VV286 Review 2

JIANG Yicheng

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Evaluation of Real Integrals

$$\int_{\mathbb{R}/\mathbb{R}^+} f(x)dx = A$$

1. Extend the real domain to complex domain.
   1.1 Usually you only need to change $x \in \mathbb{R}$ to $z \in \mathbb{C}$
   1.2 For $\sin x, \cos x$, do integral for e^{iz}
2. Find poles for the function $f(z)$
3. Decide the contour and the branch (for $x^\alpha, \ln x$) if needed.
4. Calculate the residue for poles in the contour.
   4.1 During an exam, you may calculate residue for all poles if you cannot decide the contour at first.
5. Apply residue theorem or Cauchy's theorem.

$$\int_{\mathcal{C}} f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f$$

For right hand side

$$\mathrm{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right)$$

$$\mathrm{RHS} = a_1 + b_1 i$$

For left hand side

$$\mathrm{LHS} = A + 0 + a_2 + b_2 i + (a_3 + b_3 i)A$$

$$\mathrm{LHS} = \mathrm{RHS} \Rightarrow A = \frac{a_1 - a_2}{1 + a_3}$$

Singularities

Let $\Omega \subset \mathbb{C}$ be ***open***, $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic. Then f is said to have a point singularity or isolated singularity at z_0.

1. The singularity is said to be removable if there exists an analytic continuation $\tilde{f} : \Omega \to \mathbb{C}$. (such $\tilde{f}$ is unique)
2. The singularity is said to be a ***pole*** if $g = 1/f$ is holomorphic on $\Omega \setminus \{z_0\}$ and has a removable singularity at z_0 such that the analytic continuation $\tilde{g}$ of g satisfies $\tilde{g}(z_0) = 0$.
3. The singularity is said to be essential if it is neither removable nor a pole.

How to judge?

Removable Singularity

Whether $\lim_{z \to z_0} f$ exists.

e.g.

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

So $f(z) = \dfrac{\sin z}{z}$ has removable singularity at $z = 0$.

Pole (Informal way)

$f(z) = \dfrac{g(z)}{h(z)}$, then all z_0 such that $h(z_0) = 0$ may be a pole.

(All z_0 such that $g(z_0) = 0$ may be a zero.)

Multiplicity of Poles

If $f : \Omega \mapsto \mathbb{C}$ has a pole at $z_0 \in \Omega$, then in a neighborhood U of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(z) = (z - z_0)^{-n} h(z) \quad \text{for all } z \in U$$

The integer n is called the multiplicity or order of the pole of f .
If $n = 1$, we say that the pole is ***simple***.

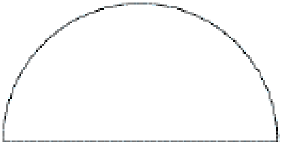
$$f(p) = \frac{p + e^{-\frac{\pi}{2}p}}{(1 + p^2)^2}$$

Principle Part

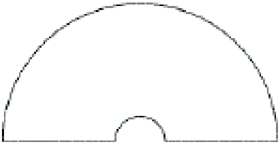
$$
\begin{aligned}
f(z) &= (z-z_0)^{-n}\sum_{m=0}^{\infty} b_m (z-z_0)^m \\
&= \frac{b_0}{(z-z_0)^n} + \frac{b_1}{(z-z_0)^{n-1}} + \cdots + \frac{b_{n-1}}{z-z_0} + \sum_{m=n}^{\infty} b_m (z-z_0)^{m-n} \\
&= \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}}_{\text{Principle part}} + \sum_{m=0}^{\infty} a_0 (z-z_0)^m
\end{aligned}
$$

a_{-1} is the residue of f at z_0

Contour 1—-Semi-circle



Semicircle



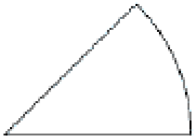
Indented semicircle

Most common ones.
You have used them to solve

$$\int_0^\infty \frac{\sin x}{x}dx, \quad \int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}dx, \quad \int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx, \quad \int_{-\infty}^\infty \frac{dx}{1+x^4}$$

$$\int_0^\infty \frac{x\sin x}{(x^2+4)^2}dx, \quad \int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}}dx, \quad \int_0^\infty \frac{1-\cos x}{x^2}dx$$

Contour 2—-Sector

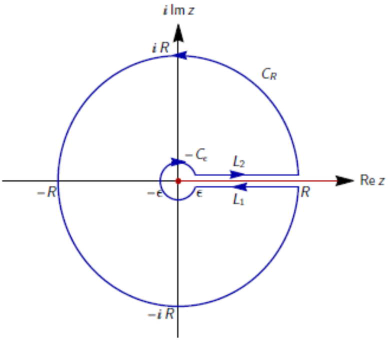


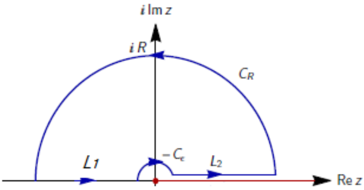
Sector

Similar to Semi-circle.
May be useful for integral containing $\sin(x^n)$, $\cos(x^n)$
(choose central angle=$\frac{\pi}{2n}$)
You have used it to solve

$$\int_0^\infty \sin x^2 dx, \quad \int_0^\infty \cos x^2 dx$$

Contour 3





Used for integral containing $\sqrt{x}$, $\ln x$ (For these two contours, the branch we choose is $\mathbb{C} \setminus \mathbb{R}_{-}^{0}$, so $\phi \in (0, 2\pi)$)

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx, \quad \int_0^\infty \frac{\ln x}{x^2 + a^2} dx$$

Residue Calculus for Functions with Branch Points

Let P and Q be polynomials of degree m and n, respectively, where $n \geqslant m+2$. If $Q(x) \neq 0$ for $x > 0$, if Q has a zero of order at most 1 at the origin and if

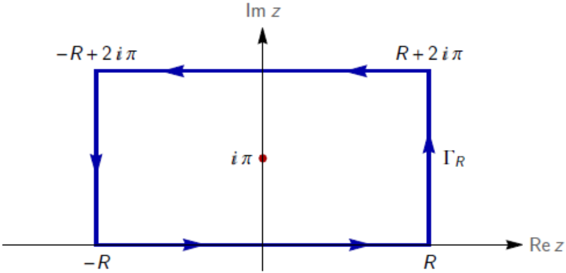
$$f(z) = \frac{z^\alpha P(z)}{Q(z)}, \quad 0 < \alpha < 1$$

then

$$\text{p.v.} \int_0^\infty \frac{x^\alpha P(x)}{Q(x)} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{j=1}^{k} \operatorname{res}_{z_j} f$$

This theorem is obtained by using the contour in the left on last slide. Pay attention to the branch. Also pay attention to its requirement.

Contour 4



$$\int_0^\infty \frac{e^{ax}}{1+e^x}dx$$

Lapalace Transform

$$(\mathscr{L}f)(p) = \int_0^\infty e^{-pt} f(t)dt$$

$$\text{Convolution : } (\mathscr{L})^{-1}\Big((\mathscr{L}f)\cdot(\mathscr{L}g)\Big) = f * g = \int_0^t f(t-s)g(s)ds$$

$$\text{Bromwich Integral: } (\mathscr{M}F)(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pt}(\mathscr{L}f)(p)dp$$

Fourier Transform

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}d\xi$$

Solving an ODE with the Laplace Transform

To deal with discontinuous inhomogeneities and even inhomogeneities that are not functions at all.

$$ay'' + by' + cy = f(x), \;\; y(0) = y_0, \;\; y'(0) = y_1$$

1. Apply the Laplace transform to both sides of the ODE/IVP;

$$(\mathscr{L}f')(p) = p \cdot (\mathscr{L}f)(p) - f(0)$$
$$(\mathscr{L}f'')(p) = p^2(\mathscr{L}f)(p) - p \cdot f(0) - f'(0)$$

$$(ap^2 + bp + c)Y - (ap + b)y_0 - ay_1 = (\mathscr{L}f)(p)$$

2.

$$Y = (\mathscr{L}f)(p) \cdot \frac{1}{ap^2 + bp + c} + \frac{ay_0 p + by_0 + ay_1}{ap^2 + bp + c}$$

Find $g(x)$ such that $(\mathscr{L}g)(p) = \dfrac{1}{ap^2 + bp + c}$

The function g is called a Green's function for the differential equation.

3.

Use transform table and apply convolution to find inverse Laplace transform.

Note

1. Since we usually need to use convolution to find inverse Lapalace transform, we don't need to take Lapalace transform for those on the right hand side of the equation.
2. If f or g contains $H(x)$, don't insert $x - s$ into $H(x)$

Calculate the Fourier transform

1. Do direct integral
2. Do direct integral through contour integration in the complex plane
3. Construct ODE

Solving an ODE with the Fourier Transform

$$ay'' + by' + cy = f(x)$$

We do not impose any initial conditions but instead require that

$$\lim_{x \to \pm\infty} y(x) = 0$$

and assume that y is absolutely integrable.

Decay

Let $\Omega \subset \mathbb{R}$ be bounded and $f : \mathbb{R} \setminus \Omega \to \mathbb{C}$.

1. If $f(x) = O(x^{-n})$ as $|x| \to \infty$ for some $n > 0$, then f is said to have polynomial decay at infinity.
2. If $f(x) = O(x^{-n})$ as $|x| \to \infty$ for all $n > 0$, then f is said to have faster-than-polynomial decay at infinity.
3. If $f(x) = O(e^{-b|x|})$ as $|x| \to \infty$ for some $b > 0$, then f is said to have exponential decay at infinity.