VV471— Introduction to Numerical Methods

Assignment 5

Jing and Manuel — UM-JI (Summer 2019)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Metric space*

Let X be a metric space.

1. Show that $\emptyset$ and X are closed in X.
2. Prove that the intersection of a collection of closed subsets of X is closed in X.
3. Prove that the union of a finite collection of closed subsets of X is closed in X.

Ex. 2 — *Continuity*

For each of the following question prove your result.

1. Give an example of function which is continuous but not uniformly continuous;
2. Give an example of function which is uniformly continuous but not Lipschitz continuous;

Ex. 3 — *Cardinality*

* 1. Prove that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same number of elements.

* 2. Prove that $[0, 1]$ has as many elements as $\mathbb{R}$.

3. Prove that $[0, 1]$ has more elements than $\mathbb{N}$.

   Hint: understand Cantor's diagonal argument.

Ex. 4 — *Slides*

1. In the lecture the Cauchy-Schwarz inequality (4.20) was proven for real numbers, prove it for the complex numbers.
2. Show that a distance is always positive.

Ex. 5 — *Linear algebra*

* 1. Let f be a linear map from a vector space V_1 into a vector space V_2. Show that the dimension of V_1 is the sum of the dimensions of the kernel and of the image of f. This result is called the rank nullity theorem.

2. Prove that the composition of two linear maps is a linear map.
3. Prove that the inverse of a linear map is a linear map.

Ex. 6 — *Discontinuous linear maps*

Finding discontinuous linear maps is not a simple task when the domain of definition is complete. In this exercise we show that the differential of a function is not always a continuous linear map.

Let$C^\infty(\mathbb{R})$, the set of the functions that are infinitely differentiable over $\mathbb{R}$. We equip $C^\infty(\mathbb{R})$ of the supremum norm.

1. Show that $f_n(x) = \sin(nx)/n$, $n > 0$, belongs to $C^\infty(\mathbb{R})$.
2. Determine the differential of f_n.

   Hint: refer to box 1 for a simple clarification on the differential.
3. * Observing the behavior of the differential when n tends to infinity, prove that it is not continuous.
4. Reach the same conclusion by applying theorem 4.28.

Ex. 7 — *Pi*

1. Write the pseudocode for at least one the following strategy to approximate π
   a) The polygons method;
   b) Machin's formula $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$ and Taylor series;
2. Implement at least one of the previous algorithms in MATLAB.

Box 1: A simple non-mathematical note on differentials and derivatives

Although differential and derivative are extremely similar, especially in dimension 1, they are in practice slightly different.

This can be seen intuitively by observing that the differential of a function f is denoted $\mathrm{d}f$ and its derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$. In other words, while the differential "represents" a small variation of f, the derivative "measures" a small variation of f with respect to a small variation of x, i.e. some kind of rate.

It is not too hard to prove that the existence of a differential implies the existence of a derivative following "any vector". However as seen in the following example, the existence of derivatives does not imply the existence of a differential.

Example:
Let $f : (x, y) = \frac{xy}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and 0 otherwise.
We quickly see that for $x \neq 0$ $f(x, x) = 1/2$, meaning that f is discontinuous at $(0, 0)$, and as such non differentiable in $(0, 0)$.

However for all $x, y \in \mathbb{R}$ we have $f(0, y) - f(0, 0) = f(x, 0) - f(0, 0) = 0$, that is the partial derivatives of f exist in $(0, 0)$ and they are equal to 0.