1.

(a)

For $f(x) = \dfrac{1}{x}$, $x > 0$,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h} = -\lim_{h\to 0}\frac{1}{(x+h)x} = -\frac{1}{x^2}$$

which holds for all $x \in (0, +\infty)$, so it is differentiable. While we see that

$$\lim_{x\searrow 0^+}\frac{d\,f(x)}{dx} = \lim_{x\searrow 0^+} -\frac{1}{x^2} \to -\infty$$

which is not bounded on $(0, +\infty)$.

(b)

Since f is differentiable and f' is bounded in some open interval $\mathcal{I}$, according to Lagrange mean value theorem, $\forall x_1, x_2 \in \mathcal{I}, x_1 < x_2, \exists \xi \in (x_1, x_2)$

$$\begin{aligned} &f(x_2) = f'(\xi)(x_2 - x_1) + f(x_1) \\ \Rightarrow &|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1| \leqslant C|x_2 - x_1| \end{aligned}$$

So f is Lipschitz continuous in the interval $\mathcal{I}$.

(c)

For $f(x) = \dfrac{1}{x}, x \in (0, +\infty)$, it is differentiable. While $\forall C > 0$, we can find $x_1 = \dfrac{1}{C+2}, x_2 = 1$ which satisfies that

$$|f(x_1) - f(x_2)| = |\frac{1}{x_1} - \frac{1}{x_2}| = |C + 2 - 1| = C + 1 > C$$

so it is not Lipschitz continuous.

(d)

For $f(x) = |x|$, $x \in (-1, 1)$, $\forall x_1, x_2 \in (-1, 1)$,

$$|f(x_1) - f_(x_2)| = \Big||x_1| - |x_2|\Big| \leqslant |x_1 - x_2|$$

So it is Lipschitz continuous. While it is not differentiable at $x = 0$ and therefore it is not differentiable in $(-1, 1)$.

2.

(a)

Since $g, g' \in \mathcal{C}[a,b]$, according to Lagrange mean value theorem, $\forall x_1, x_2 \in [a,b], x_1 < x_2, \exists \xi \in (x_1, x_2)$,

$$g(x_2) = g'(\xi)(x_2 - x_1) + g(x_1) \Rightarrow |g(x_2) - g(x_1)| = |g'(\xi)||x_2 - x_1|$$

Then using fixed point iteration $x_{k+1} = g(x_k)$, $\forall k \in \mathbb{N}, \exists \xi_k$

$$E_{k+1} := |x_{k+1} - x^*| = |g(x_k) - g(x^*)| = |g'(\xi_k)||x_k - x^*| = |g'(\xi_k)| \cdot E_k$$

where x^* is the unique fixed point of g in $[a,b]$. So $E_k = \prod_{i=0}^{k} |g'(\xi_k)| \cdot E_0$. Since $0 \leqslant |g'(x)| < 1$,

$$\lim_{k\to\infty} \prod_{i=0}^{k} |g'(\xi_k)| \to 0 \Rightarrow \lim_{k\to\infty} |x_k - x^*| = \lim_{k\to\infty} E_k = 0$$

and therefore $\lim\limits_{k\to\infty} x_k = x^*$, i.e. the fixed-point iteration will converge to the unique fixed point $x^* \in [a,b]$.

(b)

Using fixed point iteration $x_{k+1} = g(x_k)$, $\forall k \in \mathbb{N}, \exists \xi_k$

$$E_{k+1} := |x_{k+1} - x^*| = |g(x_k) - g(x^*)| = |g'(\xi_k)||x_k - x^*| = |g'(\xi_k)| \cdot E_k$$

where x^* is the unique fixed point of g in $[a,b]$. So $E_k = \prod_{i=0}^{k} |g'(\xi_k)| \cdot E_0$. Since $|g'(x)| > 1$,

$$\lim_{k\to\infty} \prod_{i=0}^{k} |g'(\xi_k)| \to \infty$$

and therefore the fixed-point iteration will never converge to x^*.

3.

bisection.m

```
function [root] = bisection(a, b)
% required a < b
    r = (a + b)/2;
    if (b - a < 0.0001)
        root = r;
    elseif ((tan(a)-4 * a) * (tan(r) - 4 * r) < 0)
        root = bisection(a, r);
    else
        root = bisection(r, b);
    end
end
```

命令行窗口

```
>> bisection(0,1.57)

ans =

    1.3933

>> bisection(3.14,4.71)

ans =

    4.6588
```

So the first positive root is $x = 1.393$, and the second one is $x = 4.659$.

I choose bisection method. Because I can draw draft figure for $f(x) = \tan x$ and $g(x) = 4x$, the range where root would appear can be inferred. And I have to find the smallest and second smallest positive root of the equation, using bisection method with relatively exact range is quite convenient and convincing.

4.

(a)

Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, and

$$\lim_{n\to\infty} \frac{|a_{n+1} - 0|}{|a_n - 0|} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

So the order of convergence of $a_n = \frac{1}{n}$ is 1, and the rate of convergence is 1.

(b)

Since $\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \frac{1}{\ln k} = 0$, $\lim_{k\to\infty} b_{2k+1} = \lim_{k\to\infty} \frac{1}{k} = 0$, we obtain that $\lim_{n\to\infty} b_n = 0$. $\forall \alpha > 0$,

$$\lim_{x\to\infty} \frac{\ln(x+1)}{x^\alpha} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{\frac{x}{x+1}}{\alpha x^\alpha} = 0$$

and therefore $\lim_{k\to\infty} \frac{\ln(k+1)}{k^\alpha} = \lim_{x\to\infty} \frac{\ln(x+1)}{x^\alpha} = 0$. Then

$$\frac{|b_{2k+2} - 0|}{|b_{2k+1} - 0|^\alpha} = \frac{k^\alpha}{\ln(k+1)} \xrightarrow{k\to\infty} \infty$$

Since $\{\frac{b_{2k+2}}{b_{2k+1}}\}$ is the subsequence of $\{\frac{b_{n+1}}{b_n}\}$, and it does not converge to some constant $\lambda \in (0,\infty)$ for all $\alpha > 0$. And $\forall \alpha \leqslant 0$, $\lim_{n\to\infty} \frac{|b_{n+1} - 0|}{|b_n - 0|^\alpha} = 0$. So there does not exist a constant $0 < \lambda < \infty$ such that for some constant α

$$\lim_{n\to\infty} \frac{|b_{n+1} - 0|}{|b_n - 0|^\alpha} = \lambda$$

(c)

To use fixed point iteration, we assume that the function g is Lipschitz with $0 < c < 1$, and $x^* = g(x^*)$ is its only fixed point in interval $\mathcal{I}$.

Using fixed point iteration, $x_o \in \mathcal{I}$, $x_{k+1} = g(x_k)$, $\forall k \in \mathbb{N}$, we know that our method would lead to an error sequence which satisfies that

$$E_{k+1} := |x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leqslant c|x_k - x^*| = c \cdot E_k$$

so $E_k \leqslant c^k \cdot E_0$ holds for all $k \in \mathbb{N}$. And we can find that $\lim_{n\to\infty} c^k \cdot E_0 = 0$,

$$\lim_{k\to\infty} \frac{|c^{k+1} \cdot E_0 - 0|}{|c^k \cdot E_0 - 0|} = \lim_{k\to\infty} \frac{|c^{k+1} \cdot E_0 - 0|}{|c^k \cdot E_0 - 0|} = c$$

So the error sequence led by fixed-point iteration has at least linear convergence.

6.

(a)

For the equation

$$\tan x = 4x$$

I have solved two positive roots for it by using bisection method. To use fixed point method, we should use $g(x) = \tan x - 3x$. While

$$g'(x) = sec^2 x - 3 \in [-2, \infty)$$

which means that using fixed point iteration, it may not converge and therefore we cannot use it.