Ex. 1

1.

$\forall n, m \in \mathbb{N}, m < n$, since -1 and 1 are two order-n roots of $f_n(x) = (x^2-1)^n$,

$$(f_n(x))^{(m)}|_{x=-1} = (f_n(x))^{(m)}|_{x=1} = 0$$

So for all $i, j \in \mathbb{N}$, assume $i \geqslant j$

$$\begin{aligned}
&\langle Q_i, Q_j \rangle = \langle Q_j, Q_i \rangle \\
=&\int_{-1}^{1} Q_i(x)Q_j(x)w(x)dx \\
=&\int_{-1}^{1} \frac{1}{2^i i!}\big((x^2-1)^i\big)^{(i)} \cdot \frac{1}{2^j j!}\big((x^2-1)^j\big)^{(j)} \cdot 1dx \\
=&\frac{1}{2^{i+j}i!j!}\Big(\big((x^2-1)^j\big)^{(j)} \cdot \big((x^2-1)^i\big)^{(i-1)}\Big|_{-1}^{1} - \int_{-1}^{1} \big((x^2-1)^i\big)^{(i-1)}\big((x^2-1)^j\big)^{(j+1)}dx\Big) \\
=&-\frac{1}{2^{i+j}i!j!}\int_{-1}^{1} \big((x^2-1)^i\big)^{(i-1)}\big((x^2-1)^j\big)^{(j+1)}dx \\
=&\frac{(-1)^i}{2^{i+j}i!j!}\int_{-1}^{1} (x^2-1)^i\big((x^2-1)^j\big)^{(j+i)}dx
\end{aligned}$$

1. If $i > j$, $\big((x^2-1)^j\big)^{(j+i)} = \Big(\big((x^2-1)^j\big)^{(j+j+1)}\Big)^{(i-j-1)} = 0^{(i-j-1)}$, so $\langle Q_i, Q_j \rangle = 0$.
2. If $i = j$,

$$\begin{aligned}
&\int_{-1}^{1} (x^2-1)^i\big((x^2-1)^j\big)^{(j+i)}dx \\
=&(2i)!\int_{-1}^{1} (x^2-1)^i dx = (2i)!\int_{-1}^{1}(x-1)^i(x+1)^i dx \\
=&\frac{(2i)!}{i+1}\Big((x+1)^i(x-1)^{i+1}\Big|_{-1}^{1} - i\int_{-1}^{1}(x-1)^{i+1}(x+1)^{i-1}dx\Big) \\
=&-\frac{(2i)!\cdot i}{i+1}\int_{-1}^{1}(x-1)^{i+1}(x+1)^{i-1}dx \\
=&(-1)^i\frac{(2i)!\cdot i!i!}{(2i)!}\int_{-1}^{1}(x-1)^{2i}dx \\
=&(-1)^i\frac{i!i!}{2i+1}(x-1)^{2i+1}|_{-1}^{1} \\
=&(-1)^i\frac{(i!)^2}{2i+1}2^{2i+1}
\end{aligned}$$

So $\langle Q_i, Q_i \rangle = \dfrac{(-1)^i}{2^{2i}i!i!} \cdot \Big((-1)^i\dfrac{(i!)^2}{2i+1}2^{2i+1}\Big) = \dfrac{2}{2i+1} \neq 0$

So $(Q_n)_{n\in\mathbb{N}}$ defines a sequence of orthogonal polynomials.

2.

If $n = 2k, k \in \mathbb{N}$

$$\begin{aligned}
Q_n(-x) =& \frac{1}{2^n n!}\big((t^2-1)^n\big)^{(n)}|_{t=-x}\\
=& \frac{1}{2^n n!}(\sum_{i=0}^{n} a_i t^{2i})^{(n)}|_{t=-x}\\
=& \frac{1}{2^n n!}(\sum_{i=0}^{k} b_i t^{2i})|_{t=-x}\\
=& \frac{1}{2^n n!}(\sum_{i=0}^{k} b_i t^{2i})|_{t=x}\\
=& Q_n(x)
\end{aligned}$$

If $n = 2k+1, k \in \mathbb{N}$

$$\begin{aligned}
Q_n(-x) =& \frac{1}{2^n n!}(\sum_{i=0}^{k} b_i t^{2i+1})|_{t=-x}\\
=& -\frac{1}{2^n n!}(\sum_{i=0}^{k} b_i t^{2i})|_{t=x}\\
=& -Q_n(x)
\end{aligned}$$

To sum up, $Q_n(-x) = (-1)^n Q_n(x)$.

3.

$Q_n(x) = \frac{1}{2^n n!}\big((x^2-1)^n\big)^{(n)}$ is a polynomial of degree at most n, and the highest-degree term is given by

$$\frac{1}{2^n n!}\big(x^{2n}\big)^{(n)} = \frac{(2n)!}{2^n (n!)^2}x^n = \frac{\big(2(n-1)\big)!}{2^{n-1}\big((n-1)!\big)^2}x^{n-1}\cdot \underbrace{\frac{2n(2n-1)}{2n^2}x}_{=\frac{2n-1}{n}x}$$

so $Q(x) = Q_{n+1}(x) - \frac{2n+1}{n+1}x\cdot Q_n(x)$ is a polynomial of degree at most n, and therefore $\exists c_k \in \mathbb{R}, k = 0, 1, \cdots, n$ such that

$$Q(x) = \sum_{k=0}^{n} c_k Q_k(x)$$

$\forall i \leqslant n-2$,

$$\begin{aligned}
\langle Q(x), Q_i(x)\rangle =& \sum_{k=0}^{n} c_k \langle Q_k(x), Q_i(x)\rangle = c_i ||Q_i(x)||^2\\
\langle Q(x), Q_i(x)\rangle =& \langle Q_{n+1}(x), Q_i(x)\rangle - \frac{2n+1}{n+1}\langle xQ_n(x), Q_i(x)\rangle = -\frac{2n+1}{n+1}\langle xQ_n(x), Q_i(x)\rangle
\end{aligned}$$

since $\langle xQ_n(x), Q_i(x)\rangle = \int_{-1}^{1} xQ_n(x)Q_i(x)dx = \langle Q_n(x), xQ_i(x)\rangle$, and $xQ_i(x)$ is a polynomial of degree at most $i+1 \leqslant n-1$,

$$c_i||Q_i(x)||^2 = \langle Q(x), Q_k(x)\rangle = -\frac{2n+1}{n+1}\langle Q_n(x), xQ_i(x)\rangle = -\frac{2n+1}{n+1}\sum_{j=0}^{i-1} b_j\langle Q_n(x), Q_j(x)\rangle = 0$$

So $c_i = 0$ for $i \leqslant n-2$.

For $i = n-1$, on one hand

$$\langle Q(x), Q_{n-1}(x)\rangle = \sum_{k=0}^{n} c_k\langle Q_k(x), Q_{n-1}(x)\rangle = c_{n-1}||Q_{n-1}(x)||^2 = c_{n-1}\cdot\frac{2}{2n-1}$$

on the other hand

$$\begin{aligned}\langle Q(x), Q_{n-1}(x)\rangle &= -\frac{2n+1}{n+1}\langle Q_n(x), xQ_{n-1}(x)\rangle \\ &= -\frac{2n+1}{n+1}\sum_{j=0}^{n} b_j\langle Q_n(x), Q_j(x)\rangle \\ &= -\frac{2n+1}{n+1}\cdot b_n\frac{(-1)^{n+1}\cdot 2}{2n+1} \\ &= -\frac{2}{n+1}\cdot\frac{n}{2n-1}\end{aligned}$$

So $c_{n-1} = -\dfrac{n}{n+1}$. For $i = n$, on one hand

$$\langle Q(x), Q_n(x)\rangle = \sum_{k=0}^{n} c_k\langle Q_k(x), Q_n(x)\rangle = c_n||Q_n(x)||^2 = c_n\cdot\frac{2}{2n+1}$$

on the other hand

$$\langle Q(x), Q_n(x)\rangle = -\frac{2n+1}{n+1}\langle xQ_n(x), Q_n(x)\rangle = -\frac{2n+1}{n+1}\int_{-1}^{1}\underbrace{x(Q_n(x))^2}_{\text{odd}}dx$$

since $f(x) = x(Q_n(x))^2 = -(-x)((-1)^nQ_n(x))^2 = -(-x)(Q_n(-x))^2 = -f(-x)$. Therefore

$$c_n = \frac{2n+1}{2}\langle Q(x), Q_n(x)\rangle = 0$$

So $Q_{n+1}(x) - \dfrac{2n+1}{n+1}xQ_n(x) = -\dfrac{n}{n+1}Q_{n-1}(x)$, i.e.

$$(n+1)Q_{n+1}(x) = (2n+1)xQ_n(x) - nQ_{n-1}(x)$$

4.

Since

$$\begin{aligned}
Q_n(x) =& \frac{1}{2^n n!}\left((x^2-1)^n\right)^{(n)} = \frac{1}{2^n n!}\left((x+1)^n(x-1)^n\right)^{(n)} \\
=& \frac{1}{2^n n!}\sum_{i=0}^{n}\binom{n}{i}\left((x+1)^n\right)^{(i)}\left((x-1)^n\right)^{(n-i)} \\
=& \frac{1}{2^n n!}\sum_{i=0}^{n}\binom{n}{i}\frac{n!}{(n-i)!}(x+1)^{n-i}\cdot\frac{n!}{(n-n+i)!}(x-1)^{n-n+i} \\
=& \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}^2(x+1)^{n-i}(x-1)^i \\
=& \sum_{i=0}^{n}\binom{n}{i}^2(-1)^i\left(\frac{x+1}{2}\right)^{n-i}\left(\frac{1-x}{2}\right)^i
\end{aligned}$$

we see that

$$Q_n(x) = \sum_{i=0}^{n}\binom{n}{i}^2(-1)^i\left(\frac{x+1}{2}\right)^{n-i}\left(\frac{1-x}{2}\right)^i$$

Ex. 2

Use Lagrange interpolation,

```
function [res] = Lagrange_interpolation(c)
    res = 0;
    x = [-5, -1, 0, 1, 3, 5, 10, 12];
    f = [781, 5, 1, 1, 61, 521, 9091, 19141];
    for i = 1 : 8
        l(i) = f(i);
        for j = 1 : 8
            if j ~= i
                l(i) = l(i) * (c - x(j))/(x(i) - x(j));
            end
        end
        res = res + l(i);
    end
end
```

and get the result $f(2) = 11$.

Ex. 3

1.

a)

Known that $P^1(x_0) = f(x_0), P^1(x_1) = f(x_1)$,

$$P^1(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$

Since $P^0(x) = f(x_0)$,

$$P^1(x) = P^0(x) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

b)

Since $P^2(x) = P^1(x) + R(x)$,

$$\begin{cases} R(x_0) = P^2(x_0) - P^1(x_0) = f(x_0) - f(x_0) = 0 \\ R(x_1) = P^2(x_1) - P^1(x_1) = f(x_1) - f(x_1) = 0 \\ R(x_2) = P^2(x_2) - P^1(x_2) = f(x_2) - f(x_0) - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) \end{cases}$$

assume $R(x) = a(x - x_0)(x - x_1)$, then

$$\begin{aligned} a(x_2 - x_0)(x_2 - x_1) &= f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) \\ \Rightarrow a &= \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_1)(x_1 - x_0)} \end{aligned}$$

So

$$R(x) = \left(\frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_1)(x_1 - x_0)}\right)(x - x_0)(x - x_1)$$

c)

Use induction to prove that

$$P^j(x) = P^{j-1}(x) + a_j \prod_{k=0}^{j-1}(x - x_k)$$

1. For $j = 1$, we know that $P^1(x) = P^0(x) + \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$, so statement holds.

2. Assume that for $j \leqslant m$ the statement holds. Denote $R(x) = P^{m+1}(x) - P^m(x)$, then

$$R(x_i) = P^{m+1}(x_i) - P^m(x_i) = f(x_i) - f(x_i) = 0, \quad \text{for } i = 0, 1, \cdots, m$$

Since $R(x)$ is a polynomial of degree at most m, $R(x) = a \cdot \prod_{i=0}^{m}(x - x_i)$, where a is a constant. Since $R(x_m + 1) = P^{m+1}(x_{m+1}) - P^m(x_{m+1})$,

$$a = \frac{P^{m+1}(x_{m+1}) - P^m(x_{m+1})}{\prod_{i=0}^{m}(x_{m+1} - x_i)} = \frac{f(x_{m+1}) - P^m(x_{m+1})}{\prod_{i=0}^{m}(x_{m+1} - x_i)}$$

On the other hand,

$$\begin{aligned}
P^m(x) =& P^{m-1}(x) + a_m \prod_{k=0}^{m-1}(x - x_k) \\
=& P^0(x) + \sum_{i=1}^{m}\left(a_i \prod_{k=0}^{i-1}(x - x_k)\right) \\
=& f(x_0) + \sum_{i=1}^{m}\left(a_i \prod_{k=0}^{i-1}(x - x_k)\right)
\end{aligned}$$

since $\forall i \in [1, m] \cap \mathbb{N}, a_i$ only depends on $x_0, x_1, \cdots, x_i$, $P^m(x_{m+1})$ only depends on $x_0, x_1, \cdots, x_{m+1}$. Therefore, a only depends on $x_0, x_1, \cdots, x_{m+1}$. So the statement also holds for $j = m + 1$.

To sum up, $\forall j \in \mathbb{N}^*$

$$P^j(x) = P^{j-1}(x) + a_j \prod_{k=0}^{j-1}(x - x_k)$$

2.

According to Ex.3.1.c),

$$\begin{aligned}
P^n(x) =& \sum_{j=1}^{n}(P^j(x) - P^{j-1}(x)) + P^0(x) \\
=& \sum_{j=1}^{n} a_j \prod_{k=0}^{j-1}(x - x_k) + f(x_0)
\end{aligned}$$

so the statement holds.

3.

Let $f[x_0] = f(x_0)$ since it is arbitrary. And define

$$f[x_k] = f(x_k), \quad f[x_m, x_{m+1}, \cdots, x_{m+n}] = \frac{f[x_{m+1}, \cdots, x_{m+n}] - f[x_m, \cdots, x_{m+n-1}]}{x_{m+n} - x_m}$$

for $k > 0, m > 0, n > 0$. Assume $Q^{k-1}(x)$ is constructed in the same way as $P^{k-1}(x)$ with $x_1, f_1, \cdots, x_k, f_k$, assume

$$P^k(x) = a(x - b)P^{k-1}(x) + c(x - d)Q^{k-1}(x)$$

Then

$$P_k(x_0) = f_0 \Rightarrow a(x_0 - b)P^{k-1}(x_0) + c(x_0 - d)Q^{k-1}(x_0) = f_0$$

$$P_k(x_k) = f_k \Rightarrow a(x_k - b)P^{k-1}(x_k) + c(x_k - d)Q^{k-1}(x_k) = f_k$$

These two equations should hold for any x_0, x_k. So we can obtain that $b = x_k, d = x_0, a = \dfrac{1}{x_0 - x_k}, c = \dfrac{1}{x_k - x_0}$, and

$$P_k(x) = \frac{x - x_k}{x_0 - x_k}P^{k-1}(x) + \frac{x - x_0}{x_k - x_0}Q^{k-1}(x)$$

Now use induction to prove that $a_j = f[x_0, x_1, \cdots, x_j]$

1. For $j = 1$, $P^1(x) = P^0(x) + \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$, we see that

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1]$$

   so the statement holds for $j = 1$.

2. Assume that the statement also holds for $j = n$, i.e.

$$a_n = f[x_0, x_1, \cdots, x_n]$$

   we can also obtain the coefficient of x^{n-1} in Q^{n-1} is $b_n = f[x_1, x_2, \cdots, x_{n+1}]$, then

$$\begin{aligned} a_{n+1} =& \frac{1}{x_0 - x_{n+1}}a_n + \frac{1}{x_{n+1} - x_0}b_n \\ =& \frac{f[x_1, x_2, \cdots, x_{n+1}] - f[x_0, x_1, \cdots, x_n]}{x_{n+1} - x_0} \\ =& f[x_0, x_1, \cdots, x_{n+1}] \end{aligned}$$

   So the statement also holds for $j = n + 1$.

To sum up, $a_j = f[x_0, x_1, \cdots, x_j]$ holds for all $j \in \mathbb{N}$.

4.

Algorithm 1: Computing f(test) with Newton interpolation

Input: number n of nodes (x,f), and an input value test

Output: The value of function at test given by Newton interpolation

```
Function Newton_Interpolation(x,f,n,test):
    res ← f[1];
    for i ← 2 to n do
        prod ← Divided_Difference (1, i, f, x);
        for k ← 1 to i-1 do
            prod ← prod·(test-x[k]);
        end
        res ← res + prod;
    end
    return res;
end
Function Divided_Difference (i,k,f,x):
    if k - i =1 then
        prod ← (f[k]-f[i])/(x[k]-x[i]);
    end
    else
        prod ← (Divided_Difference (i+1,k,f,x)-Divided_Difference
          (i,k-1,f,x))/(x[k]-x[i]);
    end
    return prod;
end
```

5.

Use induction to prove that $\forall i,k \in \mathbb{N}, f[x_i,\cdots,x_{i+k}] = \dfrac{1}{k!h^k}\nabla^k f_i$

1. For $k=0$, $\forall i \in \mathbb{N}, f[x_i] = f_i = \nabla^0 f_i = \dfrac{1}{0!h^0}\nabla^0 f_i$
2. Assume that for $k=m$, $\forall i \in \mathbb{N}$, the statement holds, then

$$
\begin{aligned}
f[x_i,\cdots,x_{i+m+1}] =& \frac{f[x_{i+1},\cdots,x_{i+m+1}] - f[x_i,\cdots,x_{i+m}]}{x_{i+m+1}-x_i} \\
=& \frac{1}{(m+1)h}(f[x_{i+1},\cdots,x_{i+m+1}] - f[x_i,\cdots,x_{i+m}]) \\
=& \frac{1}{(m+1)h}(\frac{1}{m!h^m}\nabla^m f_{i+1} - \frac{1}{m!h^m}\nabla^m f_i) \\
=& \frac{1}{(m+1)!h^{m+1}}\nabla^{m+1} f_i
\end{aligned}
$$

So the statement also holds for $k=m+1, i \in \mathbb{N}$.

To sum up, $\forall i,k \in \mathbb{N}, f[x_i,\cdots,x_{i+k}] = \dfrac{1}{k!h^k}\nabla^k f_i$.

6.

Set $s = \dfrac{x - x_0}{h}$, and denote $\dbinom{s}{j} = \dfrac{1}{j!}\sum_{k=0}^{j-1}(s-k)$

$$\begin{aligned}
P^n(x) =& f(x_0) + \sum_{j=1}^{n} f[x_0, \cdots, x_j] \prod_{k=0}^{j-1}(x - x_k) \\
=& f_0 + \sum_{j=1}^{n} \frac{1}{j!h^j} \nabla^j f_0 \prod_{k=0}^{j-1}(x - x_0 - kh) \\
=& f_0 + \sum_{j=1}^{n} \frac{1}{j!} \nabla^j f_0 \prod_{k=0}^{j-1} \left(\frac{x - x_0}{h} - k\right) \\
=& f_0 + \sum_{j=1}^{n} \binom{s}{j} \nabla^j f_0
\end{aligned}$$

So the statement holds.

7.

Algorithm 2: Known t_i, f_i for $i = 0, 1, \cdots, n$, compute $f(x)$ by Newton interpolation with equidistant nodes

Input: t_i, f_i for $i = 0, 1, \cdots, n$, a step h, a value x
Output: The value of function at x given by Newton interpolation

Function `Newton_Interpolation`(t,f,n,x):
    res $\leftarrow f_0$;
    s $\leftarrow (x - t_0)/h$;
    for $i \leftarrow 1$ **to** n **do**
        prod $\leftarrow$ `Divided_Difference` (0,k, f);
        for $k \leftarrow 0$ **to** j-1 **do**
            prod $\leftarrow$ prod·(s-k)/(k+1);
        end
        res $\leftarrow$ res + prod;
    end
    return res;
end
Function `Divided_Difference` (i,k,f):
    if $k = 0$ **then**
        return f_i;
    end
    else
        return `Divided_Difference` $(i+1, k-1, f)$-`Divided_Difference` $(i, k-1, f)$;
    end
end