-1}AU)^k = \underbrace{(U^{-1}AU)(U^{-1}AU)\cdots(U^{-1}AU)}_{k \text{ times}} = U^{-1}A^kU$$

$$A^k = UD^kU^{-1}$$

Functional Calculus

For a power series has infinite redius of convergence

$$f(x) = \sum_{j=0}^{\infty} c_j x^j$$

$$f(A) = \sum_{j=0}^{\infty} c_j A^j = \sum_{j=0}^{\infty} c_j (U D^j U^{-1}) = U \left(\lim_{N \to \infty} \sum_{j=0}^{N} c_j D^j \right) U^{-1}$$

$$= U \begin{pmatrix} \sum_{j=0}^{\infty} c_j \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \sum_{j=0}^{\infty} c_j \lambda_n^j \end{pmatrix} U^{-1}$$

$$= U \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} U^{-1}$$

Matrix Exponential

For a diagonlizable matrix A,

$$e^{At} = U \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} U^{-1}$$

The Spectral Theorem

Every self-adjoint matrix A is diagonalizable.

Self-adjoint

For a inner product space $(V, \langle x, Ay \rangle)$ and $A : V \to V$, the adjoint of A is defined through the relation

$$\langle x, Ay \rangle = \langle A^* x, y \rangle \quad \text{for all } x, y \in V$$

If $A = A^*$, the map A is said to be self-adjoint. In the case of matrices, if $A \in \mathrm{Mat}(n \times n, \mathbb{C})$

$$A = A^* = \overline{A}^T$$

Non-Diagonalizable Matrices

"Bottom-up" Method

For some eigenvalue λ, $\dim V_\lambda < a_\lambda$,

$$E_1 = V_\lambda = \ker(A - \lambda\mathbb{1})$$

$$E_k = ker(A - \lambda\mathbb{1})^k = \{v \in V : (A - \lambda\mathbb{1})^k v = 0\}$$

Choose $(\dim V_\lambda - 1)$ different eigenvector for the eigenvalue λ. Then start from $k = 1$. Choose a suitable $v^{(k)} \in E_k$ and solve $(A - \lambda\mathbb{1})v^{(k+1)} = v^{(k)}$ to find one $v^{(k+1)} \in E_{k+1} \setminus E_k$. $k = k + 1$. Repeat until $\dim V_\lambda + k = a_\lambda$.

U is formed with all independent (generalized) eigenvectors. For some eigenvalue λ, $\dim V_\lambda < a_\lambda$, assume its generalized eigenvectors are at $i_\lambda th$ to $(i_\lambda + a_\lambda - 1)th$ column in U

For $i_\lambda \leqslant j \leqslant i_\lambda + \dim V_\lambda - 1$, $Ue_j \in V_\lambda$, so $U^{-1}AUe_j = \lambda e_j$.

For $i_\lambda + \dim V_\lambda \leqslant j \leqslant i_\lambda + a_\lambda - 1$, set $k = j - i_\lambda - \dim V_\lambda + 2$

$$\begin{aligned} U^{-1}AUe_j = U^{-1}Av^{(k)} &= U^{-1}(\lambda \mathbb{1} v^{(k)} + v^{(k-1)}) \\ &= \lambda e_j + e_{j-1} \end{aligned}$$

(Slide 200)

"Top-down" Method

For some eigenvalue λ, $\dim V_\lambda < a_\lambda$, set $m = a_\lambda - \dim V_\lambda + 1$, then solve

$$(A - \lambda \mathbb{1})^m v = 0, (A - \lambda \mathbb{1})^{m-1} v \neq 0 \quad \text{as } v^{(m)}$$

For $2 \leqslant k \leqslant m - 1$, set $v^{(k)} = (A - \lambda \mathbb{1}) v^{(k+1)}$ which (naturally) satisfies

$$(A - \lambda \mathbb{1})^k v^{(k)} = 0, (A - \lambda \mathbb{1})^{k-1} v^{(k)} \neq 0$$

then $v^{(1)} = (A - \lambda \mathbb{1}) v^{(2)} \in V_\lambda$.

Choose $(\dim V_\lambda - 1)$ independent eigenvector for the eigenvalue λ.

Jordan Matrices

For $\lambda \in \mathbb{C}$ we define the Jordan block of size $k \in \mathbb{N} \setminus \{0\}$

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in \mathrm{Mat}(k \times k, \mathbb{C})$$

A block matrix of the form

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_m}(\lambda_m) \end{pmatrix}$$

with not necessarily distinct $\lambda_1, \cdots, \lambda_m \in \mathbb{C}$ and $k_1, \cdots, k_m \in \mathbb{N}$ is called Jordan matrix.

$J = D + N$

There exists some $k \in \mathbb{N}$ such that $N^k = 0$. (Nipotent Matrices)
Assume $D, N \in$Mat$(n \times n, \mathbb{C})$.

For $1 \leqslant i \leqslant j \leqslant n$

$$(DN)_{ji} = D_{j\cdot} \cdot N_{\cdot i} = D_{jj}N_{ji} + D_{j,i-1}N_{i-1,i} = 0$$

$$(ND)_{ji} = N_{j\cdot} \cdot D_{\cdot i} = N_{j,j+1}D_{j+1,i} + N_{ji}D_{ii} = 0$$

For $1 \leqslant j \leqslant n-2, j+2 \leqslant i \leqslant n$

$$(DN)_{ji} = D_{j\cdot} \cdot N_{\cdot i} = D_{jj}N_{ji} + D_{j,i-1}N_{i-1,i} = 0$$

$$(ND)_{ji} = N_{j\cdot} \cdot D_{\cdot i} = N_{j,j+1}D_{j+1,i} + N_{ji}D_{ii} = 0$$

For $1 \leqslant i \leqslant n-1, j=i+1$

$$(DN)_{i,i+1} = D_{i\cdot} \cdot N_{\cdot,i+1} = D_{ii} N_{i,i+1}$$

$$(ND)_{i,i+1} = N_{i\cdot} \cdot D_{\cdot,i+1} = N_{i,i+1} D_{i+1,i+1}$$

If $D_{ii} \neq D_{i+1,i+1}$, $N_{i,i+1} = 0$ and therefore $(DN)_{i,i+1} = (ND)_{i,i+1}$

So $\forall i, j \in [1, n] \cap \mathbb{N}$, $(DN)_{ij} = (ND)_{ij}$. So $DN = ND$ and we can prove that

$$e^J = e^{D+N} = e^D \cdot e^N$$

For $e^N = \sum_{i=0}^{\infty} \frac{1}{i!} N^i = \sum_{i=0}^{k-1} \frac{1}{i!} N^i$,

$$e^{At} = Ue^{Jt}U^{-1} = U(e^{Dt} \cdot \sum_{i=0}^{k-1} \frac{1}{i!} (Nt)^i) U^{-1}$$

Homogeneous Solution

For any basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of $\mathbb{R}^n$, the systems of functions

$$\mathcal{F} = \{e^{At}v_1, \cdots, e^{At}v_n\}$$

is a fundamental system for $\dfrac{dx}{dt} = Ax$.

1. Take $v_i = e_i$, the fundamental matrix is given by

$$X(t) = e^{At}$$

2. Take $v_i = u_i$, the fundamental matrix is given by

$$X(t) = Ue^{Jt}$$

   2.1 If A is diagonalizable, $X(t) = (e^{\lambda_1 t}u_1, \cdots, e^{\lambda_n t}u_n)$

Particular Solution

$$
\begin{aligned}
&\frac{dx}{dt} = Ax + b(t) \\
\Rightarrow & e^{-At}\frac{dx}{dt} = Ae^{-At} + e^{-At}b(t) \\
\Rightarrow & \frac{d}{dt}(e^{-At}x) = e^{-At}b(t) \\
\Rightarrow & x_{\text{part}} = e^{At}\int e^{-As}b(s)ds
\end{aligned}
$$

Short Summary

To solve linear explicit higher order ODE with constant coefficient,

1. Transform into systems of linear ODEs with constant coefficients $\frac{dx}{dt} = Ax + b(t)$
2. Find all eigenvalues of A, calculate their eigenvectors
   2.1 For those $\dim V_\lambda < a_\lambda$, find $a_\lambda - \dim V_\lambda$ more generalized eigenvectors $v_\lambda^{(2)}, \cdots, v_\lambda^{(a_\lambda - \dim V_\lambda + 1)}$
3. Form U with all eigenvectors in order, write down $J(= U^{-1}AU)$ directly. Obtain the fundamental matrix $X(t)$.
4. Find particular solution,
   4.1 $x_{\text{part}} = e^{At} \int e^{-As} b(s) ds \ (X(t) = e^{At})$
   4.2 Use Cramer's rule to solve $X(t)c'(t) = b(t)$, then $x_{\text{part}}(t) = \sum_{k=1}^{n} c_k(t) x^{(k)}(t)$

Linear Systems with Variable Coefficients

$$\frac{dx}{dt} = A(t)x + b(t)$$

No general method to solve variable-coefficient homogeneous, linear systems in terms of elementary functions.

Given a fundamental systems of solutions to an associated homogeneous equation, a solution to an inhomogeneous equation can be found.

Variation of Parameters for Linear Systems

$$\frac{dx}{dt} = A(t)x + b(t), A : \mathbb{R} \to \mathrm{Mat}(n \times n, \mathbb{R}), b : \mathbb{R} \to \mathbb{R}^n$$

Given a fundamental system $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$, then

$$x_{\mathrm{hom}}(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t), \quad c_1, \cdots, c_n \in \mathbb{R}$$

Set

$$x_{\mathrm{part}}(t) = c_1(t)x^{(1)}(t) + \cdots + c_n(t)x^{(n)}(t)$$

$$\frac{dx_{\mathrm{part}}}{dt} = \sum_{k=1}^{n}(c_k'(t)x^{(k)}(t) + c_k(x^{(k)})'(t)) = A(t)x_{\mathrm{part}} + b(t)$$

$$\Rightarrow \sum_{k=1}^{n} c_k'(t)x^{(k)}(t) = b(t)$$

$$x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}, \quad c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

$$\begin{aligned}
\sum_{k=1}^{n} (c_k'(t) x^{(k)}(t) &= \begin{pmatrix} c_1'(t)x_1^{(1)}(t) + \cdots + c_n'(t)x_1^{(n)}(t) \\ \vdots \\ c_1'(t)x_n^{(1)}(t) + \cdots + c_n'(t)x_n^{(n)}(t) \end{pmatrix} \\
&= \begin{pmatrix} x_1^{(1)}(t) & x_2^{(2)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ \vdots \\ c_n'(t) \end{pmatrix} \\
&= X(t)c'(t)
\end{aligned}$$

Use Cramer's rule to solve $X(t)c'(t) = b(t)$,

$$c_k'(t) = \frac{\det X^{(k)}(t)}{\det X(t)}$$

where $X^{(k)}$ is the fundamental matrix with the kth column replaced with b

The Wronskian of n Solutions of a System

$$W(t) = \det(x^{(1)}(t), \cdots, x^{(n)}(t)) = \det X(t)$$

$$\frac{dW}{dt} = \operatorname{tr} A(t) \cdot W, \quad W(t) = W(t_0) e^{\int_{t_0}^{t} \operatorname{tr} A(s) ds}$$

$$W(t) = 0 \text{ for all } t \text{ or } W(t) \neq 0 \text{ for all } t$$