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# VV286 RC7

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# Series Methods for Second-Order Equations

Set $x(t) = \sum_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k$$

Insert to find relation among coefficients $a_k, a_{k+1}, a_{k+2}$. If we can find two independent solutions (usually no $t$ before $x''$, the easiest case), then we have done; else

Set $x(t) = t^r \sum_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=0}^{\infty} (k+r) a_k t^{k+r-1}$$

$$x''(t) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r-2}$$

Insert to solve $r$.

1. Two distinct real roots $r_1$ and $r_2$, $r_1 - r_2 \notin \mathbb{Z}$

$$x_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n, \quad x_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n$$

2. Two distinct real roots $r_1$ and $r_2$, $r_1 - r_2 \in \mathbb{Z}$

$$\begin{aligned} & x_2(t) \\ =& \frac{d}{dr}\left(t^r \sum_{n=0}^{\infty} a_n(r) t^n\right)\Bigg|_{r=r_2} = t^r \ln t \sum_{n=0}^{\infty} a_n(r) t^n + t^r \sum_{n=0}^{\infty} a_n'(r) t^n \Bigg|_{r=r_2} \\ =& c \cdot x_1(t) \ln t + t^{r_2} \sum_{n=0}^{\infty} a_n'(r_2) t^n \end{aligned}$$

Especially $c = 1$ for $r_1 = r_2$.

## 3. Complex root

$$x_1(t) = \mathrm{Re}\left(t^{r_1}\sum_{n=0}^{\infty} a_n t^n\right), \quad x_2(t) = \mathrm{Im}\left(t^{r_2}\sum_{n=0}^{\infty} b_n t^n\right)$$

## Singular Point

The equation

$$x'' + p(t)x' + q(t)x = 0$$

is said to have a ***regular singular point*** at $t_0$ if the functions $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic in a neighborhood of $t_0$. A singular point which is not regular is said to be ***irregular*** .

# Bessel Equation

The Bessel equation of order $\nu \geqslant 0$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$$
\begin{aligned}
0 =& x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r)a_k x^{k+r-1} + (x^2-\nu^2)x^r \sum_{k=0}^{\infty} a_k x^k \\
=& \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)a_k x^{k+r} \\
&+ \sum_{k=0}^{\infty} a_k x^{k+r+2} - \nu^2 \sum_{k=0}^{\infty} a_k x^{k+r} \\
=& r(r-1)a_0 x^r + (1+r)ra_1 x^{1+r} + ra_0 x^r + (1+r)a_1 x^{1+r} - \nu^2 a_0 x^r - \nu^2 a_1 x^{1+r} \\
&+ \sum_{k=0}^{\infty} ((k+r+2)(k+r+1)a_{k+2} + (k+r+2)a_{k+2} + a_k - \nu^2 a_{k+2}) x^{x+r+2}
\end{aligned}
$$

So

$$\begin{cases} (r^2 - \nu^2)a_0 = 0 \\ ((r+1)^2 - \nu^2)a_1 = 0 \\ ((k+r+2)^2 - \nu^2)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

Choose

$$\begin{cases} r = \nu \vee r = -\nu \\ ((k+r+2)^2 - \nu^2)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

For $r = \nu$, $a_1 = 0$

$$a_{2k+1} = 0, k \in \mathbb{N}$$

$$a_{2k} = -\frac{1}{(2k)(2k+2\nu)} a_{2k-2} = -\left(\frac{1}{2}\right)^2 \frac{1}{k(k+\nu)} a_{2k-2}$$

$$= \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod_{i=1}^{k} (i+\nu)} a_0, k \in \mathbb{N}^*$$

Set $a_0 = \dfrac{1}{2^\nu \Gamma(\nu+1)}$, then

$$a_{2k} = \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod_{i=1}^{k} (i+\nu)} \frac{1}{2^\nu \Gamma(\nu+1)} = \left(\frac{1}{2}\right)^{2k+\nu} \frac{(-1)^k}{k!\Gamma(1+k+\nu)}$$

So one solution (*Bessel function of the first kind*) is

$$J_\nu(x) = x^r \sum_{k=0}^{\infty} a_k x^k = x^\nu \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n+\nu} \frac{(-1)^n}{n!\Gamma(1+n+\nu)} x^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}$$

For $r = -\nu$,

1. $2\nu \in 2\mathbb{N} + 1$

$$\begin{cases} a_1 = 0 \\ (k+2)(k+2-2\nu)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

So $a_{2\nu-2} = 0 \Rightarrow \forall k \in [0, 2\nu - 2] \cap 2\mathbb{N} + 1, a_k = 0$
And $a_{2\nu}$ is arbitrary.

For $k \geqslant \nu + 1/2$

$$
\begin{aligned}
& a_{2k+1}(-\nu) \\
=& -\left(\frac{1}{2}\right)^2 \frac{1}{(k+1/2)(k+1/2-\nu)} a_{2k-1}(-\nu) \\
=& \left(\frac{1}{2}\right)^{2k+1-2\nu} \frac{(-1)^{k+1/2-\nu}}{\prod\limits_{i=\nu+1/2}^{k}(i+1/2)\prod\limits_{i=\nu+1/2}^{k}(i+1/2-\nu)} a_{2\nu}(-\nu) \\
=& \left(\frac{1}{2}\right)^{2k+1-2\nu} \frac{(-1)^{k+1/2-\nu}}{\prod\limits_{i=1}^{k-\nu+1/2}(i+\nu)\cdot(k+1/2-\nu)!} a_{2\nu}(-\nu)
\end{aligned}
$$

Choose $a_{2\nu}(-\nu) = a_0(\nu)$, then $a_{2k+1}(-\nu) = a_{2k-2\nu+1}(\nu)$

So

$$
\begin{aligned}
x^{-\nu}\sum_{k=0}^{\infty} a_{2k+1}(-\nu)x^{2k+1} &= x^{-\nu}\sum_{k=\nu-1/2}^{\infty} a_{2k+1}(-\nu)x^{2k+1}\\
=&x^{-\nu}\sum_{k=0}^{\infty} a_{2k+2\nu}(-\nu)x^{2k+2\nu} = \sum_{k=0}^{\infty} a_{2k}(\nu)x^{2k+\nu}\\
=&y_1(x)
\end{aligned}
$$

is not independent of $y_1(x)$.

$$a_{2k}(-\nu) = \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod_{i=1}^{k} (i+\nu)} a_0(-\nu)$$

If $\nu \notin \mathbb{N} \wedge \lceil \nu \rceil = m$, we can set $a_0(-\nu) = \dfrac{2^\nu}{\Gamma(-\nu)} = \dfrac{2^\nu \prod_{i=1}^{m} (i-\nu)}{\Gamma(m-\nu)}$,

then

$$a_{2k}(-\nu) = \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod_{i=1}^{k} (i-\nu)} \frac{2^\nu \prod_{i=1}^{m} (i-\nu)}{\Gamma(m-\nu)} = \left(\frac{1}{2}\right)^{2k-\nu} \frac{(-1)^k}{k!\Gamma(1+k-\nu)}$$

which is also hold for $k = 0$.

So

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n-\nu)} \left(\frac{x}{2}\right)^{2n-\nu}$$

is an independent solution of $y_1(x) = J_\nu(x)$.

## 3. $\nu \in \mathbb{N}$

$$\begin{cases} a_1 = 0 \\ (k+2)(k+2-2\nu)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

And therefore

$$a_{2k+1}(-\nu) = 0, k \in \mathbb{N}; a_{2k}(-\nu) = 0, k \leqslant \nu - 1$$

$$a_{2k}(-\nu) = -\frac{1}{(2k)(2k-2\nu)} a_{2k-2}(-\nu)$$

$$= \left(\frac{1}{2}\right)^{2k-2\nu} \frac{(-1)^{k-\nu}}{\prod_{i=\nu+1}^{k} i \cdot \prod_{i=\nu+1}^{k} (i-\nu)} a_{2\nu}(-\nu)$$

Choose $a_{2\nu}(-\nu) = a_0(\nu)$, then

$$a_{2k}(-\nu) = \left(\frac{1}{2}\right)^{2k-2\nu} \frac{(-1)^{k-\nu}}{\prod_{i=1}^{k-\nu}(i+\nu)\cdot(k-\nu)!} a_0(\nu) = a_{2k-2\nu}(\nu)$$

$$\begin{aligned} y_2(x) &= x^{-\nu}\sum_{k=0}^{\infty} a_k(-\nu)x^k = x^{-\nu}\sum_{k=\nu}^{\infty} a_{2k}(-\nu)x^{2k} \\ &= x^{-\nu}\sum_{k=0}^{\infty} a_{2k+2\nu}(-\nu)x^{2k+2\nu} = \sum_{k=0}^{\infty} a_{2k}(\nu)x^{2k+\nu} \\ &= y_1(x) \end{aligned}$$

Fail to find an independent solution!

## Reduction of Order

Set $y_2(x) = c(x) \cdot J_\nu(x)$, then

$$
\begin{aligned}
& x^2 y_2'' + x y_2' + (x^2 - \nu^2) y_2 = 0 \\
\Rightarrow & x^2 \left( c''(x) J_\nu(x) + 2c'(x) J_\nu'(x) + c(x) J_\nu''(x) \right) \\
& \quad + x \left( c'(x) J_\nu(x) + c(x) J_\nu(x) \right) + (x^2 - \nu^2) c(x) \cdot J_\nu(x) = 0 \\
\Rightarrow & x^2 J_\nu(x) c''(x) + (2x^2 J_\nu'(x) + x J_\nu(x)) c'(x) = 0 \\
\Rightarrow & \ln |c'(x)| = (-2 \ln |J_\nu(x)| - \ln |x|) \\
\Rightarrow & c'(x) = \frac{1}{x \cdot J_\nu^2(x)} \\
\Rightarrow & c(x) = \int \frac{dx}{x \cdot J_\nu^2(x)}
\end{aligned}
$$

$$((k+r+2)^2-\nu^2)a_{k+2}=-a_k \quad (k\in 2\mathbb{N}+1)$$

$$\Rightarrow a_k=-\frac{1}{(k+r-\nu)(k+r+\nu)}a_{k-2}=\frac{(-1)^{k/2}}{\prod_{i=1}^{k/2}(2i+r-\nu)(2i+r+\nu)}a_0$$

$$\Rightarrow \frac{a_k'(r)}{a_k(r)}=\frac{d\ln|a_k(r)|}{dr}$$

$$=\frac{d}{dr}(\ln|a_0(r)|-\sum_{i=1}^{k/2}\ln|2i+r+\nu|-\sum_{i=1}^{k/2}\ln|2i+r-\nu|)$$

$$=\frac{a_0'(r)}{a_0(r)}-\sum_{i=1}^{k/2}\frac{1}{2i+r+\nu}-\sum_{i=1}^{k/2}\frac{1}{2i+r-\nu}$$

$$\Rightarrow a_k'(-\nu)=(\frac{a_0'(-\nu)}{a_0(-\nu)}-\sum_{i=1}^{k/2}\frac{1}{2i}-\sum_{i=1}^{k/2}\frac{1}{2i-2\nu})\frac{(-1)^{k/2}}{\prod_{i=1}^{k/2}2i(2i-2\nu)}a_0(-\nu)$$

$$
\begin{aligned}
&c \cdot y_1(x)\ln x + x^{-\nu}\sum_{k=0}^{\infty} a_k'(-\nu)x^k \\
=&c \cdot J_\nu(x)\ln x - a_0'(-\nu)J_{-\nu}(x) \\
&-\sum_{k=0}^{\infty} a_0(-\nu)\frac{(-1)^{k/2}(\sum_{i=1}^{k/2}\frac{1}{2i}+\sum_{i=1}^{k/2}\frac{1}{2i-2\nu})}{\prod_{i=1}^{k/2}2i(2i-2\nu)}\left(\frac{x}{2}\right)^{2k-\nu}
\end{aligned}
$$

## Bessel function of the second kind

$$
Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}
$$

As $\nu \to 0$, $J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x) \to 0, \sin(\nu\pi) \to 0$, so we can use l'Hospital's rule,

$$\begin{aligned}
Y_n(x) =& \lim_{\nu\to n} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \\
=& \lim_{\nu\to n} \frac{J_\nu(x)(-\pi\sin(\nu\pi)) + \frac{dJ_\nu(x)}{d\nu}\cos(\nu\pi) - \frac{dJ_{-\nu}(x)}{d\nu}}{\pi\cos(\nu\pi)} \\
=& \lim_{\nu\to n} \frac{J_\nu(x)(-\pi\sin(\nu\pi)) + \frac{dJ_t(x)}{dt}|_{t=\nu}\cos(\nu\pi) + \frac{dJ_t(x)}{dt}|_{t=-\nu}}{\pi\cos(\nu\pi)} \\
=& \frac{1}{\pi}\left(\frac{dJ_t(x)}{dt}|_{t=n} + (-1)^n\frac{dJ_t(x)}{dt}|_{t=-n}\right) \\
=& \frac{2}{\pi}\frac{dJ_\nu(x)}{d\nu}|_{\nu=0}
\end{aligned}$$

So $Y_0(x) = \dfrac{2}{\pi}\dfrac{dJ_\nu(x)}{d\nu}|_{\nu=0}$.

$$\frac{dJ_\nu(x)}{d\nu}\Big|_{\nu=n} = \frac{d}{d\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\Big|_{\nu=n}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(-\Gamma'(k+\nu+1))}{k!(\Gamma(k+\nu+1))^2}\left(\frac{x}{2}\right)^{2k+\nu}\Big|_{\nu=n}$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\ln(\frac{x}{2})\Big|_{\nu=n}$$

$$=-\sum_{k=0}^{\infty}\frac{(-1)^k(\psi(k+\nu+1))}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\Big|_{\nu=n}+J_n(x)\ln(\frac{x}{2})$$

$$=-\sum_{k=0}^{\infty}\frac{(-1)^k(\psi(k+n+1))}{k!\Gamma(k+n+1)}\left(\frac{x}{2}\right)^{2k+n}+J_n(x)\ln(\frac{x}{2})$$

$$=-\sum_{k=0}^{\infty}\frac{(-1)^k(-\gamma+\sum_{m=1}^{n+k}\frac{1}{m})}{k!\Gamma(k+n+1)}\left(\frac{x}{2}\right)^{2k+n}+J_n(x)\ln(\frac{x}{2})$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k \sum_{m=1}^{n+k} \frac{1}{m}}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n} + J_n(x) \ln(\frac{x}{2}) + \gamma J_n(x)$$

$$= J_n(x)\left(\ln\left(\frac{x}{2}\right) + \gamma\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{m=1}^{n+k} \frac{1}{m}}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

Similarily,

$$\left.\frac{dJ_\nu(x)}{d\nu}\right|_{\nu=-n} = J_{-n}(x)\left(\ln\left(\frac{x}{2}\right) + \gamma\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{m=1}^{k} \frac{1}{m}}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$- (-1)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{x}\right)^{2k-n}$$

$$
\begin{aligned}
Y_n(x) =& \frac{2}{\pi} J_n(x) \left( \ln\left(\frac{x}{2}\right) + \gamma \right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{x}\right)^{2k-n} \\
& - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m} \right)}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}
\end{aligned}
$$

## Solution to Bessel Equation

The Bessel equation of order $\nu \geqslant 0$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

1. $\nu \notin \mathbb{N}$

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

2. $\nu \in \mathbb{N}$

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$\nu = \frac{2n+1}{2}, n \in \mathbb{N}$$

$$J_{1/2} = \sqrt{\frac{2}{\pi x}} \sin x, J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$$

$$\frac{d}{dx}(x^{-\nu} J_\nu(x)) = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx}(x^{\nu} J_\nu(x)) = x^{\nu} J_{\nu-1}(x)$$