VE471 Introduction to Numerical Methods
Assignment 1

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Question 1.

(a).

The function $f : (0, \infty) \to \mathbb{R}, x \mapsto \ln(x)$ is differentiable in its domain, but the derivative

$$f'(x) = \frac{1}{x}$$

is not bounded in the interval $(0, \infty)$. It diverges to minus infinity when $x \to 0$.

(b).

Suppose f is differentiable and f' is bounded in some open interval $\mathcal{I}$. If $x_1 = x_2$, the statement holds. Otherwise, according to mean value theorem, there exist $x_0 \in (x_1, x_2)$ such that

$$f'(x_0) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \qquad |f'(x_0)| \leq c,$$

since f' is bounded in the interval. Therefore, we have

$$|f(x_1) - f(x_2)| \leq c|x_1 - x_2|,$$

which verifies the Lipschitz continuity.

(c).

The function defined by

$$f : (0, \infty) \to \mathbb{R}, \qquad x \mapsto \frac{1}{x}$$

is differentiable but not Lipschitz continuous in its domain. The derivative is given by

$$f'(x) = -\frac{1}{x^2}, \qquad x \in (0, \infty).$$

To prove that it is not Lipschitz continuous, consider the points with relation

$$x_2 = \frac{1}{2}x_1$$

and when $x_1 \to 0$, the Lipschitz condition gives

$$\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| = \left|\frac{\frac{1}{x_1} - \frac{2}{x_2}}{x_1 - \frac{1}{2}x_1}\right| = \frac{2}{x_1^2}$$

is not bounded. Therefore, the function is not Lipschitz continuous.

(d).

The function

$$f : (-1, 1) \to \mathbb{R}, \qquad x \to |x|$$

is Lipschitz continuous but not differentiable in its domain.

For Lipschitz continuity, suppose $x_1 \neq x_2$, we then have

$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| = 1$$

if $x_1 \leq 0, x_2 \leq 0$ or $x_1 \geq 0, x_2 \geq 0$. And

$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| < 1$$

if $x_1 \leq 0 \leq x_2$ or $x_2 \leq 0 \leq x^1$. Therefore, there exists a Lipschitz constant $c = 1$ such that Lipschitz continuity is satisfied. But f is not differentiable at $x = 0$, which is noticed from

$$\lim_{\varepsilon \to 0^+} \frac{|\varepsilon| - 0}{\varepsilon - 0} = 1, \qquad \lim_{\varepsilon \to 0^-} \frac{|\varepsilon| - 0}{\varepsilon - 0} = -1.$$

Question 2.

(a).

The fixed point iteration is defined by

$$x_{k+1} = g(x_k).$$

From mean value theorem, we know that given $0 \leq |g'(x)| \leq 1$, then the Lipschitz constant $c < 1$.

We have

$$|g(x_{k+1}) - g(x_k)| = |g(x_{k+1}) - x_{k+1}| \leq c|x_{k+1} - x_k|,$$

which means that

$$|x_{k+2} - x_{k+1}| \leq c|x_{k+1} - x_k| \quad \Rightarrow \quad |x_{k+2} - x_{k+1}| < c^{k+1}|b - a| \to 0 \text{ as } k \to \infty.$$

Therefore,

$$|g(x_k) - x_k| \to 0 \text{ as } k \to \infty,$$

meaning that the sequence converges to a fixed point.

Then we show that the fixed point is unique. Suppose there exist $x_1 < x_2$ such that $g(x_1) = x_1, g(x_2) = x_2$. Then according to mean value theorem, there exists a point $x_0 \in (x_1, x_2)$

such that

$$g'(x_0) = \frac{g(x_1) - g(x_2)}{x_1 - x_2} = 1,$$

contradicting that $|g'(x)| < 1$ for all $x \in [a, b]$.

Therefore, the fixed-point iteration converges to the unique fixed point x^*.

(b).

Suppose $|g'(x)| > 1$. Then suppose the fixed-point iteration still converges to x^*, it should satisfy that for $k \to \infty$,

$$|x_{k+2} - x_{k+1}| = |g(x_{k+1}) - g(x_k)| < \varepsilon$$

for $\forall\varepsilon > 0$. Then let $\varepsilon = |x_{k+1} - x_k|$, then

$$|g(x_{k+1}) - g(x_k)| < |x_{k+1} - x_k|$$

as $k \to \infty$. Then there exists a point $x_0 \in (x_k, x_{k+1})$ and

$$|g'(x_0)| = \left|\frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k}\right| < 1,$$

contradicting the condition that $|g'(x)| > 1$. Therefore, the fixed-point iteration never converges to x^*.

Question 3.

We can use bisection method to find the first and second smallest positive roots. This is because from graphical knowledge, we know that the the first and second smallest intersection of $\tan(x)$ and $4x$ are in the interval $(0, \pi/2)$ and $(\pi, 3\pi/2)$, as is shown in Figure 1. Therefore, we can apply two runs of bisection iteration with initial boundaries as each range mentioned above.

Newton's method and fixed-point iteration are not optimal in this situation. It is difficult to find a proper initial point for Newton's iteration (possible for x_1 but not for x_2). Moreover, it does not satisfy Lipschitz continuity for fixed-point iteration.

Using MATLAB, we are able to find the first root $x_1 = 1.393$ and the second root $x = 4.659$. The MATLAB code is shown below.

```
% find the first and second smallest positive root of tan(x) - 4x = 0;

f = @(x) tan(x) - 4 * x;
accuracy = 0.0001;

% first root;
a = 1;
```

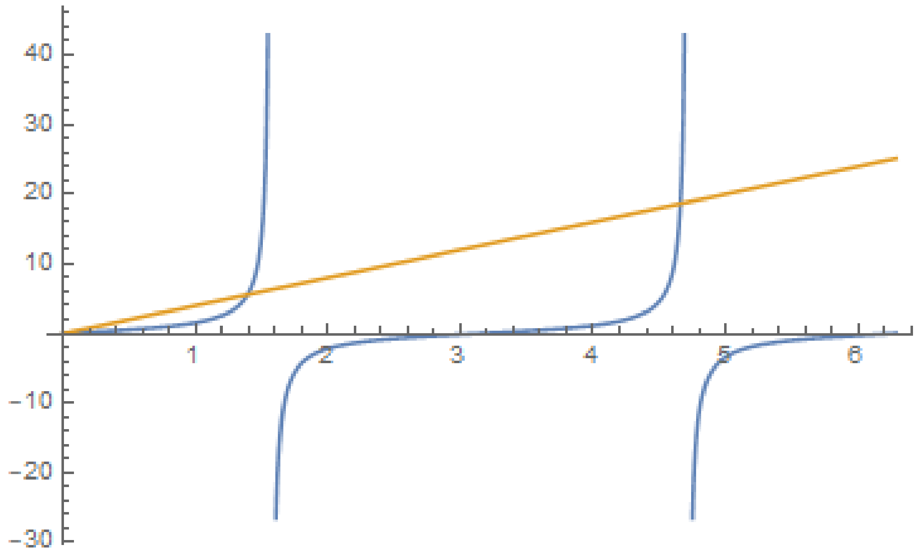

Figure 1: Intersection of $\tan(x)$ and $4x$.

```
b = pi / 2 - accuracy;

% second root;
a = pi;
b = 3 * pi / 2 - accuracy;

while (abs(a - b) >= accuracy)
    mid = (a + b) / 2;
    if (f(a) * f(mid) > 0)
        a = mid;
    else
        b = mid;
    end
end

result = (a + b) / 2;
fprintf('The root is %.3f\n', result)
```

Question 4

(a).

The sequence converges to $L = 0$. Then the order and rate are given by

$$\lim_{n\to\infty} \left| \frac{1/(n+1)}{1/n} \right| = 1, \qquad \alpha = 1$$

(b).

We know that the sequence converges to $L = 0$. Suppose there exists $\alpha \geq 1, \lambda$ such that the order of convergence is α and the rate of convergence is λ. Then we have

$$\lim_{k\to\infty}\left|\frac{\ln k}{k^\alpha}\right| = 0, \qquad \lim_{k\to\infty}\left|\frac{k}{(\ln k)^\alpha}\right| = \infty.$$

Therefore, the rate and order of convergence does not exist for this sequence.

(c).

The fixed-point iteration is given by

$$x_{k+1} = g(x_k)$$

to find the fixed point $x^* = g(x^*)$. We know that

$$|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leq c|x_k - x^*|,$$

and $0 < c < 1$, which means that

$$|x_k - x^*| \leq Mc^k,$$

where M is a constant, and

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|} \leq c.$$

Then using the definitions above, define a sequence $\{a_k\}$ as

$$a_0 = x_0, \qquad a_{k+1} = ca_k.$$

Then we have

$$E_k = |x_{k+1} - x_k| \leq a_k \qquad \text{for all } k$$

and a_k converges to 0.

Since

$$\lim_{k\to\infty}\frac{|a_{k+1} - 0|}{|a_k - 0|} = c,$$

the error sequence has at least linear convergence.

(d).

The fixed-point iteration gives a sequence that converges to x^*. When the convergence is quadratic, then we have

$$\begin{aligned}\lim_{k\to\infty}\frac{|x_{k+1}-x^*|}{|x_k-x^*|^2} &= \lim_{k\to\infty}\frac{|g(x_k)-g(x^*)|}{|x_k-x^*|^2}\\ &= \lim_{x\to x^*}\frac{|g(x_k)-g(x^*)|}{|x_k-x^*|^2}\\ &= \lim_{x\to x^*}\frac{|g'(x_k)-0|}{2|x_k-x^*|}.\end{aligned}$$

For this limit to exist, we require $g'(x^*)=0$ and $\lambda=\dfrac{1}{2}g''(x^*)$ in this case.

(e).

The secant iteration is given by

$$x_{k+1}=x_k-\frac{(x_k-x_{k-1})\cdot f(x_k)}{f(x_k)-f(x_{k-1})}.$$

Denote the error sequence as

$$e_k=x^*-x_k.$$

Then from the iteration relation, we have

$$e_{k+1}=e_k-\frac{(e_k-e_{k-1})f(e_k+x^*)}{f(x^*+e_k)-f(x^*+e_{k-1})}.$$

The error sequence converges to 0, and we have

$$\begin{aligned}f(x^*+e_k) &= f(x^*)+f'(x^*)e_k+\frac{f''(x^*)}{2}e_k^2+o(e_k^2),\\ f(x^*+e_{k-1}) &= f(x^*)+f'(x^*)e_{k-1}+\frac{f''(x^*)}{2}e_{k-1}^2+o(e_{k-1}^2).\end{aligned}$$

Therefore,

$$\begin{aligned}
e_{k+1} &= e_k - \frac{f(e_k + x^*)}{f'(x^*) + \dfrac{f''(x^*)}{2}(e_k + e_{k-1})} + o(e_k^2) \\
&= e_k - \frac{f(x^*) + f'(x^*)e_k + \dfrac{f''(x^*)}{2}e_k^2 + o(e_k^2)}{f'(x^*) + \dfrac{f''(x^*)}{2}(e_k + e_{k-1})} + o(e_k^2) \\
&= \frac{f''(x^*)e_k e_{k-1} - 2f(x^*)}{2f'(x^*) + f''(x^*)(e_k + e_{k-1})} + o(e_k^2) \\
&= \frac{f''(x^*)e_k e_{k-1}}{2f'(x^*) + f''(x^*)(e_k + e_{k-1})} + o(e_k^2)
\end{aligned}$$

To ensure $e_{k+1} \approx \lambda e_k^\alpha$ and prove that $\alpha = \dfrac{1+\sqrt{5}}{2}$, we have

$$e_{k+1} = \frac{f''(x^*)\lambda e_{k-1}^\alpha \cdot e_{k-1}}{2f'(x^*) + f''(x^*)(\lambda e_{k-1}^\alpha + e_{k-1})} + o((\lambda e_{k-1}^\alpha)^2),$$

which gives

$$e_{k+1} \approx \lambda e_k^\alpha \approx \frac{\lambda f''(x^*)}{2f'(x^*)} e_{k-1}^{\alpha+1}$$

when $k \to \infty$. Then

$$\alpha^2 = \alpha + 1 \quad \Rightarrow \quad \alpha = \frac{1+\sqrt{5}}{2}.$$

Question 5.

(a).

For the base case when $m = 1$, since f is continuous, there exists a continuous function h such that $f(x) = (x - x^*)h(x)$. For the case of m, suppose we have f has a root of order m then

$$f(x) = (x - x^*)^m h(x), \qquad h(x^*) \neq 0$$

and h is continuous. If f has a root of order $m+1$, then $f^{(m)}$ has has a simple root. Therefore, there exists a continuous function g such that

$$f^{(m)} = (x - x^*)g(x),$$

where g has a root of order m at x^*. Therefore, from the hypothesis, we know that there exists a continuous function h such that

$$g(x) = (x - x^*)^m h(x) \quad \Rightarrow \quad f(x) = (x - x^*)^{m+1} h(x),$$

verifying the statement for the case $m+1$.

(b).

We define the error sequence

$$E_k = |x_k - x^*|,$$

and from the previous proof,

$$f(x) = (x - x^*)^m h(x)$$

for some continuous function h. Moreover, since we require that f is differentiable, h is also differentiable, and

$$f'(x) = m(x - x^*)^{m-1} h(x) + (x - x^*)^m h'(x).$$

Therefore,

$$\begin{aligned}
E_{k+1} &= |x_{k+1} - x^*| \\
&= \left| x_k - m\frac{f(x_k)}{f'(x_k)} - x^* \right| \\
&= \left| x_k - \frac{m \cdot (x_k - x^*)^m h(x_k)}{m(x_k - x^*)^{m-1} h(x_k) + (x_k - x^*)^m h'(x_k)} - x^* \right| \\
&= \left| x_k - \frac{m \cdot (x_k - x^*) h(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} - x^* \right| \\
&= \left| \frac{(x_k - x^*)^2 h'(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} \right| \\
&= \left| \frac{h'(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} \right| \cdot E_k^2 \leq M \cdot E_k^2,
\end{aligned}$$

Then if the sequence converges linearly to the root x^* using the original Newton's method, this means that originally,

$$\begin{aligned}
E_{k+1} &= |x_{k+1} - x^*| \\
&= \left| x_k - \frac{f(x_k)}{f'(x_k)} - x^* \right| \\
&= \left| x_k - \frac{(x_k - x^*)^m h(x_k)}{m(x_k - x^*)^{m-1} h(x_k) + (x_k - x^*)^m h'(x_k)} - x^* \right| \\
&= \left| x_k - \frac{(x_k - x^*) h(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} - x^* \right| \\
&= \left| \frac{(m-1)(x_k - x^*) h(x_k) + (x_k - x^*)^2 h'(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} \right| \\
&= \left| \frac{(m-1) h(x_k) + (x_k - x^*) h'(x_k)}{m h(x_k) + (x_k - x^*) h'(x_k)} \right| \cdot E_k \leq M' \cdot E_k,
\end{aligned}$$

where $M' < 1$. Then

$$M = \frac{h'(x_k)}{(m-1)h(x_k) + (x_k - x^*)h'(x_k)} M'$$

is bounded. Therefore, the sequence for the modified Newton's method converges quadratically to x^*.

(c).

At the original root x^*, we have $f(x^*) = \dfrac{f(x^*)}{f'(x^*)} = 0$. Then instead of finding a simple root of f, we can instead find the root of u. Therefore, according to the original Newton's method, the modified method still yields a sequence converging to x^*.

We first calculate

$$\begin{aligned} u'(x) &= \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}, \\ x_{k+1} &= x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}. \end{aligned}$$

Then we have

$$\begin{aligned} E_{k+1} &= |x_{k+1} - x_k| \\ &= \left| \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)} \right| \\ &= \left| \frac{f'(x_k)\left[f'(x_k)e_k + \frac{1}{2}f''(x_k)e_k^2 + o(e_k^2)\right]}{f'(x_k)^2 - f(x_k)f''(x_k)} \right|, \end{aligned}$$

which implies at least linear convergence. If the root of f has higher order, then $f'(x_k)$, and even higher order derivatives can be further approximated by $f'(x^*)$ and so on. Then the order of convergence implied by the equation above becomes higher than the non-modified method.

(d).

First we calculate

$$f'(x) = \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

Based on the iteration in part (b), the iteration is defined as

$$x_{k+1} = x_k - \frac{\sin(x_k^3)}{x_k^2 \cos(x_k^3)}.$$

The values are given by

$$\begin{aligned}
x_1 &= x_0 - \frac{\sin(x_0^3)}{x_0^2\cos(x_0^3)} = -0.5574,\\
x_2 &= x_1 - \frac{\sin(x_1^3)}{x_1^2\cos(x_1^3)} = 0.0056,\\
x_3 &= x_2 - \frac{\sin(x_2^3)}{x_2^2\cos(x_2^3)} = 0.
\end{aligned}$$

Then we have

$$x_{k+1} = x_k - \frac{x_k\sin(2x_k^3)}{6x_k^3 - 2\sin(2x_k^3)}$$

according to iterations defined in part (c).

Therefore, the values are given by

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0\sin(2x_0^3)}{6x_0^3 - 2\sin(2x_0^3)} = 0.7825,\\
x_2 &= x_1 - \frac{x_1\sin(2x_1^3)}{6x_1^3 - 2\sin(2x_1^3)} = 0.2656,\\
x_3 &= x_2 - \frac{x_2\sin(2x_2^3)}{6x_2^3 - 2\sin(2x_2^3)} = 0.0002,\\
x_4 &= x_3 - \frac{x_3\sin(2x_3^3)}{6x_3^3 - 2\sin(2x_3^3)} = 0.
\end{aligned}$$

The results are calculated in MATLAB using the following script. We can conclude that the theoretically convergence rate is merely an approximated measure of the practical convergence rate. It does not necessarily suggest that one iteration rule converges faster than the other in practice if its theoretical convergence rate is higher.

```
x = ones(1, 7);
for i = 2:7
    x(i) = x(i - 1) - sin(x(i - 1)^3) / (x(i - 1)^2 * cos(x(i - 1)^3));
end
disp(x);

for i = 2:7
    x(i) = x(i - 1) - x(i - 1) * sin(2 * x(i - 1)^3) ...
        / (6 * x(i - 1)^3 - 2 * sin(2 * x(i - 1)^3));
end
disp(x);
```

(e).

Starting from initial value 0, Newton's method returns 0.999649 in 13 iterations, while modified Newton's method returns 1.000020 in 4 iterations.

Changing the initial value to 2, Newton's method returns 3.000000 in 1 iteration, while modified Newton's method returns 2.995443 in 4 iterations. The MATLAB script is shown as below.

```
x0 = 0;

f   = @(x) x^5 - 11 * x^4 + 46 * x^3 - 90 * x^2 + 81 * x - 27;
df  = @(x) 5 * x^4 - 44 * x^3 + 138 * x^2 - 180 * x + 81;
ddf = @(x) 20 * x^3 - 132 * x^2 + 276 * x - 180;

accuracy = 0.000001;

xc = Inf;
xn = x0;

iters = 0;

while abs(f(xn)) > accuracy
    iters = iters + 1;
    xc = xn;
    xn = xc - f(xc) / df(xc);
    if isnan(xn)
        xn = xc;
        break;
    end
end
fprintf('Newton returns %f in %d iterations.\n', xn, iters);

xc = Inf;
xn = x0;

iters = 0;

while abs(f(xn)) > accuracy
    iters = iters + 1;
    xc = xn;
    xn = xc - f(xc) * df(xc) / (df(xc)^2 - f(xc) * ddf(xc));
    if isnan(xn)
        xn = xc;
        break;
    end
end
fprintf('Modified Newton returns %f in %d iterations.\n', xn, iters);
```

Question 6.

(a).

Fixed-point iteration requires that the function f in $f(x^*) = 0$ has the form of $f(x) = g(x) - x$, with g is Lipschitz continuous in an open interval containing the root with Lipschitz constant less than 1. Therefore, an example of this is

$$f : (0, 4) \to \mathbb{R}, \qquad x \mapsto x^2 - x - 6.$$

It has derivative greater than 1 at both of its two roots. It cannot be solved using fixed-point iteration, but bisection works for this function.

(b).

The root of function

$$f : [-1, 1] \to \mathbb{R}, \qquad x \mapsto \cos(x) - x$$

can be found using fixed-point iteration by setting

$$f(x) = g(x) - x, \qquad g(x) = \cos(x),$$

and when starting from initial value $x_0 = -\frac{\pi}{2}$, Newton's method will fail because $f'(x_0) = 0$. But fixed-point iteration still works.

Question 7.

(a).

We calculate

$$f'(z) = 7z^6, \qquad f''(z) = 42z^5.$$

The iterative relation is given by

$$z_{k+1} = z_k - \frac{z_k(z_k^7 - 1)}{4z_k^7 + 3}.$$

Then the iteration stops when

$$\left| \frac{z_k(z_k^7 - 1)}{4z_k^7 + 3} \right| < \epsilon,$$

namely

$$\frac{z_k}{4}\left[1-\frac{7}{4z_k^7+3}\right]<\epsilon.$$

When the convergence criterion is defined to be

$$|z_k-z_{k-1}|^2<\epsilon,$$

the iteration stops when the two points z_k and z_{k-1} are close to each other on the complex plane.

The depicted surface between the relationship and the position of initial point on the complex plane is shown in Figure 2. The MATLAB plot is appended below, where `findRootComplex` is the implementation of the iteration described above.

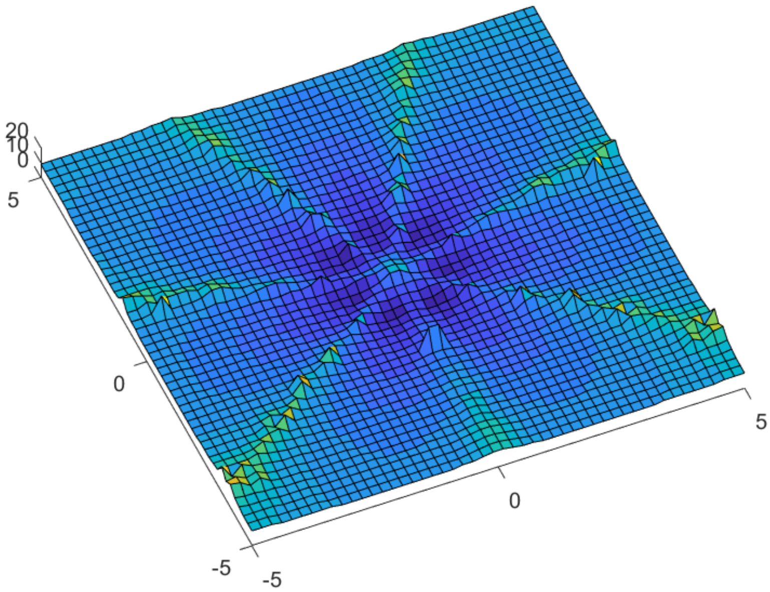

Figure 2: Number of Iterations for Different Initial Values (a).

```
clearvars;
f = @(z) z^7 - 1;

reals = linspace(-5, 5, 50);
imags = linspace(-5, 5, 50);

roots = zeros(50);
iters = zeros(50);

epsilon = 0.0001;

for ik = 1:50
```

```
    for jk = 1:50
        [roots(ik, jk), iters(ik, jk)] = findRootComplex(f, reals(ik) + ...
            imags(jk) * 1i, epsilon);
    end
end

figure('Name', 'Roots Real');
surface(reals, imags, real(roots));
view(3);
figure('Name', 'Roots Imaginary');
surface(reals, imags, imag(roots));
view(3);
figure('Name', 'Iterations');
surface(reals, images, iters);
view(3);
```

(b).

When the convergence criterion is defined to be

$$\left||z_k|^2 - |z_{k-1}|^2\right| < \epsilon,$$

the iteration stops when the two points z_k and z_{k-1} have relatively the same distance from the origin of the complex plane.

The relation between the number of iterations and the initial point is depicted in Figure 3. The general shape of this figure is similar to that of part (a), but the number of iterations are a few more than the case in part (a).

The change described above can be seen from the $y - z$ view of the figure, which is shown in Figure 4 and Figure 5.

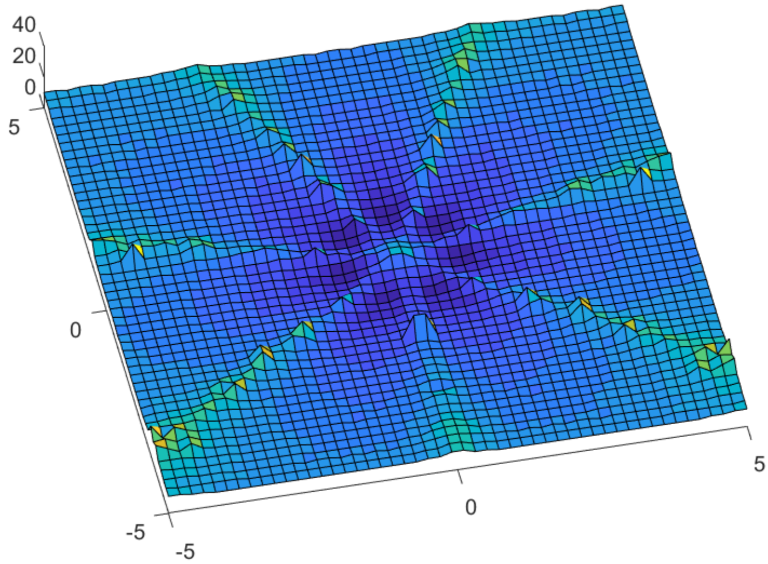

Figure 3: Number of Iterations for Different Initial Values (b).

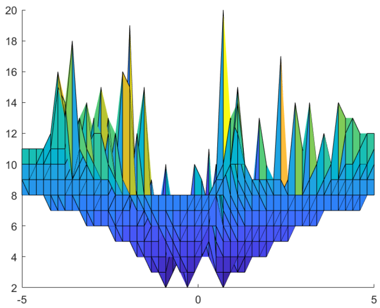

Figure 4: $y - z$ View (a).

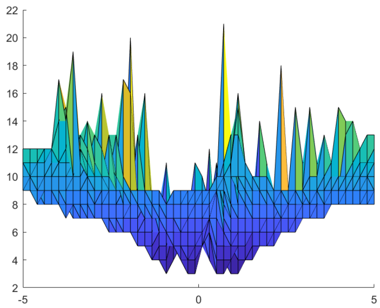

Figure 5: $y - z$ View (b).