VV286 RC1

JIANG Yicheng

September 20, 2017

Separable Equations

Initial Value Problem (I.V.P.)

f is continuous in an interval $I_x \subset \mathbb{R}$; g is continuous in an interval $I_y \subset \mathbb{R}$; $\xi \in I_x, \eta \in I_y$

$$\frac{dy}{dx} = f(x)g(y), \qquad y(\xi) = \eta$$

Solution

1. $g(\eta) \neq 0$

$$\int_{\eta}^{y} \frac{ds}{g(s)} = \int_{\xi}^{x} f(t)dt \qquad \text{(Unique solution)}$$

2. $g(\eta) = 0$

   2.1 Obvious solution

   $$y(x) = \eta$$

   2.2 Check

   $$\int_{\eta}^{y} \frac{ds}{g(s)}$$

   in a small neighbourhood of η

Example

$$\frac{dy}{dx} = x^4 y + x^4 y^4, \qquad y(0) = 1$$

$$y(0) = 0? \;\; y(0) = -\frac{1}{2}? \;\; y(0) = -1? \;\; y(0) = -2?$$

Solution

$$\int x^4 dx = \int \frac{1}{y + y^4} dy$$
$$\frac{1}{5}x^5 = \int \frac{1}{y} - \frac{y^2}{1 + y^3} dy$$
$$\frac{1}{5}x^5 = \ln|y| - \frac{1}{3}\ln|1 + y^3| + C$$
$$e^{3x^5/5} = C\left(\frac{y^3}{1 + y^3}\right)$$

$$C = 2 \;\; (y(0) = 1); C = \frac{7}{8} \;\; (y(0) = -2); C = -7 \;\; (y(0) = -\frac{1}{2})$$

Equilibrium solution

$$x_{equi}(t) = constant$$

Steady-state solution

$$x_{ss}(t) = \lim_{t \to \infty} x(t)$$

Transient solution

$$x(t) - x_{ss}$$

Linear Equations

A general linear, first-order ODE on an open interval $I \in \mathbb{R}$

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$

where a_0, a_1, f is continuous, real-valued functions on I.

Solution

$$y_{\text{inhom}} = y_{\text{part}} + C \cdot y_{\text{hom}}$$

where y_{hom} is one solution of $a_1(x)y' + a_0(x)y = 0$

How to find a y_{part}?

Variation of Parameters

Set $y_{\text{inhom}} = c(x)y_{\text{hom}}(x)$, then

$$a_1(x)c'(x)y_{\text{hom}}(x)+\underbrace{a_1(x)c(x)y'_{\text{hom}}(x) + a_0(x)c(x)y_{\text{hom}}(x)}_{=0} = f(x)$$

solve $c(x)$.

Duhamel's Principle

Let $I \subset \mathbb{R}$ be an open interval, $x_0 \in \bar{I}$, and a_0, a_1, f continuous, real-valued functions on $\bar{I}$, where $a_1(x) \neq 0$ for all $x \in \bar{I}$. Let y_ξ solve the initial value problem

$$a_1(x)y' + a_0(x)y = 0, \quad y_\xi(\xi) = \frac{1}{a_1(\xi)}$$

for $x \in \bar{I}$. Then

$$y(x) = \int_{x_0}^{x} f(\xi) y_\xi(x) d\xi$$

solves

$$a_1(x)y' + a_0(x)y = f(x), \quad y(x_0) = 0$$

(Choose $c(\xi)$ which leads to $y_\xi(\xi) = c(\xi)y_{hom}(\xi) = \dfrac{1}{a_1(\xi)}$.)

Example

$$\frac{dy}{dx} = 3y + x$$

Solution

1. $\frac{dy}{dx} - 3y = 0 \Rightarrow y_{\text{hom}} = c \cdot e^{3x}$
2. Set $y_{\text{part}} = c(x)e^{3x}$, then $c'(x)e^{3x} = x$. So

$$\begin{aligned} c(x) &= \int xe^{-3x}dx = -\frac{1}{3}\int xd(e^{-3x}) \\ &= -\frac{1}{3}\left(xe^{-3x} - \int e^{-3x}dx\right) \\ &= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} \end{aligned}$$

Finally, $y = c \cdot e^{3x} - \frac{1}{3}x - \frac{1}{9}$

Transformable Equations

$y' = f(ax + by + c); b \neq 0$

$$u(x) = ax + by(x) + c$$

$y' = f(y/x)$

$$u(x) = \frac{y(x)}{x}$$

$$y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$$

$$u(x) = a_1 x + b_1 y(x) + c_1, v(x) = a_2 x + b_2 y(x) + c_2$$

$$x = \frac{b_2(u - c_1) - b_1(v - c_2)}{a_1 b_2 - a_2 b_1}$$

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = (a_1 + b_1 \frac{dy}{dx}) \frac{b_2(du/dv) - b_1}{a_1 b_2 - a_2 b_1}$$

$$\frac{du}{dv} = (a_1 + b_1 f\left(\frac{u}{v}\right)) \frac{b_2(du/dv) - b_1}{a_1 b_2 - a_2 b_1}$$

$$\frac{du}{dv} = b_2 g\left(\frac{u}{v}\right) \frac{du}{dv} - b_1 g\left(\frac{u}{v}\right)$$

$$\frac{du}{dv} = h\left(\frac{u}{v}\right)$$

Example

$$y' = \frac{x - y}{x + y}$$

Solution

Set $u(x) = \dfrac{y(x)}{x}$, then $\dfrac{1-u}{1+u} = y' = u'x + u$. So

$$\int \frac{1+u}{1-2u-u^2} du = \int \frac{1}{x} dx$$

$$\ln(u+1+\sqrt{2}) + \ln(u+1-\sqrt{2}) = -2\ln x + C$$

$$(y/x + (1+\sqrt{2}))(y/x + (1-\sqrt{2})) = \frac{C}{x^2}$$

$$y^2 + 2xy - x^2 = C$$

$$y = -x \pm \sqrt{2x^2 - C}$$

$y' + gy + hy^\alpha = 0, \alpha \neq 1$ (Bernoulli's equation)

$$u(x) = (y(x))^{1-\alpha}$$

$$\begin{aligned} y' + gy + hy^\alpha = 0 &\Rightarrow (1-\alpha)y^{-\alpha}y' + (1-\alpha)gy^{1-\alpha} + (1-\alpha)h = 0 \\ &\Rightarrow (y^{1-\alpha})' + (1-\alpha)gy^{1-\alpha} + (1-\alpha)h = 0 \\ &\Rightarrow u' + (1-\alpha)gu + (1-\alpha)h = 0 \end{aligned}$$

We assume $y(x) > 0$ for all $x \in I$ and each strictly positive solution $u(x)$ can yield a strictly positive solution

$$y_+(x) = (u(x))^{1/(1-\alpha)}$$

Note

1. $\alpha > 0$, $y = 0$
2. $\alpha \in \mathbb{Z}, \alpha \equiv 1 \pmod 2$, $y_- = -y_+$
3. $\alpha \in \mathbb{Z}, \alpha \equiv 0 \pmod 2$, $y_- = -|u(x)|^{1/(1-\alpha)}$

$y' + gy + hy^2 = k$ (Ricatti's equation)

1. Guess or given a solution ϕ
2. For other solution y, set $u = y - \phi$, then

$$\begin{cases} y' + gy + hy^2 = k \\ \phi' + g\phi + h\phi^2 = k \end{cases}$$

$$\Rightarrow (y' - \phi') + g(y - \phi) + h(y - \phi)(y + \phi) = 0$$

$$\Rightarrow u' + gu + hu(u + 2\phi) = 0$$

$$\Rightarrow u' + (g + 2\phi h)u + hu^2 = 0$$

Example

$$\frac{dy}{dx} = x^4 y + x^4 y^4, \qquad y(0) = 1$$

JOINT INSTITUTE 交大密西根学院

Solution

$y(x) = 0$ is not a solution.

$$y' - x^4y - x^4y^4 = 0 \xrightarrow{\cdot(-3y^{-4})} (y^{-3})' + 3x^4(y^{-3}) + 3x^4 = 0$$

Set $u = y^{-3}$, then $y = u^{-1/3}$.

$$u' + 3x^4u = 0 \Rightarrow u_{\text{hom}} = c \cdot e^{-\frac{3}{5}x^5}$$

Set $u_{\text{part}} = c(x) \cdot e^{-\frac{3}{5}x^5}$, then

$$c'(x) = -3x^4e^{\frac{3}{5}x^5} \Rightarrow c(x) = -e^{\frac{3}{5}x^5}$$

So $u(x) = c \cdot e^{-\frac{3}{5}x^5} - 1$. Since $y(0) = 1$,

$$y = \frac{1}{\sqrt[3]{2e^{-3x^5/5} - 1}}$$

$$h(x,y)y' + g(x,y) = 0$$

Another view

$$h(x,y)y' + g(x,y) = 0 \Rightarrow \left\langle \begin{pmatrix} 1 \\ y' \end{pmatrix}, \begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix} \right\rangle = 0$$

$\begin{pmatrix} 1 \\ y' \end{pmatrix}$: tangent vector of integral curve

Integral curve is perpendicular to the vector field

$$F^{\perp} : \mathbb{R}^2 \mapsto \mathbb{R}^2, \quad F^{\perp}(x,y) = \begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix}$$

Equipotential Line

Solution is $U(x, y)$=constant, where $U : \mathbb{R}^2 \mapsto \mathbb{R}$ is a potential function of the conservation vector field

What do we need to do?

Find a potential function $U(x, y)$ whose gradient at each point is parallel to the vector $\begin{pmatrix} g(x, y) \\ h(x, y) \end{pmatrix}$ i.e.

$$\nabla U(x, y) = M(x, y) \cdot F^{\perp}(x, y)$$

Integrating factors (Euler Multipliers)

Let g, h be continuous functions on an open set $D \subset \mathbb{R}^2$. A function M with $M(x, y) \neq 0$ defined on D is said to be an integrating factor or Euler multiplier for the differential equation

$$h(x, y)y' + g(x, y) = 0$$

if the vector field

$$F^{\perp}(x, y) = \begin{pmatrix} M(x, y)g(x, y) \\ M(x, y)h(x, y) \end{pmatrix}$$

has a potential function.

Requirement

If D is open, simply connected and $g, h, M \in C^1(D)$,

$$\frac{\partial M(x,y)g(x,y)}{\partial y} = \frac{\partial M(x,y)h(x,y)}{\partial x} \quad \text{(Rotation is zero)}$$

i.e.

$$\frac{\partial M}{\partial y} g + M \frac{\partial g}{\partial y} = \frac{\partial M}{\partial x} h + M \frac{\partial h}{\partial x}$$

Assumption

1. M depends only on x or only on y
2. M depends only on $x \cdot y$

Example

$$y' = \frac{x - y}{x + y}$$

Solution

$$M_y(y-x)+M=M_x(y+x)+M \Rightarrow M=\text{constant.}$$

$$\begin{aligned}
&\frac{\partial U}{\partial x}=y-x,\frac{\partial U}{\partial y}=y+x\\
\Rightarrow &U=\int(y-x)dx=yx-\frac{1}{2}x^2+C(y),\frac{\partial U}{\partial y}=y+x\\
\Rightarrow &x+\frac{\partial C(y)}{\partial y}=y+x\\
\Rightarrow &C(y)=\frac{1}{2}y^2\\
\Rightarrow &U(x,y)=\frac{1}{2}y^2+xy-\frac{1}{2}x^2\\
\Rightarrow &y^2+2xy-x^2=C
\end{aligned}$$