Theory of Complex Systems CW1

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(a) The branching probabilities for the process is

$$p_i = \begin{cases} 1-p & \text{for i} = 0 \\ p & \text{for i} = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where i is $Z_{\geq 0}$.

(b) The generator function for the process is

$$g(s) = \sum_{k=0}^{\infty} p_k s^k = (1-p) + ps^2.$$

To determine the average branching ratio, we differentiate and set $s = 1$:

$$g'(s) = 2ps$$

$$\mu = g'(1) = 2p.$$

The process is critical when $\mu = 1$:

$$1 = 2p \Rightarrow p_c = 1/2.$$

Hence, the value p_c for which the process is critical is $\frac{1}{2}$.

(c) If $p > p_c = \frac{1}{2}$, then the probability to have an infinite avalanche is nonzero. Given the definition of the time to extinction $\tau \coloneqq min\{n \geq 1 | Z_n = 0\}$, τ may not be defined when $p > p_c = \frac{1}{2}$. So we only consider the case $p < p_c$, where all avalanches are finite and $\mu < 1$. In the lecture notes, we have that for $\mu < 1 : P(\tau > n) \sim \mu^n$ for large n. To show that $P(\tau > n) \propto exp(-\frac{n}{n_0})$, we show that

$$\frac{\mu^n}{exp(-\frac{n}{n_0})} = C,$$

where C is a non-zero constant.

$$\frac{\mu^n}{exp(-\frac{n}{n_0})} = \frac{\mu^n}{exp(-\frac{1}{n_0})^n} = \Big[\frac{\mu}{exp(-\frac{1}{n_0})}\Big]^n.$$

For $\left[\frac{\mu}{exp(-\frac{1}{n_0})}\right]^n$ to be a non-zero constant, we require that $\mu = \exp -\frac{1}{n_0}$. Hence,

$$n_0 = -\frac{1}{ln\mu}.$$

As μ is < 1, $ln\mu < 0 \Rightarrow n_0 \in R^+$.

(d) Let the average branching ratio be of the form $\mu = 1 - \Delta$, then

$$n_0 = -\frac{1}{ln(1-\Delta)}.$$

If we Taylor expand $ln(1-\Delta)$ we get

$$n_0 = -\frac{1}{(-\Delta) - \frac{(-\Delta)^2}{2} + \frac{(-\Delta)^3}{3} - ...} = -\frac{1}{(-\Delta) + O(\Delta^2)}.$$

As $\Delta \to 0^+$, the higher order terms $\Delta^2, \Delta^3, \Delta^4, ...$ will be $<< \Delta$. Hence,

$$n_0 \to -\frac{1}{-\Delta} \propto \frac{1}{\Delta^a},$$

where $a = 1$.

(e) A site in the system will relax if the energy on the site reaches a critical value. The probability that an active site relaxes will depend on the energy in the system and the size of the system. Instead of considering the energy at each site separately, Eq. (1) models the probability that any site will relax by using the average energy of the system at each site. This single averaged approximation greatly simplifies the problem.

The term $\frac{1-\sigma_n(p,t)}{N}$ in Eq. (1) describes the change in total energy of the system averaged over the total number of sites, N. This term assumes that one unit of energy is added to the system at the initial active site, and loses one unit of energy for each $\sigma_n(p,t)$ which leaves the system.

If no particles leave the system, $\sigma_n(p,t) = 0$, then energy has been added to the system and hence the probability that an active site relaxes will increase. If $\sigma_n(p,t) > 0$, then energy has left the system and hence the probability an active site relaxes will decrease. Both these statements are described by Eq. (1).

If we assume that each relaxing boundary site lost two units of energy then Eq. (1) becomes

$$p(t+1) = p(t) + \frac{1 - 2\sigma_n(p,t)}{N}.$$

For a fixed value of p, the average value of σ_n is $(2p)^n$. We can write $o_n(p,t) = (2p)^n + \eta(p,t)$, where η is the fluctuations around the average.

Inserting this expression in Eq. (1) and taking the continuum time limit, we obtain

$$\frac{dp}{dt} = \frac{1 - 2(2p)^n}{N} + \frac{2\eta(p,t)}{N}.$$

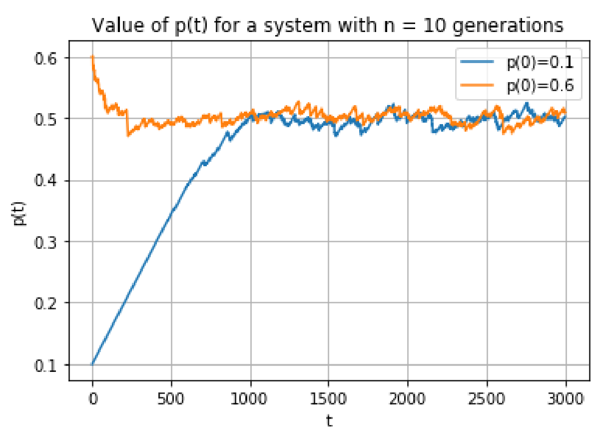
If we ignore the last term and solve for the fixed point we get

$$\frac{dp}{dt} = \frac{1 - 2(2p)^n}{N} = 0$$
$$1 - 2(2p)^n = 0$$
$$\frac{1}{2} = (2p)^n$$
$$\frac{1}{2^{1/n}} = 2p$$
$$p = \frac{1}{2(2^{1/n})}.$$

Hence the fixed point value is $p = \frac{1}{2(2^{1/n})}$.

As n $\rightarrow \infty$, $p \rightarrow \frac{1}{2}$.

(f) Simulating the process described by the dynamical equation $p(t+1) = p(t) + \frac{1-\sigma_n(p,t)}{N}$ with $n = 10$ generations and plotting $p(t)$ against t for $p(0) = 0.1$ and $p(0) = 0.6$. We can see that the $p(t)$ self organises into its critical value $p_c = 0.5$.

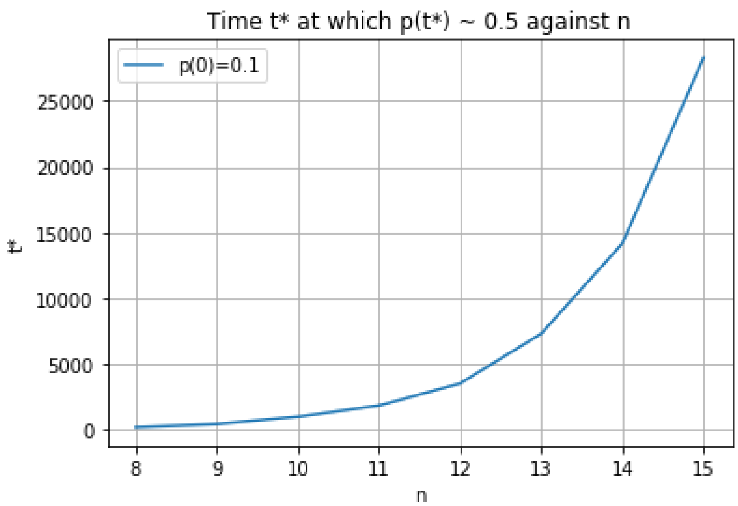


Here is my code written in Python:

```python
import numpy as np
import matplotlib.pyplot as plt

def SOBP(p,n,t):
    d = dict()
    p_list=[p]
    N = 2**(n+1)-1
    for i in range(t):
        L=[]
        for j in range(n):
            L.append([])

        initial = np.random.choice([0,1],None,p=[1-p,p])
        if initial == 1:
            #two active sites to start
            L[0].append(1)
            L[0].append(1)
            for b in range(n-1):
                for a in L[b]:
                    if a == 1:
                        site_status = np.random.choice([0,1],None,p=[1-p,p])
                        L[b+1].append(site_status)
                        L[b+1].append(site_status)
        sigma = sum(L[n-1])
        p = p + (1-sigma)/N
        p_list.append(p)

        s=1
        for i1 in range(n):
            s = s + sum(L[i1])

        key = s
        if key in d:
            d[key] += 1
        else:
            d[key] = 1

    return p_list , d

#part f
x_axis = range(3001)
y1,_ = SOBP(0.1,10,3000)
y2,_ = SOBP(0.6,10,3000)
plt.figure()
plt.plot(x_axis,y1,label="p(0)=0.1")
plt.plot(x_axis,y2,label="p(0)=0.6")
```

```
plt.title("Value of p(t) for a system with n = 10 generations")
plt.xlabel('t')
plt.ylabel('p(t)')
plt.legend()
plt.grid()
plt.show()
```

(g) To demonstrate that the time t^* at which p(t) reaches the asymptotic value $p(t^*) \simeq 1/2$ depends exponentially on n. I have plotted a graph for n against t^*. $p(0)$ is assumed to be 0.1.



Looking at the graph, we can see that t^* increases exponentially with n.

Here is my code:

```
#part g
time=[]
grads=[]
for n in range(8,16):
    p_list,_ = SOBP(0.1,n,30000)
    grad = (p_list[10]-p_list[0])/11
    grads.append(grad)
    t = 0
    while 0.5 - p_list[t] > 0.01:
        t = t+1
    time.append(t)
```

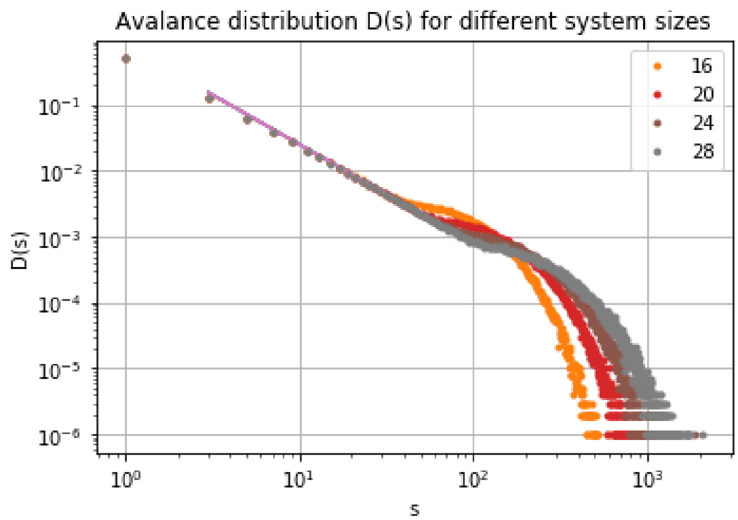
```
x_axis = range(8,16)
plt.figure()
plt.plot(x_axis,time,label="p(0)=0.1")
plt.title("Time t* at which p(t) ~ 0.5 against n")
plt.xlabel('n')
plt.ylabel('t*')
plt.legend()
plt.grid()
plt.show()
print(grads)
```

To estimate of the gradient of $p(t)$ at $t = 0$ for different values of n, I calculate $\frac{p(10)-p(0)}{11}$ for $n = 8, ..., 15$. Here, I have a table of the estimated gradients.

n	gradient at t = 0
8	1.78 e-03
9	8.89 e-04
10	4.44 e-04
11	2.22 e-04
12	1.11 e-04
13	5.55 e-05
14	2.77 e-05
15	1.39 e-05

We can see that the gradient roughly halves every time n increases by 1. As the gradient decreases exponentially with values halving at each n, we can expect for the time t^* for which $p(t^*) \simeq \frac{1}{2}$ will increase exponentially with base 2. This is what we see in our graph.

(h) Creating a graph of the avalanche distributions $D(s)$ for avalanches of systems with $n = 16, 20, 24, 28$ generations and $p = \frac{1}{2}$. The line $\sqrt{\frac{2}{\pi}} s^{-\frac{3}{2}}$ is plotted for reference.



Here is my code for plotting the graph:

```
#part h
plt.figure()
for n in [16,20,24,28]:
    _,d = SOBP(0.5,n,1000000)

    sizes=[]
    for key in d.keys():
        sizes.append(d[key])
    dist = np.asarray(sizes)/sum(sizes)

    plt.plot([s for s in range(3,100)], [np.sqrt(2/np.pi)*s**-1.5
    for s in range(3,100)])
    plt.loglog(d.keys(),dist,'.',label = n)

plt.title("Avalance distribution D(s) for different system sizes")
plt.xlabel('s')
plt.ylabel('D(s)')
plt.legend()
plt.grid()
plt.show()
```

The probability an avalanche of size s in a system with n generations is given by Eq. (8)

$$P_n(s,p) = \frac{\sqrt{2(1-p)/\pi p}}{s^{\frac{3}{2}}} exp(-s/s_c(p),$$

where $s_c(p) = \frac{-2}{ln 4p(1-p)}$. As $p \to \frac{1}{2}, s_c(p) \to \infty$, hence

$$P_n(s,p) \to \frac{\sqrt{2(1-\frac{1}{2})/\pi\frac{1}{2}}}{s^{\frac{3}{2}}} = Cs^{-\frac{3}{2}},$$

where $C = \sqrt{\frac{2}{\pi}}$. Hence, the mean-field exponent for $p_c = \frac{1}{2}$ is $\tau = \frac{3}{2}$.