VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 9

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December 1, 2016

Exercise 9.1

Set $y(x) = x^r \sum\limits_{k=0}^{\infty} a_k x^k$, since

$$x^2 y'' + xy' + (x^2 - 9/4)y = 0$$

then we can obtain that

$$\begin{cases} (r^2 - 9/4)a_0 = 0 \\ ((r+1)^2 - 9/4)a_1 = 0 \\ ((k+r+2)^2 - 9/4)a_{k+2} = -a_k, k \geqslant 2 \end{cases} \Leftrightarrow \begin{cases} r = 3/2 \vee r = -3/2 \\ a_1 = 0 \\ ((k+r+2)^2 - 9/4)a_{k+2} = -a_k, k \geqslant 0 \end{cases}$$

For $r = 3/2$, we get one solution to the Bessel equation

$$x^2 y'' + xy' + (x^2 - 9/4)y = 0$$

is

$$J_{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+3/2)} \left(\frac{x}{2}\right)^{2n+3/2}$$

For $r = -3/2$,

$$\begin{aligned} a_{2k+1} &= 0, k \in \mathbb{N} \\ a_{2k} &= -\frac{1}{(2k)(2k-3)} a_{2k-2} = -\left(\frac{1}{2}\right)^2 \frac{1}{k(k-3/2)} a_{2k-2} \\ &= \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod\limits_{i=1}^{k} (i - 3/2)} a_0, k \in \mathbb{N}^* \end{aligned}$$

we can set $a_0 = \dfrac{2^{3/2}}{\Gamma(-3/2)} = \dfrac{2^{3/2}(1-3/2)}{\Gamma(2-3/2)}$, then

$$a_{2k} = \left(\frac{1}{2}\right)^{2k} \frac{(-1)^k}{k! \prod\limits_{i=1}^{k} (i-3/2)} \frac{2^{3/2}(1-3/2)}{\Gamma(2-3/2)} = \left(\frac{1}{2}\right)^{2k-3/2} \frac{(-1)^k}{k!\Gamma(1+k-3/2)}$$

this is also hold for $k = 0$. So one solution is

$$J_{-3/2}(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n-3/2)} \left(\frac{x}{2}\right)^{2n-3/2}$$

So two independent solutions to the Bessel equation

$$x^2 y'' + xy' + (x^2 - 9/4)y = 0$$

are

$$J_{3/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+3/2)} \left(\frac{x}{2}\right)^{2n+3/2}$$

$$J_{-3/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n-3/2)} \left(\frac{x}{2}\right)^{2n-3/2}$$

Exercise 9.2

i)

Since

$$\begin{aligned}
f(2m,2n) :=& \int_0^\pi \cos^{2m}\theta \sin^{2n}\theta d\theta \\
=& -\frac{1}{2m+1}\int_0^\pi \sin^{2n-1}\theta d(\cos^{2m+1}\theta) \\
=& -\frac{1}{2m+1}(\sin^{2n-1}\theta\cos^{2m+1}\theta|_0^\pi - \int_0^\pi \cos^{2m+1}\theta d(\sin^{2n-1}\theta)) \\
=& \frac{2n-1}{2m+1}\int_0^\pi \cos^{2m+2}\theta \sin^{2n-2}\theta d\theta \\
=& \frac{2n-1}{2m+1}\int_0^\pi \cos^{2m}\theta(1-\sin^2\theta)\sin^{2n-2}\theta d\theta \\
=& \frac{2n-1}{2m+1}(f(2m,2(n-1)) - f(2m,2n))
\end{aligned}$$

then we can get that

$$f(2m,2n) = \frac{2n-1}{2m+2n}f(2m,2(n-1)) = \frac{\prod\limits_{i=1}^{n}(2i-1)}{\prod\limits_{i=1}^{n}(2m+2i)}f(2m,0)$$

Also since

$$\begin{aligned}
f(2m,0) =& \int_0^\pi \cos^{2m}\theta \sin^0\theta d\theta \\
=& \int_0^\pi \cos^{2m-2}\theta(1-\sin^2\theta)d\theta \\
=& \int_0^\pi \cos^{2m-2}\theta d\theta + \frac{1}{2m-1}\int_0^\pi \sin\theta d(\cos^{2m-1}\theta) \\
=& f(2(m-1),0) + \frac{1}{2m-1}(\sin\theta\cos^{2m-1}\theta|_0^\pi - \int_0^\pi \cos^{2m}\theta d\theta) \\
=& f(2(m-1),0) - \frac{1}{2m-1}f(2m,0)
\end{aligned}$$

then

$$f(2m,0) = \frac{2m-1}{2m}f(2(m-1),0) = \frac{\prod\limits_{i=1}^{m}(2i-1)}{\prod\limits_{i=1}^{m}(2i)}f(0,0) = \frac{\prod\limits_{i=1}^{m}(2i-1)}{\prod\limits_{i=1}^{m}(2i)}\pi$$

So

$$\begin{aligned}
&\int_0^\pi \cos^{2m}\theta\sin^{2n}\theta d\theta\\
=&\frac{\prod\limits_{i=1}^{n}(2i-1)}{\prod\limits_{i=1}^{n}(2m+2i)}\frac{\prod\limits_{i=1}^{m}(2i-1)}{\prod\limits_{i=1}^{m}(2i)}\pi=\frac{(2n!)(2m)!}{2^{m+n}(m+n)!}\frac{1}{\prod\limits_{i=1}^{m}(2i)\prod\limits_{i=1}^{n}(2i)}\pi\\
=&\frac{(2n!)(2m)!}{2^{m+n}(m+n)!}\frac{1}{(2^m m!)(2^n n!)}\pi\\
=&\frac{(2m)!}{2^{2m}m!}\frac{(2n)!}{2^{2n}n!}\frac{\pi}{(m+n)!}
\end{aligned}$$

To sum up

$$\int_0^\pi \cos^{2m}\theta\sin^{2n}\theta d\theta=\frac{(2m)!}{2^{2m}m!}\frac{(2n)!}{2^{2n}n!}\frac{\pi}{(m+n)!}$$

ii)

Since

$$\cos(x\cos\theta)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(x\cos\theta)^{2k}$$

$$\begin{aligned}
&\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi\cos(x\cos\theta)\sin^{2n}\theta d\theta\\
=&\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(x\cos\theta)^{2k}\sin^{2n}\theta d\theta=\frac{(2x)^n n!}{\pi(2n)!}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\int_0^\pi(\cos\theta)^{2k}\sin^{2n}\theta d\theta\\
=&\frac{(2x)^n n!}{\pi(2n)!}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\frac{(2k)!}{2^{2k}k!}\frac{(2n)!}{2^{2n}n!}\frac{\pi}{(k+n)!}\\
=&\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\left(\frac{x}{2}\right)^{2k+n}\\
=&J_n(x)
\end{aligned}$$

So

$$J_n(x)=\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi\cos(x\cos\theta)\sin^{2n}\theta d\theta$$

iii)

Since

$$\begin{aligned}
\int_0^\pi\sin(x\cos\theta)\sin^{2n}\theta d\theta&=\int_0^{\frac{\pi}{2}}\sin(x\cos\theta)\sin^{2n}\theta d\theta+\int_{\frac{\pi}{2}}^{\pi}\sin(x\cos\theta)\sin^{2n}\theta d\theta\\
&=\int_0^{\frac{\pi}{2}}\sin(x\cos\theta)\sin^{2n}\theta d\theta+\int_{\frac{\pi}{2}}^{0}\sin(x\cos t)\sin^{2n}t dt\\
&=0
\end{aligned}$$

then

$$
\begin{aligned}
&\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi e^{i(x\cos\theta)}\sin^{2n}\theta d\theta\\
=&\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi cos(x\cos\theta)\sin^{2n}\theta d\theta + i\frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi sin(x\cos\theta)\sin^{2n}\theta d\theta\\
=&J_n(x)+0
\end{aligned}
$$

To sum up,

$$
J_n(x) = \frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi e^{i(x\cos\theta)}\sin^{2n}\theta d\theta
$$

iv)

Set $\xi = \cos\theta$, then

$$
\frac{d\theta}{d\xi} = \frac{d\arccos\xi}{d\xi} = -\frac{1}{\sqrt{1-\xi^2}}
$$

So

$$
\begin{aligned}
J_n(x) &= \frac{(2x)^n n!}{\pi(2n)!}\int_0^\pi e^{i(x\cos\theta)}\sin^{2n}\theta d\theta = \frac{(2x)^n n!}{\pi(2n)!}\int_1^{-1} e^{i(x\xi)}(1-\xi^2)^n(-\frac{d\xi}{\sqrt{1-\xi^2}})\\
&= \frac{(2x)^n n!}{\pi(2n)!}\int_{-1}^1 e^{i(x\xi)}(1-\xi^2)^{n-1/2}d\xi
\end{aligned}
$$

Exercise 9.3

In this section, $\forall\nu\in\mathbb{R}\setminus\mathbb{Z}^-$,we denote

$$
J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}
$$

$\forall\nu\in\mathbb{Z}^-$ we denote

$$
J_\nu(x) = (-1)^\nu J_{-\nu}(x)
$$

i)

$\forall\nu\in\mathbb{R}$

$$
\begin{aligned}
\frac{d}{dx}(x^\nu J_\nu(x)) =& \frac{d}{dx}(x^\nu\sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu})\\
=&\sum_{k=0}^\infty \frac{(-1)^k(2k+2\nu)}{k!(k+\nu)\Gamma(k+\nu)}\left(\frac{x}{2}\right)^{2k+2\nu-1}\cdot\frac{1}{2})\\
=&x^\nu\sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu-1+1)}\left(\frac{x}{2}\right)^{2k+\nu-1}\\
=&x^\nu J_{\nu-1}
\end{aligned}
$$

So $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}$

ii)

$\forall \nu \in \mathbb{R}$

$$
\begin{aligned}
\frac{d}{dx}(x^{-\nu}J_\nu(x)) =& \frac{d}{dx}(x^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu})\\
=& \sum_{k=1}^{\infty}\frac{(-1)^k(2k)}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k-1}\cdot\frac{1}{2})\\
=& x^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!\Gamma(k+(\nu+1)+1)}\left(\frac{x}{2}\right)^{2k+\nu+1}\\
=& x^{-\nu}J_{\nu+1}
\end{aligned}
$$

So $\frac{d}{dx}(x^{-\nu}J_\nu(x)) = x^{-\nu}J_{\nu+1}$

iii)

$$
\begin{aligned}
&2\nu J_\nu(x) - xJ_{\nu+1}(x) - xJ_{\nu-1}(x)\\
=&2\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} - x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1+1)}\left(\frac{x}{2}\right)^{2k+\nu+1}\\
&- x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu-1+1)}\left(\frac{x}{2}\right)^{2k+\nu-1}\\
=&2\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1+\nu+1)}\left(\frac{x}{2}\right)^{2(k+1)+\nu}\\
&- 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu)}\left(\frac{x}{2}\right)^{2k+\nu}\\
=&2\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} + 2\sum_{k=0}^{\infty}\frac{(-1)^k k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\\
&- 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu)}\left(\frac{x}{2}\right)^{2k+\nu}\\
=&2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu)}\left(\frac{x}{2}\right)^{2k+\nu} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu)}\left(\frac{x}{2}\right)^{2k+\nu}\\
=&0
\end{aligned}
$$

So $2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x)$

iv)

$$
\begin{aligned}
&J_\nu'(x)\\
=&\frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}\\
=&\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+\nu)}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}\\
=&\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(k+\nu)}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}\\
=&\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu-1+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}\\
=&\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu-1+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!\Gamma(k+\nu+1+1)}\Big(\frac{x}{2}\Big)^{2k+\nu+1}\\
=&\frac{1}{2}(J_{\nu-1}(x)-J_{\nu+1}(x))
\end{aligned}
$$

So $J_\nu'(x)=\dfrac{1}{2}(J_{\nu-1}(x)-J_{\nu+1}(x))$

Exercise 9.4

For a suspended chain of constant density, the tension is given by

$$T(x)=\int\rho\cdot g\cdot dx=\int_0^x\rho_0\cdot g\cdot x^\mu=\rho_0 g\frac{1}{\mu+1}x^{\mu+1}$$

from gravity and we have no additional external forces, so the model equation is

$$
\begin{aligned}
&\rho u_{tt}(x,t)=\frac{\partial}{\partial x}(Tu_x)\\
\Leftrightarrow&\rho_0\cdot x^\mu u_{tt}(x,t)=\rho_0\cdot g\cdot x^\mu u_x+\frac{1}{\mu+1}\rho_0\cdot g\cdot x^{\mu+1}u_{xx}\\
\Leftrightarrow&\frac{\mu+1}{g}u_{tt}(x,t)=(\mu+1)u_x+xu_{xx}
\end{aligned}
$$

Set $u(x,t)=y(x)\cdot e^{i\omega t}$, then

$$\frac{(\mu+1)}{g}y(x)\cdot(-\omega^2e^{i\omega t})=(\mu+1)y'(x)\cdot e^{i\omega t}+x\cdot y''(x)e^{i\omega t}$$

Set $f(z)=x^{\mu/2}y(x), x=gz^2/(4(\mu+1)\omega^2)$, so $y(x)=\Big(\dfrac{gz^2}{4(\mu+1)\omega^2}\Big)^{-\mu/2}f(z)$, $z=2\sqrt{\mu+1}\omega\sqrt{x/g}$, then

$$
\begin{aligned}
y'(x)&=\frac{dy}{dz}\frac{dz}{dx}=\frac{\sqrt{\mu+1}\omega}{\sqrt{gx}}\Big(\frac{g}{4(\mu+1)\omega^2}\Big)^{-\mu/2}(-\mu z^{-\mu-1}f(z)+z^{-\mu}f'(z))\\
&=\frac{2(\mu+1)\omega^2}{g}\Big(\frac{g}{4(\mu+1)\omega^2}\Big)^{-\mu/2}(-\mu z^{-\mu-2}f(z)+z^{-\mu-1}f'(z))
\end{aligned}
$$

$$\begin{aligned}y''(x) = \frac{dy'(x)}{dz}\frac{dz}{dx} =& \frac{4(\mu+1)^2\omega^4}{zg^2}\Big(\frac{g}{4(\mu+1)\omega^2}\Big)^{-\mu/2}\\ &\cdot(\mu(\mu+2)z^{-\mu-3}f(z)-(2\mu+1)z^{-\mu-2}f'(z)+z^{-\mu-1}f''(z))\end{aligned}$$

Since $xy''(x)+(\mu+1)y'(x)+\dfrac{(\mu+1)\omega^2}{g}y(x)=0$, we get that

$$\begin{aligned}0 =& \frac{gz^2}{4(\mu+1)\omega^2}\frac{4(\mu+1)^2\omega^4}{zg^2}\Big(\frac{g}{4(\mu+1)\omega^2}\Big)^{-\mu/2}\\ &\cdot(\mu(\mu+2)z^{-\mu-3}f(z)-(2\mu+1)z^{-\mu-2}f'(z)+z^{-\mu-1}f''(z))\\ &+(\mu+1)\frac{2(\mu+1)\omega^2}{g}\Big(\frac{g}{4(\mu+1)\omega^2}\Big)^{-\mu/2}(-\mu z^{-\mu-2}f(z)+z^{-\mu-1}f'(z))\\ &+\frac{(\mu+1)\omega^2}{g}\Big(\frac{gz^2}{4(\mu+1)\omega^2}\Big)^{-\mu/2}f(z)\end{aligned}$$

So

$$z^2f''(z)+zf'(z)+(z^2-\mu^2)f(z)=0$$

This is precisely the Bessel equation of order μ, since we are looking for solutions that are finite at $x=0$ (the end of the chain), so we obtain

$$f(z)=c\cdot J_\mu(z)$$

Substituting back,

$$y(x)=-c\cdot x^{-\mu/2}\cdot J_\mu(2\sqrt{\mu+1}\omega\sqrt{x/g})$$

The boundary condition at the upper end implies

$$y(l)=0=-c\cdot x^{-\mu/2}\cdot J_\mu(2\sqrt{\mu+1}\omega\sqrt{x/g})$$

So we see that

$$\omega=\frac{1}{2\sqrt{\mu+1}}\sqrt{\frac{g}{l}}\cdot\alpha_{\mu,n}$$

where $\alpha_{\mu,n}, n=1,2,\cdots$ is the n^{th} zero of J_μ, the Bessel function of order μ.

Exercise 9.5

i)

Since $u(x,t)=X(x)e^{i\omega t}, \dfrac{T}{\rho}u_{xx}=u_{tt}, u(0)=u(l)=0$

$$\frac{T}{\rho_0}X''(x)e^{i\omega t}=X(x)(-\omega^2)e^{i\omega t}, X(0)=X(l)=0$$

i.e.

$$X''(x)=-\frac{\omega^2\rho_0}{T}X(x)$$

the genenral solution to it is

$$X(x)=A\cos(\sqrt{\frac{\omega^2\rho_0}{T}}x+\varphi)$$

so

$$0 = X(0) = A\cos(\varphi), 0 = X(l) = A\cos(\sqrt{\frac{\omega^2\rho_0}{T}}l + \varphi)$$

so

$$\varphi = \frac{\pi}{2} + k_1\pi, \sqrt{\frac{\rho_0}{T}}\omega l + \varphi = \frac{\pi}{2} + k_2\pi, k_1, k_2 \in \mathbb{Z}$$

so

$$\omega = \frac{k\pi}{l}\sqrt{\frac{T}{\rho_0}}, k \in \mathbb{Z}$$

is the frequencies of the normal modes.

ii)

Since $u(x,t) = X(x)e^{i\omega t}, \dfrac{T}{\rho}u_{xx} = u_{tt}, u(0) = u(l) = 0$

$$\frac{T}{\rho_0\xi}X''(x)e^{i\omega t} = X(x)(-\omega^2)e^{i\omega t}, X(0) = X(l) = 0$$

so

$$X''(x) + \frac{\omega^2\rho_0\xi}{T}X(x) = 0$$

since $\xi = 1 + \dfrac{kx}{l}$, set $Y(\xi) = X(x)$, then

$$Y'(\xi) = \frac{dX(x)}{dx}\frac{dx}{d\xi} = \frac{l}{k}X'(x), Y''(\xi) = \frac{dY'(\xi)}{dx}\frac{dx}{d\xi} = \frac{l^2}{k^2}X''(x)$$

So

$$Y''(\xi) + \frac{\rho_0\omega^2 l^2}{k^2 T}\xi Y(\xi) = \frac{l^2}{k^2}(X''(x) + \frac{\rho_0\omega^2}{T}\xi X(x)) = 0$$

Set $\kappa^2 = \dfrac{\rho_0\omega^2 l^2}{k^2 T}$ then $Y''(\xi) + \kappa^2\xi Y(\xi) = 0$. Set $Z(t) = Y(\xi), t = \kappa^{2/3}\xi$, then

$$Y'(\xi) = \frac{dZ(t)}{dt}\frac{dt}{d\xi} = \kappa^{2/3}Z'(t), Y''(\xi) = \frac{dZ'(t)}{dt}\frac{dt}{d\xi} = \kappa^{4/3}Z''(t)$$

So

$$Z''(t) + tZ(t) = 0$$

and the solution to this Airy equation

$$Z(t) = t^{1/2}(c_1 J_{1/3}(\frac{2}{3}t^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}t^{3/2}))$$

So

$$\begin{aligned} Y(\xi) =& Z(t) = (\kappa^{2/3}\xi)^{1/2}(c_1 J_{1/3}(\frac{2}{3}(\kappa^{2/3}\xi)^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}(\kappa^{2/3}\xi)^{3/2})) \\ =& \kappa^{1/3}\xi^{1/2}(c_1 J_{1/3}(\frac{2}{3}\kappa\xi^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}\kappa\xi^{3/2})) \end{aligned}$$

So

$$X(x) = Y(\xi) = \kappa^{1/3}(1+\frac{kx}{l})^{1/2}(c_1 J_{1/3}(\frac{2}{3}\kappa(1+\frac{kx}{l})^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}\kappa(1+\frac{kx}{l})^{3/2}))$$

Since $X(0) = X(l) = 0$, we obtain that

$$\begin{cases} c_1 J_{1/3}(\frac{2}{3}\kappa) + c_2 J_{-1/3}(\frac{2}{3}\kappa) = 0 \\ c_1 J_{1/3}(\frac{2}{3}\kappa(1+k)^{3/2}) + c_2 J_{-1/3}(\frac{2}{3}\kappa(1+k)^{3/2}) = 0 \end{cases}$$

To get non-trival solution for c_1, c_2,

$$J_{1/3}(\frac{2}{3}\kappa)J_{-1/3}(\frac{2}{3}\kappa(1+k)^{3/2}) = J_{-1/3}(\frac{2}{3}\kappa)J_{1/3}(\frac{2}{3}\kappa(1+k)^{3/2})$$

Denote $\mu = \frac{2}{3}\kappa$ is the solution to the equation

$$J_{1/3}(\mu)J_{-1/3}(\mu(1+k)^{3/2}) = J_{-1/3}(\mu)J_{1/3}(\mu(1+k)^{3/2})$$

Then

$$\frac{9}{4}\mu^2 = \kappa^2 = \frac{\rho_0\omega^2 l^2}{k^2 T}$$

So

$$\omega^2 = \frac{9\mu^2 k^2 T}{4\rho_0 l^2}$$

Exercise 9.6

In the lecture we have seen that

$$l_{max} = \sqrt[3]{\frac{9\alpha_{-1/3,1}^2 EI}{4q}}$$

Use mathematica we find that $\alpha_{-1/3,1} = 1.86635$.

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N[BesselJZero[-1/3, 1]]
1.86635
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From wikipedia, we find $E_{steel} = 200 \times 10^9 Pa$, also we know that for a hollow pole with inner radius r and outer radius $0.1m$ its moment of inertia is

$$I = \frac{1}{2}m(r^2+0.01) = \frac{1}{2}\rho\pi(0.01-r^2)l_{max}(r^2+0.01)$$

and its uniform load is

$$q = \frac{mg}{l} = g\rho\pi(0.01-r^2)$$

So

$$l_{max} = \sqrt[3]{\frac{9\alpha_{-1/3,1}^2 El_{max}(r^2+0.01)}{8g}} \Rightarrow l_{max} = \sqrt{\frac{9\cdot 1.86635^2\cdot 200\times 10^9 \times 0.01\times 2}{8\times 9.80665}} = 39980m$$

So the maximum height to which I can build the flagpole is 39980m.

Exercise 9.7

i)

For $x > 0$, since $\Gamma(x+1) = x\Gamma(x)$, then

$$\Gamma'(x+1) = \frac{d\Gamma(x+1)}{dx} = \frac{dx\Gamma(x)}{dx} = \Gamma(x) + x\Gamma'(x)$$

So

$$\psi(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\Gamma(x) + x\Gamma'(x)}{x\Gamma(x)} = \frac{1}{x} + \psi(x)$$

i.e.

$$\psi(x+1) = \frac{1}{x} + \psi(x)$$

ii)

Since $\psi(x+1) = \dfrac{1}{x} + \psi(x)$,

$$\psi(n+1) = \frac{1}{n} + \psi(n) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k}$$

Since

$$\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \frac{-\gamma}{\int_0^\infty e^{-y} dy} = \frac{-\gamma}{-e^{-y}|_0^\infty} = -\gamma$$

So

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

iii)

According to Stirling's formula, $n! \sim \sqrt{2\pi n}\Big(\dfrac{n}{e}\Big)^n$, also we know that $\Gamma(n+1) = n!$, so $\Gamma(x+1) \sim \sqrt{2\pi x}\Big(\dfrac{x}{e}\Big)^x$ as $x \to \infty$. So

$$\Gamma'(x+1) = \sqrt{\frac{\pi}{2x}}\Big(\frac{x}{e}\Big)^x + \sqrt{2\pi x}(-e^{-x}x^x + e^{-x}x^x(\ln x + 1))$$

So as $x \to \infty$

$$\psi(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\sqrt{2\pi x}\Big(\frac{x}{e}\Big)^x(\frac{1}{2x} + \ln x)}{\sqrt{2\pi x}\Big(\frac{x}{e}\Big)^x} = \frac{1}{2x} + \ln x + O\Big(\frac{1}{x^2}\Big)$$

So

$$-\gamma + \sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) = \ln(n) + \frac{1}{2n} + O\Big(\frac{1}{n^2}\Big)$$

i.e.

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln(n) + \frac{1}{2n} + O\Big(\frac{1}{n^2}\Big)$$

Exercise 9.8

i)

As $\nu \to 0,\ J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x) \to 0, \sin(\nu\pi) \to 0$, so we can use l'Hospital's rule,

$$\begin{aligned}
Y_0(x) =& \lim_{\nu\to 0} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \\
=& \lim_{\nu\to 0} \frac{J_\nu(x)(-\pi\sin(\nu\pi)) + \frac{dJ_\nu(x)}{d\nu}\cos(\nu\pi) - \frac{dJ_{-\nu}(x)}{d\nu}}{\pi\cos(\nu\pi)} \\
=& \lim_{\nu\to 0} \frac{J_\nu(x)(-\pi\sin(\nu\pi)) + \frac{dJ_t(x)}{dt}|_{t=\nu}\cos(\nu\pi) + \frac{dJ_t(x)}{dt}|_{t=-\nu}}{\pi\cos(\nu\pi)} \\
=& \frac{1}{\pi}(\frac{dJ_t(x)}{dt}|_{t=0} + \frac{dJ_t(x)}{dt}|_{t=0}) \\
=& \frac{2}{\pi}\frac{dJ_\nu(x)}{d\nu}|_{\nu=0}
\end{aligned}$$

So $Y_0(x) = \dfrac{2}{\pi}\dfrac{dJ_\nu(x)}{d\nu}|_{\nu=0}$.

ii)

$$\begin{aligned}
Y_0(x) =& \frac{2}{\pi}\frac{dJ_\nu(x)}{d\nu}\Big|_{\nu=0} = Y_0(x) = \frac{2}{\pi}\frac{d}{d\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}\Big|_{\nu=0} \\
=& \frac{2}{\pi}\Big(\sum_{k=0}^{\infty}\frac{(-1)^k(-\Gamma'(k+\nu+1))}{k!(\Gamma(k+\nu+1))^2}\Big(\frac{x}{2}\Big)^{2k+\nu}\Big|_{\nu=0} + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}\ln(\frac{x}{2})\Big|_{\nu=0}\Big) \\
=& \frac{2}{\pi}\Big(-\sum_{k=0}^{\infty}\frac{(-1)^k(\psi(k+\nu+1))}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}\Big|_{\nu=0} + J_0(x)\ln(\frac{x}{2})\Big) \\
=& \frac{2}{\pi}\Big(-\sum_{k=0}^{\infty}\frac{(-1)^k(\psi(k+1))}{k!\Gamma(k+1)}\Big(\frac{x}{2}\Big)^{2k} + J_0(x)\ln(\frac{x}{2})\Big) \\
=& \frac{2}{\pi}\Big(-\sum_{k=0}^{\infty}\frac{(-1)^k(-\gamma + \sum\limits_{k=1}^{n}\frac{1}{k})}{k!\Gamma(k+1)}\Big(\frac{x}{2}\Big)^{2k} + J_0(x)\ln(\frac{x}{2})\Big) \\
=& \frac{2}{\pi}\Big(-\sum_{k=0}^{\infty}\frac{(-1)^k\sum\limits_{k=1}^{n}\frac{1}{k}}{(k!)^2}\Big(\frac{x}{2}\Big)^{2k} + J_0(x)\ln(\frac{x}{2}) + \gamma J_0(x)\Big) \\
=& \frac{2}{\pi}J_0(x)\Big(\ln\Big(\frac{x}{2}\Big) + \gamma\Big) - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\sum\limits_{k=1}^{n}\frac{1}{k}}{(k!)^2}\Big(\frac{x}{2}\Big)^{2k}
\end{aligned}$$

So

$$Y_0(x) = \frac{2}{\pi}J_0(x)\Big(\ln\Big(\frac{x}{2}\Big) + \gamma\Big) - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\Big(\frac{x}{2}\Big)^{2n}H_n$$

where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$