VV286 RC4

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November 11, 2017

Complex Analysis

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, or holomorphic, at $z \in \mathbb{C}$ if

$$f'(z) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

exists.

We say that a function is holomorphic on an open set $\Omega \subset \mathbb{C}$ if it is holomorphic at every $z \in \Omega$. A function that is holomorphic on $\mathbb{C}$ is called ***entire***.

Three Main Properties

1. A holomorphic function is automatically infinitely often differentiable (***Cauchy Integral Formulas P331/P334***)
2. A holomorphic function is automatically analytic (has a power series expansion)(***P337***);
3. Any closed curve integral of a holomorphic function is vanishes.(***Cauchy's Theorem***)

The Cauchy-Riemann Differential Equations

For a complex function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x+iy) = u(x,y) + iv(x,y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$, if f is complex differentiable, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Suppose that the partial derivatives of u and v exist, are continuous and satisfy the Cauchy-Riemann equations. Then f is holomorphic.

Define differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big)$$

For holomorphic function,

$$f'(z) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}, \quad \frac{\partial f}{\partial \overline{z}} = 0$$

If f is a holomorphic function given by $f(x+iy) = u(x,y) + v(x,y)i$, then u and v are harmonic, i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Relation to Vector Field

For a vector field $F : \Omega \to \mathbb{R}^2$, if it's divergence and rotation are zero, i.e.

$$\exists U : \mathbb{R}^2 \to \mathbb{R}, F = \nabla U, \quad \text{i.e. } F_1 = \frac{\partial U}{\partial x}, F_2 = \frac{\partial U}{\partial y}$$

$$\exists V : \mathbb{R}^2 \to \mathbb{R}, \; F_1 = \frac{\partial V}{\partial y}, F_2 = -\frac{\partial V}{\partial x}$$

Then for function $G = U(x, y) + iV(x, y)$,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

it's holomorphic.

Sets in the Complex Plane

1. A set $\Omega \subset \mathbb{C}$ is called open if for every $z \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\} \subset \Omega$. A set is called closed if its complement is open.
2. A set $\Omega \subset \mathbb{C}$ is called bounded if $\Omega \subset B_R(0)$ for some $R > 0$.
3. A set $K \subset \mathbb{C}$ is called compact if every sequence in K has a subsequence that converges in K. A set $K \in \mathbb{C}$ is compact if and only if it is closed and bounded.
4. An open (closed) set $\Omega \subset \mathbb{C}$ is called disconnected if there exist two open (closed) sets $\Omega_1, \Omega_2 \subset \mathbb{C}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2$. If Ω is not disconnected, Ω is called connected. A set $\Omega \subset \mathbb{C}$ is conneted if and only if for any two points in Ω there exists a curve joining them.
5. A ***region*** or ***domain*** in $\mathbb{C}$ is an open and conneted set.

We define the diameter of a set $\Omega \subset \mathbb{C}$ by

$$\operatorname{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|$$

If (Ω_n) is a sequence of non-empty compact sets such that $\Omega_{n+1} \subset \Omega_n$ for $n \in \mathbb{N}$ and diam $\Omega_n \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Primitive

Let $\Omega \subset \mathbb{C}$ be an open set, $f : \Omega \to \mathbb{C}$. A primitive for f is a holomorphic function $F : \Omega \to \mathbb{C}$ such that $f(z) = F'(z)$ for all $z \in \Omega$.

If a continuous function f has a primitive F in Ω, and $\mathcal{C}^*$ is a curve in Ω that begins at w_1 and ends at w_2, then

$$\int_{\mathcal{C}^*} f(z)dz = F(w_2) - F(w_1)$$

Especially, if $\mathcal{C}^*$ is a closed curve,

$$\int_{\mathcal{C}^*} f(z)dz = 0$$

If f is holomorphic in a region Ω and $f' = 0$, then f is constant.

Integrals along Complex Curves

Let $\Omega \subset \mathbb{C}$ be an open set, f holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve. We then define the integral of f along $\mathcal{C}^*$ by

$$\int_{\mathcal{C}^*} f(z)dz = \int_I f(\gamma(t)) \cdot \gamma'(t)dt$$

where $\gamma : I \to \mathcal{C}^*$ is a parametrization of the parameterized curve $\mathcal{C}^*$.

Curve Length

$$\ell(\mathcal{C}) = \left| \int_{\mathcal{C}} dz \right|$$

$$\int_{-\mathcal{C}^*} f(z)dz = -\int_{\mathcal{C}^*} f(z)dz, \qquad \left| \int_{\mathcal{C}^*} f(z)dz \right| \leqslant \ell(\mathcal{C}) \cdot \sup_{z \in \mathcal{C}} |f(z)|$$

Cauchy's Theorem

If f is holomorphic in an open (simple) connected set Ω, and $\mathcal{C}^*$ is a closed curve in Ω,

$$\int_{\mathcal{C}^*} f(z)dz = 0$$

Singularities

Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Then f is said to have a point singularity or isolated singularity at z_0.

1. The singularity is said to be removable if there exists an analytic continuation $\tilde{f} : \Omega \to \mathbb{C}$. (such $\tilde{f}$ is unique)
2. The singularity is said to be a ***pole*** if $g = 1/f$ is holomorphic on $\Omega \setminus \{z_0\}$ and has a removable singularity at z_0 such that the analytic continuation $\tilde{g}$ of g satisfies $\tilde{g}(z_0) = 0$.
3. The singularity is said to be essential if it is neither removable nor a pole.

How to judge?

Removable Singularity

Whether $\lim_{z \to z_0} f$ exists.

e.g.

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

So $f(z) = \dfrac{\sin z}{z}$ has removable singularity at $z = 0$.

Pole (Informal way)

$f(z) = \dfrac{g(z)}{h(z)}$, then all z_0 such that $h(z_0) = 0$ may be a pole.

(All z_0 such that $g(z_0) = 0$ may be a zero.)

Multiplicity of Poles

If $f : \Omega \mapsto \mathbb{C}$ has a pole at $z_0 \in \Omega$, then in a neighborhood U of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(z) = (z - z_0)^{-n} h(z) \quad \text{for all } z \in U$$

The integer n is called the multiplicity or order of the pole of f . If $n = 1$, we say that the pole is ***simple***.

$$
\begin{aligned}
f(z) &= (z-z_0)^{-n}\sum_{m=0}^{\infty} b_m (z-z_0)^m \\
&= \frac{b_0}{(z-z_0)^n} + \frac{b_1}{(z-z_0)^{n-1}} + \cdots + \frac{b_{n-1}}{z-z_0} + \sum_{m=n}^{\infty} b_m (z-z_0)^{m-n} \\
&= \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}}_{\text{Principle part}} + \sum_{m=0}^{\infty} a_0 (z-z_0)^m
\end{aligned}
$$

Residue

Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ have a pole of order n at z_0. Then

$$a_{-1} = \operatorname{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z)\right)$$

Residue Theorem

Suppose that f is holomorphic in an open set containing a (positively oriented) toy contour $\mathcal{C}$ and its interior, except for ***poles*** at the points $z_1, \cdots, z_N$ inside $\mathcal{C}$. Then

$$\int_{\mathcal{C}} f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f$$

Complex Logarithm

On any simply connected open set Ω, $(1 \in \Omega, 0 \notin \Omega)$, set

$$\ln 1 = 0, \quad \ln z = \int_{\mathcal{C}} \frac{dz}{z}$$

where $\mathcal{C} \subset \Omega$ is any simple curve joining $1 \in \Omega$ to $z \in \Omega$.

1. $\ln(re^{i\phi}) = \ln r + \varphi i, (r > 0, -\pi < \phi < \pi)$
2. $\ln(re^{i\phi}) = \ln r + \varphi i, (r > 0, 0 < \phi < 2\pi)$

Complex Powers

$$z^\alpha = e^{\alpha \ln z}, \quad \alpha \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}^0_-$$

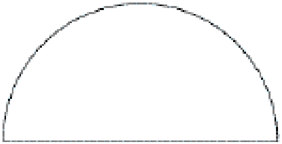
Complex Roots

$$\sqrt[n]{\alpha} = z^{1/n}$$

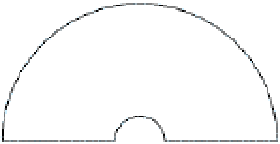
Evaluation of Real Integrals

1. Extend the real domain to complex domain.
   1.1 Usually you only need to change $x \in \mathbb{R}$ to $z \in \mathbb{C}$
   1.2 For $\sin x, \cos x$, do integral for e^{iz}
2. Find poles for the function $f(z)$
3. Decide the countour and the branch if needed.
4. Calculate the residue for poles in the countour.
   4.1 During an exam, you may calculate residue for all poles if you cannot decide the countour at first.
5. Apply residue theorem or Cauchy's theorem.
6. Save the integral part we need and solve other parts one by one.
   6.1 You may need to use Jordan's Lemma or do similar operation

Countour 1—-Semi-circle



Semicircle



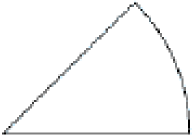
Indented semicircle

Most common ones.
You have use them to solve

$$\int_0^\infty \frac{\sin x}{x}dx, \quad \int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}dx, \quad \int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx, \quad \int_{-\infty}^\infty \frac{dx}{1+x^4}$$

$$\int_0^\infty \frac{x\sin x}{(x^2+4)^2}dx, \quad \int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}}dx, \quad \int_0^\infty \frac{1-\cos x}{x^2}dx$$

Countour 2—-Sector

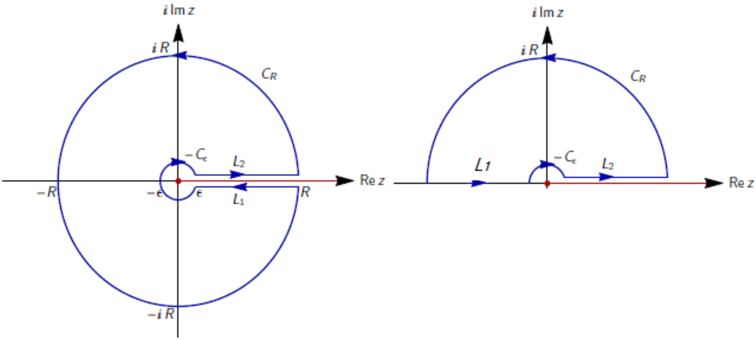


Sector

Similar to Semi-circle.
May be useful for integral containing $\sin(x^n)$, $\cos(x^n)$ (choose central angle=$\frac{\pi}{2n}$)
You have use it to solve

$$\int_0^{\infty} \sin^2 x dx, \quad \int_0^{\infty} \cos^2 x dx$$

Countour 3



Used for integral containing $\sqrt{x}$, $\ln x$ (For these two countours, the branch we choose is $\mathbb{C} \setminus \mathbb{R}^0_-$, so $\phi \in (0, 2\pi)$)

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx, \quad \int_0^\infty \frac{\ln x}{x^2 + a^2} dx$$

Residue Calculus for Functions with Branch Points

Let P and Q be polynomials of degree m and n, respectively, where $n \geqslant m+2$. If $Q(x) \neq 0$ for $x > 0$, if Q has a zero of order at most 1 at the origin and if

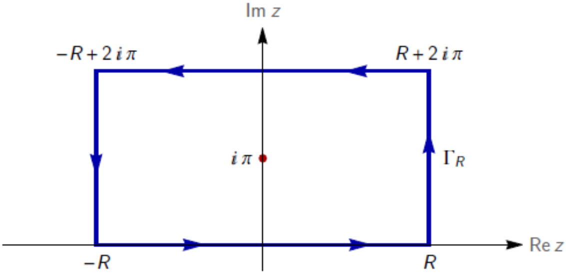
$$f(z) = \frac{z^\alpha P(z)}{Q(z)}, \quad 0 < \alpha < 1$$

then

$$\text{p.v.} \int_0^\infty \frac{x^\alpha P(x)}{Q(x)} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{j=1}^{k} \operatorname{res}_{z_j} f$$

This theorem is obtained by using the countour in the left on last slide. Pay attention to the branch. Also pay attention to its requirement.

Countour 4



$$\int_0^\infty \frac{e^{ax}}{1+e^x}dx$$

Jordan's Lemma

Assume that for some $R_0 > 0$ the function $g : \mathbb{C} \setminus B_{R_0}(0) \to \mathbb{C}$ is holomorphic. Let

$$f(z) = e^{iaz} g(z), \quad \text{for some } a > 0$$

Let

$$C_R = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leqslant \theta \leqslant \pi\}$$

be a semi-circle segement in the upper half-plane and assume that

$$\sup_{0 \leqslant \theta \leqslant \pi} |g(Re^{i\theta})| \xrightarrow{R \to \infty} 0$$

Then

$$\lim_{R \to \infty} \int_{C'_R} f(z) dz = 0$$

$(C'_R \subset C_R)$

The following theorem is used to prove Cauchy's Theorem.

Goursat's Theorem

Let $\Omega \subset \mathbb{C}$ be an open and f ***holomorphic*** on Ω. Let $T \subset \Omega$ be a ***triangle*** whose interior is also contained in Ω. Then

$$\oint_T f(z)dz = 0$$

and therefore, for a ***rectangle*** R,

$$\oint_R f(z)dz = 0$$

Morera's Theorem

Let $\Omega \subset \mathbb{C}$ be an open, connected set and let $f : \Omega \to \mathbb{C}$ be continuous. Suppose that

$$\oint_T f(z)dz = 0 \quad \text{for any triangle } T \text{ wholly contained in } \Omega$$

1. f has a primitive on Ω
2. (f is holomorphic.)

For the proof of this theorem, it's similar to the one in below.

Local Existence of Primitives

A holomorphic function in an open disc has a primitive in that disc. (P316)

Proof

$\forall z \in \Omega$, choose some z_0 such that $z \in B_\delta(z_0)$ for some $\delta > 0$. Then define the function

$$F : B_\delta(z_0) \to \mathbb{C}, \quad F(z) = \int_{\mathcal{C}(z_0,z)} f(\zeta)d\zeta$$

where $\mathcal{C}(z_0, z)$ is parametrized by

$$\gamma : [0, 1] \to \mathbb{C}, \quad \gamma(t) := (1 - t)z_0 + tz$$

Suppose that $z + h \in B_\delta(z_0)$ for some h. Any integral along the triangle with vertices z_0, z and $z + h$ vanishes, so that

$$\begin{aligned} F(z+h) - F(z) &= \int_{\mathcal{C}(z,z+h)} f(\zeta)d\zeta = \int_0^1 f((1-t)z + t(z+h)) \cdot h dt \\ &= h \int_0^1 f(z + th)dt \end{aligned}$$

Proof (continued)

Since f is continuous, $f(z+th) = f(z) + \psi(h)$, where $\psi(h) \to 0$ as $h \to 0$, so

$$F(z+h) = F(z) + hf(z) + o(h)$$

So $F'(z) = f(z)$. A holomorphic function is automatically infinitely often differentiable. So $f(z)$ is holomorphic.

Cauchy Integral Formulas

Suppose f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$. If D is an open disc whose closure is contained in Ω then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $C = \partial D$ is the (positively oriented) boundary circle of D. (It holds for all toy contours.)

If f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$, then f has infinitely many complex derivatives in Ω. Moreover, if D is an open disc whose closure is contained in Ω,

$$f^{(n)} = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

where $C = \partial D$ is the (positively oriented) boundary circle of D.

Holomorphic Functions are Analytic

Suppose f is a holomorphic function in an open set Ω. If D is an open disc centered at z_0 whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$ and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N}$$

Uniqueness of Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be a region and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions. Suppose that $S \subset \Omega$ has an accumulation point that is contained in Ω and that

$$f(z) = g(z) \quad \text{for all } z \in S$$

Then $f(z) = g(z)$ for all $z \in \Omega$.

Analytic Continuation

Let $M \subset \mathbb{C}$ be a any set and $f : M \to \mathbb{C}$ any function. Let Ω be a region with $M \subset \Omega$ and $g : \Omega \to \mathbb{C}$ a holomorphic function such that $g(z) = f(z)$ for $z \in M$. Then g is called an analytic continuation of f to Ω.