VE203
Assignment 5

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Exercise 5.1 Binary Insertion Sort

i)

Algorithm 1: Binary Insertion Sort

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Input  : a_1, ..., a_n, n unsorted elements
Output: all the a_i , 1 ⩽ i ⩽ n in increasing order
for j ← 2 to n do
    i ← 1 ;
    k ← j;
    while i < k do
        m ← ⌊(i + k)/2⌋ ;
        if a_m > a_j then
            k ← m;
        else
            i ← m + 1;
        end if
    end while
    l ← a_j ;
    for k ← 0 to j − i − 1 do
        a_{j−k} ← a_{j−k−1};
    end for
    a_i ← l;
end for
return (a_1, ..., a_n) in increasing order
```

ii)

For insertion sort (according to the algorithm in slides),

1. $4 : 4 - 7 \to 4, 7, 3, 8, 1, 5, 4, 2$
2. $3 : 3 - 4 \to 3, 4, 7, 8, 1, 5, 4, 2$
3. $8 : 8 - 3, 8 - 4, 8 - 7, 8 - 8 \to 3, 4, 7, 8, 1, 5, 4, 2$
4. $1 : 1 - 3 \to 1, 3, 4, 7, 8, 5, 4, 2$
5. $5 : 5 - 1, 5 - 3, 5 - 4, 5 - 7 \to 1, 3, 4, 5, 7, 8, 4, 2$
6. $4 : 4 - 1, 4 - 3, 4 - 4 \to 1, 3, 4, 4, 5, 7, 8, 2$
7. $2 : 2 - 1, 2 - 3 \to 1, 2, 3, 4, 4, 5, 7, 8$

So the number of comparisons of elements used by the insertion sort is 16.

For binary insertion sort,

1. $4 : 4 - 7 \to 4, 7, 3, 8, 1, 5, 4, 2$
2. $3 : 3 - 7, 3 - 4 \to 3, 4, 7, 8, 1, 5, 4, 2$
3. $8 : 8 - 4, 8 - 7 \to 3, 4, 7, 8, 1, 5, 4, 2$
4. $1 : 1 - 7, 1 - 4, 1 - 3 \to 1, 3, 4, 7, 8, 5, 4, 2$

5. $5 : 5-4, 5-8, 5-7 \to 1, 3, 4, 5, 7, 8, 4, 2$
6. $4 : 4-5, 4-3, 4-4 \to 1, 3, 4, 4, 5, 7, 8, 2$
7. $2 : 2-4, 2-3, 2-1 \to 1, 2, 3, 4, 4, 5, 7, 8$

So the number of comparisons of elements used by the binary insertion sort is 17.

iii)

During the $j^{th}(j = 2, 3, \cdots, n)$ pass, less than or equal to j comparisons are used among these elements. The total number of comparisons N is then

$$N \leqslant \sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 = \frac{n^2}{2} + \frac{n}{2} - 1 \leqslant \frac{n^2}{2} + \frac{n^2}{2} = n^2$$

So $N = O(n^2)$ as $n \to \infty$. So the insertion sort uses $O(n^2)$ comparisons of elements.

iv)

During the $j^{th}(j = 2, 3, \cdots, n)$ pass, less than or equal to $1 + log_2 j$ comparisons are used among these elements. The total number of comparisons N is then

$$N \leqslant \sum_{j=2}^{n}(1 + log_2 j) = n - 1 + log_2 n! \leqslant n - 1 + log_2 n^n \leqslant 2n log_2 n$$

While for the number of total comparisons N',

$$N' \leqslant \sum_{j=2}^{n}(1 + j - 1 - 1 + 2log_2 j) = \frac{n(n-1)}{2} + 2log_2 n! \leqslant \frac{n^2}{2} + 2n^2$$

So $N' = O(n^2)$ as $n \to \infty$. So the complexity of binary insertion sort is $O(n^2)$. So it is really faster during the comparison of elements, while it is not significantly faster than the insertion sort in total.

Exercise 5.2 M,I,C,H,I,G,A,N

Merge Sort

1. $M, I, C, H; I, G, A, N$
2. $M, I; C, H; I, G; A, N$
3. $M - I, C - H, I - G, A - N \to I, M; C, H; G, I; A, N$
4. $I - C, I - H, G - A, G - N, I - N \to C, H, I, M; A, G, I, N$
5. $C - A, C - G, H - G, H - I, I - I, I - N, M - N \to A, C, G, H, I, I, M, N$

So the number of comparisons of elements used by the merge sort is 16.

Insertion Sort

(according to the algorithm in slides)

1. $I - M \to I, M, C, H, I, G, A, N$
2. $C - I \to C, I, M, H, I, G, A, N$
3. $H - C, H - I \to C, H, I, M, I, G, A, N$
4. $I - C, I - H, I - I \to C, H, I, I, M, G, A, N$
5. $G - C, G - H \to C, G, H, I, I, M, A, N$
6. $A - C \to A, C, G, H, I, I, M, N$
7. $N - A, N - C, N - G, N - H, N - I, N - I, N - M, N - N \to A, C, G, H, I, I, M, N$

So the number of comparisons of elements used by the insertion sort is 18.

Bubble Sort

1. (a) $M - I \to I, M, C, H, I, G, A, N$
   (b) $M - C \to I, C, M, H, I, G, A, N$
   (c) $M - H \to I, C, H, M, I, G, A, N$
   (d) $M - I \to I, C, H, I, M, G, A, N$
   (e) $M - G \to I, C, H, I, G, M, A, N$
   (f) $M - A \to I, C, H, I, G, A, M, N$
   (g) $M - N \to I, C, H, I, G, A, M, N$
2. (a) $I - C \to C, I, H, I, G, A, M, N$
   (b) $I - H \to C, H, I, I, G, A, M, N$
   (c) $I - I \to C, H, I, I, G, A, M, N$
   (d) $I - G \to C, H, I, G, I, A, M, N$
   (e) $I - A \to C, H, I, G, A, I, M, N$
   (f) $I - M \to C, H, I, G, A, I, M, N$
3. (a) $C - H \to C, H, I, G, A, I, M, N$
   (b) $H - I \to C, H, I, G, A, I, M, N$
   (c) $I - G \to C, H, G, I, A, I, M, N$
   (d) $I - A \to C, H, G, A, I, I, M, N$
   (e) $I - I \to C, H, G, A, I, I, M, N$
4. (a) $C - H \to C, H, G, A, I, I, M, N$
   (b) $H - G \to C, G, H, A, I, I, M, N$
   (c) $H - A \to C, G, A, H, I, I, M, N$
   (d) $H - I \to C, G, A, H, I, I, M, N$
5. (a) $C - G \to C, G, A, H, I, I, M, N$

   (b) $G - A \to C, A, G, H, I, I, M, N$
   (c) $G - H \to C, A, G, H, I, I, M, N$

6. (a) $C - A \to A, C, G, H, I, I, M, N$
   (b) $C - G \to A, C, G, H, I, I, M, N$

7. (a) $A - C \to A, C, G, H, I, I, M, N$

So the number of comparisons of elements used by the bubble sort is 28.

Exercise 5.3 Application to Feng Shui

i)

For $\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} \leqslant n < \underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k+1)\times 10}$, the number of decimal digits of n is $\lfloor log_{10}n\rfloor + 1$. So after one operation, the sum of these decimal digits at most

$$n_1 = 9 \times (\lfloor log_{10}n\rfloor + 1) \in [9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-1)\times 10} + 1), 9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} + 1)) \subset [\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-1)\times 10}, 9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} + 2))$$

so the number of decimal digits of n_1 is $\lfloor log_{10}n_1\rfloor + 1$. So after one operation, the sum of these decimal digits at most

$$\begin{aligned}
n_2 = 9 \times (\lfloor log_{10}n_1\rfloor + 1) &\in (9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-2)\times 10} + 1), 9 \cdot (\lfloor log_{10}9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} + 2)\rfloor) + 1) \\
&\subset [\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-2)\times 10}, 9 \cdot (1 + \lfloor log_{10}(\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} + 2)\rfloor) + 1) \\
&\subset [\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-2)\times 10}, 9 \cdot (1 + \lfloor log_{10}(\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} + 2)\rfloor) + 1) \\
&= [\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-2)\times 10}, 9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-1)\times 10} + 2))
\end{aligned}$$

then with the similar process, we can prove that

$$n_i \in [\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k-i)\times 10}, 9 \cdot (\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k+1-i)\times 10} + 2))$$

so after k times,

$$n_k \in [1, 9 \cdot (10 + 2)) = [1, 108)$$

then

$$n_{k+1} \in [1, 18], n_{k+2} \in [1, 9]$$

So the worst-case number of additions that need to be performed to calculate the iterated integer sum of some n that $\underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{k\times 10} \leqslant n < \underbrace{10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}}_{(k+1)\times 10}$ in this way is the $k + 2$.

ii)

The iterated integer sum of a number n is equal to $n \ mod \ 9$.

Proof:

We first prove that the sum of decimal digits of n is congruent to n modulo 9.

$\forall n \in \mathbb{N}$, we can set $n = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots a_1 \cdot 10 + a_0, a_0, a_1, \cdots, a_k \in [0,9] \cap \mathbb{N}$. Then

$$\begin{aligned} n &\equiv a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots a_1 \cdot 10 + a_0 \equiv a_k \cdot 1^k + a_{k-1} \cdot 1^{k-1} + \cdots a_1 \cdot 1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots a_1 + a_0 \ (mod \ 9) \end{aligned}$$

So the sum of decimal digits of n (n_1) is congruent to n modulo 9. So

$$n \equiv n_1 \equiv n_2 \equiv \cdots \equiv the \ iterated \ integer \ sum \ of \ n \ (mod \ 9)$$

Since the iterated integer sum of n is in $[0,9]$, then the sum is just equal to $n \ mod \ 9$

To sum up, the iterated integer sum of a number n is equal to $n \ mod \ 9$.

iii)

The iterated integer sum of a number n represented in arbitrary base b is equal to $n \, mod \, b-1$.

Proof:

We first prove that the sum of digits of n represented in arbitrary base b is congruent to n modulo $b-1$.

$\forall n \in \mathbb{N}$, we can set $n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots a_1 \cdot b + a_0, a_0, a_1, \cdots, a_k \in [0, b-1] \cap \mathbb{N}$. Then

$$\begin{aligned} n &\equiv a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots a_1 \cdot b + a_0 \equiv a_k \cdot 1^k + a_{k-1} \cdot 1^{k-1} + \cdots a_1 \cdot 1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots a_1 + a_0 \ (mod \ b-1) \end{aligned}$$

So the sum of digits of n represented in arbitrary base b is congruent to n modulo $b-1$. So

$$n \equiv n_1 \equiv n_2 \equiv \cdots \equiv the \ iterated \ integer \ sum \ of \ n \ represented \ in \ base \ b (mod \ b-1)$$

Since the iterated integer sum of n is in $[0, b-1]$, then the sum is just equal to $n \ mod \ b-1$

To sum up, the iterated integer sum of a number n represented in arbitrary base b is equal to $n \ mod \ b-1$.

Exercise 5.4 Modular Exponentiation

$1042 = 1024 + 16 + 2 = (10000010010)_2, 4102 \equiv 74 \ (mod \ 2014)$. Let $power = 74, x = 1$.

1. $i = 0, n_i = 0$, $power \cdot power \equiv 1448 \ (mod \ 2014)$, let $power = 1448$
2. $i = 1, n_i = 1$, let $x = 1 \cdot 1448 = 1448$, $power \cdot power \equiv 2096704 \equiv 130 \ (mod \ 2014)$, let $power = 130$
3. $i = 2, n_i = 0$, $power \cdot power \equiv 16900 \equiv 788 \ (mod \ 2014)$, let $power = 788$
4. $i = 3, n_i = 0$, $power \cdot power \equiv 620944 \equiv 632 \ (mod \ 2014)$, let $power = 632$

5. $i = 4, n_i = 1$, let $x = (1448 \cdot 632)$ mod $2014 = 780$, $power \cdot power \equiv 399424 \equiv 652 \pmod{2014}$, let $power = 652$

6. $i = 5, n_i = 0$, $power \cdot power \equiv 425104 \equiv 150 \pmod{2014}$, let $power = 150$

7. $i = 6, n_i = 0$, $power \cdot power \equiv 22500 \equiv 346 \pmod{2014}$, let $power = 346$

8. $i = 7, n_i = 0$, $power \cdot power \equiv 119716 \equiv 890 \pmod{2014}$, let $power = 890$

9. $i = 8, n_i = 0$, $power \cdot power \equiv 792100 \equiv 598 \pmod{2014}$, let $power = 598$

10. $i = 9, n_i = 0$, $power \cdot power \equiv 357604 \equiv 1126 \pmod{2014}$, let $power = 1126$

11. $i = 10, n_i = 1$, let $x = (780 \cdot 1126)$ mod $2014 = 176$

To sum up, 4102^{1042} mod $2014 = 176$.

Exercise 5.5 Stein's Algorithm for the GCD in base 2

i)

For multiplication by 2, it's just add a 0 after the first digit from right of a number in base 2; for division, it's just add a decimal point between the first and second digit from right of a number in base 2 (or move the decimal point to left for a digit).

ii)

According to Lemma 1.6.22, if a and b are both even, then

$$gcd(a, b) = 2 \cdot gcd(\frac{a}{2}, \frac{b}{2})$$

iii)

If a and b are both odd, then

$$gcd(a - b, b) = gcd(a, b)$$

iv)

Algorithm 2: Algorithm to calculate the gcd of two natural numbers in base 2.

Input : Two natrual number a, b

Output: The greatest common divisor of a, b

```
Function StGCD(a, b):
    if a = b then
        return a
    end if
    if a = 0 ∧ b = 0 then
        return 0
    end if
    if a = 0 then
        return b
    end if
    if b = 0 then
        return a
    end if
    if a is even and b is even then
        return 2×StGCD(a/2, b/2)
    end if
    if a is even and b is odd then
        return StGCD(a/2, b)
    end if
    if a is odd and b is even then
        return StGCD(a, b/2)
    end if
    if a is odd and b is odd then
        if a > b then
            return StGCD((a − b)/2, b)
        else
            return StGCD(a, (b − a)/2)
        end if
    end if
end
```

Exercise 5.6

i) $a_n = a_{n-1} + 6a_{n-2}$

Solve $r^2 - r - 6 = 0$ and we get $r_1 = 3, r_2 = -2$. So set the solution of the recurrence relation is

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot (-2)^n$$

Since $a_0 = 3, a_1 = 6$, then

$$\begin{cases} \alpha_1 + \alpha_2 = 3 \\ 3\alpha_1 - 2\alpha_2 = 6 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \dfrac{12}{5} \\ \alpha_2 = \dfrac{3}{5} \end{cases}$$

So the solution of the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ is

$$a_n = \frac{12}{5} \cdot 3^n + \frac{3}{5} \cdot (-2)^n$$

ii)$a_{n+2} = -4a_{n+1} + 5a_n$

Solve $r^2 + 4r - 5 = 0$ and we get $r_1 = -5, r_2 = 1$. So set the solution of the recurrence relation is

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot (-5)^n$$

Since $a_0 = 2, a_1 = 8$, then

$$\begin{cases} \alpha_1 + \alpha_2 = 2 \\ \alpha_1 - 5\alpha_2 = 8 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 3 \\ \alpha_2 = -1 \end{cases}$$

So the solution of the recurrence relation $a_{n+2} = -4a_{n+1} + 5a_n$ is

$$a_n = 3 - (-5)^n$$

Exercise 5.7

First, we show that $a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot nr_0^n, \alpha_1, \alpha_2 \in \mathbb{R}, n \in \mathbb{N}$ solves the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, where $r_0^2 - c_1 r_0 - c_2 = 0, c_1^2 + 4c_2 = 0$. So $r_0 = \dfrac{c_1}{2}$

$\forall n \in \mathbb{N}$,

$$\begin{aligned} & a_{n+2} - c_1 a_{n+1} - c_2 a_n \\ =& \alpha_1 \cdot r_0^n \underbrace{(r_0^2 - c_1 r_0 - c_2)}_{0} + \alpha_2 \cdot r_0^n((n+2)r_0^2 - c_1(n+1)r_0 - c_2 n) \\ =& \alpha_2 \cdot r_0^n(n \cdot \underbrace{(r_0^2 - c_1 r_0 - c_2)}_{0} + (2(c_1/2)^2 - c_1 \cdot c_1/2)) \\ =& 0 \end{aligned}$$

so the recurrence relation is satisfied.

Now let (a_n) be a solution to the recurrence relation. By Lemma 2.3.4 this sequence is unique and determined by a_0 and a_1. We thus need to show that we can find α_1 and α_2 such that

$$a_0 = \alpha_1, a_1 = \alpha_1 r_0 + \alpha_2 r_0$$

and we get that

$$\alpha_1 = a_0, \alpha_2 = \frac{a_1}{r_0} - \alpha_1 \ \ (r_0 \neq 0)$$

$$\alpha_1 = a_0, \alpha_2 \ \textit{is no need} \ (r_0 = 0)$$

To sum up, all solution to a linear homogeneous recurrence relation of degree two are of the form

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot nr_0^n, \alpha_1, \alpha_2 \in \mathbb{R}, n \in \mathbb{N}$$

if there is only a single characteristic root r_0.

Exercise 5.8

i) $a_n = 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

Since $-336 \cdot (5 \cdot 4^{n-1} - 6 \cdot 4^{n-2} - 4^n) = -42 \cdot 8 \cdot 4^{n-2} \cdot (5 \cdot 4 - 6 - 16) = 42 \cdot 4^n$,

$$a_n = 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n \Leftrightarrow (a_n - 336 \cdot 4^n) = 5(a_{n-1} - 336 \cdot 4^{n-1}) - 6(a_{n-2} - 336 \cdot 4^{n-2})$$

Set $b_n = a_n - 336 \cdot 4^n$, then $b_n = 5b_{n-1} - 6b_{n-2}$. Solve $r^2 - 5r + 6 = 0$ and we get $r_1 = 2, r_2 = 3$. So set the solution of the recurrence relation is

$$b_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$$

then

$$\begin{cases} \alpha_1 + \alpha_2 = b_0 \\ 2\alpha_1 + 3\alpha_2 = b_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 3b_0 - b_1 \\ \alpha_2 = b_1 - 2b_0 \end{cases}$$

So

$$\begin{aligned} a_n &= b_n + 336 \cdot 4^n = (3b_0 - b_1) \cdot 2^n + (b_1 - 2b_0) \cdot 3^n + 336 \cdot 4^n \\ &= (3(a_0 - 336) - (a_1 - 336 \cdot 4)) \cdot 2^n + ((a_1 - 336 \cdot 4) - 2(a_0 - 336)) \cdot 3^n + 336 \cdot 4^n \\ &= (3a_0 - a_1 + 336) \cdot 2^n + (a_1 - 2a_0 - 672) \cdot 3^n + 336 \cdot 4^n \end{aligned}$$

So the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$ is

$$a_n = (3a_0 - a_1 + 336) \cdot 2^n + (a_1 - 2a_0 - 672) \cdot 3^n + 336 \cdot 4^n$$

ii) $a_n = -5a_{n-1} - 6a_{n-2} + 2^n + 3n$

Since

$$\begin{aligned} & -5(-\frac{1}{5} \cdot 2^{n-1} - \frac{1}{4}(n-1) - \frac{17}{48}) - 6(-\frac{1}{5} \cdot 2^{n-2} - \frac{1}{4}(n-2) - \frac{17}{48}) - (-\frac{1}{5} \cdot 2^n - \frac{1}{4}n - \frac{17}{48}) \\ =& 2^{n-2}(2 + 6/5 + 4/5) + n(5/4 + 3/2 + 1/4) + (-5/4 + 85/48 - 3 + 17/8 + 17/48) \\ =& 2^n + 3n \end{aligned}$$

then

$$\begin{aligned} & a_n = -5a_{n-1} - 6a_{n-2} + 2^n + 3n \\ \Leftrightarrow & (a_n - \frac{1}{5} \cdot 2^n - \frac{1}{4}n - \frac{17}{48}) = -5(a_{n-1} - \frac{1}{5} \cdot 2^{n-1} - \frac{1}{4}(n-1) - \frac{17}{48}) \\ & - 6(a_{n-2} - \frac{1}{5} \cdot 2^{n-2} - \frac{1}{4}(n-2) - \frac{17}{48}) \end{aligned}$$

Set $b_n = a_n - \frac{1}{5} \cdot 2^n - \frac{1}{4}n - \frac{17}{48}$, then $b_n = -5b_{n-1} - 6b_{n-2}$. Solve $r^2 + 5r + 6 = 0$ and we get $r_1 = -2, r_2 = -3$. So set the solution of the recurrence relation is

$$b_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (-3)^n$$

then

$$\begin{cases} \alpha_1 + \alpha_2 = b_0 \\ -2\alpha_1 - 3\alpha_2 = b_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 3b_0 + b_1 \\ \alpha_2 = -b_1 - 2b_0 \end{cases}$$

So

$$\begin{aligned}
a_n =& b_n + \frac{1}{5} \cdot 2^n + \frac{1}{4} n + \frac{17}{48} \\
=& (3b_0 + b_1) \cdot (-2)^n - (b_1 + 2b_0) \cdot (-3)^n + \frac{1}{5} \cdot 2^n + \frac{1}{4} n + \frac{17}{48} \\
=& (3(a_0 - 1/5 - 17/48) + (a_1 - 2/5 - 1/4 - 17/48)) \cdot (-2)^n \\
& - ((a_1 - 2/5 - 1/4 - 17/48) + 2(a_0 - 1/5 - 17/48)) \cdot (-3)^n + \frac{1}{5} \cdot 2^n + \frac{1}{4} n + \frac{17}{48} \\
=& (3a_0 + a_1 - \frac{8}{3}) \cdot (-2)^n - (a_1 + 2a_0 - \frac{169}{80}) \cdot 3^n + \frac{1}{5} \cdot 2^n + \frac{1}{4} n + \frac{17}{48}
\end{aligned}$$

So the solution of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 2^n + 3n$ is

$$a_n = (3a_0 + a_1 - \frac{8}{3}) \cdot (-2)^n - (a_1 + 2a_0 - \frac{169}{80}) \cdot 3^n + \frac{1}{5} \cdot 2^n + \frac{1}{4} n + \frac{17}{48}$$

iii) $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$

Since

$$\begin{aligned}
& 7(-16(n-1) + 80) \cdot 4^{n-1} - 16(-16(n-2) + 80) \cdot 4^{n-2} + 12(-16(n-3) + 80) \cdot 4^{n-3} \\
& - (-16n + 80) \cdot 4^n \\
=& 4^{n-3}(n \cdot (-7 \cdot 16^2 + 16^2 \cdot 4 - 12 \cdot 16 + 16 \cdot 64) + 7 \cdot 16 \cdot 96 - 64 \cdot 112 + 12 \cdot 128 - 80 \cdot 64) \\
=& n4^n
\end{aligned}$$

then

$$\begin{aligned}
& a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n \\
\Leftrightarrow & (a_n + (-16n + 80) \cdot 4^n) = 7(a_{n-1} + (-16(n-1) + 80) \cdot 4^{n-1}) \\
& - 16(a_{n-2} + (-16(n-2) + 80) \cdot 4^{n-2}) + 12(a_{n-3} + (-16(n-3) + 80) \cdot 4^{n-3}) +
\end{aligned}$$

Set $b_n = a_n + (-16n + 80) \cdot 4^n$, then $b_n = 7b_{n-1} - 16b_{n-2} + 12b_{n-3}$. Solve $r^3 - 7r^2 + 16r - 12 = 0$ and we get $r_1 = 2, r_2 = 2, r_3 = 3$. So set the solution of the recurrence relation is

$$b_n = \alpha_1 \cdot 2^n + \alpha_2 n \cdot 2^n + \alpha_3 \cdot 3^n$$

then

$$\begin{cases} \alpha_1 + \alpha_3 = b_0 \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 = b_1 \\ 4\alpha_1 + 8\alpha_2 + 9\alpha_3 = b_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -3b_0 + 4b_1 - b_2 = -3a_0 + 4a_1 - a_2 + 16 \\ \alpha_2 = \dfrac{-6b_0 + 5b_1 - b_2}{2} = \dfrac{-6a_0 + 5a_1 - a_2 + 32}{2} \\ \alpha_3 = 4b_0 - 4b_1 + b_2 = 4a_0 - 4a_1 + a_2 + 64 \end{cases}$$

So

$$\begin{aligned}
a_n =& b_n - (-16n + 80) \cdot 4^n = (3b_0 - b_1) \cdot 2^n + (b_1 - 2b_0) \cdot 3^n + 336 \cdot 4^n \\
=& (-3a_0 + 4a_1 - a_2 + 16) \cdot 2^n + \frac{-6a_0 + 5a_1 - a_2 + 32}{2} n \cdot 2^n + (4a_0 - 4a_1 + a_2 + 64) \cdot 3^n \\
& - (-16n + 80) \cdot 4^n
\end{aligned}$$

So the solution of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ is

$$a_n = ((-3a_0 + 4a_1 - a_2 + 16) + \frac{-6a_0 + 5a_1 - a_2 + 32}{2} n) \cdot 2^n + (4a_0 - 4a_1 + a_2 + 64) \cdot 3^n - (-16n + 80) \cdot 4^n$$