VV471— Introduction to Numerical Methods

Assignment 8

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Reminders

- Write in a neat and legible handwriting or use $\LaTeX$
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Legendre polynomials*

Let $(Q_n)_{n\in\mathbb{N}}$ be the sequence of polynomials defined by $Q_n(x) = \frac{1}{2^n n!}\left((x^2-1)^n\right)^{(n)}$ over $[-1,1]$. This sequence defines the Legendre polynomials.

1. Using the constant weight function $w(x) = 1$ over $(-1,1)$, show that $(Q_n)_{n\in\mathbb{N}}$ defines a sequence of orthogonal polynomials.
2. Show that $Q_n(-x) = (-1)^n Q_n(x)$.
3. Show that for any $x \in [-1,1]$, the Legendre polynomials obey the recurrence relation

$$(n+1)Q_{n+1}(x) = (2n+1)xQ_n(x) - nQ_{n-1}(x).$$

   Hint: use elements from the proof of proposition 5.53.

* 4. Prove that

$$Q_n(x) = \sum_{i=0}^{n}(-1)^i \binom{n}{i}^2 \left(\frac{1+x}{2}\right)^{n-i}\left(\frac{1-x}{2}\right)^i.$$

Ex. 2 — *Interpolation*

Let f be a continuous function for which we know the eight points deifned in the following table.

x	-5	-1	0	1	3	5	10	12
$f(x)$	781	5	1	1	61	521	9091	19141

Determine $f(2)$.

Ex. 3 — *Newton's form of the interpolation polynomial*

Let f be a continuous function and P^n be its interpolation polynomial in the points $x_0, \cdots, x_n$.

1. Let $P^0(x) = f(x_0)$ be the interpolation polynomial in a single point x_0.
   a) Show that for two points x_0 and x_1, $P^1(x) = P^0(x) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$.
   b) Determine a polynomial R of degree at most two, such that $P^2(x) = P^1(x) + R(x)$, for three nodes x_0, x_1, and x_2.

c) Prove by induction that

$$P^j(x) = P^{j-1}(x) + a_j \prod_{k=0}^{j-1}(x - x_k), \quad \text{where } a_j \text{ only depends on } x_0, \cdots, x_j.$$

2. Show that

$$P^n(x) = f(x_0) + \sum_{j=1}^{n} a_j \prod_{k=0}^{j-1}(x - x_k).$$

* 3. Denoting a_j by $f[x_0, \cdots, x_j]$, prove that for any $k > 0$,

$$\begin{cases} f[x_k] = f(x_k) \\ f[x_0, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0} \end{cases}$$

4. Write the pseudocode of a clear algorithm to compute $P^n(x)$ when given $n+1$ nodes $x_0, \cdots, x_n$ and the value of f at those nodes.

We now consider the case of equidistant nodes, i.e. $x_i = x_0 + ih$, for any $0 \leq i \leq n$, and some $h \in \mathbb{R}_+^*$. Denoting $f[x_i]$ by f_i, $0 \leq i \leq n$, we recursively define the operator ∇ such that

$$\begin{cases} \nabla^0 f_i = f_i \\ \nabla^{k+1} f_i = \nabla^k f_{i+1} - \nabla^k f_i. \end{cases}$$

5. Show that for all $i, k \in \mathbb{N}$, $f[x_i, \cdots, x_{i+k}] = \frac{1}{k!h^k} \nabla^k f_i$.

* 6. Observing that $\binom{s}{k} = \frac{1}{k!} \prod_{j=0}^{k-1}(s - j)$, prove that $P^n(x) = \sum_{k=1}^{n} \binom{s}{k} \nabla^k f_0 + f_0$, where $s = \frac{x - x_0}{h}$.

7. Write the pseudocode of an algorithm which takes a step h as input, a number of nodes, the value of f at each of those nodes, and a value x. The algorithm should return $P^n(x) \approx f(x)$.