VV286
Honors Mathematics IV
Ordinary Differential Equations
Assignment 2

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1

Proof: Set $u(x) = e^{\int h(x)y(x)dx}$, then

$$\frac{du}{dx} = \frac{d}{dx} e^{\int h(x)y(x)dx} = e^{\int h(x)y(x)dx} \cdot h(x)y(x)$$

$$\begin{aligned}\frac{d^2u}{dx^2} &= \frac{d}{dx}(e^{\int h(x)y(x)dx} \cdot h(x)y(x)) \\ &= e^{\int h(x)y(x)dx} \cdot (h(x)y(x))^2 + e^{\int h(x)y(x)dx}(h'(x)y(x) + h(x)y'(x))\end{aligned}$$

Then

$$\begin{aligned}&u'' + \Big(g - \frac{h'}{h}\Big)u' - khu \\ =&e^{\int h(x)y(x)dx} \cdot (h(x)y(x))^2 + e^{\int h(x)y(x)dx}(h'(x)y(x) + h(x)y'(x)) \\ &+ \Big(g(x) - \frac{h'(x)}{h(x)}\Big)e^{\int h(x)y(x)dx} \cdot h(x)y(x) - k(x)h(x)e^{\int h(x)y(x)dx} \\ =&e^{\int h(x)y(x)dx}((h(x)y(x))^2 + h(x)y'(x) + g(x)h(x)y(x) - k(x)h(x)) \\ =&e^{\int h(x)y(x)dx}h(x)(\underbrace{y'(x) + g(x)y(x) + h(x)(y(x))^2 - k(x)}_{0}) \\ =&0\end{aligned}$$

So, the Ricatti differential equation

$$y' + g(x)y + h(x)y^2 = k(x) \qquad \text{(on an open interval } I \subset \mathbb{R})$$

with $g, h \in C(I), h \in C^1(I), h \neq 0$ on I, can be transformed into the linear differential equation of second order,

$$u'' + \Big(g - \frac{h'}{h}\Big)u' - khu = 0,$$

using transformation

$$u(x) = e^{\int h(x)y(x)dx}.$$

2

Proof: We first prove that if the equation has an intergrating factor of the form $M(x, y) = M(x \cdot y)$, then $\frac{h_x - g_y}{xg - hy}$ is a function of $x \cdot y$ only.

This is because

$$\begin{aligned}&M_y g + M g_y = M_x h + M h_x \\ \Rightarrow&\frac{dM}{dxy}\frac{dxy}{dy}g + Mg_y = \frac{dM}{dxy}\frac{dxy}{dx}h + Mh_x \\ \Rightarrow&\frac{1}{M}\frac{dM}{dxy} = \frac{h_x - g_y}{xg - hy}\end{aligned}$$

Since $M(x, y) = M(x \cdot y)$, $\frac{dM}{dxy}$ will also be a function of $x \cdot y$ only. So $\frac{h_x - g_y}{xg - hy}$ is a function of $x \cdot y$ only.

Next we prove that if $\dfrac{h_x - g_y}{xg - hy}$ is a function of $x \cdot y$ only, then the equation has an intergrating factor of the form $M(x, y) = M(x \cdot y)$

Set $\dfrac{h_x - g_y}{xg - hy} = F(x \cdot y)$. Moreover set $\int F(x \cdot y)d(x \cdot y) = G(x \cdot y)$ then

$$\begin{aligned}
&(\frac{d}{dy}e^{G(x\cdot y)}g + e^{G(x\cdot y)}g_y) - (\frac{d}{dx}e^{G(x\cdot y)}h + e^{G(x\cdot y)}h_x) \\
=&e^{G(x\cdot y)}\Big(\frac{dG(x\cdot y)}{d(x\cdot y)}\frac{d(x\cdot y)}{dy}g + g_y - \frac{dG(x\cdot y)}{d(x\cdot y)}\frac{d(x\cdot y)}{dx}h - h_x\Big) \\
=&e^{G(x\cdot y)}\Big(F(x\cdot y)xg + g_y - F(x\cdot y)yh - h_x\Big) \\
=&e^{G(x\cdot y)}\Big(h_x - g_y + g_y - h_x\Big) \\
=&0
\end{aligned}$$

So $M(x, y) = e^{G(x\cdot y)}$ is an intergrating factor for the equation $h(x, y)y' + g(x, y) = 0$ and it's in the form $M(x, y) = M(x \cdot y)$.

To sum up, the equation

$$h(x, y)y' + g(x, y) = 0$$

has an intergrating factor of the form $M(x, y) = M(x \cdot y)$ if and only if

$$\frac{h_x - g_y}{xg - hy}$$

is a function of $x \cdot y$ only.

For the equation

$$\Big(\frac{x^2}{y} + 3\frac{y}{x}\Big)y' + \Big(3x + \frac{6}{y}\Big) = 0$$

we have that

$$\frac{h_x - g_y}{xg - hy} = \frac{\dfrac{2x}{y} - \dfrac{3y}{x^2} + \dfrac{6}{y^2}}{3x^2 + \dfrac{6x}{y} - x^2 - 3\dfrac{y^2}{x}} = \frac{2x^3y - 3y^3 + 6x^2}{xy(2x^3y - 3y^3 + 6x^2)} = \frac{1}{xy}$$

then according to former proof, set $M(x, y) = e^{\int \frac{1}{xy}d(xy)} = e^{ln(xy)} = xy$. Then we can further set

$$F^{\perp}(x, y) = \begin{pmatrix} xy(3x + \dfrac{6}{y}) \\ xy(\dfrac{x^2}{y} + 3\dfrac{y}{x}) \end{pmatrix} = \begin{pmatrix} 3x^2y + 6x \\ x^3 + 3y^2 \end{pmatrix}$$

This is a potential field since for the potential function $U(x, y) = x^3y + 3x^2 + y^3$,

$$\frac{\partial U}{\partial x} = 3x^2y + 6x, \frac{\partial U}{\partial y} = x^3 + 3y^2$$

so all integral curves are given by $x^3y + 3x^2 + y^3 = C$, C is a constant.

3

Solution: Let's start with checking whether $M(x) = \frac{1}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx}$ is an intergrating factor for the equation

$$a_1(x)y' + a_0(x)y = f(x)$$

Since

$$\begin{aligned}
&M_y g + M g_y - M_x h - M h_x \\
=&\frac{e^{\int \frac{a_0(x)}{a_1(x)} dx}}{a_1(x)}(a_0(x)) - \frac{\frac{a_0(x)}{a_1(x)} \cdot a_1(x) - a_1'(x)}{(a_1(x))^2} e^{\int \frac{a_0(x)}{a_1(x)} dx} \cdot a_1(x) - a_1'(x) \frac{1}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} \\
=&0
\end{aligned}$$

then $M(x) = \frac{1}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx}$ is an intergrating factor for the equation.

Set

$$F^{\perp}(x,y) = \begin{pmatrix} (a_0(x)y - f(x)) \frac{1}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} \\ a_1(x) \frac{1}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} \end{pmatrix} = \begin{pmatrix} \frac{a_0(x)y - f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} \\ e^{\int \frac{a_0(x)}{a_1(x)} dx} \end{pmatrix}$$

Set $U = ye^{\int \frac{a_0(x)}{a_1(x)} dx} - \int \frac{f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} dx$ is a potential function, then

$$\frac{dU}{dy} = e^{\int \frac{a_0(x)}{a_1(x)} dx}$$

$$\frac{dU}{dx} = \frac{ya_0(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} - \frac{f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} = \frac{a_0(x)y - f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx}$$

so $F^{\perp}(x,y)$ is a potential field, and all integral curves are given by

$$ye^{\int \frac{a_0(x)}{a_1(x)} dx} - \int \frac{f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} dx = C$$

We can change this equation into

$$y = C \cdot e^{-\int \frac{a_0(x)}{a_1(x)} dx} + e^{-\int \frac{a_0(x)}{a_1(x)} dx} \int \frac{f(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)} dx} dx$$

which is just the formular obtained from Duhamel's principle.

4

The solution of Clairaut's equation obtained from a slope parametrization of the integral curve is

$$x(p) = -g'(p), y(p) = -pg'(p) + g(p)$$

then for each point $(-g'(p), -pg'(p) + g(p))$, it's also on the line

$$y = px + g(p)$$

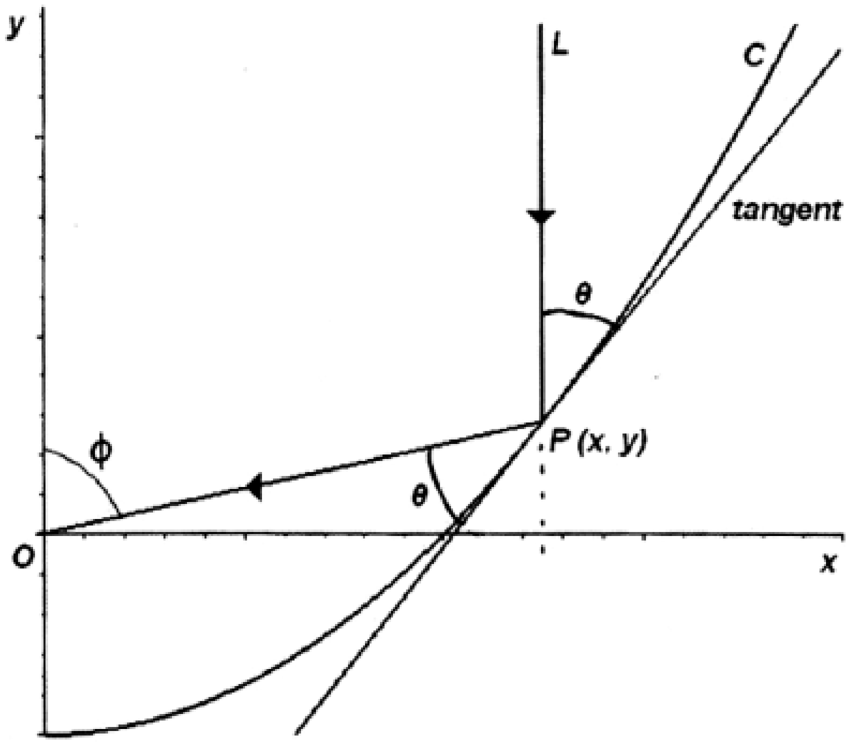
and the slope at this point of the intergral curve is

$$\frac{dy}{dx} = \frac{dy}{dp} \div \frac{dx}{dp} = \frac{-pg''(p) - g'(p) + g'(p)}{-g''(p)} = p$$

which is also the slope of the line $y = px + g(p)$. Then according to the difinition of tangential, $y = px + g(p)$ tangent to the intergral curve at point $(-g'(p), -pg'(p) + g(p))$. So for each point on the integral curve, there exist a curve in $\{y = cx + g(c), c \in I\}$ such that it is tangent to the integral curve at that point. And we can see that these lines are just the straight-line solutions of Clairaut's equation, so the integral curve is just the envelope of the straight-line solutions.

To sum up, the solution of Clairaut's equation obtained from a slope parametrization of the integral curve is always the envelope of the straight-line solutions.

5



5.1

Since beam L parallel to the y-axis, $\phi + \angle LPO = \pi$. Since $\angle LPO + 2\theta = \pi$, $\phi = 2\theta$. Since $tan(\frac{\pi}{2} - \phi) = tan\angle POx = \frac{y}{x}$, when $\phi \neq \frac{\pi}{2}$, $tan\phi = \frac{x}{y}$. Also the slope of the tangent line is $\frac{dy}{dx}$, which is also equal to $tan(\frac{\pi}{2} - \theta)$.

To sum up, $\phi = 2\theta, tan\phi = \frac{x}{y}, tan(\pi/2 - \theta) = \frac{dy}{dx}$.

5.2

Since the beam L parallel to y-axis, $\theta \neq \frac{\pi}{2}$. So $tan\theta = \frac{1}{tan(\pi/2 - \theta)} = \frac{1}{dy/dx} = \frac{dx}{dy}$.

To sum up, $tan\theta = \frac{dx}{dy}$

5.3

When $\phi \neq \frac{\pi}{2}$,

$$\begin{aligned} &\phi = 2\theta \\ \Rightarrow &tan\phi = tan(2\theta) = \frac{2tan\theta}{1 - tan^2\theta} \\ \Rightarrow &\frac{x}{y}\left(1 - \left(\frac{dx}{dy}\right)^2\right) = 2\left(\frac{dx}{dy}\right) \\ \Rightarrow &\ x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) = x \end{aligned}$$

When $\phi = \frac{\pi}{2}, y = 0, \theta = \frac{\pi}{4}, \frac{dx}{dy} = tan(\pi/4) = 1$. So $x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) - x = x - x = 0$

To sum up, the equation

$$x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) = x$$

always holds.

5.4

Set $w = x^2$, then

$$\begin{aligned} &x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) = x \\ \Rightarrow &\left(2x \cdot \frac{dx}{dy}\right)^2 + 4y\left(2x \cdot \frac{dx}{dy}\right) = 4x^2 \\ \Rightarrow &\left(\frac{dw}{dx} \cdot \frac{dx}{dy}\right)^2 + 4y\left(\frac{dw}{dx} \cdot \frac{dx}{dy}\right) = 4w \\ \Rightarrow &w = y\left(\frac{dw}{dy}\right) + \frac{1}{4}\left(\frac{dw}{dy}\right)^2 \end{aligned}$$

The straight-line solution to this equation is given by

$$w = c \cdot y + \frac{1}{4}c^2$$

the family of lines are parametrized by

$$\gamma(y, c) = \begin{pmatrix} y \\ c \cdot y + \frac{1}{4}c^2 \end{pmatrix}$$

And we can obtain that

$$\begin{aligned} \frac{\partial \gamma_1}{\partial y}\frac{\partial \gamma_1}{\partial c} &= \frac{\partial \gamma_2}{\partial y}\frac{\partial \gamma_2}{\partial c} \\ \Rightarrow & 1 \cdot (y + 0.5c) = 0 \\ \Rightarrow & c = -2y \end{aligned}$$

then $\gamma(y, c)\Big|_{c=-2y} = \begin{pmatrix} y \\ -2y \cdot y + \frac{1}{4}(-2y)^2 \end{pmatrix} = \begin{pmatrix} y \\ -y^2 \end{pmatrix}$

So the envelope of the straight line family is $w = -y^2$. While since $w = x^2$, this solution only give a point $(0, 0)$ and it cannot be a solution to $x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) = x$. So all possible solution to $w = y\left(\frac{dw}{dy}\right) + \frac{1}{4}\left(\frac{dw}{dy}\right)^2$ are

$$w = c \cdot y + \frac{1}{4}c^2$$

resubstitute $w = x^2$ we can obtain that the solution to $x\left(\frac{dx}{dy}\right)^2 + 2y\left(\frac{dx}{dy}\right) = x$ is

$$y = c(x^2 - \frac{1}{4c^2}) \quad ,(c > 0 \ \textit{according to diagram})$$

5.5

Parabola can be used to focus rays into a single point (its focus).

6

6.1

$$y_1(x) = 0 + \int_0^x (y_0(s))^2 + s^2 ds = \int_0^x (0)^2 + s^2 ds = \frac{x^3}{3}$$

$$y_2(x) = 0 + \int_0^x (y_1(s))^2 + s^2 ds = \int_0^x (\frac{s^6}{9}) + s^2 ds = \frac{x^7}{63} + \frac{x^3}{3}$$

$$y_3(x) = 0 + \int_0^x (y_2(s))^2 + s^2 ds = \int_0^x (\frac{s^7}{63} + \frac{s^3}{3})^2 + s^2 ds = \frac{x^{15}}{59535} + \frac{2x^{11}}{2079} + \frac{x^7}{63} + \frac{x^3}{3}$$

$$\begin{aligned} y_4(x) = 0 + \int_0^x (y_2(s))^2 + s^2 ds = \int_0^x (\frac{s^{15}}{59535} + \frac{2s^{11}}{2079} + \frac{s^7}{63} + \frac{s^3}{3})^2 + s^2 ds \\ = \frac{x^{31}}{109876902975} + \frac{4x^{27}}{3341878155} + \frac{662x^{23}}{10438212015} + \frac{82x^{19}}{37328445} + \frac{13x^{15}}{218295} + \frac{2x^{11}}{2079} + \frac{x^7}{63} + \frac{x^3}{3} \end{aligned}$$

6.2

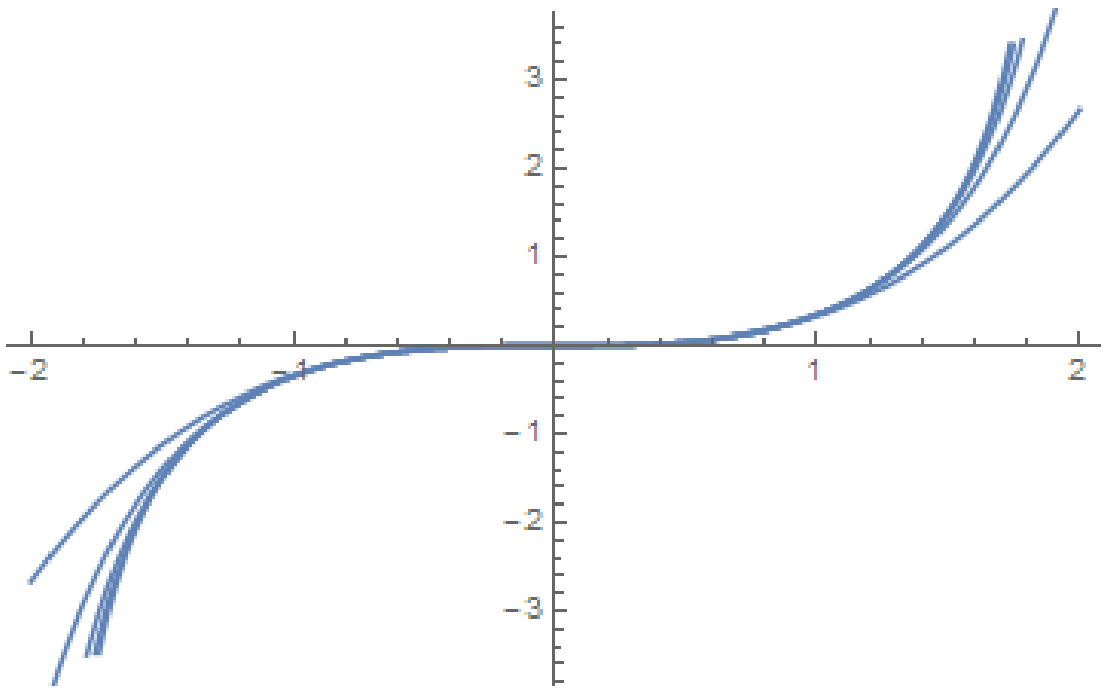


Figure 1: Numerical solution to (**) and y_1, y_2, y_3, y_4

Since numerical solution will have great error when x goes far away from 0, I just choose $[-2, 2]$ to have a look.(On the right-hand side of y-axis, from the bottom to top are numerical solution, y_4, y_3, y_2, y_1)