VE203
Assignment 2

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1 $2 + 2 = 4$

1.1

We define the function $\cdot + \cdot : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ has the following properties:

1. $n + 0 = n$
2. For $m \neq 0$, set $m = succ(m')$, $n + m = succ(n) + m'$

1.2

Since $n + 1 = succ(n), 2 = succ(1), 3 = succ(2), 4 = succ(3)$, then

$$2 + 2 = 2 + succ(1) = succ(2) + 1 = 3 + 1 = succ(3) = 4$$

So we get that $2 + 2 = 4$

1.3

We first prove that $\forall m, n \in \mathbb{N}, succ(n + m) = succ(n) + m$
$\forall n \in \mathbb{N}$,

1. When $m = 0$, $succ(n + m) = succ(n + 0) = succ(n) = succ(n) + 0 = succ(n) + m$ so the statement holds when $m = 0$
2. Assume that the statement holds when $m = m_0$, i.e. $succ(n + m_0) = succ(n) + m_0$, then according to definition of addition

$$\begin{aligned} succ(n + succ(m_0)) &= succ(succ(n) + m_0) \\ &= succ(succ(n)) + m_0 \\ &= succ(n) + succ(m_0) \end{aligned}$$

so the statement also holds when $m = succ(m_0)$.
To sum up, $\forall m, n \in \mathbb{N}, succ(n + m) = succ(n) + m$. Then we try to prove that $\forall n \in \mathbb{N}, n + 0 = 0 + n$

1. When $n = 0$, $n + 0 = 0 + 0 = 0 + n$ so the statement holds when $n = 0$
2. Assume that the statement holds when $n = n_0$, i.e. $n_0 + 0 = 0 + n_0$, then according to definition of addition and the statement we just prove

$$\begin{aligned} 0 + succ(n_0) &= succ(0) + n_0 = succ(0 + n_0) \\ &= succ(n_0 + 0) = succ(n_0) \\ &= succ(n_0) + 0 \end{aligned}$$

so the statement also holds when $n = succ(n_0)$.
To sum up, $\forall n \in \mathbb{N}, n + 0 = 0 + n$.
Now we start to prove that $\forall m, n \in \mathbb{N}, n + m = m + n$. $\forall n \in \mathbb{N}$

1. When $m = 0$, $n + 0 = 0 + n$ has been proved , so the statement holds when $m = 0$

2. Assume that the statement holds when $m = m_0$, i.e. $n + m_0 = m_0 + n$, then according to definition of addition and the statement we just prove

$$\begin{aligned} n + succ(m_0) &= succ(n) + m_0 = succ(n + m_0) \\ &= succ(m_0 + n) \\ &= succ(m_0) + n \end{aligned}$$

so the statement also holds when $m = succ(m_0)$.
In conclusion, $\forall m, n \in \mathbb{N}, n + m = m + n$.

2 Straightforward Induction

Proof:

1. When $n = 1, 2$, since $a_1 = 1 = 3 \cdot 2^{1-1} + 2 \cdot (-1)^1$, $a_2 = 8 = 3 \cdot 2^{2-1} + 2 \cdot (-1)^2$, so $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ holds for $n = 1, 2$

2. Assume that $\forall n \leqslant k (k \geqslant 2)$, $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ holds, then

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 2 \cdot (3 \cdot 2^{k-1-1} + 2 \cdot (-1)^{k-1}) \\ &= (3+3) \cdot 2^{k-1} + (2-4) \cdot (-1)^k \\ &= 3 \cdot 2^{k+1-1} + 2 \cdot (-1)^{k+1} \end{aligned}$$

so the equation also holds for $n = k + 1$.
To sum up, $\forall n > 0, a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$

3 The Fifth Peano Axiom

Proof: Assume that there exists some non-empty set $S \subset \mathbb{N}$ doesn't have a least element, that is $\forall m \in S, \exists m_0 \in S$ such that $m_0 < m$.

Set $T = \mathbb{N} \setminus S$. Since 0 is not the successor of any natural number, then if $0 \in S$, it would be the least number in S. So $0 \in T$.

Since 1 is the successor of 0, and since $0 \notin S$, then if $1 \in S$, 1 would be the least number in S. So $1 \in T$.

Since 2 is the successor of 1, and since $0, 1 \notin S$, then if $2 \in S$, 2 would be the least number in S. So $2 \in T$.

Repeat this way, we can prove that for each natural number n, its successor is in T. And since $0 \in T$, then according to induction axiom $T = \mathbb{N}$, and therefore $S = \varnothing$. This leads to contradiction.

So such kind of set doesn't exist, and *Well-Ordering Principle* holds.

4 Is a direct induction approach always successful?

First, we use induction to prove that $(1 + x)^n \geqslant 1 + nx$ holds for any n $\in \mathbb{N}$, where $x > -1$

Proof:

1. When n $= 0$, $(1+x)^n = (1+x)^0 = 1, 1+nx = 1+0\cdot x = 1$. So $(1+x)^n \geqslant 1+nx$ holds when n=0

2. Assume that $(1+x)^n \geqslant 1+nx$ holds when n=k, where k $\in \mathbb{N}$, i.e. $(1+x)^k \geqslant 1+kx$. Then

$$\begin{aligned}(1+x)^{k+1} &\geqslant (1+x)\cdot(1+kx)\\ &= 1+kx+x+kx^2\\ &= (1+(k+1)x+kx^2\\ &\geqslant 1+(k+1)x\end{aligned}$$

   This is because $k \in \mathbb{N}, x > -1$

   So $(1+x)^n \geqslant 1+nx$ also holds when $n = k+1$.

According to $1, 2, \forall n \in \mathbb{N}$, $(1+x)^n \geqslant 1+nx$.
So $\forall n \in \mathbb{N}, (1+x)^n \geqslant nx$

5 Strong Induction

Proof:

1. When $n = 1, 2$, since $1 = 2^0, 2 = 2^1$, then we can see that the statement holds for $n = 1, 2$.

2. Assume that $\forall n \in \mathbb{N}, 1 \leqslant n \leqslant k (k \geqslant 2)$, it can be written as a sum of distinct powers of 2. Then for $n = k+1$:

   (a) If $k+1$ is even, then $k \geqslant \frac{k+1}{2} \in \mathbb{N}^*$, according to assumption we can set that $\frac{k+1}{2} = p_0 2^0 + p_1 2^1 + \cdots + p_{(k+1)/2} 2^{(k+1)/2} (p_i \in \{0,1\}, i = 0, 1, \cdots, (k+1)/2)$. This is practical since $2^n = (1+1)^n > n \cdot 1 = n$. Then $k+1 = p_0 2^1 + p_1 2^2 + \cdots + p_{(k+1)/2} 2^{(k+1)/2+1}$ which is a sum of distinct powers of 2.

   (b) If $k+1$ is odd, then $\frac{k}{2} \in \mathbb{N}^*, 1 \leqslant \frac{k}{2} \leqslant k$, according to assumption we can set that $\frac{k}{2} = p_0 2^0 + p_1 2^1 + \cdots + p_{k/2} 2^{k/2} (p_i \in \{0,1\}, i = 0, 1, \cdots, k/2)$. Then $k+1 = 2^0 + p_0 2^1 + p_1 2^2 + \cdots + p_{k/2} 2^{k/2+1}$ which is a sum of distinct powers of 2.

   according to (a)(b), $k+1$ always can be written as a sum of distinct powers of 2. So, the statement also holds for $n = k+1$.

From 1,2, every $n \in \mathbb{N} \setminus \{0\}$ can be written as a sum of distinct powers of 2.

6 Structural Induction

Proof: Use structural induction to prove

1. Since $(0,0) \in S$, and $5 \big| 0+0$, the base case is established.

2. Assume that $\forall a, b \in S, 5\big|(a+b)$, so we can set $a+b=5k, k \in \mathbb{N}$. Then $(a+2, b+3), (a+3, b+2) \in S$, and $(a+2)+(b+3) = (a+3)+(b+2) = a+b+5 = 5(k+1)$. Since $k+1 \in \mathbb{N}$, $5\big|((a+2)+(b+3)), 5\big|((a+3)+(b+2))$.

According to 1,2, we can see that $\forall(a,b) \in S, 5\big|(a+b)$

7 Some easy practice of relation properties

7.1 x+y=0

1. Since $1 \in \mathbb{Z}, 1+1=2 \neq 0$, this shows that the relation is not reflexive.

2. $\forall(x,y) \in R, y+x = x+y = 0$, so $(y,x) \in R$. So the relation is symmetric.

3. Since $1+(-1)=0, (-1)+1=0$, then $(1,-1), (-1,1) \in R$, while $1+1=2 \neq 0$, so the relation is not transitivity.

7.2 $2\big|$(x-y)

1. $\forall x \in \mathbb{Z}, (x,x) \in R$ since $2\big|0 = x-x$, which shows that the relation is reflexive.

2. $\forall(x,y) \in R$, set $x-y=2k, k \in \mathbb{Z}$. Then $y-x=-2k=2(-k)$. Since $-k \in \mathbb{Z}, 2\big|(y-x)$. So $(y,x) \in R$. So the relation is symmetric.

3. $\forall(x,y),(y,z) \in R$, set $x-y=2k_1, y-z=2k_2 (k_1,k_2 \in \mathbb{Z})$. Then $x-z = x-y+y-z = 2(k_1+k_2)$. Since $k_1+k_2 \in \mathbb{Z}$, $2\big|(x-z)$. So $(x,z) \in R$. So the relation is transitivity.

7.3 xy=0

1. Since $1 \in \mathbb{Z}, 1 \cdot 1 = 1 \neq 0$, this shows that the relation is not reflexive.

2. $\forall(x,y) \in R, yx = xy = 0$, so $(y,x) \in R$. So the relation is symmetric.

3. Since $1 \cdot 0 = 0, 0 \cdot 2 = 0$, then $(1,0), (0,2) \in R$, while $1 \cdot 2 = 2 \neq 0$, so the relation is not transitivity.

7.4 x=1 or y=1

1. Since $2 \in \mathbb{Z}, 2 \neq 1$, then $(2,2) \notin R$. This shows that the relation is not reflexive.

2. $\forall(x,y) \in R, x=1 \vee y=1$, so $(y,x) \in R$. So the relation is symmetric.

3. Since $(3,1),(1,2) \in R$, while $(3,2) \notin R$, then the relation is not transitivity.

7.5 x=± y

1. $\forall x \in \mathbb{Z}, (x, x) \in R$ since $x = x$, which shows that the relation is reflexive.
2. $\forall (x, y) \in R, x = y \vee x = -y$. So $y = x \vee y = -x$. So $(y, x) \in R$. So the relation is symmetric.
3. $\forall (x, y), (y, z) \in Rx = y \vee x = -y, y = z \vee y = -z$. Then $x = z \vee x = -z$. So $(x, z) \in R$. So the relation is transitivity.

7.6 x=2y

1. Since $1 \in \mathbb{Z}, 1 \neq 2 = 2 \cdot 1$, then $(1, 1) \notin R$. This shows that the relation is not reflexive.
2. $(2, 1) \in R$, while $(1, 2) \notin R$. So the relation is not symmetric.
3. Since $(4, 2), (2, 1) \in R$, while $(4, 1) \notin R$, then the relation is not transitivity.

7.7 xy$\geqslant 0$

1. $\forall x \in \mathbb{Z}, (x, x) \in R$ since $x \cdot x = x^2 \geqslant 0$, which shows that the relation is reflexive.
2. $\forall (x, y) \in R, yx = xy \geqslant 0$. So $(y, x) \in R$. So the relation is symmetric.
3. Since $1 \cdot 0 = 0 \geqslant 0, 0 \cdot (-1) = 0 \geqslant 0$, then $(1, 0), (0, -1) \in R$, while $1 \cdot (-1) = -1 < 0$, so the relation is not transitivity.

7.8 x=1

1. Since $2 \in \mathbb{Z}, 2 \neq 1$, then $(2, 2) \notin R$. This shows that the relation is not reflexive.
2. $(1, 2) \in R$, while $(2, 1) \notin R$, so the relation is not symmetric.
3. $\forall x, y, z \in \mathbb{Z}, (x, y), (y, z) \in R$, then $x = 1$ and therefore $(x, z) \in R$, then the relation is transitivity.