Ex. 1

1.

Since $C = \bigcap_{i=0}^{\infty} C_i \subset C_0 = [0,1]$, C is bounded. Consider its complementary set in $\mathbb{R}$

$$\mathbb{R}\backslash C = \mathbb{R}\backslash\left(\bigcap_{i=0}^{\infty} C_i\right) = \bigcup_{i=0}^{\infty} \mathbb{R}\backslash C_i$$

$\forall i \in \mathbb{N}$, according to the construction of C_i, it is composed of 2^i closed interval. Then its complementary set is composed of $2^i + 1$ open intervals, and therefore they are open. So $\forall i \in \mathbb{N}$, $\mathbb{R}\backslash C_i$ is open and their union $\mathbb{R}\backslash C$ is open. So C is closed in $\mathbb{R}$.

To sum up C is closed and bounded on $\mathbb{R}$ and therefore C is compact.

2.

Assume that $\exists x, y \in C$ such that there is no element z satisfying that $x < z < y$ and $z \notin C$, then $\forall z \in (x, y)$, $z \in C$, i.e. $[x, y] \subset C$. Since $C = \bigcap_{i=0}^{\infty} C_i$, $[x, y] \subset C_i$ for all $i \in \mathbb{N}$.

Consider the length of this interval which is $0 < y - x \leqslant 1$, then $0 \leqslant \log_{\frac{1}{3}}(y - x) < \infty$. So $\exists N \in \mathbb{N}$ such that $\log_{\frac{1}{3}}(y-x) < N \Rightarrow y - x > \dfrac{1}{3^N}$. So for all interval $\mathcal{I}$ with length less or equal to $\dfrac{1}{3^N}$, $[x, y]$ cannot be covered by $\mathcal{I}$. Consider the set C_N, which is composed of 2^N disjoint closed intervals with length $\dfrac{1}{3^N}$, it cannot cover $[x, y]$, i.e. $\exists z \in [x, y]$ such that $z \notin C_N$. So $[x, y] \not\subset C_N$ which is contradiction.

So for any two elements $x < y$ of C, there exists $z \notin C$ such that $x < z < y$.

3.

a)

For any countable set A, we have

$$\lambda(A) = \sum_{x \in A} [x, x] = 0$$

So any countable set of real numbers has Lebesgue measure 0.

b)

$\forall i \in \mathbb{N}$, consider the set C_i, which is composed of 2^i disjoint closed intervals with length $\dfrac{1}{3^i}$, then

$$\lambda(C) = \sum_{j=0}^{n} |\ell(\mathcal{I}_j)| = 2^i \cdot \frac{1}{3^i} = \left(\frac{2}{3}\right)^i$$

Since $C \subset C_i$,

$$0 \leqslant \lambda(C) \leqslant \lambda(C_i) = \left(\frac{2}{3}\right)^i \xrightarrow{i\to\infty} 0 \text{ for any } i \in \mathbb{N}$$

So $\lambda(C) = 0$.

4.

a)

According to the construction of C_i, we can see that $\forall i \in \mathbb{N}, 0 \in C_i$ and therefore $0 \in C$. So C is not empty.

b)

For $C_i = \bigcup_{j=0}^{2^{i-1}} \left[\sum_{k=1}^{i} \left(\frac{2}{3^{i+1-k}} \cdot \lambda(k,j) \right), \sum_{k=1}^{i} \left(\frac{2}{3^{i+1-k}} \cdot \lambda(k,j) \right) + \frac{1}{3^i} \right]$ where $\lambda(k,j)$ satisfies that

$$\lambda(k,j) \in \{0,1\}, \quad \sum_{k=1}^{i} \lambda(k,j) 2^k = j$$

So the element in C_i can be represented as

$$x = \sum_{k=1}^{i} \frac{2b_k}{3^k} + \sum_{k=i+1}^{\infty} \frac{a_k}{3^k}, \qquad b_k \in \{0,1\}, \quad a_k \in \{0,1,2\}$$

c)

We assume that each element can be mapped to a unique natural number, then list them in an order and express them in base 2. Now consider an element, whose i^{th} number to the right of digital point is different from that of the i^{th} element. On one hand, such element should be one in the set, however, it is different from any of element in that set which means that it cannot be an element of that set. So there doesn't exist a map such that each element corresponding to a unique natural number, and therefore the set is not countable.

d)

As $i \to \infty$, we can see that each $x \in C$ can be represented as

$$x = \sum_{k=1}^{\infty} \frac{2b_k}{3^k}, \qquad b_k \in \{0,1\}$$

If we denote each x by its b_k, each element is formed of zero and one only, and each compose of zero and one corresponding to an $x \in C$. According to Cantor's diagonal argument, such kind of set is uncountable.

5.

These two measurement ways are different by definition. The Lebesgue measurement gives the set a "volume". Even when this volume is zero, it may be formed by uncountable "tiny" elements. Cantor's set is such a special case.

Ex. 2

1.

Consider f_n, it is continuous according to the construction, and

1. For $x \in C_n$, $f_n(x) = ax + b$ is monotonically increasing
2. For $x \notin C_n$, since $\mathbb{R}\backslash C_n$ is open, there exist ε for each x such that $B(x, \varepsilon) \subset \mathbb{R}\backslash C_n$, and according to the construction, $f(x) =$ constant in this region.

To sum up, $(f_n)_{n\in\mathbb{N}}$ define a sequence of monotonically increasing continuous functions.

2.

Known $f_0(x) = x, x \in [0,1]$, $f_1(x) = \begin{cases} 0.5 \cdot 3x, & x \in [0, \frac{1}{3}] \\ 0.5, & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0.5 \cdot (3x-2) + 0.5, & x \in [\frac{2}{3}, 1] \end{cases}$, we find that

$$f_1(x) = \begin{cases} 0.5 \cdot f_0(3x), & x \in [0, \frac{1}{3}] \\ 0.5, & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0.5 \cdot f_0(3x-2) + 0.5, & x \in [\frac{2}{3}, 1] \end{cases}$$

and such kind of iteration can be continued, i.e.

$$f_{n+1}(x) = \begin{cases} 0.5 \cdot f_n(3x), & x \in [0, \frac{1}{3}] \\ 0.5, & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0.5 \cdot f_n(3x-2) + 0.5, & x \in [\frac{2}{3}, 1] \end{cases}$$

so

$$\begin{aligned} \max_{x\in[0,1]} |f_{n+1}(x) - f_n(x)| &= \max\{\max_{x\in[0,\frac{1}{3}]} 0.5|f_n(3x) - f_{n-1}(3x)|, 0, \max_{x\in[\frac{2}{3},1]} 0.5|f_n(3x-2) - f_{n-1}(3x-2)|\} \\ &\leqslant \max_{x\in[0,1]} 0.5|f_n(x) - f_{n-1}(x)| \\ &= 0.5 \max_{x\in[0,1]} |f_n(x) - f_{n-1}(x)| \end{aligned}$$

from which we see that $\max_{x\in[0,1]} |f_{n+1}(x) - f_n(x)| \xrightarrow{n\to\infty} 0$. So f_n converges uniformly to f_C.

3.

a)

We have seen that $f_{n+1}(x)$ is a linear transformation of $f_n(x)$, as $n \to \infty$, the continuous property will hold. Since $[0,1]$ is compact, the continuous function over a compact interval is uniformly continuous. So $f_C(x)$ is uniformly continuous.

b)

We have seen that $f_{n+1}(x)$ is a linear transformation of $f_n(x)$, and no negative sign appears to change the growth of f_n, as $n \to \infty$, the monotonically increasing property will hold.

c)

For $f_C(x)$, except at $x = \sum\limits_{n=1}^{\infty} \frac{2b_n}{3^n}, b_n \in \{0, 1\}$, over the interval formed with two nearest x, f_C is differentiable and equal to a constant, and therefore $f_C'(x) = 0$ over these intervals. For those part in C, since C has Lebesgue measure 0, we can conclude that $f_C'(x)$ is differentiable and $f_C'(x) = 0$ almost everywhere.

4.

For $\varepsilon < 1$, for any $\eta > 0$, we can find $N \in \mathbb{N}$ such that $\left(\frac{2}{3}\right)^N < \eta$, then consider $f_N(x)$, there are $2^N - 1$ intervals over which $f_N(x)$ is a constant, the distance between two neighbour intervals $[a_i, b_i]$ is $\frac{1}{3^N}$. These value will not change as $n \to \infty$, i.e. $f_n \to f_C$. Now define

$$[x_1, y_1] = [0, a_1], [x_2, y_2] = [b_1, a_2], \cdots, [x_{2^N-1}, y_{2^N-1}] = [b_{2^N-2}, a_{2^N-1}], [x_{2^N}, y_{2^N}] = [b_{2^N-1}, 1]$$

with length $\ell_i = y_i - x_i = \frac{1}{3^N}$, then

$$\sum_{i=1}^{2^N} |y_i - x_i| = \sum_{i=1}^{2^N} \ell_i = \frac{2^N}{3^N} < \eta$$

while

$$\sum_{i=1}^{2^N} |f_C(y_i) - f_C(x_i)| = \sum_{i=1}^{2^N} \frac{1}{2^N} = 1 > \varepsilon$$

so f_C is not absolutely continuous.

Ex. 3

1.

We use induction to prove: Given a function f such that $f^{(n)}$ is absolutely continuous on the compact $[a, x] \subset \mathbb{R}$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$$

1. For $n = 0$, according to the fundamental theorem for calculus

   $$f(x) = f(a) + \int_a^x f'(t) dt$$

   the statement holds.

2. Assume that the statement holds for $n = m$. Then given a function f such that $f^{(m+1)}$ is absolutely continuous on the compact $[a, x] \subset \mathbb{R}$, and we obtain that

$$\begin{aligned}
&\frac{1}{(m+1)!}\int_a^x f^{(m+1+1)}(t)(x-t)^{m+1}dt \\
=&\frac{1}{(m+1)!}\left(f^{(m+1)}(t)(x-t)^{m+1}\Big|_a^x - (m+1)\int_a^x f^{(m+1)}(t)(x-t)^m dt\right) \\
=&-\frac{1}{(m+1)!}f^{(m+1)}(a)(x-a)^{m+1} + \frac{1}{m!}\int_a^x f^{(m+1)}(t)(x-t)^m dt
\end{aligned}$$

so by the induction,

$$\begin{aligned}
f(x) =& \sum_{k=0}^{m}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{m!}\int_a^x f^{(m+1)}(t)(x-t)^m dt \\
=& \sum_{k=0}^{m+1}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(m+1)!}\int_a^x f^{(m+1+1)}(t)(x-t)^{m+1}dt \\
& \underbrace{-\frac{f^{(m+1)}(a)}{(m+1)!} + \frac{1}{m!}\int_a^x f^{(m+1)}(t)(x-t)^m dt - \frac{1}{(m+1)!}\int_a^x f^{(m+1+1)}(t)(x-t)^{m+1}dt}_{=0} \\
=& \sum_{k=0}^{m+1}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(m+1)!}\int_a^x f^{(m+1+1)}(t)(x-t)^{m+1}dt
\end{aligned}$$

we see that the statement also holds for $n = m + 1$.

To sum up, for all $n \in \mathbb{N}$, Taylor theorem holds.

2.

Given a function f such that $f^{(n)}$ is absolutely continuous on the compact $[a, x] \subset \mathbb{R}$, we define

$$R_n(x) = f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

we can see that

$$R_n(a) = R_n'(a) = \cdots = R_n^{(n)}(a) = 0$$

then according to Cauchy mean value theorem, $\exists \xi_1 \in [a, x]$ such that

$$\begin{aligned}
\frac{R_n'(\xi_1)}{((\xi_1-a)^{n+1})'} &= \frac{R_n(x) - R_n(a)}{(x-a)^{n+1} - (a-a)^{n+1}} \\
\Rightarrow \frac{R_n(x)}{(x-a)^{n+1}} &= \frac{R_n'(\xi_1)}{(n+1)(\xi_1-a)^n}
\end{aligned}$$

and $\exists \xi_2 \in [a, \xi_1]$ such that

$$\begin{aligned}
&\frac{R_n''(\xi_2)}{((\xi_2-a)^n)'} = \frac{R_n'(\xi_1)-R_n'(a)}{(\xi_1-a)^n-(a-a)^n} \\
\Rightarrow &\frac{R_n'(\xi_1)}{(\xi_1-a)^n} = \frac{R_n''(\xi_2)}{n(\xi_2-a)^{n-1}} \\
\Rightarrow &\frac{R_n(x)}{(x-a)^{n+1}} = \frac{R_n'(\xi_1)}{(n+1)(\xi_1-a)^n} = \frac{R_n''(\xi_2)}{(n+1)n(\xi_2-a)^{n_1}}
\end{aligned}$$

Recursively, we will obtain $\exists \xi_{n+1} \in [a, \xi_n] \subset [a, \xi_{n-1}] \subset \cdots \subset [a, \xi_1] \subset [a, x]$ such that

$$\frac{R_n(x)}{(x-a)^{n+1}} = \frac{R_n^{(n+1)}(\xi_2)}{(n+1)!}$$

And we know that

$$R_n^{(n+1)}(x) = f^{(n+1)}(x) - \frac{d^n}{dx^n}\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k = f^{(n+1)}(x)$$

So we see that

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi_2)}{(n+1)!}(x-a)^{n+1}$$

Ex. 4

1.

Since $4! = 24 > 2^4$, for $n \geqslant 4, n! > 2^n$, and therefore

$$\begin{aligned}
e = &\sum_{i=0}^{\infty}\frac{1}{i!} < 1+1+\frac{1}{2}+\frac{1}{6}+\sum_{i=4}^{\infty}\frac{1}{2^i} = 2+\frac{2}{3}+\frac{\frac{1}{2^4}}{1-\frac{1}{2}} = 2+\frac{19}{24} < 3 \\
e = &\sum_{i=0}^{\infty}\frac{1}{i!} > 1+1 = 2
\end{aligned}$$

which means that e cannot be an integer. Assume that e is a rational number, then $\exists p, q \in \mathbb{N}, p > 1$, the greatest common factor of p, q is 1 such that $e = \frac{q}{p} = \sum\limits_{i=0}^{\infty}\frac{1}{i!}$. Multiple $p!$ on both sides andwe obtain that

$$\underbrace{q \cdot p!}_{integer} = p! \cdot \sum_{i=0}^{\infty}\frac{1}{i!} = \underbrace{p! + \sum_{i=1}^{p} i!}_{integer} + \sum_{i=p+1}^{\infty}\frac{p!}{i!}$$

So $\sum\limits_{i=p+1}^{\infty} \frac{p!}{i!}$ should also be an integer, however,

$$0 < \sum_{i=p+1}^{\infty} \frac{p!}{i!} = \sum_{i=1}^{\infty} \frac{1}{\prod\limits_{j=1}^{i}(p+j)} < \sum_{i=1}^{\infty} \frac{1}{3^i} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

which means that it cannot be an integer. So e is a irrational number.

2.

Define $a^n := \exp\left(\frac{1}{n}\right)^n$, and set $b^n = (1+\frac{1}{n})^n$, since

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

For $a-b$,

$$a - b = \exp\left(\frac{1}{n}\right) - \left(1+\frac{1}{n}\right) = \sum_{i=0}^{\infty} \frac{1}{i!} - \left(1+\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{1}{n}\right)^i$$

as for $\sum\limits_{k=0}^{n-1} a^k b^{n-1-k}$,

$$\begin{aligned}
\sum_{k=0}^{n-1} a^k b^{n-1-k} &\leqslant \sum_{k=0}^{n-1} \exp\left(\frac{1}{n}\right)^k \left(1+\frac{1}{n}\right)^{n-1-k} \\
&\leqslant \sum_{k=0}^{n-1} \exp\left(\frac{1}{n}\right)^k \left(\exp\left(\frac{1}{n}\right)\right)^{n-1-k} \\
&= n\left(\exp\left(\frac{1}{n}\right)\right)^{n-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
a^n - b^n &\leqslant \frac{1}{n}\sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{1}{n}\right)^i \cdot n\left(\exp\left(\frac{1}{n}\right)\right)^{n-1} \\
&\leqslant \sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{1}{n}\right)^i \cdot \left(\exp\left(\frac{1}{n}\right)\right)^n \\
&= e \cdot \sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{1}{n}\right)^i \\
&\xrightarrow{n\to\infty} 0
\end{aligned}$$

so $\lim\limits_{n\to} \left(1+\frac{1}{n}\right)^n = \lim\limits_{n\to\infty} \exp\left(\frac{1}{n}\right)^n = e$, and therefore, for all $\frac{\varepsilon}{2} > 0$, $\exists N \in \mathbb{N}$, such that $\forall n > N$,

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon$$

so $\forall\varepsilon > 0.\exists N \in \mathbb{N}$ such that $\forall m, k > N$

$$\begin{aligned}\left|\left(1+\frac{1}{m}\right)^m - \left(1+\frac{1}{k}\right)^k\right| =& \left|\left(1+\frac{1}{m}\right)^m - e + e - \left(1+\frac{1}{k}\right)^k\right| \\ \leqslant& \left|\left(1+\frac{1}{m}\right)^m - e\right| + \left|\left(1+\frac{1}{k}\right)^k - e\right| \\ <& \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ =& \varepsilon\end{aligned}$$

Thus $u_n = \left(1+\frac{1}{n}\right)^n$ is a Cauchy sequence converging to e.

3.

We see that each term in sequence $u_n = \left(1+\frac{1}{n}\right)^n$ is a product of rational numbers and therefore it is a rational number. However, the sequence converges to e, which is not a rational number. And therefore $\mathbb{Q}$ is not complete.