VV286 RC3

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Linear Systems with Variable Coefficients

$$\frac{dx}{dt} = A(t)x + b(t)$$

No general method to solve variable-coefficient homogeneous, linear systems in terms of elementary functions.

Given a fundamental systems of solutions to an associated homogeneous equation, a solution to an inhomogeneous equation can be found.

Variation of Parameters for Linear Systems

$$\frac{dx}{dt} = A(t)x + b(t), A : \mathbb{R} \to \mathrm{Mat}(n \times n, \mathbb{R}), b : \mathbb{R} \to \mathbb{R}^n$$

Given a fundamental system $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$, then

$$x_{\text{hom}}(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t), \quad c_1, \cdots, c_n \in \mathbb{R}$$

Set

$$x_{\text{part}}(t) = c_1(t)x^{(1)}(t) + \cdots + c_n(t)x^{(n)}(t)$$

$$\frac{dx_{\text{part}}}{dt} = \sum_{k=1}^{n}(c_k'(t)x^{(k)}(t) + c_k(x^{(k)})'(t)) = A(t)x_{\text{part}} + b(t)$$

$$\Rightarrow \sum_{k=1}^{n} c_k'(t)x^{(k)}(t) = b(t)$$

$$x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}, \quad c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

$$\begin{aligned}
\sum_{k=1}^{n} (c_k'(t) x^{(k)}(t) &= \begin{pmatrix} c_1'(t) x_1^{(1)}(t) + \cdots + c_n'(t) x_1^{(n)}(t) \\ \vdots \\ c_1'(t) x_n^{(1)}(t) + \cdots + c_n'(t) x_n^{(n)}(t) \end{pmatrix} \\
&= \begin{pmatrix} x_1^{(1)}(t) & x_2^{(2)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ \vdots \\ c_n'(t) \end{pmatrix} \\
&= X(t) c'(t)
\end{aligned}$$

Use Cramer's rule to solve $X(t)c'(t) = b(t)$,

$$c_k'(t) = \frac{\det X^{(k)}(t)}{\det X(t)}$$

where $X^{(k)}$ is the fundamental matrix with the kth column replaced with b

The Wronskian of n Solutions of a System

$$W(t) = \det(x^{(1)}(t), \cdots, x^{(n)}(t)) = \det X(t)$$

$$\frac{dW}{dt} = \mathrm{tr}A(t) \cdot W, \quad W(t) = W(t_0)e^{\int_{t_0}^{t} \mathrm{tr}A(s)ds}$$

$$W(t) = 0 \text{ for all } t \text{ or } W(t) \neq 0 \text{ for all } t$$

Linear Second-Order Equations and Vibrations

Definition

A linear differential equation of order 2 is an equation of the form

$$r(t)y'' + p(t)y' + q(t)y = g(t)$$

We only focus on $y'' + p(t)y' + q(t)y = g(t)$ now.

Reduction of Order

Given a solution y_1 to the equation $y'' + p(t)y' + q(t)y = 0$, set $y_2(t) = v(t)y_1(t)$, then

$$0 = y_1 v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{=0} v = y_1 v'' + (2y_1' + py_1)v'$$

Example

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1$$

$$y(x) = x$$

Solution

$(1-x^2)\cdot 0 - 2x + 2x = 0$, so $y_1(x) = x$ is a solution. Set $y_2(x) = c(x)x$, then

$$
\begin{aligned}
&(1-x^2)(c''(x)x + 2c'(x)) - 2x(c'(x)x + c(x)) + 2xc(x) = 0\\
\Rightarrow & x(1-x^2)c''(x) + (2-4x^2)c'(x) = 0\\
\Rightarrow & c''(x) = -\frac{4-4x^2-2}{x(1-x^2)}c'(x) = -\frac{1}{x}(4 - \frac{1}{1-x} - \frac{1}{1+x})c'(x)\\
\Rightarrow & \ln|c'(x)| = -4\ln|x| + \ln|x| - \ln|x-1| + \ln|x| - \ln|1+x| + C\\
\Rightarrow & c'(x) = \frac{C}{x^2(x^2-1)} = C\cdot(\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2})\\
\Rightarrow & c(x) = C_1\cdot(\frac{2}{x} + \ln|x-1| - \ln|x+1|)(+C_2)\\
\Rightarrow & y_2(x) = C_1(2 + x\ln\frac{1-x}{1+x})
\end{aligned}
$$

Linear Second-Order ODEs with Constant Coefficients

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, a \neq 0$$

1. $b^2 \neq 4ac$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{C}$$

2. $b^2 = 4ac$

$$y(t) = (c_1 + c_2 t)e^{\lambda t}, \quad c_1, c_2 \in \mathbb{R}$$

(Similar to the result of $ax_{n+2} + bx_{n+1} + cx_n = 0$.)

Amplitude Modulation

The atmosphere will propagate signals at a higher frequency range over longer distances.

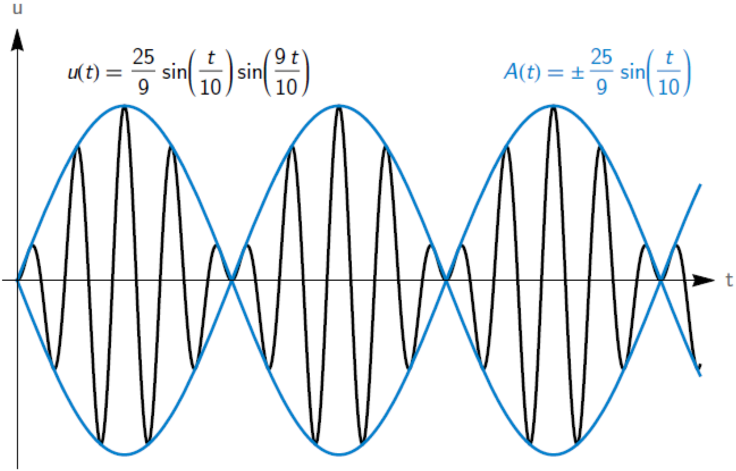
Sinusoidal Carrier

$$y(t) = x(t)c(t) = x(t)\cos(\omega_c t + \theta_c)$$

Synchronous Demodulation

$$\begin{aligned} w(t) &= x(t)\cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c) \\ &= \frac{1}{2}x(t)(\cos(\theta_c - \phi_c) + \cos(2\omega_c t + \theta_c + \phi_c)) \\ &\overset{\theta_c = \phi_c}{=\!=\!=} \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_c t + 2\theta_c) \end{aligned}$$

Asynchronous Demodulation



$$y(t) = (A + x(t))\cos(\omega_c t + \theta_c), \quad A > |x(t)|_{\max}$$