Ex. 2

1.

We see that $w(x) = 1$ is a weight function and $\frac{a+b}{2} \in [a, b]$, and therefore formula 2.1 fall under Peano's method.

2.

We see that

$$\int_a^b x^0 dx = b - a = (b-a) \cdot (\frac{a+b}{2})^0$$

$$\int_a^b x^1 dx = \frac{b^2 - a^2}{2} = (b-a) \cdot (\frac{a+b}{2})^1$$

always hold, while

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = (b-a) \cdot (\frac{a+b}{2})^2$$

hold conditionally and therefore $N = 1$.

$$K_N(t) = \int_a^b [(x-t)_+]^N dx - (b-a)[(\frac{a+b}{2} - t)_+]^N$$

1. If $t > b$,

$$K_N(t) = 0 - 0 = 0$$

.

2. If $\frac{a+b}{2} < t \leqslant b$,

$$K_N(t) = \int_t^b (x-t) dx = \frac{1}{2}(b-t)^2 \geqslant 0$$

3. If $a < t \leqslant \frac{a+b}{2}$,

$$K_N(t) = \int_t^b (x-t)dx - (b-a)(\frac{a+b}{2} - t) = \frac{1}{2}(t-a)^2 \geqslant 0$$

4. If $t \leqslant a$,

$$K_N(t) = \int_a^b (x-t)dx - (b-a)(\frac{a+b}{2} - t) = 0$$

So the Peano kernel for this formula is

$$K_N(t) = \begin{cases} 0 & , t \leqslant a \vee t > b \\ \frac{1}{2}(b-t)^2 & , \frac{a+b}{2} < t \leqslant b \\ \frac{1}{2}(t-a)^2 & , a < t \leqslant \frac{a+b}{2} \end{cases}$$

and it keeps a constant sign.

3.

According to Peano kernel theorem,

$$E(f) = \frac{1}{1!} \int_a^b K_N(t) f^{''}(t) dt$$

since $K_N(t)$ keeps a constant sign, according to first mean value theorem, $\exists \imath \in (a, b)$ such that

$$\begin{aligned} E(f) =& \frac{1}{1!} \int_a^b K_N(t) f^{''}(t) dt = f''(\xi) \int_a^b K_N(t) dt \\ =& f''(\xi) \Big(\frac{1}{6}(t-a)^3 \Big|_a^{\frac{a+b}{2}} + \frac{1}{6}(t-b)^3 \Big|_{\frac{a+b}{2}}^b \Big) \\ =& f''(\xi) \cdot \frac{1}{24}(b-a)^3 \end{aligned}$$

Ex. 3

1.

a)

$\forall x \in (-1, 1)$, $\sqrt{1-x^2} > 0$ and

$$\int_{-1}^1 \sqrt{1-x^2} \xlongequal{x=\cos t} \int_\pi^0 \sin t \cdot (-\sin t) dt = \int_0^\pi \frac{1}{2}(1 - \cos(2t)) dt = \frac{\pi}{2} < \infty$$

So $w(x) = \sqrt{1-x^2}$ is a weight function.

b)

$\forall i, j \in \mathbb{N}$,

$$\begin{aligned} \int_{-1}^1 q_i(x) q_j(x) w(x) dx \xlongequal{x=\cos\theta} & \int_0^\pi \frac{\sin(i+1)\theta}{\sin\theta} \frac{\sin(j+1)\theta}{\sin\theta} \sin^2\theta d\theta \\ =& \int_0^\pi \frac{1}{2} \Big(\cos(i-j)\theta - \cos(i+j+2)\theta \Big) d\theta \end{aligned}$$

1. For $i = j$,

$$\int_{-1}^1 q_i(x) q_j(x) w(x) dx = \frac{\pi}{2}$$

2. For $i \neq j$

$$\int_{-1}^1 q_i(x) q_j(x) w(x) dx = 0$$

So $(q_k)_{k\in\mathbb{N}}$ define a sequence of orthogonal polynomials for the weight function w.

c)

For $p_k(x) = \sqrt{\frac{2}{\pi}} q_k(x)$, we can see that

$$\int_{-1}^{1} p_i(x)p_j(x)w(x)dx = \begin{cases} 1 & , i = j \\ 0, & i \neq j \end{cases}$$

and therefore it is the orthonormal polynomials associated to q_k

2.

a)

Since q_k is a sequence of orthogonal polynomials for the weight function w,

$$q_{n+1}(x_k) = 0 \Rightarrow \frac{\sin((n+2)\arccos x_k)}{\sin(\arccos x_k)} = 0 \Rightarrow \arccos x_k = \frac{(k+1)\pi}{n+2}$$

So $x_k = \cos\left(\frac{(k+1)\pi}{n+2}\right)$

b)

$\forall k \geqslant 1$,

$$\begin{aligned} q_{k+1}(x) + q_{k-1}(x) =& \frac{\sin(k+1)\theta + \sin(k-1)\theta}{\sin\theta} = \frac{2\sin k\theta\cos 2\theta}{\sin\theta} \\ =& 2\cos\theta\frac{\sin k\theta}{\sin\theta} = 2xq_k(x) \end{aligned}$$

denote a_k as the coefficient of x^k in $q_k(x)$, then $a_k = 2a_{k-1}$, so

$$A_k = \frac{a_{n+1}}{a_n}\frac{\int_{-1}^{1} w(x)q_n(x)^2 dx}{q'_{n+1}(x_k)q_n(x_k)} = 2\cdot\frac{\pi}{2}\cdot\frac{1}{q'_{n+1}(x_k)q_n(x_k)}$$

1.

$$\begin{aligned} q'_{n+1}(x_k) =& \frac{dq_{n+1}(\cos\theta)}{d\theta}\frac{d\theta}{d\cos\theta}\Big|_{\theta=\theta_k} \\ =& \frac{(n+2)\cos(n+2)\theta\sin\theta - \cos\theta\sin(n+2)\theta}{\sin^2\theta}\frac{1}{-\sin\theta}\Big|_{\theta=\theta_k} \\ =& \frac{(-1)^k(n+2)}{\sin^2\frac{(k+1)\pi}{n+2}} \end{aligned}$$

2.

$$q_n(x_k) = \frac{\sin\frac{(n+1)(k+1)\pi}{n+2}}{\sin\frac{(k+1)\pi}{n+2}} = \frac{\sin\left((k+1)\pi - \frac{(k+1)\pi}{n+2}\right)}{\sin\frac{(k+1)\pi}{n+2}} = (-1)^k$$

And therefore, $A_k = \frac{\pi}{n+1}\sin^2\frac{(k+1)\pi}{n+2}$

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c)

According to Theorem 7.179, we see that here $k = n, l = n + 1$ and therefore the statement holds.