Computability Theory XII

Creative Set

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CREATIVE SET

Most Difficult Semi-Decidable Problems?

An r.e. set is very difficult if it is very non-recursive.

An r.e. set is very non-recursive if its complement is very non-r.e..

A set is very non-r.e. if it is easy to distinguish it from any r.e. set.

Most Difficult Semi-Decidable Problems?

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A set is very non-r.e. if it is easy to distinguish it from any r.e. set.

These sets are creative respectively productive.

Synopsis

1. Productive Set
2. Creative Set
3. Simple Sets

PRODUCTIVE SET

Productive Set

Suppose $W_x \subseteq \overline{K}$. Then $x \in \overline{K} \setminus W_x$.

Productive Set

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So x witnesses the strict inclusion $W_x \subseteq \overline{K}$.

In other words the identity function is an effective proof that $\overline{K}$ differs from every r.e. set.

Productive Set

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A productive set is not r.e. by definition.

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$\{x \mid c \notin E_x\}$ is productive.

$\{x \mid \phi_x(x) \neq 0\}$ is productive.

Productive Set

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By S-m-n Theorem one gets a primitive recursive function $p(x)$ such that $\phi_{p(x)}(y) = 0$ if and only if $\phi_x(y)$ is defined. Then

$$p(x) \in W_x \Leftrightarrow p(x) \notin A.$$

So if $W_x \subseteq A$ we must have $p(x) \in A \setminus W_x$.
Thus p is a productive function for A.

Productive Set

Lemma. If $A \leq_m B$ and A is productive, then B is productive.

Productive Set

Theorem. Suppose that $\mathcal{B}$ is a set of unary computable functions with $f_\emptyset \in \mathcal{B}$ and $\mathcal{B} \neq \mathcal{C}$. Then $B = \{x \mid \phi_x \in \mathcal{B}\}$ is productive.

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Proof. Suppose $g \notin \mathcal{B}$. Consider the function f defined by

$$f(x,y) = \begin{cases} g(y), & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is a primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

Clearly $x \notin W_x$ iff $\phi_{k(x)} = f_\emptyset$ iff $\phi_{k(x)} \in \mathcal{B}$ iff $k(x) \in B$.

Hence $k : \overline{K} \leq_m B$.

Property of Productive Set

Lemma. Suppose that g is a total computable function. Then there is a primitive recursive function p such that for all x, $W_{p(x)} = W_x \cup \{g(x)\}$.

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Proof. Using S-m-n Theorem, take $p(x)$ to be a primitive recursive function such that

$$\phi_{p(x)}(y) = \begin{cases} 1, \text{if } y \in W_x \vee y = g(x), \\ \uparrow, \text{otherwise} \end{cases}$$

Property of Productive Set

Theorem. A productive set contains an infinite r.e. subset.

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Theorem. A productive set contains an infinite r.e. subset.

Proof. Suppose p is a production function for A.

Take e_0 to be some index for $\emptyset$. Then $p(e_0) \in A$ by definition.

By the Lemma there is a primitive recursive function k such that for all x, $W_{k(x)} = W_x \cup \{p(x)\}$.

Apparently $\{e_0, \ldots, k^n(e_0), \ldots\}$ is r.e.

Consequently $\{p(e_0), \ldots, p(k^n(e_0)), \ldots\}$ is a r.e. subset of A, which must be infinite by the definition of k.

Productive Function via a Partial Function

Proposition. A set A is productive iff there is a partial recursive function p such that

$$\forall x.(W_x \subseteq A \Rightarrow (p(x)\downarrow \wedge p(x) \in A \setminus W_x)). \tag{1}$$

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Proposition. A set A is productive iff there is a partial recursive function p such that

$$\forall x.(W_x \subseteq A \Rightarrow (p(x)\downarrow \wedge p(x) \in A \setminus W_x)). \tag{1}$$

Proof. Suppose p is a partial recursive function satisfying (1). Let s be a primitive recursive function such that

$$\phi_{s(x)}(y) = \begin{cases} y, & p(x)\downarrow \wedge\, y \in W_x, \\ \uparrow, & \text{otherwise.} \end{cases}$$

A productive function q can be defined by running $p(x)$ and $p(s(x))$ in parallel and stops when either terminates.

Productive Function Made Injective

Proposition. A productive set has an injective productive function.

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Proof. Suppose p is a productive function of A. Let

$$W_{h(x)} = W_x \cup \{p(x)\}$$

Clearly

$$W_x \subseteq A \Rightarrow W_{h(x)} \subseteq A$$

Define $q(0) = p(0)$.

If there is a least $y \in \{p(x+1), ph(x+1), ph^2(x+1), \ldots\}$ such that $y \notin \{q(0), \ldots, q(x)\}$, let $q(x+1)$ be y;

otherwise let $q(x+1)$ be $\mu y.y \notin \{q(0), \ldots, q(x)\}$.

It is easily seen that q is an injective production function for A.

Myhill's Characterization of Productive Set

Fact. $\overline{K} \leq_m A$ iff $\overline{K} \leq_1 A$.

Myhill's Characterization of Productive Set

Theorem. (Myhill, 1955) A is productive iff $\overline{K} \leq_1 A$ iff $\overline{K} \leq_m A$.

Creative Set

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Intuitively a creative set A is effectively non-recursive in the sense that the non-recursiveness of $\overline{A}$, hence the non-recursiveness of A, can be effectively demonstrated.

Creative Set

K is creative.

$\{x \mid c \in W_x\}$ is creative.

$\{x \mid c \in E_x\}$ is creative.

$\{x \mid \phi_x(x) = 0\}$ is creative.

Creative Set

Theorem. Suppose that $\mathcal{A} \subseteq \mathcal{C}$ and let $A = \{x \mid \phi_x \in \mathcal{A}\}$. If A is r.e. and $A \neq \emptyset, \mathbb{N}$, then A is creative.

Creative Set

Theorem. Suppose that $\mathcal{A} \subseteq \mathcal{C}$ and let $A = \{x \mid \phi_x \in \mathcal{A}\}$. If A is r.e. and $A \neq \emptyset, \mathbb{N}$, then A is creative.

Proof. Suppose A is r.e. and $A \neq \emptyset, \mathbb{N}$. If $f_\emptyset \in \mathcal{A}$, then A is productive by a previous theorem. This is a contradiction.

So $\overline{A}$ is productive by the same theorem. Hence A is creative.

Creative Set

The set $K_0 = \{x \mid W_x \neq \emptyset\}$ is creative. It corresponds to the set $\mathcal{A} = \{f \in \mathcal{C} \mid f \neq f_\emptyset\}$.

Discussion

Question. Are all non-recursive r.e. sets creative?

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The answer is negative. By a special construction we can obtain r.e.sets that are neither recursive nor creative.

SIMPLE SETS

Simple Sets

A set A is simple if

1. A is r.e.,
2. $\overline{A}$ is infinite,
3. $\overline{A}$ contains no infinite r.e. subset.

Simple Sets

Theorem. A simple set is neither recursive nor creative.

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Proof. Since $\overline{A}$ can not be r.e., A can not be recursive.

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(3) implies that A can not be creative.

Simple Sets

Theorem. There is a simple set.

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Proof. Define $f(x) = \phi_x(\mu z(\phi_x(z) > 2x))$. Let A be $Ran(f)$.

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1. A is r.e.
2. $\overline{A}$ is infinite. This is because $A \cap \{0, 1, \ldots, 2n\}$ contains at most the elements $\{f(0), f(1), \ldots, f(n-1)\}$.

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1. A is r.e.
2. $\overline{A}$ is infinite. This is because $A \cap \{0, 1, \ldots, 2n\}$ contains at most the elements $\{f(0), f(1), \ldots, f(n-1)\}$.
3. Suppose B is an infinite r.e. set. Then there is a total computable function ϕ_b such that $B = E_b$. Since ϕ_b is total, $f(b)$ is defined and $f(b) \in A$. Hence $B \not\subseteq \overline{A}$.