Solutions for HW3

4.1. ☺

Suppose that you are testing the following hypotheses where the variance is known:

$H_0 : \mu = 100$
$H_1 : \mu \neq 100$

Find the P-value for the following values of the test statistic.

(a) $Z_0 = 2.75$

From Appendix II, $\Phi(Z_0) = \Phi(2.75) = 0.99702$
For a two-sided test: $P = 2[1 - \Phi(|Z_0|)] = 2[1 - 0.99702] = 0.00596$

(b) $Z_0 = 1.86$

From Appendix II, $\Phi(Z_0) = \Phi(1.86) = 0.96856$
For a two-sided test: $P = 2[1 - \Phi(|Z_0|)] = 2[1 - 0.96856] = 0.06289$

(c) $Z_0 = -2.05$

From Appendix II, $\Phi(Z_0) = \Phi(|-2.05|) = \Phi(2.05) = 0.97982$
For a two-sided test: $P = 2[1 - \Phi(|Z_0|)] = 2[1 - 0.97982] = 0.04036$

(d) $Z_0 = -1.86$

Same answer as for (b)
From Appendix II, $\Phi(Z_0) = \Phi(|-1.86|) = \Phi(1.86) = 0.96856$
For a two-sided test: $P = 2[1 - \Phi(|Z_0|)] = 2[1 - 0.96856] = 0.06289$

4.4. ☺

Suppose that you are testing the following hypotheses where the variance is unknown:

$H_0: \mu = 100$

$H_1: \mu \neq 100$

The sample size is n = 20. Find bounds on the P-value for the following values of the test statistic.

(a) $t_0 = 2.75$

From Appendix IV and $v = n - 1 = 10 - 1 = 9$,

α	0.025		0.01
$T_{\alpha,9}$	2.262	2.75	2.821

For a two-sided test: $2(0.01) < P < 2(0.025)$, or $0.02 < P < 0.05$

(b) $t_0 = 1.86$

From Appendix IV and $v = 9$,

α	0.05		0.025
$T_{\alpha,9}$	1.833	1.86	2.262

For a two-sided test: $2(0.025) < P < 2(0.05)$, or $0.05 < P < 0.10$

(c) $t_0 = -2.05$

From Appendix IV and $v = 9$,

α	0.05		0.025
$T_{\alpha,9}$	1.833	2.05	2.262

For a two-sided test and by symmetry of the *t* distribution: $2(0.025) < P < 2(0.05)$, or $0.05 < P < 0.10$

(d) $t_0 = -1.86$

Same answer as for (b)

4.8.

The tensile strength of a fiber used in manufacturing cloth is of interest to the purchaser. Previous experience indicates that the standard deviation of tensile strength is 2 psi. A random sample of eight fiber specimens is selected, and the average tensile strength is found to be 127 psi.

(a) Test the hypothesis that the mean tensile strength equals 125 psi versus the alternative that the mean exceeds 125 psi. Use $\alpha = 0.05$.

$\mu_0 = 125;\ \alpha = 0.05$

Test H_0: $\mu = 125$ vs. H_1: $\mu > 125$. Reject H_0 if $Z_0 > Z_\alpha$.

$$Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{127 - 125}{2/\sqrt{8}} = 2.828$$

$Z_\alpha = Z_{0.05} = 1.645$

Reject H_0: $\mu = 125$, and conclude that the mean tensile strength exceeds 125 psi.

(b) What is the P-value for this test?

P-value $= 1 - \Phi(Z_0) = 1 - \Phi(2.828) = 1 - 0.99766 = 0.00234$

(c) Discuss why a one-sided alternative was chosen in part (a).

In strength tests, we usually are interested in whether some minimum requirement is met, not simply that the mean does not equal the hypothesized value. A one-sided hypothesis test lets us do this.

(d) Construct a 95% lower confidence interval on the mean tensile strength.

$$\bar{x} - Z_\alpha\left(\sigma/\sqrt{n}\right) \le \mu$$

$$127 - 1.645\left(2/\sqrt{8}\right) \le \mu$$

$$125.8 \le \mu$$

MTB > Stat > Basic Statistics > 1-Sample Z

One-Sample Z

```
Test of mu = 125 vs > 125
The assumed standard deviation = 2

                        95% Lower
N     Mean  SE Mean      Bound     Z      P
8  127.000    0.707    125.837  2.83  0.002
```

4.12.

A machine is used to fill containers with a liquid product. Fill volume can be assumed to be normally distributed. A random sample of ten containers is selected, and the net contents (oz) are as follows: 12.03, 12.01, 12.04, 12.02, 12.05, 11.98, 11.96, 12.02, 12.05, and 11.99.

(a) Suppose that the manufacturer wants to be sure that the mean net contents exceeds 12 oz. What conclusions can be drawn from the data (use $\alpha = 0.01$).

$x \sim N(\mu, \sigma)$, $\mu_0 = 12$, $\alpha = 0.01$

$n = 10$, $\bar{x} = 12.015$, $s = 0.030$

Test H_0: $\mu = 12$ vs. H_1: $\mu > 12$. Reject H_0 if $t_0 > t_\alpha$.

$$t_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} = \frac{12.015 - 12}{0.0303/\sqrt{10}} = 1.5655$$

$t_{\alpha/2,\, n-1} = t_{0.005,\, 9} = 3.250$

Do not reject H_0: $\mu = 12$, and conclude that there is not enough evidence that the mean fill volume exceeds 12 oz.

MTB > Stat > Basic Statistics > 1-Sample t

One-Sample T: Ex 4-12

```
Test of mu = 12 vs > 12
                                          99% Lower
Variable   N     Mean   StDev  SE Mean      Bound     T      P
Ex 4-12   10  12.0150  0.0303   0.0096    11.9880  1.57  0.076
```

(b) Construct a 95% two-sided confidence interval on the mean fill volume.

$\alpha = 0.05$

$t_{\alpha/2,\, n-1} = t_{0.025,\, 9} = 2.262$

$$\bar{x} - t_{\alpha/2,n-1}\left(S/\sqrt{n}\right) \le \mu \le \bar{x} + t_{\alpha/2,n-1}\left(S/\sqrt{n}\right)$$

$$12.015 - 2.262\left(S/\sqrt{10}\right) \le \mu \le 12.015 + 2.62\left(S/\sqrt{10}\right)$$

$$11.993 \le \mu \le 12.037$$

4.12. (b) continued

MTB > Stat > Basic Statistics > 1-Sample t

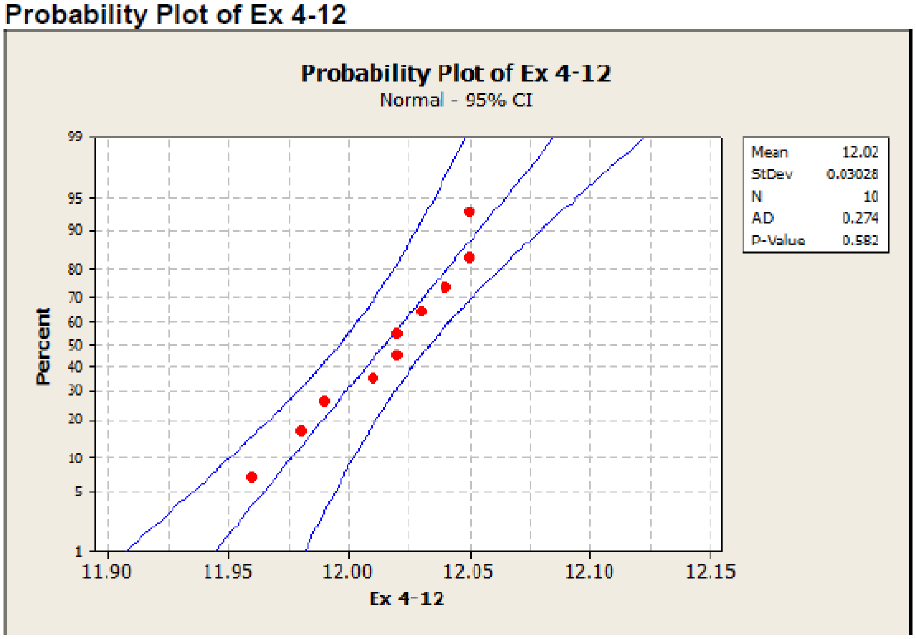
One-Sample T: Ex 4-12

```
Test of mu = 12 vs not = 12

Variable   N     Mean   StDev  SE Mean          95% CI             T      P
Ex 4-12   10  12.0150  0.0303   0.0096  (11.9933, 12.0367)  1.57  0.152
```

(c) Does the assumption of normality seem appropriate for the fill volume data?

MTB > Graph > Probability Plot > Single and ensure Distribution = Normal

Probability Plot of Ex 4-12



The plotted points fall approximately along a straight line, so the assumption that fill volume is normally distributed is appropriate.

4.13.

Ferric chloride is used as a flux in some types of extraction metallurgy processes. This material is shipped in containers, and the container weight varies. It is important to obtain an accurate estimate of mean container weight. Suppose that from long experience a reliable value for the standard deviation of flux container weight is determined to be 4 lb. How large a sample would be required to construct a 95% two-sided confidence interval on the mean that has a total width of 1 lb?

σ = 4 lb, α = 0.05, $Z_{\alpha/2} = Z_{0.025}$ = 1.9600, total confidence interval width = 1 lb, find n

$$2\left[Z_{\alpha/2}\left(\sigma/\sqrt{n}\right)\right] = \text{total width}$$

$$2\left[1.9600\left(4/\sqrt{n}\right)\right] = 1$$

$$n = 246$$

4.15.

The output voltage of a power supply is assumed to be normally distributed. Sixteen observations taken at random on voltage are as follows: 10.35, 9.30, 10.00, 9.96, 11.65, 12.00, 11.25, 9.58, 11.54, 9.95, 10.28, 8.37, 10.44, 9.25, 9.38, and 10.85.

(a) Test the hypothesis that the mean voltage equals 12 V against a two-sided alternative using $\alpha = 0.05$.

$x \sim N(\mu, \sigma)$, $n = 16$, $\bar{x} = 10.259$ V, $s = 0.999$ V, $\mu_0 = 12$, $\alpha = 0.05$

Test H_0: $\mu = 12$ vs. H_1: $\mu \neq 12$. Reject H_0 if $|t_0| > t_{\alpha/2}$.

$$t_0 = (\bar{x} - \mu_0) \big/ \left(S/\sqrt{n}\right) = (10.259 - 12) \big/ \left(0.999/\sqrt{16}\right) = -6.971 \quad \text{(Equation 4.33)}$$

$t_{\alpha/2, n-1} = t_{0.025, 15} = 2.131$ (from Appendix IV)

Reject H_0: $\mu = 12$, and conclude that the mean output voltage differs from 12V.

MTB > Stat > Basic Statistics > 1-Sample t

One-Sample T: Ex 4-15

```
Test of mu = 12 vs not = 12

Variable    N    Mean  StDev  SE Mean        95% CI            T      P
Ex 4-15    16  10.259  0.999    0.250  (9.727, 10.792)  -6.97  0.000
```

(b) Construct a 95% two-sided confidence interval on μ.

Construct a 95% two-sided confidence interval on μ.

$$\bar{x} - t_{\alpha/2,n-1}\left(S/\sqrt{n}\right) \le \mu \le \bar{x} + t_{\alpha/2,n-1}\left(S/\sqrt{n}\right)$$

$$10.259 - 2.131\left(0.999/\sqrt{16}\right) \le \mu \le 10.259 + 2.131\left(0.999/\sqrt{16}\right) \quad \text{(Equation 4.34)}$$

$$9.727 \le \mu \le 10.792$$

(c) Test the hypothesis that $\sigma^2 = 11$ using $\alpha = 0.05$.

Test H_0: $\sigma^2 = 1$ vs. H_1: $\sigma^2 \neq 1$. Reject H_0 if $\chi^2_0 > \chi^2_{\alpha/2, n-1}$ or $\chi^2_0 < \chi^2_{1-\alpha/2, n-1}$.

$$\chi_0^2 = (n-1)S^2/\sigma_0^2 = \frac{(16-1)0.999^2}{1} = 14.970 \quad \text{(Equation 4.38)}$$

$\chi^2_{\alpha/2, n-1} = \chi^2_{0.025,16-1} = 27.488$ (from Appendix III)

$\chi^2_{1-\alpha/2, n-1} = \chi^2_{0.975,16-1} = 6.262$ (from Appendix III)

Do not reject H_0: $\sigma^2 = 1$, and conclude that there is insufficient evidence that the variance differs from 1.

4.19.

Two different hardening processes—(1) saltwater quenching and (2) oil quenching—are used on samples of a particular type of metal alloy. The results are shown in Table 4E.2. Assume that hardness is normally distributed.

TABLE 4E.2
Hardness Data for Exercise 4.19

Saltwater Quench	Oil Quench
145	152
150	150
153	147
148	155
141	140
152	146
146	158
154	152
139	151
148	143

Saltwater quench: $n_1 = 10$, $\overline{x}_1 = 147.6$, $s_1 = 4.97$; Oil quench: $n_2 = 10$, $\overline{x}_2 = 149.4$, $s_2 = 5.46$

14.19 continued

(a) Test the hypothesis that the mean hardness for the saltwater quenching process equals the mean hardness for the oil quenching process. Use $\alpha = 0.05$ and assume equal variances.

Test H_0: $\mu_1 - \mu_2 = 0$ vs. H_1: $\mu_1 - \mu_2 \neq 0$. Reject H_0 if $|t_0| > t_{\alpha/2,\, n1+n2-2}$.

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{(10-1)4.97^2 + (10-1)5.46^2}{10+10-2}} = 5.2217 \qquad \text{(Eqns 4.51 and 4.52)}$$

$$t_0 = (\bar{x}_1 - \bar{x}_2) \Big/ \left(s_p \sqrt{1/n_1 + 1/n_2} \right) = (147.60 - 149.40) \Big/ \left(5.2217\sqrt{1/10 + 1/10} \right) = -0.77$$

$t_{\alpha/2,\, n1+n2-2} = t_{0.025,\, 18} = 2.101$ (from Appendix IV)

Do not reject H_0, and conclude that there is not sufficient evidence of a difference between measurements produced by the two hardening processes.

MTB > Stat > Basic Statistics > 2-Sample t

Two-Sample T-Test and CI: Ex4-19SQ, Ex4-19OQ

```
Two-sample T for Ex4-19SQ vs Ex4-19OQ

          N    Mean  StDev  SE Mean
Ex4-19SQ  10  147.60   4.97      1.6
Ex4-19OQ  10  149.40   5.46      1.7

Difference = mu (Ex4-19SQ) - mu (Ex4-19OQ)
Estimate for difference:  -1.80
95% CI for difference:  (-6.71, 3.11)
T-Test of difference = 0 (vs not =): T-Value = -0.77  P-Value = 0.451  DF = 18
Both use Pooled StDev = 5.2217
```

(b) Assuming that the variances σ_1^2 and σ_2^2 are equal, construct a 95% confidence interval on the difference in mean hardness.

$\alpha = 0.05$, $t_{\alpha/2,\, n1+n2-2} = t_{0.025,\, 18} = 2.1009$

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{1/n_1 + 1/n_2} \le (\mu_1 - \mu_2) \le (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{1/n_1 + 1/n_2}$$

$$(147.6 - 149.4) - 2.1009(5.22)\sqrt{1/10 + 1/10} \le (\mu_1 - \mu_2) \le (147.6 - 149.4) + 2.1009(5.22)\sqrt{1/10 + 1/10}$$

$$-6.7 \le (\mu_1 - \mu_2) \le 3.1$$

(Eqn 4.56)

The confidence interval for the difference contains zero. We conclude that there is not sufficient evidence of a difference between measurements produced by the two hardening processes.

(c) Construct a 95% confidence interval on the ratio σ_1^2 / σ_2^2. Does the assumption made earlier of equal variances seem reasonable?

$$F_{1-\alpha/2,n_2-1,n_1-1} = F_{0.975,9,9} = 0.2484; \; F_{\alpha/2,n_2-1,n_1-1} = F_{0.025,9,9} = 4.0260$$

$$\frac{S_1^2}{S_2^2} F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\alpha/2,n_2-1,n_1-1}$$

$$\frac{4.97^2}{5.46^2}(0.2484) \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{4.97^2}{5.46^2}(4.0260)$$

$$0.21 \le \frac{\sigma_1^2}{\sigma_2^2} \le 3.34$$

MTB > Stat > Basic Statistics > 2 Variances

Test and CI for Two Variances: Ex4-19SQ, Ex4-19OQ

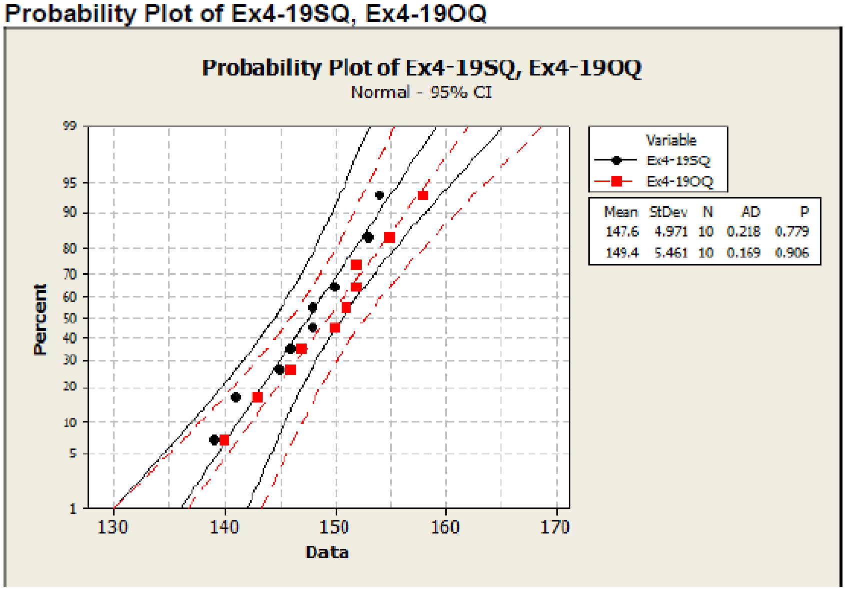
```
Method

Null hypothesis         Sigma(Ex4-19SQ) / Sigma(Ex4-19OQ) = 1
Alternative hypothesis  Sigma(Ex4-19SQ) / Sigma(Ex4-19OQ) not = 1
Significance level      Alpha = 0.05

Statistics
Variable   N  StDev  Variance
Ex4-19SQ  10  4.971    24.711
Ex4-19OQ  10  5.461    29.822

Ratio of standard deviations = 0.910
Ratio of variances = 0.829

95% Confidence Intervals
                                  CI for
Distribution   CI for StDev      Variance
of Data           Ratio           Ratio
Normal        (0.454, 1.826)  (0.206, 3.336)
Continuous    (0.390, 2.157)  (0.152, 4.652)

Tests
                                             Test
Method                          DF1  DF2  Statistic  P-Value
F Test (normal)                   9    9       0.83    0.784
Levene's Test (any continuous)    1   18       0.08    0.783
```

Since the confidence interval includes the ratio of 1, the assumption of equal variances seems reasonable.

(d) Does the assumption of normality seem appropriate for these data?

MTB > Graph > Probability Plot > Multiple and ensure Distribution = Normal

Probability Plot of Ex4-19SQ, Ex4-19OQ



The normal distribution assumptions for both the saltwater and oil quench methods seem reasonable. However, the slopes on the normal probability plots do not appear to be the same, so the assumption of equal variances does not seem reasonable.

4.28.

Suppose we wish to test the hypotheses

$H_0 : \mu = 15$

$H_1 : \mu \neq 15$

where we know that $\sigma^2 = 9.0$. If the true mean is really 20, what sample size must be used to ensure that the probability of type II error is no greater than 0.10? Assume that $\alpha = 0.05$.

What n is needed such that the Type II error, β, is less than or equal to 0.10?

$\delta = \mu_1 - \mu_2 = 20 - 15 = 5 \qquad d = |\delta|/\sigma = 5/\sqrt{9} = 1.6667$

From Figure 4.7, the operating characteristic curve for two-sided at $\alpha = 0.05$, $n = 4$. Check:

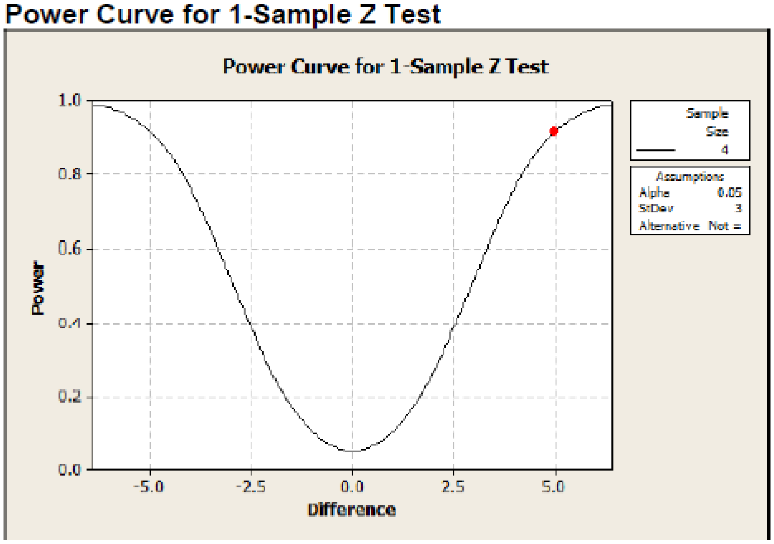
$$\beta = \Phi\left(Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right) - \Phi\left(-Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right) = \Phi\left(1.96 - 5\sqrt{4}/3\right) - \Phi\left(-1.96 - 5\sqrt{4}/3\right)$$
$$= \Phi(-1.3733) - \Phi(-5.2933) = 0.0848 - 0.0000 = 0.0848$$

MTB > Stat > Power and Sample Size > 1-Sample Z

Power and Sample Size

```
1-Sample Z Test
Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 3
            Sample  Target
Difference    Size   Power  Actual Power
         5       4     0.9      0.915181
```

Power Curve for 1-Sample Z Test



4.29.

Consider the hypotheses

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

where σ^2 is known. Derive a general expression for determining the sample size for detecting a true mean of $\mu_1 \neq \mu_0$ with probability 1 – β if the type I error is α.

Let $\mu_1 = \mu_0 + \delta$.

From equation. 4.46, $\beta = \Phi\left(Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right) - \Phi\left(-Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right)$

If δ>0, then $\Phi\left(-Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right)$ is likely to be small compared with β. So,

$$\beta \approx \Phi\left(Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right)$$

$$\Phi(\beta) \approx \Phi^{-1}\left(Z_{\alpha/2} - \delta\sqrt{n}/\sigma\right)$$

$$-Z_\beta \approx Z_{\alpha/2} - \delta\sqrt{n}/\sigma$$

$$n \approx \left[(Z_{\alpha/2} + Z_\beta)\sigma/\delta\right]^2$$