Ex. 1

1.

a)

On one hand, since $T_{n+1}(x_i) = \cos((n+1)\theta_i) = \cos\left(\frac{2i+1}{2}\pi\right) = 0$

$$\lim_{x\to x_i} \frac{T_{n+1}(x)}{(x-x_i)T'_{n+1}(x_i)} = \lim_{x\to x_i} \frac{T_{n+1}(x) - T_{n+1}(x_i)}{(x-x_i)T'_{n+1}(x_i)} = \frac{T'_{n+1}(x_i)}{T'_{n+1}(x_i)} = 1$$

On the other hand,

$$\ell_i(x_i) = \prod_{j\neq i} \frac{x_i - x_j}{x_i - x_j} = 1$$

According to Theorem 5.113, the polynomial $P(x) \in \mathbb{R}_n$ determined by $x_0, x_1, \cdots, x_n$ is unique, these two basis must be equivalent, and therefore

$$\ell_i(x) = \frac{T_{n+1}(x)}{(x-x_i)T'_{n+1}(x_i)}$$

b)

Since $T_{n+1}(x) = \cos((n+1)\arccos x)$, for $x \in [-1,1]$, set $x = \cos\theta$ where $\theta \in [0,\pi]$,

$$\begin{aligned} T'_{n+1}(x) &= -\sin((n+1)\arccos x)\cdot\left(-\frac{n+1}{\sqrt{1-x^2}}\right) \\ &\xlongequal{x=\cos\theta} \frac{n+1}{\sqrt{1-\cos^2\theta}}\sin((n+1)\theta) \\ &= \frac{n+1}{\sin\theta}\sin((n+1)\theta) \end{aligned}$$

Furthermore,

$$T'_{n+1}(x_k) = \frac{n+1}{\sin\theta_k}\cos\left(\frac{2k+1}{2}\pi\right) = (-1)^k\frac{n+1}{\sin\theta_K}$$

To sum up,

$$T'_{n+1}(x) = \frac{n+1}{\sqrt{1-\cos^2\theta}}\sin((n+1)\theta), \quad T'_{n+1}(x_k) = (-1)^k\frac{n+1}{\sin\theta_K}$$

c)

Using the conslusion above,

$$\begin{aligned}\sum_{i=0}^{n}|\ell_i(1)| =&\sum_{i=0}^{n}\left|\frac{T_{n+1}(1)}{(1-x_i)T'_{n+1}(x_i)}\right|\\ =&\sum_{i=0}^{n}\left|\frac{\cos(n\arccos(1))\cdot\sin\theta_i}{(1-\cos\theta_i)\cdot(n+1))}\right|\\ =&\frac{1}{n+1}\sum_{i=0}^{n}\left|\tan\left(\frac{\pi-\theta_i}{2}\right)\right|\\ \geqslant&\frac{1}{n+1}\sum_{i=0}^{n}\cot\left(\frac{\theta_i}{2}\right)\end{aligned}$$

So the statement holds.

2.

a)

Since $\theta_k=\dfrac{(2k+1)}{2(n+1)}\pi\in(0,\pi)$, $\dfrac{\theta_k}{2}\in(0,\dfrac{\pi}{2})$. Known that $\cot x$ decreases on $(0,\pi)$ and $\cot x>0$ on $(0,\pi)$

$$\begin{aligned}\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,dt\leqslant&\frac{\theta_{k+1}-\theta_k}{2}\cdot\max_{t\in[\frac{\theta_k}{2},\frac{\theta_{k+1}}{2}]}\cot t\\ \leqslant&\frac{\theta_{k+1}-\theta_k}{2}\cot\frac{\theta_k}{2}\end{aligned}$$

b)

$$\begin{aligned}\sum_{k=0}^{n}\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,dt\leqslant&\sum_{k=0}^{n}\frac{\theta_{k+1}-\theta_k}{2}\cot\frac{\theta_k}{2}\\ =&\frac{1}{2}\cdot\frac{2\pi}{2(n+1)}\sum_{k=0}^{n}\cot\frac{\theta_k}{2}\end{aligned}$$

So

$$\frac{\pi}{2(n+1)}\sum_{k=0}^{n}\cot\frac{\theta_k}{2}\geqslant\sum_{k=0}^{n}\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,dt$$

c)

For $i=n$, we observed that

$$\int_{\frac{\theta_n}{2}}^{\frac{\pi}{2}}\cot t\,dt<\frac{\pi-\theta_n}{2}\cot\frac{\theta_n}{2}=\frac{\pi}{4(n+1)}\cot\frac{\theta_n}{2}<\frac{\pi}{2(n+1)}\cot\frac{\theta_n}{2}$$

and therefore

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}\cot\frac{\theta_k}{2} \geqslant &\frac{2}{\pi}\sum_{k=0}^{n-1}\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,dt+\frac{2}{\pi}\int_{\frac{\theta_n}{2}}^{\frac{\pi}{2}}\cot t\,dt\\ =&\frac{2}{\pi}\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}}\cot t\,dt\end{aligned}$$

So we that the statement holds.

3.

$$\begin{aligned}\Lambda_n =&\max_{x\in[-1,1]}\sum_{i=0}^{n}|\ell_i(x)| \geqslant \sum_{i=0}^{n}|\ell_i(1)| \geqslant \frac{1}{n+1}\sum_{k=0}^{n}\cot\frac{\theta_k}{2}\\ \geqslant&\frac{2}{\pi}\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}}\cot t\,dt=\frac{2}{\pi}\ln|\sin t|\Big|_{\frac{\theta_0}{2}}^{\frac{\pi}{2}}\\ =&\frac{2}{\pi}\Big(0-\ln(\sin\frac{\pi}{2(n+1)})\Big)\\ >&-\frac{2}{\pi}\ln(\frac{\pi}{4(n+1)})\\ >&\frac{2}{\pi}\ln n\end{aligned}$$

So $\Lambda_n \geqslant \dfrac{2}{\pi}\ln n$

Ex. 3

1.

Use induction to prove that $(\cos\theta)^k = \sum\limits_{i=0}^{k} a_i\cos(i\theta)$

1. For $k=0,1$, it is easy to see the statement is true.
2. Assume that for $k=m$ the statement is true, then

$$\begin{aligned}(\cos\theta)^{m+1} =&\cos\theta\cdot\sum_{i=0}^{m}a_i\cos(i\theta)\\ =&a_0\cos\theta+\sum_{i=1}^{m}a_i\frac{1}{2}(\cos((i+1)\theta)+\cos((i-1)\theta))\\ =&\sum_{i=0}^{m+1}b_i\cos(i\theta)\end{aligned}$$

So the statement also holds for $k=m+1$.

To sum up, $(\cos\theta)^k = \sum\limits_{i=0}^{k} a_i\cos(i\theta)$, i.e. $(\cos\theta)^k\in T_n$, and therefore $Q_n(x)=P_n(\cos\theta)$ is a linear bijection from $\mathbb{R}_n[x]$ in to T_n by setting $x=\cos\theta$.

2.

For $f(x)$, we can find some $P(x) \in \mathbb{R}_n[x]$ such that

$$P(x_k) = f(x_k)$$

where $x_k = \cos\dfrac{(2k+1)\pi}{2(n+1)}$. Then by setting $x = \cos\theta$, according to Ex. 3.1, $P(\cos\theta)$ corresponding to a $Q_n(\theta) \in T_n$. So the existence of $P(x)$ leads to the existence of $Q_n(\theta)$.

3.

To find Q_n, we need to solve $a_0, \cdots, a_n$ such that

$$Q_{\theta_k} = f(\cos(\theta_k)) \Rightarrow \sum_{k=0}^{n} a_k \cos(k\theta_k) = f(\cos\theta_k)$$

which forms a linear equations set.

4.

Denote $P(x) = \sum\limits_{k=0}^{n} a_k \cos(k \arccos x)$, which is a polynomial with degree of n.

For $x = x_k = \cos\theta_k$, $k = 0, 1, \cdots, n$, $f(x) - P(x) = 0$ while $\dfrac{\cos((n+1)\arccos x)}{2^n(n+1)!} f^{(n+1)}(\xi) = 0$ for any ξ. So the result holds.

Assume $x \neq x_k, k = 0, 1, \cdots, n$, construct the polynomial

$$H(t) = P(t) + \frac{f(x) - P(x)}{\cos((n+1)\arccos x)} \cos((n+1)\arccos t)$$

and define

$$g(t) = f(t) - H(t) = f(t) - P(t) - \frac{f(x) - P(x)}{\cos((n+1)\arccos x)} \cos((n+1)\arccos t)$$

It is easy to see that for $k = 0, 1, \cdots, n$

$$g(x_k) = f(x_k) - H(x_k) = f(x_k) - P(x_k) = 0$$

and

$$g(x) = f(x) - H(x) = 0$$

so $g(t)$ has $n+2$ distinct roots in $(-1, 1)$, according to Roll's theorem, there exists some $\theta \in (a, b)$ such that $g^{(n+1)}(\xi) = 0$, i.e.

$$f^{(n+1)}(\xi) - P^{(n+1)}(\xi) = \frac{f(x) - P(x)}{\cos((n+1)\arccos x)} \frac{d^{n+1}}{dt^{n+1}} \cos((n+1)\arccos t)\Big|_{t=\xi}$$

Since $P(x)$ is a polynomial with degree of n, $P^{(n+1)}(\xi) = 0$. Similarly,

$$\frac{d^{n+1}}{dt^{n+1}} \cos((n+1)\arccos t)\Big|_{t=\xi} = a_{n+1}(n+1)!$$

where a_{n+1} is the coefficient of t^{n+1} in $\cos((n+1)\arccos t)$. Since

$$\begin{aligned}&\cos((n+1)\arccos t)+\cos((n-1)\arccos t)\\=&2\cos(n\arccos t)\cos(\arccos t)\\=&2t\cos(n\arccos t)\end{aligned}$$

and $\cos(1\cdot\arccos t)=t$, we can obtain that $a_{n+1}=2^n$. So

$$f^{(n+1)}(\xi)\cdot\cos((n+1)\arccos t)\cdot\frac{1}{2^n(n+1)!}=f(x)-P(x)$$

i.e. $\forall\theta\in(-\pi,\pi),\exists\xi\in(-1,1)$ such that

$$F(\theta)-Q_n(\theta)=f(\cos\theta)-P(\cos\theta)=\frac{\cos(n+1)\theta}{2^n(n+1)!}f^{(n+1)}(\xi)$$