# VV286
# Honors Mathematics IV
# Ordinary Differential Equations
# Assignment 7

*Jiang Yicheng*
*515370910224*

November 17, 2016

# Exercise 7.1

$\forall k \in \mathbb{N}$,

$$\int_{kT}^{kT+T} f(t)e^{-pt}dt \xlongequal{s=t-kT} \int_0^T f(s+kT)e^{-p(s+kT)}ds = e^{-pkT}\int_0^T f(s)e^{-ps}ds$$

Then

$$(\mathcal{L}f)(p) = \int_0^\infty f(t)e^{-pt}dt = \sum_{k=0}^{\infty} e^{-pkT}\int_0^T f(s)e^{-ps}ds = \int_0^T f(s)e^{-ps}ds \lim_{k\to\infty}\frac{1-e^{-pT(k+1)}}{1-e^{-pT}}$$
$$\xlongequal{p>0} \frac{1}{1-e^{-pT}}\int_0^T f(s)e^{-ps}ds$$

So $(\mathcal{L}f)(p) = \dfrac{1}{1-e^{-pT}}\int_0^T f(s)e^{-ps}ds$.

Then for $f(t) = at, t \in [0,1]$ with period $T = 1$,

$$\begin{aligned}(\mathcal{L}f)(p) &= \frac{1}{1-e^{-pT}}\int_0^T f(t)e^{-pt}dt = \frac{1}{1-e^{-p}}\int_0^1 ate^{-pt}dt \\ &= \frac{1}{1-e^{-p}}a\Big(-\frac{1}{p}(t+\frac{1}{p})e^{-pt}|_0^1\Big) \\ &= \frac{1}{1-e^{-p}}a\Big(-\frac{1}{p}(1+\frac{1}{p})e^{-p}+\frac{1}{p}(0+\frac{1}{p})\Big) \\ &= \frac{a(1-(p+1)e^{-p})}{p^2(1-e^{-p})}\end{aligned}$$

So the Laplace transform of the function

$$f(t) = at, a\in\mathbb{R}, t\in[0,1]$$

is $(\mathcal{L}f)(p) = \dfrac{a(1-(p+1)e^{-p})}{p^2(1-e^{-p})}$.

# Exercise 7.2

## i)

For $F(p) = \dfrac{1}{p(e^p+1)}$,

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{1}{p(e^p+1)}dp$$

Since

$$\begin{aligned}\sup_{\pi/2\leqslant\theta\leqslant3\pi/2}|F(Re^{i\theta})| &= \sup_{\pi/2\leqslant\theta\leqslant3\pi/2}\Big|\frac{1}{Re^{i\theta}(e^{Re^{i\theta}}+1)}\Big| \\ &= \frac{1}{R}\sup_{\pi/2\leqslant\theta\leqslant3\pi/2}\Big|\frac{1}{e^{R\cos\theta}(\cos(R\sin\theta)+i\sin(R\sin\theta))+1}\Big| \\ &= \frac{1}{R}\sup_{\pi/2\leqslant\theta\leqslant3\pi/2}\frac{1}{\sqrt{1+e^{2R\cos\theta}+2e^{R\cos\theta}\cos(R\sin\theta)}} \\ &\xlongequal{R\to\infty} O(\frac{1}{R})\end{aligned}$$

we can apply Jordan's Lemma and obtain that

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{1}{p(e^p+1)}dp = \sum_{k=-\infty}^{\infty} res_{p_k}e^{pt}F(p)$$

where $p_k$ is the pole of $e^{pt}F(p)$ with $p_0 = 0, p_k = i(2k+1)\pi$ and

$$\begin{aligned}
res_{p_k}e^{pt}F(p) &= \lim_{p\to i(2k+1)\pi}(p - i(2k+1)\pi)\frac{e^{pt}}{p(e^p+1)} = -\frac{e^{i(2k+1)\pi t}}{i(2k+1)\pi} \\
&= \frac{\cos((2k+1)\pi t) + i\sin((2k+1)\pi t)}{(2k+1)\pi}i \\
res_{p_0}e^{pt}F(p) &= \lim_{p\to 0}(p-0)\frac{e^{pt}}{p(e^p+1)} = \frac{1}{2}
\end{aligned}$$

So

$$\begin{aligned}
(\mathcal{M}F)(t) &= Re(\sum_{k=-\infty}^{\infty} res_{p_k}e^{pt}F(p)) \\
&= \frac{1}{2} - \sum_{k=-\infty}^{\infty}\frac{\sin((2k+1)\pi t)}{(2k+1)\pi} = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi t)}{2k+1} \\
&= \frac{1}{2} - \frac{2}{\pi}(\sin(\pi t) + \frac{\sin(3\pi t)}{3} + \frac{\sin(5\pi t)}{5} + \cdots)
\end{aligned}$$

So

$$(\mathcal{L}^{-1}F)(t) = \frac{1}{2} - \frac{2}{\pi}(\sin(\pi t) + \frac{\sin(3\pi t)}{3} + \frac{\sin(5\pi t)}{5} + \cdots)$$

## ii)

For $F(p) = \dfrac{1}{p\cosh p}$,

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{1}{p\cosh p}dp$$

Since

$$\sup_{\pi/2\leqslant\theta\leqslant 3\pi/2}|F(Re^{i\theta})| = \sup_{\pi/2\leqslant\theta\leqslant 3\pi/2}|\frac{1}{Re^{i\theta}\cosh(Re^{i\theta})}| \xlongequal{R\to\infty} O(\frac{1}{R})$$

we can apply Jordan's Lemma and obtain that

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{1}{p\cosh p}dp = \sum_{k=-\infty}^{\infty} res_{p_k}e^{pt}F(p)$$

where $p_k$ is the pole of $e^{pt}F(p)$ with $p_0 = 0, p_k = i(2k+1)\pi/2$ and

$$\begin{aligned}
res_{p_k}e^{pt}F(p) &= \lim_{p\to i(2k+1)\pi/2}(p - i(2k+1)\pi/2)\frac{e^{pt}}{p\cosh p} = \frac{e^{i(2k+1)\pi t}}{i(2k+1)\pi/2\sinh(i(2k+1)\pi/2)} \\
&= \frac{2(-1)^{k+1}(\cos((2k+1)\pi/2t) + i\sin((2k+1)\pi/2t))}{(2k+1)\pi} \\
res_{p_0}e^{pt}F(p) &= \lim_{p\to 0}(p-0)\frac{e^{pt}}{p\cosh p} = \frac{1}{2}
\end{aligned}$$

So

$$
\begin{aligned}
(\mathcal{M}F)(t) &= Re(\sum_{k=-\infty}^{\infty} res_{p_k} e^{pt} F(p)) \\
&= \frac{1}{2} - 2\sum_{k=-\infty}^{\infty} (-1)^k \frac{\cos((2k+1)\pi/2t)}{(2k+1)\pi} = \frac{1}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi/2t)}{2k+1} \\
&= \frac{1}{2} - \frac{4}{\pi}(\cos(\pi/2t) - \frac{\cos(3\pi/2t)}{3} + \frac{\cos(5\pi/2t)}{5} + \cdots)
\end{aligned}
$$

So

$$
(\mathcal{L}^{-1}F)(t) = \frac{1}{2} - \frac{4}{\pi}(\cos(\frac{\pi}{2}t) - \frac{1}{3}\cos(\frac{3\pi}{2}t) + \cdots)
$$

# Exercise 7.3

For $f(t) = \begin{cases} 2, 0 \leqslant t \leqslant 1 \\ 0, 1 \leqslant t \leqslant 2 \end{cases}$ with period $T = 2$,

$$
\begin{aligned}
(\mathcal{L}f)(p) &= \frac{1}{1-e^{-pT}} \int_0^T f(t)e^{-pt}dt = \frac{1}{1-e^{-2p}}\Big(\int_0^1 2e^{-pt}dt + \int_1^2 0 \cdot e^{-pt}dt\Big) \\
&= \frac{1}{1-e^{-2p}}\Big(-\frac{2}{p}e^{-pt}|_0^1\Big) \\
&= \frac{1}{1-e^{-2p}}\Big(-\frac{2}{p}(e^{-p}-1)\Big) \\
&= \frac{2(1-e^{-p})}{p(1-e^{-2p})} = \frac{2}{p(1+e^{-p})} \\
&= \frac{(e^{p/2}+e^{-p/2}) + (e^{p/2}-e^{-p/2})}{p(e^{p/2}+e^{-p/2})} \\
&= \frac{1+\tanh(p/2)}{p}
\end{aligned}
$$

So the Laplace transform of the function is $(\mathcal{L}f)(p) = \dfrac{1+\tanh(p/2)}{p}$.

For $F(p) = \dfrac{1+\tanh(p/2)}{p}$,

$$
(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{1+\tanh(p/2)}{p}dp
$$

Since

$$
\sup_{\pi/2\leqslant\theta\leqslant3\pi/2} |F(Re^{i\theta})| = \sup_{\pi/2\leqslant\theta\leqslant3\pi/2} |\frac{1+\tanh(Re^{i\theta}/2)}{Re^{i\theta}}| \xlongequal{R\to\infty} O(\frac{1}{R})
$$

we can apply Jordan's Lemma and obtain that

$$
(\mathcal{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}\frac{2}{p(1+e^{-p})}dp = \sum_{k=-\infty}^{\infty} res_{p_k} e^{pt} F(p)
$$

where $p_k$ is the pole of $e^{pt}F(p)$ with $p_0 = 0, p_k = i(2k+1)\pi$ and

$$res_{p_k} e^{pt}F(p) = \lim_{p \to i(2k+1)\pi} (p - i(2k+1)\pi)\frac{2e^{pt}}{p(e^{-p}+1)} = \frac{2e^{i(2k+1)\pi t}}{i(2k+1)\pi}$$
$$= -2\frac{\cos((2k+1)\pi t) + i\sin((2k+1)\pi t)}{(2k+1)\pi} i$$
$$res_{p_0} e^{pt}F(p) = \lim_{p \to 0}(p-0)\frac{2e^{pt}}{p(e^{-p}+1)} = 1$$

So

$$(\mathcal{M}F)(t) = Re(\sum_{k=-\infty}^{\infty} res_{p_k} e^{pt}F(p))$$
$$= 1 + 2\sum_{k=-\infty}^{\infty} \frac{\sin((2k+1)\pi t)}{(2k+1)\pi}$$
$$= 1 + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi t)}{2k+1}$$

```
Plot[{f,
  1 + 4 / Pi * Sum[Sin[(2 * n + 1) * Pi * t] / (2 * n + 1), {n, 0, 0, 1}],
  1 + 4 / Pi * Sum[Sin[(2 * n + 1) * Pi * t] / (2 * n + 1), {n, 0, 2, 1}],
  1 + 4 / Pi * Sum[Sin[(2 * n + 1) * Pi * t] / (2 * n + 1), {n, 0, 4, 1}],
  1 + 4 / Pi * Sum[Sin[(2 * n + 1) * Pi * t] / (2 * n + 1), {n, 0, 9, 1}],
  1 + 4 / Pi * Sum[Sin[(2 * n + 1) * Pi * t] / (2 * n + 1), {n, 0, 19, 1}]}, {t, 0, 2},
 PlotStyle → {Purple, Red, Blue, Green, Black, Pink}]
```

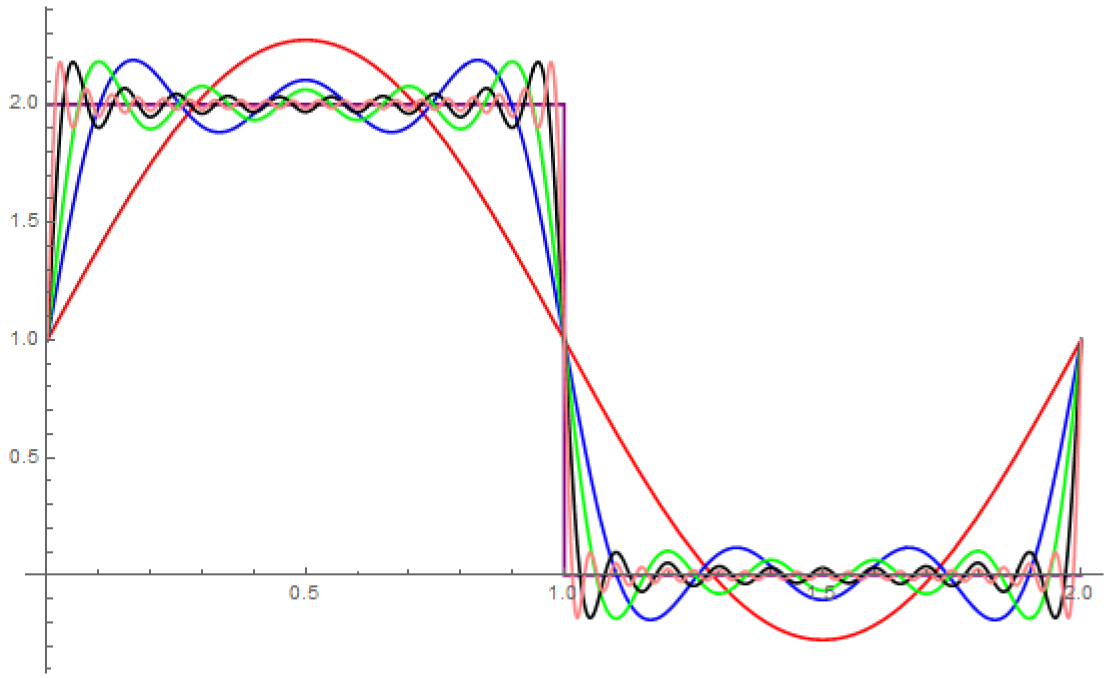


Figure 1: Function $f$ (purple) together with $n = 1$ (red), 3 (blue), 5 (green), 10 (black), 20 (pink) terms of the series $(\mathcal{M}F)(t) = 1 + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi t)}{2k+1}$.

# Exercise 7.4

## i) $y'' + y = \sin t + \delta(t-\pi), y(0) = 0, y'(0) = 0$

Set $Y(p) = (\mathcal{L}y)(p)$, then

$$(\mathcal{L}y')(p) = p \cdot (\mathcal{L}y)(p) - y(0) = pY(p)$$
$$(\mathcal{L}y'')(p) = p \cdot (\mathcal{L}y')(p) - y'(0) = p^2 Y(p)$$

and

$$\Big(\mathcal{L}(\sin t + \delta(t-\pi))\Big)(p) = \frac{1}{p^2+1} + e^{-\pi p}$$

So apply Laplace transform to the equation and we get

$$\begin{aligned}(p^2+1)Y(p) &= \frac{1}{p^2+1} + e^{-\pi p}\\ \Rightarrow Y(p) &= \frac{1}{p^2+1}\frac{1}{p^2+1} + e^{-\pi p}\frac{1}{p^2+1}\\ \Rightarrow Y(p) &= \mathcal{L}(\sin t)\mathcal{L}(\sin t) + e^{-\pi p}(\mathcal{L}(\sin t))\end{aligned}$$

Since

$$\begin{aligned}&\sin t * \sin t\\ =&\int_0^t \sin(t-s)\sin s ds = \frac{1}{2}\int_0^t \cos(t-2s) - \cos t ds\\ =&\frac{1}{2}(\frac{1}{2}\sin(2s-t) - s\cos t)\Big|_0^t\\ =&\frac{1}{2}(\sin t - t\cos t)\end{aligned}$$

Then

$$(\mathcal{L}^{-1}Y)(t) = \frac{1}{2}(\sin t - t\cos t) + \sin(t-\pi)H(t-\pi)$$

So the solution to the equation is $y(t) = \dfrac{1}{2}(\sin t - t\cos t) + \sin(t-\pi)H(t-\pi)$

## ii) $y'' + y' + y = 2\delta(t-1) - \delta(t-2), y(0) = 1, y'(0) = 0$

Set $Y(p) = (\mathcal{L}y)(p)$, then

$$(\mathcal{L}y')(p) = p \cdot (\mathcal{L}y)(p) - y(0) = pY(p) - 1$$
$$(\mathcal{L}y'')(p) = p \cdot (\mathcal{L}y')(p) - y'(0) = p^2 Y(p) - p$$

and

$$\Big(\mathcal{L}(2\delta(t-1) - \delta(t-2))\Big)(p) = 2e^{-p} - e^{-2p}$$

So apply Laplace transform to the equation and we get

$$(p^2+p+1)Y(p) = 2e^{-p} - e^{-2p} + p + 1 \Rightarrow Y(p) = \frac{2e^{-p}}{p^2+p+1} + \frac{p+1}{p^2+p+1}$$
$$\Rightarrow Y(p) = \frac{4}{\sqrt{3}}e^{-p}\frac{\frac{\sqrt{3}}{2}}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{p+\frac{1}{2}}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(p+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

Then

$$(\mathcal{L}^{-1}Y)(t) = \frac{4}{\sqrt{3}}(\mathcal{L}^{-1}e^{-p}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t) + e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$
$$= \frac{4}{\sqrt{3}}e^{-\frac{1}{2}(t-1)}\sin\frac{\sqrt{3}}{2}(t-1)H(t-1) + \frac{2e^{-\frac{1}{2}t}}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t + \frac{\pi}{3})$$

So the solution to the equation is

$$y(t) = \frac{4}{\sqrt{3}}e^{-\frac{1}{2}(t-1)}\sin\frac{\sqrt{3}}{2}(t-1)H(t-1) + \frac{2e^{-\frac{1}{2}t}}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t + \frac{\pi}{3})$$

# Exercise 7.5

**i)** $\frac{d^2y}{dt^2} + y = \sum_{j=0}^{\infty}\delta(t - j\pi), y(0) = y'(0) = 0$

Set $Y(p) = (\mathcal{L}y)(p)$, then

$$(\mathcal{L}y')(p) = p\cdot(\mathcal{L}y)(p) - y(0) = pY(p)$$
$$(\mathcal{L}y'')(p) = p\cdot(\mathcal{L}y')(p) - y'(0) = p^2Y(p)$$

and

$$\Big(\mathcal{L}(\sum_{j=0}^{\infty}\delta(t-j\pi))\Big)(p) = \sum_{j=0}^{\infty}e^{-j\pi p} = \frac{1}{1-e^{-\pi p}}, p > 0$$

So apply Laplace transform to the equation and we get

$$(p^2+1)Y(p) = \sum_{j=0}^{\infty}e^{-j\pi p} \Rightarrow Y(p) = \sum_{j=0}^{\infty}e^{-j\pi p}\frac{1}{p^2+1}$$
$$\Rightarrow \mathcal{L}^{-1}(Y(p)) = \sum_{j=0}^{\infty}\sin(t-j\pi)H(t-j\pi)$$

Since

$$\sin(t-j\pi)H(t-j\pi) = \begin{cases}(-1)^j\sin t, t \geqslant j\pi\\ 0, s < j\pi\end{cases}$$

then for $n$ is even, $\forall t \in n\pi < t < (n+1)\pi$

$$\sum_{j=0}^{\infty}\sin(t-j\pi)H(t-j\pi) = \sum_{j=0}^{n}(-1)^j\sin t = \sin t$$

and for $n$ is odd, $\forall t \in n\pi < t < (n+1)\pi$

$$\sum_{j=0}^{\infty}\sin(t-j\pi)H(t-j\pi) = \sum_{j=0}^{n}(-1)^j\sin t = 0$$

So the solution to the equation is

$$y(t) = \begin{cases}\sin t, n\ is\ even\\ 0 \quad\ , n\ is\ odd\end{cases}$$

for $t \in n\pi < t < (n+1)\pi$

**ii)** $\dfrac{d^2y}{dt^2} + y = \sum\limits_{j=0}^{\infty} \delta(t-2j\pi), y(0) = y'(0) = 0$

Set $Y(p) = (\mathcal{L}y)(p)$, then

$$(\mathcal{L}y')(p) = p \cdot (\mathcal{L}y)(p) - y(0) = pY(p)$$

$$(\mathcal{L}y'')(p) = p \cdot (\mathcal{L}y')(p) - y'(0) = p^2Y(p)$$

and

$$\Big(\mathcal{L}(\sum_{j=0}^{\infty} \delta(t-2j\pi))\Big)(p) = \sum_{j=0}^{\infty} e^{-2j\pi p} = \frac{1}{1-e^{-2\pi p}}, p > 0$$

So apply Laplace transform to the equation and we get

$$(p^2+1)Y(p) = \sum_{j=0}^{\infty} e^{-2j\pi p} \Rightarrow Y(p) = \sum_{j=0}^{\infty} e^{-2j\pi p}\frac{1}{p^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}(Y(p)) = \sum_{j=0}^{\infty} \sin(t-2j\pi)H(t-2j\pi) = \sum_{j=0}^{\infty} \sin tH(t-2j\pi)$$

Since

$$\sin tH(t-2j\pi) = \begin{cases} \sin t, t \geqslant 2j\pi \\ 0, s < 2j\pi \end{cases}$$

then $\forall t \in 2n\pi < t < 2(n+1)\pi$

$$\sum_{j=0}^{\infty} \sin tH(t-2j\pi) = \sum_{j=0}^{n} \sin t = (n+1)\sin t$$

So the solution to the equation is

$$y(t) = (n+1)\sin t$$

for $t \in 2n\pi < t < 2(n+1)\pi$

# Exercise 7.6

## i)

For $\Pi(x) = \begin{cases} 1, |x| < 1 \\ 0, |x| \geqslant 1 \end{cases}$, we have that

$$(\widehat{\Pi})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Pi(x)e^{-i\xi x}dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x}dx$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-i\xi} e^{-i\xi x}\Big|_{-1}^{1} = \sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\xi}$$

So $(\widehat{\Pi})(\xi) = \sqrt{\dfrac{2}{\pi}}\dfrac{\sin\xi}{\xi}$.

## ii)

In the former assignment, we have proved that

$$\int_{-\infty}^{\infty} \frac{e^{iz}-1}{2iz} = \frac{\pi}{2}$$

and for $x \neq \pm 1$

$$\frac{e^{ix\xi}\sin\xi}{\xi} = -(x-1)\frac{e^{i(x-1)\xi}-1}{2i(x-1)\xi} + (x+1)\frac{e^{i(x+1)\xi}-1}{2i(x+1)\xi})$$

so if $x < -1$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{e^{ix\xi}\sin\xi}{\xi}d\xi \\
=&\frac{1}{\pi}\int_{-\infty}^{\infty} -(x-1)\frac{e^{i(x-1)\xi}-1}{2i(x-1)\xi} + (x+1)\frac{e^{i(x+1)\xi}-1}{2i(x+1)\xi})d\xi \\
=&\frac{1-x}{\pi(x-1)}\int_{\infty}^{-\infty}\frac{e^{iz}-1}{2iz}dz + \frac{1+x}{\pi(x+1)}\int_{\infty}^{-\infty}\frac{e^{iz}-1}{2iz}dz \\
=&-\frac{1}{\pi}\cdot\Big(-\frac{\pi}{2}\Big) + \frac{1}{\pi}\cdot\Big(-\frac{\pi}{2}\Big) \\
=&0
\end{aligned}$$

if $-1 < x < 1$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{e^{ix\xi}\sin\xi}{\xi}d\xi \\
=&\frac{1}{\pi}\int_{-\infty}^{\infty} -(x-1)\frac{e^{i(x-1)\xi}-1}{2i(x-1)\xi} + (x+1)\frac{e^{i(x+1)\xi}-1}{2i(x+1)\xi})d\xi \\
=&\frac{1-x}{\pi(x-1)}\int_{\infty}^{-\infty}\frac{e^{iz}-1}{2iz}dz + \frac{1+x}{\pi(x+1)}\int_{-\infty}^{\infty}\frac{e^{iz}-1}{2iz}dz \\
=&-\frac{1}{\pi}\cdot\Big(-\frac{\pi}{2}\Big) + \frac{1}{\pi}\cdot\Big(\frac{\pi}{2}\Big) \\
=&1
\end{aligned}$$

if $x > 1$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{e^{ix\xi}\sin\xi}{\xi}d\xi \\
=&\frac{1}{\pi}\int_{-\infty}^{\infty} -(x-1)\frac{e^{i(x-1)\xi}-1}{2i(x-1)\xi} + (x+1)\frac{e^{i(x+1)\xi}-1}{2i(x+1)\xi})d\xi \\
=&\frac{1-x}{\pi(x-1)}\int_{-\infty}^{\infty}\frac{e^{iz}-1}{2iz}dz + \frac{1+x}{\pi(x+1)}\int_{-\infty}^{\infty}\frac{e^{iz}-1}{2iz}dz \\
=&-\frac{1}{\pi}\cdot\Big(\frac{\pi}{2}\Big) + \frac{1}{\pi}\cdot\Big(\frac{\pi}{2}\Big) \\
=&0
\end{aligned}$$

if $x = 1$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{e^{i\xi}\sin\xi}{\xi}d\xi\\
=&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi}\sin\xi-1}{\xi}+\frac{1}{\xi}d\xi\\
=&\frac{1}{\pi}\cdot\frac{\pi}{2}+0\\
=&\frac{1}{2}
\end{aligned}$$

if $x = -1$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{e^{i(-\xi)}\sin\xi}{\xi}d\xi\\
=&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i(-\xi)}\sin(-\xi)-1}{-\xi}+\frac{1}{\xi}d\xi\\
=&-\frac{1}{\pi}\int_{\infty}^{-\infty}\frac{e^{iz}\sin z-1}{z}+0\\
=&\frac{1}{2}
\end{aligned}$$

To sum up,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi}\widehat{\Pi}(\xi)d\xi=\begin{cases}1,|x|<1\\ \frac{1}{2},|x|=1\\ 0,|x|>1\end{cases}$$

# Exercise 7.7

$$\begin{aligned}
\widehat{f}(\xi+i\eta)=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(x)x^2e^{-i(\xi+i\eta)x}dx\\
=&\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}x^2e^{-i(\xi+i\eta)x}dx\\
=&\frac{1}{\sqrt{2\pi}}\left(\frac{1}{-i(\xi+i\eta)}x^2-\frac{1}{(-i(\xi+i\eta))^2}x+\frac{2}{(-i(\xi+i\eta))^3}\right)e^{-i(\xi+i\eta)x}|_0^\infty
\end{aligned}$$

To make the integral convergent, $e^{-i(\xi+i\eta)x}\xrightarrow{x\to\infty}0$. So $\eta<0$ i.e. in the region below the real axis in complex plane $\widehat{f}$ defined and

$$\widehat{f}(\xi+i\eta)=0-\frac{1}{\sqrt{2\pi}}\frac{2}{(-i(\xi+i\eta))^3}=\sqrt{\frac{2}{\pi}}\frac{1}{(\xi+i\eta))^3}i$$