Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 7

Date Due: 10:00 AM, Thursday, the 17th of November 2016

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Exercise 7.1. Let f be a piecewise continuous periodic function with period T, i.e., $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. Prove that the Laplace transform of f is given by

$$(\mathcal{L}f)(p) = \frac{1}{1-e^{-Tp}} \int_0^T f(t)e^{-tp}\,dt$$

Then find the Laplace transform of the function

$$f(t) = at, \qquad a \in \mathbb{R}, \qquad t \in [0,1],$$

extended periodically to $[0,\infty)$.
(4 + 2 Marks)

Exercise 7.2. If a function has an infinite number of poles to the left of some $\beta \in \mathbb{R}$, the inverse Laplace transform will be an infinite series. In particular, the sum of the residues when evaluating the Bromwich integral becomes an infinite sum over the residues. Use the Bromwich integral to prove that

$$F(p) = \frac{1}{p(e^p+1)} \qquad \Rightarrow \qquad (\mathcal{L}^{-1}F)(t) = \frac{1}{2} - \frac{2}{\pi}\left(\sin(\pi t) + \frac{\sin(3\pi t)}{3} + \cdots\right)$$
$$F(p) = \frac{1}{p\cosh p} \qquad \Rightarrow \qquad (\mathcal{L}^{-1}F)(t) = 1 - \frac{4}{\pi}\left(\cos\left(\frac{\pi t}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi t}{2}\right) + - \cdots\right).$$

(2 × 3 Marks)

Exercise 7.3. Consider the function

$$f(t) = \begin{cases} 2 & 0 \le t \le 1, \\ 0 & 1 \le t < 2, \end{cases}$$

extended periodically to $[0,\infty)$. Use Exercise 7.1 to show that

$$\mathcal{L}f(p) = \frac{1+\tanh(p/2)}{p}.$$

Then calculate the inverse Laplace transform of $(1+\tanh(p/2))/p$ as in Exercise 7.2, obtaining a series of sine and/or cosine functions. Use Mathematica to plot the function f together with $n = 1, 3, 5, 10$ and 20 terms of this series. Describe your observations in as detailed a manner as possible (Does the series seem to converge to f? Where? Uniformly or pointwise?).
(5 Marks)

Exercise 7.4. Use the Laplace transform to solve the following initial-value problems:

$$y'' + y = \sin t + \delta(t-\pi), \qquad y(0) = 0, \qquad y'(0) = 0,$$
$$y'' + y' + y = 2\delta(t-1) - \delta(t-2), \qquad y(0) = 1, \qquad y'(0) = 0.$$

(2 × 2 Marks)

Exercise 7.5.

i) Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = \sum_{j=0}^{\infty} \delta(t - j\pi), \qquad\qquad y(0) = y'(0) = 0$$

and show that

$$y(t) = \begin{cases} \sin t & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

in the interval $n\pi < t < (n+1)\pi$.

ii) Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = \sum_{j=0}^{\infty} \delta(t - 2j\pi), \qquad\qquad y(0) = y'(0) = 0$$

and show that $y(t) = (n+1)\sin t$ in the interval $2n\pi < t < 2(n+1)\pi$.

This example indicates why soldiers are instructed to break cadence when marching across a bridge. To wit, if the soldiers are in step with the natural frequency of the steel in the bridge, then a resonance situation of the type (ii) may be set.
(3 + 3 Marks)

Exercise 7.6. One of the "interesting" (read: annoying) properties of the Fourier transform is that it does not "see" individual values of functions. This is illustrated in the following simple example.

i) Show that the Fourier transform of the unit impulse $\Pi \colon \mathbb{R} \to \mathbb{R}$,

$$\Pi(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| \ge 1, \end{cases}$$

is given by

$$(\widehat{\Pi})(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi}.$$

ii) Now prove, using contour integration, that the inverse transform yields

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{\Pi}(\xi)\, d\xi = \begin{cases} 1 & |x| < 1, \\ 1/2 & |x| = 1, \\ 0 & |x| > 1. \end{cases}$$

It is for this reason that "jump functions" such as the unit impulse or the Heaviside function are often defined to have a value equal to the average of their left- and right-hand limits at the jump.
(1 + 4 Marks)

Exercise 7.7. Find the complex Fourier transform $\widehat{f}(\xi + i\eta)$ of $f(x) = H(x) \cdot x^2$, where H denotes the Heaviside function. In what region of $\mathbb{C}$ is $\widehat{f}$ defined?
(3 Marks)