VV286 Final Review

YUAN Ziling
JIANG Yicheng

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Series Methods for Second-Order Equations

Set $x(t) = \sum_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k$$

Insert to find relation among coefficients a_k, a_{k+1}, a_{k+2}. If we can find two independent solutions (usually no t before x'', the easiest case), then we have done; else

The Method of Frobenius

Set $x(t) = t^r \sum_{k=0}^{\infty} a_k t^k$, then

$$x'(t) = \sum_{k=0}^{\infty} (k+r) a_k t^{k+r-1}$$

$$x''(t) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k t^{k+r-2}$$

Insert to solve r.

You can try to remember

$$t^2x'' + t(tp(t))x' + t^2q(t)x = 0$$

where $tp(t)$, $t^2q(t)$ are analytic (regular singular point at $t_0 = 0$)

$$tp(t) = \sum_{j=0}^{\infty} p_j t^j, \quad t^2q(t) = \sum_{j=0}^{\infty} q_j t^j$$

$$F(x) := x(x-1) + p_0x + q_0$$

For $a_0 \neq 0$ we have the relation for coefficients

$$F(r) = 0 \qquad ,$$

$$a_mF(r+m) = \sum_{k=0}^{m-1} (q_{m-k} + (r+k)pm - k)a_k$$

1. Two distinct real roots r_1 and r_2, $r_1 - r_2 \notin \mathbb{Z}$

$$x_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n, \quad x_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n$$

2. Two distinct real roots r_1 and r_2, $r_1 - r_2 \in \mathbb{Z}$

$$\begin{aligned} & x_2(t) \\ =& \frac{d}{dr}\left(t^r \sum_{n=0}^{\infty} a_n(r) t^n\right)\Bigg|_{r=r_2} = t^r \ln t \sum_{n=0}^{\infty} a_n(r) t^n + t^r \sum_{n=0}^{\infty} a_n'(r) t^n \Bigg|_{r=r_2} \\ =& c \cdot x_1(t) \ln t + t^{r_2} \sum_{n=0}^{\infty} a_n'(r_2) t^n \end{aligned}$$

Especially $c = 1$ for $r_1 = r_2$.

3. Complex root

$$x_1(t) = \mathrm{Re}\left(t^{r_1}\sum_{n=0}^{\infty} a_n t^n\right), \quad x_2(t) = \mathrm{Im}\left(t^{r_2}\sum_{n=0}^{\infty} b_n t^n\right)$$

Singular Point

The equation

$$x'' + p(t)x' + q(t)x = 0$$

is said to have a ***regular singular point*** at t_0 if the functions $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic in a neighborhood of t_0. A singular point which is not regular is said to be ***irregular*** .

Bessel Equation

The Bessel equation of order $\nu \geqslant 0$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

Solution to Bessel Equation

The Bessel equation of order $\nu \geqslant 0$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

1. $\nu \notin \mathbb{N}$

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

2. $\nu \in \mathbb{N}$

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$\nu = \frac{2n+1}{2}, n \in \mathbb{N}$$

$$J_{1/2} = \sqrt{\frac{2}{\pi x}} \sin x, J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$$

$$\frac{d}{dx}(x^{-\nu} J_\nu(x)) = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx}(x^{\nu} J_\nu(x)) = x^{\nu} J_{\nu-1}(x)$$

Fourier Series

The orthonormal system basis in $L^2([-1,1])$

$$\mathcal{B}_{[-1,1]} = \left\{ \frac{1}{\sqrt{2}}, \cos(\pi n x), \sin(\pi n x) \right\}_{n=1}^{\infty}$$

The orthonormal system basis in $L^2([a,b])$

$$\mathcal{B}_{[a,b]} = \left\{ \tilde{e}_n(x) : \tilde{e}_n(x) = \sqrt{\frac{2}{b-a}} \cdot e_n\left(\frac{2}{b-a}\left(x - \frac{b+a}{2} \right) \right) \right\}_{n=1}^{\infty}$$

where $e_n(x) \in \mathcal{B}_{[-1,1]}$

The basis in $L^2([0, L])$

1. The Fourier-Euler Basis:

$$\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

2. The Fourier-Cosine Basis:

$$\mathcal{B}_2 = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

3. The Fourier-Sine Basis:

$$\mathcal{B}_3 = \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

The basis in $L^2([-\pi,\pi])$

The Fourier-Euler Basis:

$$\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \right\}_{n=1}^{\infty}$$

Complex Fourier-Euler Basis

Complex Fourier-Euler basis in $L^2([-\pi,\pi])$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty}$$

Complex Fourier-Euler basis in $L^2([-L,L])$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$$

Complex Fourier series for periodic function

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}x}$$

$$a_k = \frac{1}{T}\int_0^T f(x) e^{-jk\frac{2\pi}{T}x} dx$$

The Fourier Expansion

For a function $f \in L^2([a,b])$,

$$\lim_{N\to\infty} S_N(x) = \sum_{e_n \in \mathcal{B}_{[a,b]}} \langle e_n(x), f(x) \rangle e_n(x)$$
$$= \sum_{e_n \in \mathcal{B}_{[a,b]}} \int_a^b f(x)\overline{e_n(x)}dx \cdot e_n(x)$$

Pay attention to conjugate if you use complex Fourier basis

Dirichlet's rule

Let $f \in L^2([a, b])$ be piecewise continuously differentiable. Then

1. On any subinterval $[a', b'] \subset [a, b]$ with $a' > a, b' < b$ on which f is continuous the Fourier series converges uniformly towards f.
2. At any point $x \in [a, b]$, we have the pointwise limit

$$S_N(x) \xrightarrow{N\to\infty} \frac{\lim\limits_{y \nearrow x} f(y) + \lim\limits_{y \searrow x} f(y)}{2}$$

Partial Differential Equation

Example

Find the general solution of the Laplace equation $\triangle u = 0$ on the rectangle $\Omega = [0, a] \times [0, b]$ with boundary conditions

$$u(0, y) = 0, u(a, y) = 0, 0 < y < b$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, u(x, b) = g(x), 0 < x < a$$

where $g : [0, a] \to \mathbb{R}$ is a continuous function. Then find the solution if

$$g(x) = \begin{cases} x, 0 < x < a/2 \\ a - x, a/2 < x < a \end{cases}$$

Solution

We make a separation-of-variables ansatz $u(x,y) = X(x)Y(y)$, yielding

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow X''Y + XY'' = 0$$

and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

So

$$X'' = \lambda X, \, Y'' = -\lambda Y$$

with initial conditions and boundary conditions

$$X(0)Y(y) = 0, X(a)Y(y) = 0, 0 < y < b$$

$$X(x)Y'(0) = 0, X(x)Y(b) = g(x), 0 < x < a$$

To find non-trival solution, the condition can be changed to

$$X(0) = X(a) = Y'(0) = 0, X(x)Y(b) = g(x), 0 < x < a$$

For $X'' = \lambda X$, we use ansatz $X(x) = e^{\rho(\lambda)x}$, and therefore

$$\rho(\lambda)^2 = \lambda$$

1. $\lambda = 0$

$X(x) = c_0 + c_1 x$ to satisfy the initial conditions and boundary conditions, $X(x) = 0$

2. $\lambda > 0$

$$\rho(\lambda) = \pm\sqrt{\lambda} \Rightarrow X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Since $X(0) = X(a) = 0$,

$$c_1 + c_2 = 0, c_1 e^{\sqrt{\lambda}a} + c_2 e^{-\sqrt{\lambda}a} = 0 \Rightarrow c_1 = c_2 = 0$$

3. $\lambda < 0$

$$\rho(\lambda) = \pm\sqrt{-\lambda} i \Rightarrow X(x) = c_1 e^{i\sqrt{-\lambda}x} + c_2 e^{-i\sqrt{-\lambda}x}$$
$$= c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$$

Since $X(0) = X(a) = 0$,

$$c_2 = 0, c_1 \sin(\sqrt{-\lambda}a) + c_2 \cos(\sqrt{-\lambda}a) = 0 \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{a}$$

So

$$X_n(x) = c_n \sin(\frac{n\pi x}{a}), Y'' = \frac{n^2\pi^2}{a^2} Y$$

Then

$$Y_n(y) = d_n e^{n\pi y/a} + f_n e^{-n\pi y/a}$$

Considering $Y'(0) = 0$

$$d_n \cdot n\pi/a - f_n \cdot n\pi/a = 0 \Rightarrow d_n = f_n$$

So

$$u(x, y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} c_n d_n (e^{n\pi y/a} + e^{-n\pi y/a}) \sin(\frac{n\pi x}{a})$$

Expand $g(x)$ into Fourier sine series

$$\begin{aligned}
&\int_0^a g(x) \sin(n\pi x/a) dx \\
=&\int_0^{a/2} x \sin(n\pi x/a) dx + \int_{a/2}^{a} (a - x) \sin(n\pi x/a) dx \\
=&\frac{2a^2}{n^2\pi^2} \sin(n\pi/2)
\end{aligned}$$

So

$$g(x) = \sum_{n=1}^{\infty} \frac{2}{a} \left(\frac{2a^2}{n^2\pi^2} \sin(n\pi/2) \right) \sin(\frac{n\pi x}{a})$$

Since $u(x,b) = g(x)$

$$c_n d_n (e^{n\pi b/a} + e^{-n\pi b/a}) = \frac{4a}{n^2\pi^2} \sin(n\pi/2)$$

So

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4a \sin(n\pi/2)}{e^{n\pi b/a} + e^{-n\pi b/a}} (e^{n\pi y/a} + e^{-n\pi y/a}) \sin(\frac{n\pi x}{a})$$

Summary

- Use ansatz to find power series solution of non-constant coefficient 2nd-order ODE
  1. Normal one: $x(t) = \sum_{n=0}^{\infty} a_n t^n$
  2. Frobenius: $x(t) = t^r \sum_{n=0}^{\infty} a_n t^n$
     - 2.1 $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{N}$
     - 2.2 $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \in \mathbb{N}$:
       1)try;
       2)$\left.\frac{dx_1(t)}{dr}\right|_{r=r_2} \Rightarrow a_n'(r)$
     - 2.3 $r_1, r_2 \in \mathbb{C}$

- Change equation to Bessel Equation
  1. Use substitution

$$a_1 x^{b_1} y'' + a_2 x^{b_2} y' + a_3 x^{b_3} y = 0$$

$$z = g(x) = \beta x^{\gamma}, u(g(x)) = x^{\alpha} y(x)$$

$$\alpha = \frac{a_2}{2a_1} - \frac{1}{2}, \beta = \sqrt{\frac{a_3}{a_1}} \frac{2}{b_3 - b_1 + 2}, \gamma = \frac{b_3 - b_1 + 2}{2}$$

  2. If no information of order, calaulate

$$y'(x) = -\alpha x^{-\alpha-1} u(g(x)) + x^{-\alpha} \frac{du(z)}{dz}\bigg|_{z=g(x)} \frac{d(g(x))}{dx}$$

$$= -\alpha x^{-1} y(x) + x^{-\alpha} u'(z)|_{z=g(x)} g'(x)$$

$$y''(x) = \frac{d}{dx}(-\alpha x^{-1} y(x) + x^{-\alpha} u'(z)|_{z=g(x)} g'(x))$$

  3.

$$y(x) = c_1 x^{-\alpha} J_{\nu}(g(x)) + c_2 x^{-\alpha} J_{-\nu}(g(x))$$

- Evaluate series using Fouries series
  1. According to domain of given function, use proper basis for Fourier series
     1.1 $[0, L]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi nx}{L}\right) \right\}_{n=1}^{\infty}$
     1.2 $[-\pi, \pi]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}$
     1.3 $[-L, L]$: $\mathcal{B} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$
  2. Calculate the coefficients (Pay attention to constant!)
  3. Choose appropriate value for x and use ***Dirichlet's rule***

- Solve PDE (usually only two variables)
  1. Use separation of variables

$$u(x_1, \cdots, x_m) = u_1(x_1) \cdot u_2(x_2) \cdots u_m(x_m)$$

  2. Change the boundary condition
  3. According to boundary condition, choose one equation $Lu_1 = \lambda u_1$ to solve. (find proper λ)
  4. Solve other equations for each λ

$$(u_i)_n(x_i) = (c_i)_n \cdot f(\rho(\lambda)_n x_i)$$

  5. Let the whole solution satisfy boundary condition

$$u(x_1, \cdots, x_m) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{m} (u_i)_n(x_i) \right)$$

     5.1 Usually expand boundary conditions to Fourier series and determine the coefficients

Final Exam Schedule

Time:	Thursday 10:00-11:40
Exam room:	Dong Zhong Yuan 2-105

Good luck!