Computability Theory I

Introduction

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What do you think you can learn from this course?

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- The course may provide a view of computation, an overlook of what we are doing in computer science, and a basic study of theoretical computer science.
- It is rather a philosophy than a technique, although some parts are quite technically.

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- A service provider that is working on a theorem prover that is supposed to answer every question about numbers.

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  - Grandfather paradox

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What problems can be solved by computers?

Computer science is no more about computers than astronomy is about telescopes.

Edsger Dijkstra

Let us begin to learn some basic astronomical phenomena!

The technique part is quite similar to puzzles of wise men.

So, please have a fun!

Intuition is extremely important!

Reference Book

- Computability: An Introduction to Recursive Function Theory.
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- plus extra reading materials.

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  - Four assignments.
  - Each one is 5 pts.
  - Work out individually.
- 70% Final exam.
- There are also several homework. The answer may be given in exercise lectures, two or three times.

Special Requirements

A notebook and a pen.

Any questions?

0. Prologue

Effective Solutions

What problems can be solved by computers?

Famous Problems

- Diophantine equations
- Shortest path problem
- Travelling salesman problem (TSP)
- Graph isomorphism problem (GI)

Intuition

An effective procedure consists of a finite set of instructions which, given an input from some set of possible inputs, enables us to obtain an output through a systematic execution of the instructions that terminates in a finite number of steps.

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Theorem proving is in general not effective.

Proof verification is effective.

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Proof verification is effective.

Unbounded search is in general not effective.

Bounded search is effective.

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- How is the answer to a problem instance represented?
- How is an effective procedure formalized?
- Can every function from $\mathbb{N}$ to $\mathbb{N}$ be calculated by a C program?
  - Negative.

Punchline

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- In a formal theory of computability, every problem instance can be represented by a number and every number represents a problem instance.
- A problem is a function $f : \mathbb{N} \to \mathbb{N}$ from numbers to numbers.
- A problem is computable if it can be calculated by a program.

Everything is number!

Pythagoras

Decision Problem

Decision Problem

A problem $f : \mathbb{N} \to \mathbb{N}$ is a decision problem if the range $ran(f)$ of f is $\{0, 1\}$, where 1 denotes a 'yes' answer and 0 a 'no' answer.

A decision problem g can be identified with the set $\{n \mid g(n) = 1\}$.

Conversely a subset A of $\mathbb{N}$ can be seen as a decision problem via the characteristic function of A:

$$c_A(n) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Decision Problem as Predicate

A decision problem can be stated as a predicate $P(x)$ on number.

It relates to the problem-as-function viewpoint by the following characteristic function of $P(x)$:

$$c_P(n) = \begin{cases} 1, & \text{if } P(n) \text{ is valid,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{Decision Problem} &\Leftrightarrow \text{Subset of } \mathbb{N} \\ &\Leftrightarrow \text{Predicate on } \mathbb{N} \end{aligned}$$

Several Problems

Problem I

Is the function *tower*(x) defined below computable?

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Theoretically it is computable.

Problem II

Consider the function f defined as follows:

$$f(n) = \begin{cases} 1, & \text{if } n > 1 \text{ and } 2n \text{ is the sum of } 2 \text{ primes,} \\ 0, & \text{otherwise.} \end{cases}$$

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It is clearly computable even if we do not know what it is.

Problem III

Consider the function g defined as follows:

$$g(n) = \begin{cases} 1, & \text{if there is a run of exactly } n \text{ consecutive } 7'\text{s} \\ & \text{in the decimal expansion of } \pi, \\ 0, & \text{otherwise.} \end{cases}$$

It is known that π can be calculated by $4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$.
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Is g computable?

We do not know whether it is computable or not.

Problem IV

Consider the function h defined as follows:

$$h(n) = \begin{cases} 1, & \text{if } n \text{ is the machine code of a } C \text{ program that} \\ & \text{terminates in all inputs,} \\ 0, & \text{otherwise.} \end{cases}$$

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This is the Halting Problem, a well known undecidable problem. In other words there does not exist any C program calculating h.

The only general approach to check if a function is defined on all numbers is to calculate it on all inputs.

Problem V

Consider the function i defined as follows:

$$i(x, n, t) = \begin{cases} 1, & \text{if on input } x, \text{ the machine coded by } n \\ & \text{terminates in } t \text{ steps,} \\ 0, & \text{otherwise.} \end{cases}$$

There could be a number of ways to interpret "t steps".

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The function i is intuitively computable.

Next Lecture

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We will begin with a machine model, register machine.

Homework

- home reading: diagonal method.
- home reading: Presburger arithmetic.