VV471— Introduction to Numerical Methods

Assignment 10

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Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.

Ex. 1 — *Richardson extrapolation*

In this exercise we investigate Richardson extrapolation, a sequence acceleration method which can be used to improve the rate of convergence of a quadrature formula.

Let $a_0 \in \mathbb{R}$ be a value to be computed and $A(t)$, $t > 0$ be such that

(i) $\lim_{t \to 0} A(t) = a_0$;

(ii) For all $n \leq 0$, there exist $a_1, \cdots, a_n$, and c_{n+1} such that

$$A(t) = a_0 + \sum_{i=1}^{n} a_i t^i + R_{n+1}(t), \quad \text{with } |R_{n+1}(t)| \leq c_{n+1}(t);$$

Let $(A_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$\begin{cases} A_0(t) & = A(t) \\ A_n(t) & = \frac{r^n A_{n-1}(t) - A_{n-1}(rt)}{r^n - 1}, \quad n \geq 1 \text{ and } r > 1 \text{ a constant.} \end{cases}$$

* 1. Prove by induction that for all $n \in \mathbb{N}$, $A_n(t) = a_0 + O(t^{n+1})$.

2. Fixing $t_0 > 0$ and $r_0 > 1$, we define a sequence $(t_m)_{m \geq 0}$ such that $t_m = r_0^{-m} t_0$.

   a) Show that when n is fixed then $\lim_{m \to \infty} A_n(t_m) = a_0$.

   b) Show that $A_n(t_m) = a_0 + O\left(r_0^{-m(n+1)}\right)$.

3. For m and n two integers, we define a matrix M whose entry at column n and row m is $A_{m,n} = A_n(t_m)$. Write the pseudocode of a clear algorithm generating the matrix M and returning a_0.

4. Romberg integration

   a) Understand box 1 and intuitively explain how Richardson extrapolation accelerates the convergence of the trapezium rule.

   b) For a function of your choice compare the trapezium rule to Romberg method.

Box 1: Romberg integration – An application of Richardson extrapolation

We consider Romberg method, the application of Richardson extrapolation to the case of the trapezium rule.

For $f \in C[\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, the trapezium rule with constant step $h \in \mathbb{R}_+^*$ is given by

$$\int_\alpha^\beta f(x)\,dx \approx T_h(f) = h\left(\frac{1}{2}f(\alpha) + f(\alpha + h) + \cdots + f(\beta - h) + \frac{1}{2}f(\beta)\right).$$

We assume the following result (which can be proven using the Bernoulli theorems).

Let $f \in C^\infty[\alpha, \beta]$. Then for all $n \in \mathbb{N}^*$, there exists $(a_j)_{j \in \mathbb{N}}$ a sequence of real numbers such that
$$T_h(f) = \int_\alpha^\beta f(x)\,dx + \sum_{j=1}^n a_j h^{2j} + O(h^2 n).$$

From this theorem it suffices to set $a_0 = \int_\alpha^\beta f(x)\,dx$, and $t = h^2$ to obtain

$$A(t) = a_0 + \sum_{j=1}^n a_j t^j + O(t^{n+1}), \forall n \geq 1.$$

Hence all the requirements for Richardson extrapolation are met. Applying it for instance with $r_0 = 4$ and $t_0 = (\beta - \alpha)^2$ yields

$$A_{m,n} = \int_\alpha^\beta f(x)\,dx + O\left(\frac{1}{4^{m(n+1)}}\right).$$

In practice one chooses m and n with respect the expected precision level.

Integration

We consider the quadrature formula

$$\int_a^b f(x)\,dx \approx (b - a)f\left(\frac{a + b}{2}\right). \tag{2.1}$$

1. Why does formula 2.1 fall under Peano's method?

2. Determine Peano kernel for this formula, and show that it keeps a constant sign.

3. Conclude on the existence of $\xi \in [a, b]$, such that for any $f \in C^2[a, b]$, the error is expressed as

$$E(f) = \frac{1}{24}f''(\xi)(b - a)^3.$$

Gauss' method

Let $(q_k)_{k \in \mathbb{N}}$ be a sequence of polynomials such that

$$q_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta.$$

1. Let $w(x) = \sqrt{1 - x^2}$ be a function over $(-1, 1)$.

a) Show that w is a weight function.

b) Show that the $(q_k)_{k \in \mathbb{N}}$ define a sequence of orthogonal polynomials for the weight function w.

c) Determine the orthonormal polynomials $(p_k)_k$ associated to $(q_k)_k$.

2. We consider the Gauss' method of order $2n + 1$ defined by

$$\int_{-1}^1 f(x)w(x)\, dx \approx \sum_{k=0}^n A_k f(x_k).$$

a) Determine all the x_k, $0 \le k \le n$.

b) Show that for all $0 \le k \le n$,

$$A_k = \frac{\pi}{n+2} \sin^2 \frac{(k+1)\pi}{n+2}.$$

c) Assuming $f \in C^{2n+2}[-1, 1]$, show that there exists $\xi \in (-1, 1)$, such that the error of the method is given by

$$E_n(f) = c \frac{f^{(2n+2)}(\xi)}{(2n+2)!}, \quad \text{where } c \text{ is a constant to be determined.}$$