VV471— Introduction to Numerical Methods

Assignment 9

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Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Lebesgue constant for Chebyshev nodes*

In the lectures (slide 6.23) we showed that the choice of the nodes is of a major importance when performing interpolation. This in fact confirmed the observations from lab 1, where for instance Runge's phenomenon appears as the number of equidistance nodes increases. In this exercise we show that interpolation on Chebyshev nodes is a good choice as in this case the Lebesgue constant grows slowly.

We recall that the Lebesgue number is defined by $\Lambda_n = \max_{x\in[a,b]} \sum_{i=0}^{n} \left|\ell_i(x)\right|$ and the Chebyshev polynomials by $T_n(x) = \cos(n \arccos x)$. The roots x_i, $0 \leq i \leq n$, of Chebyshev polynomials are given by $x_i = \cos\theta_i$, with $\theta_i = \frac{2i+1}{2(n+1)}\pi$.

1. Let ℓ_i be the Lagrange polynomials associated to the node x_i.

   a) Prove that
   $$\ell_i(x) = \frac{T_{n+1}(x)}{(x-x_i)\,T'_{n+1}(x_i)}.$$

   b) Show that $T'_{n+1}(x) = \dfrac{n+1}{\sqrt{1-\cos^2\theta}}\sin(n+1)\theta$, and $T'_{n+1}(x_k) = (-1)^k\dfrac{n+1}{\sin\theta_k}$.

   c) Conclude that
   $$\sum_{i=0}^{n}\left|\ell_i(1)\right| \geq \frac{1}{n+1}\sum_{i=0}^{n}\cot\frac{\theta_i}{2}.$$

2. We want to show that
   $$\frac{1}{n+1}\sum_{i=0}^{n}\cot\frac{\theta_i}{2} \geq \frac{2}{\pi}\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}}\cot t\,\mathrm{d}t. \tag{1.1}$$

   * a) Prove that
   $$\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,\mathrm{d}t \leq \frac{1}{2}\left(\theta_{k+1}-\theta_k\right)\cot\frac{\theta_k}{2}.$$

   b) Show that
   $$\frac{\pi}{2(n+1)}\sum_{k=0}^{n}\cot\frac{\theta_k}{2} \geq \sum_{k=0}^{n}\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}}\cot t\,\mathrm{d}t.$$

   c) Prove that equation (1.1) is true.

* 3. Using question 2, conclude that $\Lambda_n \geq \dfrac{2}{\pi}\ln n$.

Ex. 2 — *Interpolation*

In this exercise we construct an interpolation method for an abitrary continuous function over $[a,b]\subset\mathbb{R}$.

Let $C[a,b]$, $a,b\in\mathbb{R}$, be the set of the continuous functions over $[a,b]$, endowed with the usual norm for uniform convergence, $\|u\|_\infty = \max_{x\in[a,b]}|u(x)|$. For some $n\in\mathbb{N}$ we define the collection of points (x_k,y_k), $k\in\{0,\cdots,n\}$, such that $a\le x_0<y_0<x_1<y_1<\cdots<x_n<y_n\le b$ and consider the following application

$$\begin{aligned}\Phi : C[a,b] &\longrightarrow \mathbb{R}^{n+1}\\ f &\longmapsto \left(m_0(f), m_1(f),\cdots,m_n(f)\right),\end{aligned}$$

with for all $k\in\{0,\cdots,n\}$, $m_k(f)=\frac{f(x_k)+f(y_k)}{2}$. In other words the application Φ maps a function defined on $[a,b]$ onto an element of $\mathbb{R}^{n+1}$.

1. Let $f\in C[a,b]$ such that $\Phi(f)=0$. Show that, for any k there exists $\xi_k\in[x_k,y_k]$ such that $f(\xi_k)=0$.
2. Prove that if Φ is restricted to $\mathbb{R}_n[x]$, then Φ is injective. Conclude on the existence of a unique polynomial $P_n\in\mathbb{R}_n[x]$ such that $\Phi(P_n)=\Phi(f)$.
3. Assuming $f\in C^{n+1}[a,b]$, prove that P_n is the interpolation polynomial of f and conclude that

$$\|f-P_n\|_\infty \le \frac{(b-a)^{n+1}}{(n+1)!}\sup_{x\in[a,b]}|f^{(n+1)}(x)|.$$

Ex. 3 — *Trigonometric polynomials*

Let $x\in[0,1]$ and $\theta\in[-\pi,pi]$. For $n\in\mathbb{N}$, we denote by $T_n=\left\{Q_n, Q_n(\theta)=\frac{a_0}{\sqrt{2}}+\sum_{k=1}^n a_k\cos k\theta\right\}$, the set of the trigonometric polynomials of degree less than n.

* 1. Prove that for any $0\le k\le n$, $(\cos\theta)^k$ is in T_n. Conclude that Φ, which maps $P_n(x)$ into $Q_n(\theta)=P_n(\cos\theta)$ is a linear bijection from $\mathbb{R}_n[x]$ into T_n.

2. For $f\in C^{n+1}[-1,1]$, we define $F(\theta)=f(\cos\theta)$. Show the existence of $Q_n\in T_n$, such that $Q_n(\theta_i)=F(\theta_i)$, where $\theta_i=\frac{(2i+1)\pi}{2(n+1)}$, $0\le i\le n$.
3. Prove that finding $Q_n\in T_n$ in the previous question is equivalent to solving the linear system $La=b$, with $a=(a_0,\cdots,a_n)^T$ such that $Q_n(\theta)=\frac{a_0}{\sqrt{2}}+\sum_{k=1}^n a_k\cos k\theta$.
4. Show that for any $\theta\in(-\pi,\pi)$, there exists $\xi\in(-1,1)$, such that

$$F(\theta)-Q_n(\theta)=\frac{\cos(n+1)\theta}{2^n(n+1)!}f^{(n+1)}(\xi).$$