Computability Theory XI

Recursively Enumerable Set

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Assignment

Assignment 4 was announced!

The deadline is Dec. 26!

An Exercise

Let $A, B \subseteq \mathbb{N}$. Define sets of $A \oplus B$ and $A \otimes B$ by

$$\begin{aligned} A \oplus B &= \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\} \\ A \otimes B &= \{\pi(x,y) \mid x \in A \wedge y \in B\} \end{aligned}$$

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1. $A \oplus B$ is recursive iff A and B are both recursive.
2. if $A, B \neq \emptyset$, then $A \otimes B$ is recursive iff A and B are both recursive.

We have seen that many sets, although not recursive, can be effectively generated in the sense that, for any such set, there is an effective procedure that produces the elements of the set in a non-stop manner.

We shall formalize this intuition in this lecture.

Synopsis

1. Recursively Enumerable Set
2. Characterization of R.E. Set
3. Rice-Shapiro Theorem

Recursively Enumerable Set

The Definition of R.E. Set

The partial characteristic function of a set A is given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ \uparrow, & \text{if } x \notin A. \end{cases}$$

A is recursively enumerable if χ_A is computable.

We shall often abbreviate 'recursively enumerable set' to 'r.e. set'.

Partially Decidable Problem

A problem $f : \mathbb{N} \to \{0, 1\}$ is partially decidable if $dom(f)$ is r.e.

Partially Decidable Predicate

A predicate $M(\widetilde{x})$ of natural number is partially decidable if its partial characteristic function

$$\chi_M(\widetilde{x}) = \begin{cases} 1, & \text{if } M(\widetilde{x}) \text{ holds,} \\ \uparrow, & \text{if } M(\widetilde{x}) \text{ does not hold,} \end{cases}$$

is computable.

$$\begin{aligned} \text{Partially Decidable Problem} &\Leftrightarrow \text{Partially Decidable Predicate} \\ &\Leftrightarrow \text{Recursively Enumerable Set} \end{aligned}$$

Example

The halting problem is partially decidable. Its partial characteristic function is given by

$$\chi_H(x, y) = \begin{cases} 1, & \text{if } P_x(y)\downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

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$$\chi_H(x, y) = \begin{cases} 1, & \text{if } P_x(y)\downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

K, K_0, K_1 are r.e.. But none of $\overline{K}, \overline{K_0}, \overline{K_1}$ is r.e..

Index for Recursively Enumerable Set

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Every r.e. set has an infinite number of indexes.

Closure Property

Union Theorem. The recursively enumerable sets are closed under union and intersection uniformly and effectively.

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Proof. According to S-m-n Theorem there are primitive recursive functions $r(x, y), s(x, y)$ such that

$$\begin{aligned} W_{r(x,y)} &= W_x \cup W_y, \\ W_{s(x,y)} &= W_x \cap W_y. \end{aligned}$$

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Fact. If $A \leq_m B$ and B is r.e. then A is r.e..

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Theorem. A is r.e. iff $A \leq_1 K$.

Proof. Suppose A is r.e. Let $f(x, y)$ be defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x \in A, \\ \uparrow, & \text{if } x \notin A. \end{cases}$$

By S-m-n Theorem there is an injective primitive recursive function $s(x)$ s.t. $f(x, y) = \phi_{s(x)}(y)$. It is clear that $x \in A$ iff $s(x) \in K$.

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Comment. No r.e. set is more difficult than K.

Characterization of R.E. Set

Normal Form Theorem

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Conversely suppose $M(\tilde{x})$ is partially decided by P. Let $R(\tilde{x}, y)$ be

$$P(\tilde{x})\downarrow \text{ in } y \text{ steps.}$$

Then $R(\tilde{x}, y)$ is primitive recursive and $M(\tilde{x}) \Leftrightarrow \exists y.R(\tilde{x}, y)$.

Quantifier Contraction Theorem

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Proof. Let $R(\widetilde{x}, y, z)$ be a primitive recursive predicate such that

$$M(\widetilde{x}, y) \Leftrightarrow \exists z. R(\widetilde{x}, y, z)$$

Then $\exists y. M(\widetilde{x}, y) \Leftrightarrow \exists y. \exists z. R(\widetilde{x}, y, z) \Leftrightarrow \exists u. R(\widetilde{x}, (u)_0, (u)_1)$.

Examples

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$W_x \neq \emptyset$

Uniformisation Theorem

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We may think of $c(x)$ as a choice function for $R(x, y)$. The theorem states that the choice function is computable.

A is r.e. iff there is a partially decidable predicate $R(x, y)$ such that $x \in A$ iff $\exists y . R(x, y)$.

Complementation Theorem

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Proof. Suppose A and $\overline{A}$ are r.e. Then some primitive recursive predicates $R(x, y), S(x, y)$ exist such that

$$\begin{aligned} x \in A &\Leftrightarrow \exists y R(x, y), \\ x \in \overline{A} &\Leftrightarrow \exists y S(x, y). \end{aligned}$$

Now let $f(x)$ be $\mu y(R(x, y) \vee S(x, y))$.

Then $f(x)$ is total and computable, and

$$x \in A \Leftrightarrow R(x, f(x))$$

Applying Complementation Theorem

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Comment. If $\overline{K} \leq_m A$ then A is not r.e. either.

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Fact. If A is r.e. but not recursive, then $\overline{A} \not\leq_m A \not\leq_m \overline{A}$.

Comment. However A and $\overline{A}$ are intuitively equally difficult.

Graph Theorem

Graph Theorem. Let $f(x)$ be a partial function. Then $f(x)$ is computable iff the predicate '$f(x) \simeq y$' is partially decidable iff $\{\pi(x, y) \mid f(x) \simeq y\}$ is r.e.

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Proof. If $f(x)$ is computable by $P(x)$, then

$$f(x) \simeq y \Leftrightarrow \exists t.(P(x) \downarrow y \text{ in } t \text{ steps})$$

The predicate '$P(x) \downarrow y$ in t steps' is primitive recursive.

Conversely let $R(x, y, t)$ be such that

$$f(x) \simeq y \Leftrightarrow \exists t.R(x, y, t).$$

Now $f(x) = \mu y.R(x, y, \mu t.R(x, y, t))$.

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Proof. Suppose A is nonempty and its partial characteristic function is computed by P. Let a be a member of A. The total function $g(x, t)$ given by

$$g(x, t) = \begin{cases} x, & \text{if } P(x) \downarrow \text{ in } t \text{ steps}, \\ a, & \text{otherwise.} \end{cases}$$

is computable. Clearly A is the range of $h(z) = g((z)_1, (z)_2)$.

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Conversely, $x \in A$ iff $\exists y.h(y) = x$, $\exists y.h(y) = x$ is partially decidable.

Listing Theorem

The theorem gives rise to the terminology ‘recursively enumerable’.

Implication of Listing Theorem

A set is r.e. iff it is the range of a computable function.

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Corollary. For each infinite nonrecursive r.e. A, there is an injective total recursive function f such that $ran(f) = A$.

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Proof. Suppose $A = ran(f)$. An infinite recursive subset is enumerated by the total increasing computable function g given by

$$\begin{array}{rcl} g(0) & = & f(0), \\ g(n+1) & = & f(\mu y(f(y) > g(n))). \end{array}$$

Applying Listing Theorem

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Proof. If $\{x \mid \phi_x \text{ is total}\}$ were a r.e. set, then there would be a total computable function f whose range is the r.e. set.

The function $g(x)$ given by $g(x) = \phi_{f(x)}(x) + 1$ would be total and computable.

RICE-SHAPIRO THEOREM

Rice-Shapiro Theorem

Rice-Shapiro Theorem. Suppose that $\mathcal{A}$ is a set of unary computable functions such that the set $\{x \mid \phi_x \in \mathcal{A}\}$ is r.e. Then for any unary computable function f, $f \in \mathcal{A}$ iff there is a finite function $\theta \subseteq f$ with $\theta \in \mathcal{A}$.

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Comment. Intuitively a set of recursive functions is r.e. iff it is effectively generated by an r.e. set of finite functions.

Applications of the Rice-Shapiro Theorem

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Proof

We apply the Rice-Shapiro theorem on Tot. For no $f \in \mathit{Tot}$ is there a finite $\theta \subseteq f$ with $\theta \in \mathit{Tot}$.

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Proof

We apply the Rice-Shapiro theorem on Tot. For no $f \in Tot$ is there a finite $\theta \subseteq f$ with $\theta \in Tot$.

If f is any total computable function, $f \notin \overline{Tot}$; but every finite function $\theta \subseteq f$ in $\overline{Tot}$.

What Rice-Shapiro Theorem Can Do

Can we apply Rice-Shapiro Theorem to show that any of the following sets is non-r.e.:

$$
\begin{aligned}
\mathit{Fin} &= \{x \mid W_x \text{ is finite}\},\\
\mathit{Inf} &= \{x \mid W_x \text{ is infinite}\},\\
\mathit{Tot} &= \{x \mid \phi_x \text{ is total}\},\\
\mathit{Con} &= \{x \mid \phi_x \text{ is total and constant}\},\\
\mathit{Cof} &= \{x \mid W_x \text{ is cofinite}\},\\
\mathit{Rec} &= \{x \mid W_x \text{ is recursive}\},\\
\mathit{Ext} &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}
$$

Proof of Rice-Shapiro Theorem

Suppose $A = \{x \mid \phi_x \in \mathcal{A}\}$ is r.e.

($\Rightarrow$): Suppose $f \in \mathcal{A}$ but for all finite $\theta \subseteq f . \theta \notin \mathcal{A}$.

Let P be a partial characteristic function of K. Define the computable function $g(z,t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } P(z) \not\downarrow \text{ in } t \text{ steps,} \\ \uparrow, & \text{otherwise.} \end{cases}$$

According to S-m-n Theorem, there is an injective primitive recursive function $s(z)$ such that $g(z,t) \simeq \phi_{s(z)}(t)$.

By construction $\phi_{s(z)} \subseteq f$ for all z.

$z \in K \Rightarrow \phi_{s(z)}$ is finite $\Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \in A$.

Proof of Rice-Shapiro Theorem

($\Leftarrow$): Suppose f is a computable function and there is a finite $\theta \in \mathcal{A}$ such that $\theta \subseteq f$ and $f \notin \mathcal{A}$.

Define the computable function $g(z, t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } t \in Dom(\theta) \vee z \in K, \\ \uparrow, & \text{otherwise.} \end{cases}$$

According to S-m-n Theorem, there is an injective primitive recursive function $s(z)$ such that $g(z, t) \simeq \phi_{s(z)}(t)$.

$z \in K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = \theta \Rightarrow s(z) \in A$.

Reversing Rice-Shapiro Theorem

$\{x \mid \phi_x \in \mathcal{A}\}$ is r.e. if the following hold:

1. $\Theta = \{e(\theta) \mid \theta \in \mathcal{A} \text{ and } \theta \text{ is finite}\}$ is r.e., where e is a canonical effective encoding of the finite functions.
2. $\forall f \in \mathcal{A}.\exists \text{ finite } \theta \in \mathcal{A}.\theta \subseteq f$.

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Comment. We cannot take e as the Gödel encoding function of the recursive functions. Why?

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2. $\forall f \in \mathcal{A}.\exists \text{ finite } \theta \in \mathcal{A}.\theta \subseteq f$.

Comment. We cannot take e as the Gödel encoding function of the recursive functions. Why? How would you define e?

Homework

- Homework 6: Exercise 6.14, pp. 119 of the textbook.