Computability Theory VIII

Universal Program

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Enumeration Theorem

General Remark

There are universal programs that embody all the programs.

A program is universal if upon receiving the Gödel number of a program it simulates the program indexed by the number.

Intuition

Consider the function $\psi(x, y)$ defined as follows

$$\psi(x, y) \simeq \phi_x(y)$$

In an obvious sense $\psi(x, \_)$ is a universal function for the unary functions

$$\phi_0, \phi_1, \phi_2, \phi_3, \ldots.$$

Universal Function

The universal function for n-ary computable functions is the $(n+1)$-ary function $\psi_U^{(n)}$ defined by

$$\psi_U^{(n)}(e, x_1, \ldots, x_n) \simeq \phi_e^{(n)}(x_1, \ldots, x_n).$$

We write ψ_U for $\psi_U^{(1)}$.

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Question: Is $\psi_U^{(n)}$ computable?

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Proof

Given a number e, decode the number to get the program P_e; and then simulate the program P_e. If the simulation ever terminates, then return the number in R_1. By Church-Turing Thesis, $\psi_U^{(n)}$ is computable.

Undecidability

Proposition

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Proof

If 'ϕ_x is total' were decidable, then by Church-Turing Thesis

$$f(x) = \begin{cases} \psi_U(x, x) + 1, & \text{if } \phi_x \text{ is total,} \\ 0, & \text{if } \phi_x \text{ is not total.} \end{cases}$$

would be a total computable function that differs from every total computable function.

Effectiveness of Function Operation

Proposition

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Proof

Let $f(x, y, z) \simeq \phi_x(z)\phi_y(z) \simeq \psi_U(x, z)\psi_U(y, z)$.

By S-m-n Theorem there is a total function $s(x, y)$ such that $\phi_{s(x,y)}(z) \simeq f(x, y, z)$.

Effectiveness of Set Operation

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There is a total computable function $s(x, y)$ such that $W_{s(x,y)} = W_x \cup W_y$.

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Proof

Let

$$f(x,y,z) = \begin{cases} 1, & \text{if } z \in W_x \text{ or } z \in W_y, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

By S-m-n Theorem there is a total function $s(x,y)$ such that $\phi_{s(x,y)}(z) \simeq f(x,y,z)$. Clearly $W_{s(x,y)} = W_x \cup W_y$.

Effectiveness of Recursion

Consider f defined by the following recursion

$$f(e_1, e_2, \widetilde{x}, 0) \simeq \phi_{e_1}^{(n)}(\widetilde{x}) \simeq \psi_U^{(n)}(e_1, \widetilde{x})$$

and

$$\begin{aligned} f(e_1, e_2, \widetilde{x}, y+1) &\simeq \phi_{e_2}^{(n+2)}(\widetilde{x}, y, f(e_1, e_2, \widetilde{x}, y)) \\ &\simeq \psi_U^{(n+2)}(e_2, \widetilde{x}, y, f(e_1, e_2, \widetilde{x}, y)). \end{aligned}$$

By S-m-n Theorem, there is a total computable function $r(e_1, e_2)$ such that

$$\phi_{r(e_1,e_2)}^{(n+1)}(\widetilde{x}, y) \simeq f(e_1, e_2, \widetilde{x}, y)$$

Non-Primitive Recursive Total Function

Theorem

There is a total computable function that is not primitive recursive.

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Proof

1. The primitive recursive functions have a universal function.
2. Such a function cannot be primitive recursive by diagonalisation.

RECURSION THEOREM

Recursion Theorem

Recursion Theorem

Let f be a total unary computable function. Then there is a number n such that $\phi_{f(n)} = \phi_n$.

Proof

By S-m-n Theorem there is an injective primitive recursive function $s(x)$ such that for all x

$$\phi_{s(x)}(y) \quad \simeq \quad \begin{cases} \phi_{\phi_x(x)}(y), & \text{if } \phi_x(x)\downarrow; \\ \uparrow, & \text{otherwise.} \end{cases}$$

Let v be such that $\phi_v = s \circ f$. Obviously ϕ_v is total and $\phi_v(v)\downarrow$.

$$\phi_{s(v)} = \phi_{\phi_v(v)} = \phi_{f(s(v))}$$

We are done by letting n be $s(v)$.

Exercise I

Show that there is a total computable function k such that for each n, $E_{k(n)} = W_n$.

Exercise II

Show that there is a total computable function $k(x, y)$ such that for each x, y, $E_{k(x,y)} = E_x \cup E_y$.

Exercise III

Suppose $f(n)$ is computable, show that there is a total computable function $k(n)$ such that for each n, $W_{k(n)} = f^{-1}(W_n)$.