1.

(a)

Since f f is continuously differentiable in an open neighbourhood of a local minimiser $\mathbf{x}^*$, $\forall \mathbf{x} \in B_\varepsilon(\mathbf{x}^*)$, $f(\mathbf{x}^*) \leqslant f(\mathbf{x})$ and $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \cdots, \dfrac{\partial f}{\partial x_n}$ all exist.

Consider $\left.\dfrac{\partial f}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x}^*}$, the partial derivative exists means that

$$\lim_{h\searrow 0} \frac{f(x_1^*+h, x_2^*, \cdots, x_n^*) - f(x_1^*, x_2^*, \cdots, x_n^*)}{x_1^*+h-x_1^*} = \lim_{h\nearrow 0} \frac{f(x_1^*+h, x_2^*, \cdots, x_n^*) - f(x_1^*, x_2^*, \cdots, x_n^*)}{x_1^*+h-x_1^*}$$

then for h small enough such that $(x_1^*+h, x_2^*, \cdots, x_n^*) \in B_\varepsilon(\mathbf{x}^*)$,

$$\lim_{h\searrow 0} \frac{f(x_1^*+h, x_2^*, \cdots, x_n) - f(x_1^*, x_2^*, \cdots, x_n^*)}{x_1^*+h-x_1^*} \geqslant 0$$

$$\lim_{h\nearrow 0} \frac{f(x_1^*+h, x_2^*, \cdots, x_n^*) - f(x_1^*, x_2^*, \cdots, x_n^*)}{x_1^*+h-x_1^*} \leqslant 0$$

So the two limits have to be equal to 0, i.e. $\left.\dfrac{\partial f}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x}^*} = 0$.

Similarly, $\left.\dfrac{\partial f}{\partial x_2}\right|_{\mathbf{x}=\mathbf{x}^*} = \cdots = \left.\dfrac{\partial f}{\partial x_n}\right|_{\mathbf{x}=\mathbf{x}^*} = 0$, and therefore $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

(b)

On one hand, through $1(a)$, we have known that if $\mathbf{x}^*$ is a local minimiser, then $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

On the other hand, since $\mathbf{x}^*$ is a local minimiser, $\forall \mathbf{x} \in B_\varepsilon(\mathbf{x}^*)$, $f(\mathbf{x}) - f(\mathbf{x}^*) \geqslant 0$, so $\forall \mathbf{h}$ such that $\mathbf{x}^* + \mathbf{h} \in B_\varepsilon(\mathbf{x}^*)$,

$$f(\mathbf{x}^* + \mathbf{h}) - f(\mathbf{x}^*) \geqslant 0$$

Since Hessian $\mathbf{H}$ of f is continuous,

$$f(\mathbf{x}^* + \mathbf{h}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*) \cdot \mathbf{h} + \frac{1}{2}\mathbf{H}\mathbf{h} \cdot \mathbf{h} + o(\mathbf{h}^2) \quad \text{as } \mathbf{h} \to \mathbf{0}$$

we know that $\nabla f(\mathbf{x}^*) = \mathbf{0}$, then

$$f(\mathbf{x}^* + \mathbf{h}) - f(\mathbf{x}^*) = \frac{1}{2}\mathbf{H}\mathbf{h} \cdot \mathbf{h} + o(\mathbf{h}^2) \quad \text{as } \mathbf{h} \to \mathbf{0}$$

if $\mathbf{H}$ is indefinite, i.e. $\exists \mathbf{h}_0, \mathbf{k}_0$ such that

$$\mathbf{H}\mathbf{h}_0 \cdot \mathbf{h}_0 \geqslant 0, \quad \mathbf{H}\mathbf{k}_0 \cdot \mathbf{k}_0 < 0$$

then $\forall \lambda \neq 0$,

$$\mathbf{H}(\lambda \mathbf{h}_0) \cdot (\lambda \mathbf{h}_0) = \lambda^2 \mathbf{H}\mathbf{h}_0 \cdot \mathbf{h}_0 \geqslant 0$$

$$\mathbf{H}(\lambda\mathbf{k}_0)\cdot(\lambda\mathbf{k}_0) = \lambda^2\mathbf{H}\mathbf{k}_0\cdot\mathbf{k}_0 < 0$$

so for sufficiently small $\lambda > 0$, we have

$$f(\mathbf{x}^* + \lambda\mathbf{h}_0) - f(\mathbf{x}^*) = \frac{1}{2}\mathbf{H}(\lambda\mathbf{h}_0)\cdot(\lambda\mathbf{h}_0) + o((\lambda\mathbf{h}_0)) \geqslant 0$$

$$f(\mathbf{x}^* + \lambda\mathbf{k}_0) - f(\mathbf{x}^*) = \frac{1}{2}\mathbf{H}(\lambda\mathbf{k}_0)\cdot(\lambda\mathbf{k}_0) + o((\lambda\mathbf{k}_0)) < 0$$

then $\mathbf{x}^*$ cannot be local minimiser. So $\mathbf{H}$ is either positive semi-definite or negtive semi-definite. If it is negative definite, from the previous proof, we can see that it would lead to contradiction to $\mathbf{x}^*$ is local minimiser. And therefore, $\mathbf{H}$ is positive semi-definite.

(c)

On one hand, through 1(a), we have known that if $\mathbf{x}^*$ is a local minimiser, then $\nabla f(\mathbf{x}^*) = \mathbf{0}$. Now, if $\mathbf{x}^*$ is a global minimiser, then certainly $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

On the other hand, since f is convex, for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, it satisfies

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x_2}) \leqslant \alpha f(\mathbf{x}_1) + (1-\alpha)f(\mathbf{x}_2)$$

for all $\alpha \in [0,1]$. Given $\nabla f(\mathbf{x}^*) = \mathbf{0}$, if there exits a x^{**} such that $f(x^{**}) < f(x^*)$, then $\forall\alpha \in (0,1)$

$$\begin{aligned} &f(\alpha\mathbf{x}^{**} + (1-\alpha)\mathbf{x}^*) \leqslant \alpha f(\mathbf{x}^{**}) + (1-\alpha)f(\mathbf{x}^*) \\ \Rightarrow &\frac{f(\mathbf{x}^* + \alpha(\mathbf{x}^{**} - \mathbf{x}^*)) - f(\mathbf{x}^*)}{\alpha} \leqslant f(\mathbf{x}^{**}) - f(\mathbf{x}^*) < 0 \end{aligned}$$

Take limit for both side, since $\nabla f(\mathbf{x}^*) = \mathbf{0}$, the left hand side is equal to 0, which leads to contradiction. Therfore, $\mathbf{x}^*$ is the global minimiser.

(d)

If f is convex, then $\forall\mathbf{x}_1, \mathbf{x}_2$, $\forall\alpha \in (0,1)$,

$$\begin{aligned} &f(\alpha\mathbf{x}_2 + (1-\alpha)\mathbf{x}_1) \leqslant \alpha f(\mathbf{x}_2) + (1-\alpha)f(\mathbf{x}_1) \\ \Rightarrow &f(\mathbf{x}_1) + \frac{f(\mathbf{x}_1 + \alpha(\mathbf{x}_2 - \mathbf{x}_1)) - f(\mathbf{x}_1)}{\alpha} \leqslant f(\mathbf{x}_2) \end{aligned}$$

Take limit for both side and we obtain that

$$f(\mathbf{x}_2) \geqslant f(\mathbf{x}_1) + \nabla f(\mathbf{x}_1)\cdot(\mathbf{x}_2 - \mathbf{x}_1)$$

We also know that

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla f(\mathbf{x}_1)\cdot(\mathbf{x_2} - \mathbf{x_1}) + \frac{1}{2}\mathbf{H}(\mathbf{x_2} - \mathbf{x_1})\cdot(\mathbf{x_2} - \mathbf{x_1}) + o((\mathbf{x_2} - \mathbf{x_1})^2) \quad \text{as } (\mathbf{x_2} - \mathbf{x_1}) \to \mathbf{0}$$

So we can see that $\frac{1}{2}\mathbf{H}(\mathbf{x_2} - \mathbf{x_1})\cdot(\mathbf{x_2} - \mathbf{x_1}) \geqslant 0$. Since $\mathbf{x_2}, \mathbf{x_1}$ is chosen randomly, i.e. $\forall\mathbf{x}, \mathbf{H}\mathbf{x}\cdot\mathbf{x} \geqslant 0$, so $\mathbf{H}$ is positive semi definite. And all these procedure can be reversed.

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2.

Since $(x-2\pi)^2+\pi>0$, $\sin(\frac{1}{x})\in[-1,1]$, the minimum of $f(x)$ should be less than 0, with $\sin(\frac{1}{x})<0$.

For $x<\frac{1}{\frac{3\pi}{2}}=\frac{2}{3\pi}$, $(x-2\pi)^2+\pi>(\frac{2}{3\pi}-2\pi)^2+\pi$, then for x with $-1\leqslant\sin(\frac{1}{x})<0$,

$$f(x)=\frac{\sin(\frac{1}{x})}{(x-2\pi)^2+\pi}>-\frac{1}{(\frac{2}{3\pi}-2\pi)^2+\pi}=f(\frac{2}{3\pi})$$

For $x\geqslant\frac{1}{\pi}$, $0<x\leqslant\pi$, so $\sin(\frac{1}{x})\geqslant 0$, and therefore

$$f(x)=\frac{\sin(\frac{1}{x})}{(x-2\pi)^2+\pi}\geqslant 0>f(\frac{2}{3\pi})$$

So the minimum of $f(x)$ should be reached within $[\frac{2}{3\pi},\frac{1}{\pi})$. So we consider $f'(x)$ within the interval $(\frac{2}{3\pi},\frac{1}{\pi})$,

$$\begin{aligned}
&f'(x)=0\\
\Leftrightarrow&\frac{d}{dx}\frac{\sin(\frac{1}{x})}{(x-2\pi)^2+\pi}=0\\
\Leftrightarrow&\frac{\cos(\frac{1}{x})\frac{-1}{x^2}((x-2\pi)^2+\pi)-2(x-2\pi)\sin(\frac{1}{x})}{((x-2\pi)^2+\pi)^2}=0\\
\Leftrightarrow&\cos(\frac{1}{x})\frac{1}{x^2}((x-2\pi)^2+\pi)+2(x-2\pi)\sin(\frac{1}{x})=0
\end{aligned}$$

```
function [root] = bisection(a, b)
% required a < b
    r = (a + b)/2;
    f = @(x)cos(1/x)/x^2*((x-2*pi)^2+pi)+2*(x-2*pi)*sin(1/x);
    if (b - a < 0.0001)
        root = r;
    elseif (f(a) * f(r) < 0)
        root = bisection(a, r);
    else
        root = bisection(r, b);
    end
end
```

```
>> g=@(x)sin(1/x)/((x-2*pi)^2+pi)

g =

  包含以下值的 function_handle:

    @(x)sin(1/x)/((x-2*pi)^2+pi)

>> vpa(bisection(2/3/pi+0.00000001,1/pi),6)

ans =

0.212802

>> vpa(g(bisection(2/3/pi+0.00000001,1/pi)),6)

ans =

-0.0250034
```

```
>> vpa(g(2/3/pi),6)

ans =

-0.025001

>> vpa(2/3/pi,6)

ans =

0.212207
```

We find that for $x=0.212802$, $f'(x)=0$, and

$$f(0.212802)=-0.0250034<-0.025001=f(\frac{2}{3\pi})$$

Now, let's examine whether there exist other $x \in (\frac{2}{3\pi}, \frac{1}{\pi})$ such that $f'(x) = 0$.

For $x \in (\frac{4}{5\pi}, \frac{1}{\pi})$, $\cos(\frac{1}{x}) < \sin(\frac{1}{x}) < 0$, and therefore

$$\begin{aligned}
&\frac{1}{x^2}((x-2\pi)^2+\pi)+2(x-2\pi)\\
>&\frac{1}{x}((x-2\pi)^2+\pi)+2(x-2\pi) \qquad (\frac{1}{x}>\pi>1)\\
>&\frac{1}{x}(3x^2-8\pi x+4\pi^2+\pi)\\
>&\frac{1}{x}((3x-2\pi)(x-2\pi)+\pi)\\
>&0
\end{aligned}$$

So

$$\begin{aligned}
&\cos(\frac{1}{x})\frac{1}{x^2}((x-2\pi)^2+\pi)+2(x-2\pi)\sin(\frac{1}{x})\\
<&\sin(\frac{1}{x})(\frac{1}{x^2}((x-2\pi)^2+\pi)+2(x-2\pi))\\
<&0
\end{aligned}$$

which means that $f'(x) > 0$ for $x \in (\frac{4}{5\pi}, \frac{1}{\pi})$. So the local minimum of f cannot locate in the interval $x \in (\frac{4}{5\pi}, \frac{1}{\pi})$, where $\frac{4}{5\pi} \approx 0.2546$.

On the other hand,

$$\begin{aligned}
&f''(x)\\
=&\underbrace{\frac{2\cos(\frac{1}{x})}{x^3((x-2\pi)^2+\pi)}}_{①}-\underbrace{\frac{2\sin(\frac{1}{x})}{((x-2\pi)^2+\pi)^2}}_{②}-\underbrace{\frac{\sin(\frac{1}{x})}{x^4((x-2\pi)^2+\pi)}}_{③}\\
&+\underbrace{\frac{2\sin(\frac{1}{x})(2x-4\pi)^2}{((x-2\pi)^2+\pi)^3}}_{④}+\underbrace{\frac{2\cos(\frac{1}{x})(2x-4\pi)}{x^2((x-2\pi)^2+\pi)^2}}_{>0}
\end{aligned}$$

where

$$\begin{aligned}
① - ③ > 0 \Leftrightarrow& 2x\cos(\frac{1}{x}) - \sin\frac{1}{x} > 0 \Leftrightarrow 2x < \tan(\frac{1}{x}) \quad (x \in (\frac{2}{3\pi}, \frac{1}{\pi}))\\
\Leftarrow& 2x < \frac{1}{x} - \pi < \tan(\frac{1}{x})\\
\Leftrightarrow& \frac{2}{3\pi} < x < \frac{-\pi+\sqrt{\pi^2+8}}{4} \approx 0.2714
\end{aligned}$$

$$\begin{aligned}
-② + ④ =& \frac{2\sin(\frac{1}{x})}{((x-2\pi)^2+\pi)^3}((2x-4\pi)^2-(x-2\pi)^2-\pi)\\
=& \frac{2\sin(\frac{1}{x})}{((x-2\pi)^2+\pi)^3}(3(x-2\pi)^2-\pi) > 0
\end{aligned}$$

so for $x \in (\frac{2}{3\pi}, 0.2714)$, $f''(x) > 0$ and therefore $f(0.212802)$ is a local minimum in this interval.

To sum up, $f(0.212802)$ is the local minimum of f in the interval $[\frac{2}{3\pi}, \pi)$, where the global minimum of f locates in. So the global minimum of f is

$$f_{\min} = f(0.212802) = -0.025$$

.

3.

(a)

If $f'(c_{k-1}) > 0$, it means that f is increasing near $x = c_{k-1}$, since f is unimodal, it means that c_{k-1} is on the right hand side of the minimiser, so we should squeeze our region to (a_{k-1}, c_{k-1}); on the other hand, if $f'(c_{k-1}) < 0$, it means that f is decreasing near $x = c_{k-1}$, since f is unimodal, it means that c_{k-1} is on the left hand side of the minimiser, so we should squeeze our region to (c_{k-1}, b_{k-1}).

(b)

Consider $P'(x) = 0$ and $P''(x) > 0$,

$$\begin{aligned}\frac{d\,P(x)}{d\,x} = 0 \Leftrightarrow & 3\alpha_{k-1}(x - a_{k-1})^2 + 2\beta_{k-1}(x - a_{k-1}) + \gamma_{k-1} = 0 \\ \Leftrightarrow & x = a_{k-1} + \frac{-\beta_{k-1} \pm \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}\end{aligned}$$

$$\begin{aligned}\frac{d^2\,P(x)}{d\,x^2} > 0 \Leftrightarrow & 6\alpha_{k-1}(x - a_{k-1}) + 2\beta_{k-1} > 0 \\ \Leftrightarrow & x > a_{k-1} + \frac{-\beta_{k-1}}{3\alpha_{k-1}}, \alpha_{k-1} > 0 \vee x < a_{k-1} + \frac{-\beta_{k-1}}{3\alpha_{k-1}}, \alpha_{k-1} < 0\end{aligned}$$

If $\alpha_{k-1} > 0$,

$$a_{k-1} + \frac{-\beta_{k-1} + \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}} > a_{k-1} + \frac{-\beta_{k-1}}{3\alpha_{k-1}} > a_{k-1} + \frac{-\beta_{k-1} - \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}$$

so $x = a_{k-1} + \frac{-\beta_{k-1} + \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}$ is the minimiser of $P(x)$ in (a_{k-1}, b_{k-1}).

If $\alpha_{k-1} < 0$,

$$a_{k-1} + \frac{-\beta_{k-1} + \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}} < a_{k-1} + \frac{-\beta_{k-1}}{3\alpha_{k-1}} < a_{k-1} + \frac{-\beta_{k-1} - \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}$$

so $x = a_{k-1} + \frac{-\beta_{k-1} + \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}$ is the minimiser of $P(x)$ in (a_{k-1}, b_{k-1}).

To sum up, the minimiser of $P(x)$ in $[a_{k-1}, b_{k-1}]$ is

$$c_{k-1} = a_{k-1} + \frac{-\beta_{k-1} + \sqrt{\beta_{k-1}^2 - 3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}$$

(c)

Since $f(a_{k-1}) = P(a_{k-1})$, $\rho_{k-1} = f(a_{k-1})$.
Since $f'(a_{k-1}) = P'(a_{k-1})$, $\gamma_{k-1} = f'(a_{k-1})$.
Since $f(b_{k-1}) = P(b_{k-1}), f'(b_{k-1}) = P'(b_{k-1})$,

$$\begin{cases} \alpha_{k-1}(b_{k-1} - a_{k-1})^3 + \beta_{k-1}(b_{k-1} - a_{k-1})^2 + \gamma_{k-1}(b_{k-1} - a_{k-1}) + \rho_{k-1} = f(b_{k-1}) \\ 3\alpha_{k-1}(b_{k-1} - a_{k-1})^2 + 2\beta_{k-1}(b_{k-1} - a_{k-1}) + \gamma_{k-1} = f'(b_{k-1}) \end{cases}$$

$$\Rightarrow \begin{cases} \beta_{k-1} = -\dfrac{(b_{k-1} - a_{k-1})(2f'(a_{k-1}) + f'(b_{k-1})) + 3f(a_{k-1}) - 3f(b_{k-1})}{(b_{k-1} - a_{k-1})^2} \\ \alpha_{k-1} = \dfrac{(b_{k-1} - a_{k-1})(f'(a_{k-1}) + f'(b_{k-1})) + 2(f(a_{k-1}) - f(b_{k-1}))}{(b_{k-1} - a_{k-1})^3} \end{cases}$$

To sum up,

$$\alpha_{k-1} = \frac{(b_{k-1} - a_{k-1})(f'(a_{k-1}) + f'(b_{k-1})) + 2(f(a_{k-1}) - f(b_{k-1}))}{(b_{k-1} - a_{k-1})^3}, \quad \gamma_{k-1} = f'(a_{k-1})$$

$$\beta_{k-1} = -\frac{(b_{k-1} - a_{k-1})(2f'(a_{k-1}) + f'(b_{k-1})) + 3f(a_{k-1}) - 3f(b_{k-1})}{(b_{k-1} - a_{k-1})^2}, \quad \rho_{k-1} = f(a_{k-1})$$

(d)

编辑器 - D:\summer\VV471\assignment\HW2\cub_poly_min.m

bisection.m | cub_poly_min.m

```matlab
function [min] = cub_poly_min(a, b)
% required a < b
    f = @(x)exp(x) + 2 * x + x^2/2;
    g = @(x)exp(x) + 2 + x;
    alpha = ((b - a) * (g(a) + g(b)) + 2 * (f(a) - f(b)))/(b - a)^3;
    beta = -((b - a) * (2 * g(a) + g(b)) + 3 * (f(a) - f(b)))/(b - a)^2;
    gamma = g(a);
    c = a + (-beta + sqrt(beta^2 - 3 * alpha * gamma))/(3 * alpha);
    if (b - a < 0.000001)
        min = c;
    elseif (g(c) < 0)
        min = cub_poly_min(c, b);
    else
        min = cub_poly_min(a, c);
    end
end
```

命令行窗口

```
>> vpa(cub_poly_min(-2.4, -1.6))

ans =

-2.1200282389876425881425348052289

>> f = @(x)exp(x) + 2 * x + x.^2/2;
>> f(ans)

ans =

-1.8727683719351216114763936799864
```

We use Matlab and find that the minimum of $f(x) = e^x + 2x + \dfrac{x^2}{2}$ in $[-2.4, 1.4]$ is $f_{\min} = -1.87277$.

(e)

To use Newton's method, we require f to be twice continuously differentiable. For this method, we only require f to be unimodal and differentiable, but we need to know the initial interval where the minimum locates in.

$$\begin{aligned}c_{k-1}-a_{k-1}=&\frac{-\beta_{k-1}+\sqrt{\beta_{k-1}^2-3\alpha_{k-1}\gamma_{k-1}}}{3\alpha_{k-1}}\\=&\frac{-\gamma_{k-1}}{\beta_{k-1}+\sqrt{\beta_{k-1}^2-3\alpha_{k-1}\gamma_{k-1}}}\\\approx&\frac{A}{\frac{1}{(b_{k-1}-a_{k-1})^2}}\quad(\text{as } E_k=b_k-a_k\to 0)\end{aligned}$$

So it is about quadratic convergence. And Newton's method converge at best in quadratic rate.

(f)

To use Golden Section Search, we only require f to be unimodal. While, for this method, in addition, we require f to be differentiable. The Golden Section Search converge at rate $\frac{1+\sqrt{5}}{2}$ which is less than this method.

4.

For $f(x,y)=e^x(4x^2+2y^2+4xy+2y+1)$,

$$\begin{aligned}\frac{\partial f}{\partial x}=0\Leftrightarrow&e^x(4x^2+2y^2+4xy+2y+1)+e^x(8x+4y)=0\\\Leftrightarrow&4x^2+(4y+8)x+2y^2+6y+1=0\\\Leftrightarrow&x=\frac{-(y+2)\pm\sqrt{4-(y+1)^2}}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}>0\Leftrightarrow&e^x(4x^2+(4y+8)x+2y^2+6y+1)+e^x(8x+4y+8)>0\\\Leftrightarrow&4x^2+(4y+16)x+2y^2+10y+9>0\\\Leftrightarrow&x<\frac{-(y+4)-\sqrt{8-(y+1)^2}}{2}\vee x>\frac{-(y+4)+\sqrt{8-(y+1)^2}}{2}\end{aligned}$$

Denote $x_1=\frac{-(y+2)-\sqrt{4-(y+1)^2}}{2}$, $x_2=\frac{-(y+2)+\sqrt{4-(y+1)^2}}{2}$, we see that

1. $$x_1>\frac{-(y+4)-\sqrt{8-(y+1)^2}}{2}\quad\text{is always true}$$

2. $$\begin{aligned}&x_1\leqslant\frac{-(y+4)+\sqrt{8-(y+1)^2}}{2}\\\Leftrightarrow&\sqrt{8-(y+1)^2}\geqslant 2-\sqrt{4-(y+1)^2}\geqslant 0\\\Leftrightarrow&8-(y+1)^2\geqslant 4-4\sqrt{4-(y+1)^2}+4-(y+1)^2\\&\text{which is always true}\end{aligned}$$

3.

$$\begin{aligned}
&x_2 > \frac{-(y+4)+\sqrt{8-(y+1)^2}}{2} \\
\Leftrightarrow&\sqrt{8-(y+1)^2} < 2+\sqrt{4-(y+1)^2} \\
\Leftrightarrow&8-(y+1)^2 < 4+4\sqrt{4-(y+1)^2}+4-(y+1)^2 \\
\Leftrightarrow&-3 < y < 1
\end{aligned}$$

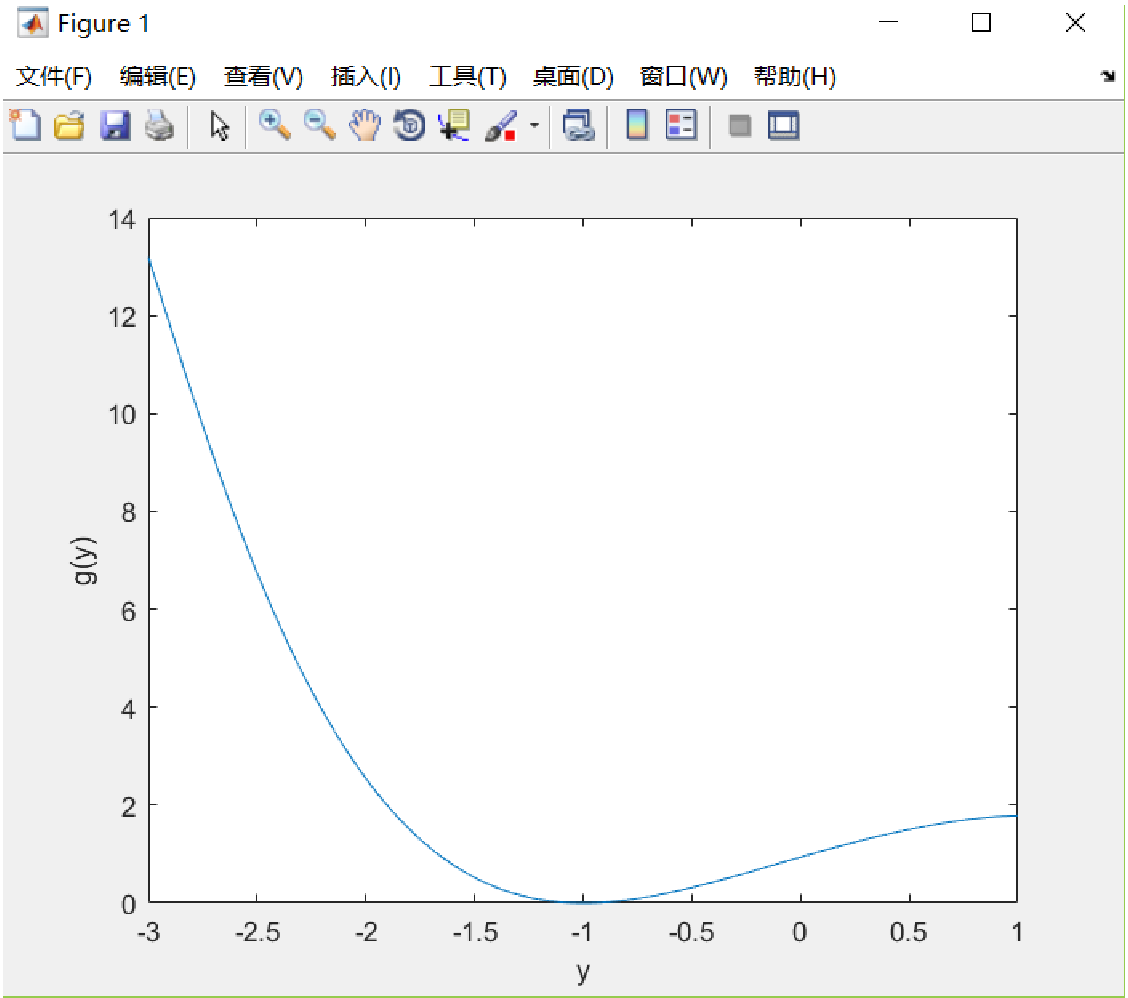
So when $-3 < y < 1$, $x = x_2 = \dfrac{-(y+2)+\sqrt{4-(y+1)^2}}{2}$ gives the only local minimum of $f(x,y)$, which means that it is the global minimum point.

When $y = 1$, $f(x) = e^x(4x^2+4x+5)$ and $f'(x) = e^x(2x+3)^2 \geqslant 0$ the equation , so f keeps growing with the increase of x.

When $y = -3$, $f(x) = e^x(4x^2-12x+13)$ and $f'(x) = e^x(2x-1)^2 \geqslant 0$, so f keeps growing with the increase of x.

For $y < -3 \vee y > 1$, $f(x,y)$ also keeps increasing with the x. So

$$g(y) = \min_x f(x,y) = \begin{cases} f(\dfrac{-(y+2)+\sqrt{4-(y+1)^2}}{2}, y), & -3 < y < 1 \\ \text{not exist} \qquad\qquad\qquad\qquad\qquad\ , & \text{otherwise} \end{cases}$$




```
>> f = @(x, y)exp(x).*(4 * x.^2 + 2 * y.^2+ 4 * x .* y + 2 * y + 1);
>> h = @(y)(-(y + 2) + sqrt(4 - (y + 1).^2))/2;
>> y=-3:0.0001:1;
>> plot(y, f(h(y), y));
```

5.

We can find the relationship between the longest ladder L that can pass the corner and the corner angle β is

$$L(\beta;\alpha) = \frac{1}{\sin(\pi-\alpha-\beta)} + \frac{1}{\sin\beta} = \frac{1}{\sin(\alpha+\beta)} + \frac{1}{\sin\beta}$$

then our goal is to minimize this function of β. For the first order derivative,

$$\begin{aligned}
&\frac{d\,L(\beta;\alpha)}{d\,\beta} = 0 \\
\Leftrightarrow &\frac{\cos(\beta+\alpha)}{\sin^2(\beta+\alpha)} + \frac{\cos\beta}{\sin^2\beta} = 0 \\
\Leftrightarrow & -\sin\alpha\tan^3\beta + (\cos^2\alpha+\cos\alpha)\tan^2\beta + 2\sin\alpha\cos\alpha\tan\beta + \sin^2\alpha = 0 \\
\Leftrightarrow &(\tan\beta - \cot\frac{\alpha}{2})(\tan^2\beta + \sin\alpha\tan\beta + 1 - \cos\alpha) = 0
\end{aligned}$$

For $\alpha \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, $\cos\alpha \leqslant \cos\frac{\pi}{4} < 1$, and therefore

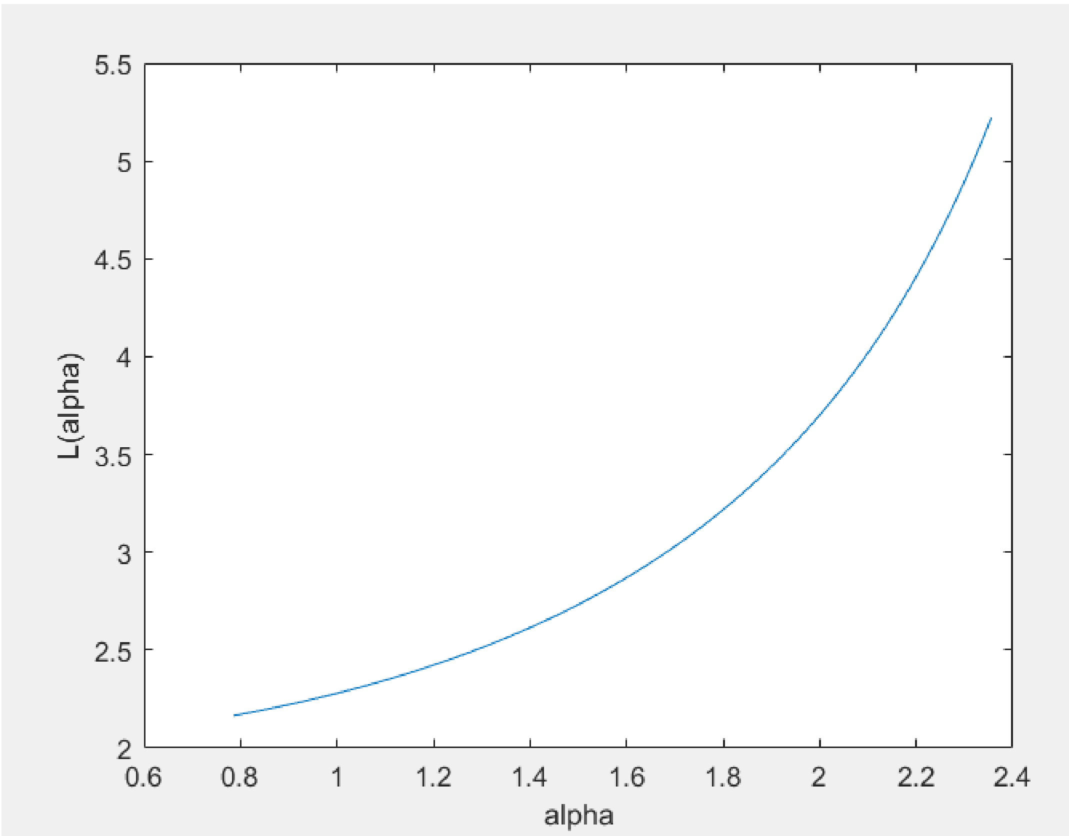
$$\Delta = \sin^2\alpha - 4(1-\cos\alpha) = -(\cos^2\alpha - 4\cos\alpha + 3) = -(\cos\alpha - 3)(\cos\alpha - 1) < 0$$

So $\frac{d\,L(\beta;\alpha)}{d\,\beta} = 0 \Leftrightarrow \tan\beta = \cot\frac{\alpha}{2}$. Given $\beta \in (0, \frac{\pi}{2})$ and $\alpha \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, $\beta = \frac{\pi-\alpha}{2}$. Then

$$\begin{aligned}\frac{d^2\,L(\beta;\alpha)}{d\,\beta^2}\Big|_{\beta=\frac{\pi-\alpha}{2}} &= -\frac{-\sin^3(\beta+\alpha) - 2\sin(\beta+\alpha)\cos^2(\beta+\alpha)}{\sin^4(\beta+\alpha)} \\ &\quad - \frac{-\sin^3\beta - 2\sin\beta\cos^2\beta}{\sin^4\beta}\Big|_{\beta=\frac{\pi-\alpha}{2}} \\ &= \frac{6}{\sin\frac{\alpha}{2}} > 0\end{aligned}$$

So $\arg\min\limits_{\beta\in(0,\frac{\pi}{2})} L(\beta;\alpha) = \frac{\pi-\alpha}{2}$, and

$$L_{\min}(\alpha) = \frac{1}{\sin(\alpha + \frac{\pi-\alpha}{2})} + \frac{1}{\sin(\frac{\pi-\alpha}{2})} = \frac{2}{\cos(\frac{\alpha}{2})}$$




```
>> alpha=pi/4:0.001:3*pi/4;
>> L=2./cos(alpha/2);
>> plot(alpha,L)
>> xlabel('alpha')
>> ylabel('L(alpha)')
```

6.

(a)

For the number of addition,

$$\begin{aligned}&\sum_{i=2}^{n}(i-1)\cdot(i-1)+\sum_{i=2}^{n}(i-1)\\=&\frac{(n-1)n(2(n-1)+1)}{6}+\frac{(n-1)n}{2}\\=&\frac{n(n-1)(n+1)}{3}\end{aligned}$$

For the number of multiplications,

$$\begin{aligned}&\sum_{i=2}^{n}((i-1)+1)+\sum_{i=2}^{n}(1+i-1)\\=&\frac{n(n+1)}{2}-1+\frac{n(n+1)}{2}-1\\=&n^2+n-2\end{aligned}$$

(b)

For the number of addition, it's unchanged

$$\frac{n(n-1)(n+1)}{3}$$

For the number of multiplications, more is needed for deciding the pivot

$$\begin{aligned}&n^2+n-2+\sum_{i=1}^{n-1}(n-i+1)\\=&n^2+n-2+\frac{n(n+1)}{2}-1\\=&\frac{3n(n+1)}{2}-3\end{aligned}$$

7.

1. For $\alpha=6$, $s_1=3, s_2=8, s_3=10$,

$$\frac{a_{11}}{s_1}=\frac{2}{3},\quad\frac{a_{21}}{s_2}=\frac{4}{8}=\frac{1}{2},\quad\frac{a_{31}}{s_3}=\frac{6}{10}=\frac{3}{5}$$

so we do not need to switch 1^{st} row with the 2^{nd} or 3^{rd} row. And

$$\frac{a_{22}}{s_2}=\frac{6}{8}=\frac{3}{4},\quad\frac{a_{32}}{s_3}=\frac{6}{10}=\frac{3}{5}$$

so we do not need to switch 2^{nd} row with the 3^{rd} row.

2. For $\alpha = 9,\ s_1 = 3, s_2 = 8, s_3 = 10$,
$$\frac{a_{11}}{s_1} = \frac{2}{3}, \quad \frac{a_{21}}{s_2} = \frac{4}{8} = \frac{1}{2}, \quad \frac{a_{31}}{s_3} = \frac{6}{10} = \frac{3}{5}$$
so we do not need to switch 1^{st} row with the 2^{nd} or 3^{rd} row. And
$$\frac{a_{22}}{s_2} = \frac{6}{8} = \frac{3}{4}, \quad \frac{a_{32}}{s_3} = \frac{9}{10}$$
so we need to switch 2^{nd} row with the 3^{rd} row.

3. For $\alpha = -3,\ s_1 = 3, s_2 = 8, s_3 = 10$,
$$\frac{a_{11}}{s_1} = \frac{2}{3}, \quad \frac{a_{21}}{s_2} = \frac{4}{8} = \frac{1}{2}, \quad \frac{a_{31}}{s_3} = \frac{6}{10} = \frac{3}{5}$$
so we do not need to switch 1^{st} row with the 2^{nd} or 3^{rd} row. And
$$\frac{a_{22}}{s_2} = \frac{6}{8} = \frac{3}{4}, \quad \frac{|a_{32}|}{s_3} = \frac{3}{10}$$
so we do not need to switch 2^{nd} row with the 3^{rd} row.

To sum up, for $\alpha = 6, -3$, no row swapping is required.

8.

(a)

For partial pivoting, we only need to find the largest one among $a_{ii}, a_{i+1,i}, \cdots, a_{ni}$ for all $i \in [1, n] \cap \mathbb{N}$, so totally
$$\sum_{i=1}^{n-1}((n-i+1)-1) = \frac{n(n-1)}{2}$$

(b)

For scaled partial pivoting, we first need to find the largest number in each row which needs
$$(n-1)\cdot n = n^2 - n$$
comparison. Then we need to compare the scaled pivoting which needs $\dfrac{n(n-1)}{2}$. So totally we need
$$n^2 - n + \frac{n(n-1)}{2} = \frac{3n(n-1)}{2}$$
comparison.

(c)

For complete pivoting, at k^{th} step we need to find the largest number among a_{ij} for all $i, j = k, k+1, \cdots, n$ so we need
$$\sum_{k=1}^{n-1}((n-k)^2-1) = \frac{(n-1)n(2(n-1)+1)}{6} - (n-1) = \frac{n(n-1)(2n-1)}{6} + 1 - n$$
comparison.