Computability Theory X

Recursive Set

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Decision Problem, Predicate, Number Set

The following emphasizes the importance of number set:

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A central theme of recursion theory is to look for sensible classification of number sets.

Classification is often done with the help of reduction.

Synopsis

1. Reduction
2. Recursive Set
3. Rice Theorem

Reduction

Reduction between Problems

A reduction is a way of defining a solution of a problem with the help of a solution of another problem.

In recursion theory we are only interested in reductions that are computable.

Reduction

There are several ways of reducing a problem to another.

The differences between different reductions from A to B consists in the manner and the extent to which information about B is allowed to settle questions about A.

Many-One Reduction

The set A is many-one reducible, or m-reducible, to the set B if there is a total computable function f such that

$$x \in A \text{ iff } f(x) \in B$$

for all x. We shall write $A \leq_m B$ or more explicitly $f : A \leq_m B$.

If f is injective, then it is a one-one reducibility, denoted by $\leq_1$.

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- The INDEPENDENT SET PROBLEM (IS) asks if there are k vertices of G with every pair of which unconnected.
- The CLIQUE Problem asks if there is a k-complete subgraph of G.

There is a simple one-one reduction from IS to CLIQUE.

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- $\mathbb{N} \leq_m A$ iff $A \neq \emptyset$; $\emptyset \leq_m A$ iff $A \neq \mathbb{N}$.

m-Degree

Two sets A, B are many-one equivalent, notation $A \equiv_m B$, if $A \leq_m B$ and $B \leq_m A$.

Similarly $A \equiv_1 B$ if $A \leq_1 B$ and $B \leq_1 A$.

Clearly both $\equiv_m$ and $\equiv_1$ an equivalence relation.

Let $d_m(A)$ be $\{B \mid A \equiv_m B\}$.

The class $d_m(A)$ is called the m-degree represented by A.

m-Degree

The set of m-degrees is ranged over by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

$\mathbf{a} \leq_m \mathbf{b}$ iff $A \leq_m B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

$\mathbf{a} <_m \mathbf{b}$ iff $\mathbf{a} \leq_m \mathbf{b}$ and $\mathbf{b} \not\leq_m \mathbf{a}$.

The relation $\leq_m$ is a partial order.

The Structure of m-Degree

Proposition

The m-degrees form a distributive lattice.

The Restriction of m-Reduction

Suppose G is a finite directed weighted graph and m is a number.

- The TRAVELLING SALESMAN PROBLEM (TSP) asks for the overall weight of a cycle with minimum weight if there are circles.
- A decision problem version asks given a budget b, whether there exists a cycle that passes through every vertex exactly once, of total cost b or less - or to report that no such tour exists.

RECURSIVE SET

Definition of Recursive Set

Let A be a subset of $\mathbb{N}$. The characteristic function of A is given by

$$c_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

A is recursive if $c_A(x)$ is computable.

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Fact. If $A \leq_m B$ and B is recursive, then A is recursive.

Fact. If $A \leq_m B$ and A is not recursive, then B is not recursive.

A Characterization of Recursive Set

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Proof. Suppose A is recursive and infinite. Then A is range of the increasing function f given by

$$\begin{array}{rcl} f(0) & = & \mu y(y \in A), \\ f(n+1) & = & \mu y(y \in A \text{ and } y > f(n)). \end{array}$$

The function is total, increasing and computable.

Conversely suppose A is the range of a total increasing computable function f. Obviously $y = f(n)$ implies $n \leq y$. Hence

$$y \in A \Leftrightarrow y \in Ran(f) \Leftrightarrow \exists n \leq y(f(n) = y).$$

Unsolvable Problem

A decision problem $f : \mathbb{N} \to \{0, 1\}$ is solvable if it is computable and $dom(f)$ is recursive.

It is unsolvable if it is not solvable.

Non-recursive $\Leftrightarrow$ Unsolvable $\Leftrightarrow$ Undecidable

Some Important Undecidable Sets

Here are some important undecidable sets:

$$
\begin{aligned}
K &= \{x \mid x \in W_x\}, \\
K_0 &= \{\pi(x, y) \mid x \in W_y\}, \\
K_1 &= \{x \mid W_x \neq \emptyset\}, \\
\mathit{Fin} &= \{x \mid W_x \text{ is finite}\}, \\
\mathit{Inf} &= \{x \mid W_x \text{ is infinite}\}, \\
\mathit{Con} &= \{x \mid \phi_x \text{ is total and constant}\}, \\
\mathit{Tot} &= \{x \mid \phi_x \text{ is total}\}, \\
\mathit{Cof} &= \{x \mid W_x \text{ is cofinite}\}, \\
\mathit{Rec} &= \{x \mid W_x \text{ is recursive}\}, \\
\mathit{Ext} &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}
$$

Rice Theorem

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Henry Rice.

Classes of Recursively Enumerable Sets and their Decision Problems. Transactions of the American Mathematical Society, **77**:358-366, 1953.

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If $\emptyset \subsetneq \mathcal{B} \subsetneq \mathcal{C}$, then $\{x \mid \phi_x \in \mathcal{B}\}$ is not recursive.

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If $\emptyset \subsetneq \mathcal{B} \subsetneq \mathcal{C}$, then $\{x \mid \phi_x \in \mathcal{B}\}$ is not recursive.

Proof. Suppose $f_\emptyset \notin \mathcal{B}$ and $g \in \mathcal{B}$. Let the function f be defined by

$$f(x,y) = \begin{cases} g(y), & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is some injective primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that k is a one-one reduction from K to $\{x \mid \phi_x \in \mathcal{B}\}$.

Applying Rice Theorem

Assume that $f(x) \simeq \phi_x(x) + 1$ could be extended to a total computable function say $g(x)$. Let e be an index of $g(x)$. Then $\phi_e(e) = g(e) = f(e) = \phi_e(e) + 1$. Contradiction.

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So we may use Rice Theorem to conclude that

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Comment: Not every partial recursive function can be obtained by restricting a total recursive function.

Remark on Rice Theorem

Rice Theorem deals with programme independent properties.

It talks about classes of computable functions rather than classes of programmes.

All non-trivial semantic problems are algorithmically undecidable.