Ve203 Discrete Mathematics (Fall 2016)

Assigment 2: Natural Numbers and Induction

Date Due: 4:00 PM, Thursday, the 29$^{\text{th}}$ of September 2016

This assignment has a total of **(25 Marks)**.

Exercise 2.1 $2+2=4$
Let $(\mathbb{N}, \text{succ})$ be a realization of the natural numbers with successor function succ. We define addition of the numbers 0 and $1 := \text{succ}(0)$ by setting

$$n + 0 := n, \qquad n + 1 := \text{succ}(n), \qquad n \in \mathbb{N}.$$

i) Formulate an inductive definition for $n+m$, where $m, n \in \mathbb{N}$.
(2 Marks)

ii) Set $2 := \text{succ}(1)$, $3 := \text{succ}(2)$, $4 := \text{succ}(3)$. Verify that[1]

$$2 + 2 = 4.$$

(2 Marks)

iii) Prove by induction that $n+m = m+n$ for all $m, n \in \mathbb{N}$.
(2 Marks)

Exercise 2.2 Straightforward Induction
Let (a_n) be the sequence defined by

$$a_1 = 1, \qquad a_2 = 8 \qquad \text{and} \qquad a_n = a_{n-1} + 2a_{n-2}, \qquad n \geq 3.$$

Prove that for all $n > 0$, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.
(2 Marks)

Exercise 2.3 The Fifth Peano Axiom
Prove that the induction axiom implies the well-ordering principle.
(3 Marks)

Exercise 2.4 Is a direct induction approach always successful?
Try to prove by induction that for any real number $x > -1$ and any $n \in \mathbb{N}$, $(1+x)^n \geq nx$. If you encounter difficulties, modify your approach.
(2 Marks)

Exercise 2.5 Strong Induction
Use strong induction to show that every $n \in \mathbb{N} \setminus \{0\}$ can be written as a sum of distinct powers of 2, i.e., as a sum of a subset of integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$ etc.

(*Hint*: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. When it is even, note that $(k+1)/2 \in \mathbb{N}$.)
(3 Marks)

Exercise 2.6 Structural Induction
Let $S \subset \mathbb{N}^2$ be defined by

- $(0,0) \in S$,
- $(a,b) \in S \Rightarrow \big(((a+2, b+3) \in S) \wedge ((a+3, b+2) \in S)\big)$.

Use structural induction to show that $(a,b) \in S$ implies $5 \mid (a+b)$.
(3 Marks)

[1] The proof of 2+2=4 is due to the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716).

Exercise 2.7 Some easy practice of relation properties

Determine whether the relation R on the set of all integers is reflexive, symmetric and/or transitive, where $(x, y) \in R$ if and only if

i) $x + y = 0$

ii) $2 \mid (x - y)$

iii) $xy = 0$

iv) $x = 1$ or $y = 1$

v) $x = \pm y$

vi) $x = 2y$

vii) $xy \geq 0$

viii) $x = 1$

(8 Marks)