Computability Theory

Preliminary

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PRELIMINARIES

SET

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  - $\mathbb{N} = \{0, 1, 2, \ldots\}$: natural numbers (nonnegative integers)
  - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$: integers
  - $\mathbb{R}$: real numbers
  - $\mathbb{E}$: even numbers
  - $\mathbb{O}$: odd numbers

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- Proper subset: a subset of T is a subset other than the empty set $\emptyset$ or T itself (Use of word proper, proper subsequence or proper substring)
- Strict Subset: S is a strict subset, $S \subset T$, if not equal to T

Set Operations

- Union: $S \cup T \rightarrow$ the set of elements that are either in S or in T.
  - $S \cup T = \{s | s \in S \text{ or } s \in T\}$
  - $\{a, b, c\} \cup \{c, d, e\} = \{a, b, c, d, e\}$
  - $|S \cup T| \leq |S| + |T|$
- Intersection: $S \cap T$
  - $S \cap T = \{s | s \in S \text{ and } s \in T\}$
  - $\{a, b, c\} \cap \{c, d, e\} = \{c\}$
- Difference: $S - T \rightarrow$ set of all elements in S not in T
  - $S - T = \{s | s \in S \text{ but not in } T\} = S \cap \overline{T}$
  - $\{1, 2, 3\} - \{1, 4, 5\} = \{2, 3\}$
- Complement:
  - Need universal set U
  - $\overline{S} = \{s | s \in U \text{ but not in } S\}$

Set Operations

- Cartesian Product
  - $S \times T = \{(s, t) | s \in S, t \in T\}$
  - In a graph $G = (V, E)$, the edge set E is the subset of Cartesian product of vertex set V. $E \subseteq V \times V$.
- Power Set
  - 2^S set of all subsets of S
  - Note: notation $|2^S| = 2^{|S|}$, meaning 2^S is a good representation for power set.
  - $S = \{a, b, c\}$, then $2^S = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
  - Indicator Vector: We can use a zero/one vector to represent the elements in power set.

	a	b	c
$\emptyset$	0	0	0
$\{a\}$	1	0	0
$\{b\}$	0	1	0
$\{a, b, c\}$	1	1	1

Ordered Pair

- (x, y): ordered pair of elements x and y; $(x, y) \neq (y, x)$.
- $(x_1, \cdots, x_n)$: ordered n-tuple $\rightarrow$ as $\widetilde{x}$.
- $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, \cdots, x_n) : x_1 \in A_1, \cdots, x_n \in A_n\}$.
- $A \times A \times \cdots \times A = A^n$.
- $A^1 = A$.

FUNCTION

Function Notation

- f is a set of ordered pairs s.t. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$, and $f(x) = y$.
- $Dom(f)$: Domain of f, $\{x : f(x) \text{ is defined}\}$.
- $f(x)$ is undefined if $x \notin Dom(f)$.
- $Ran(f)$: Range of f, $\{f(x) : x \in Dom(f)\}$.
- f is a function from A to B: $Dom(f) \subseteq A$ and $Ran(f) \subseteq B$.
- $f : A \to B$: f is a function from A to B with $Dom(f) = A$.

Mapping

- Injective: if $x, y \in Dom(f)$, $x \neq y$, then $f(x) \neq f(y)$.
- Inverse f^{-1}: the unique function g s.t. $Dom(g) = Ran(f)$, and $g(f(x)) = x$.
- Surjective: if $Ran(f) = B$.
- Bijective: both injective and surjective.

Operation

1. $f|X$: Restriction of f to X.
   Domain $X \cap Dom(f)$. Write $f(X)$ for $Ran(f|X)$.
2. $f^{-1}(Y) = \{x : f(x) \in Y\}$: inverse image of Y under f.
3. $f \subseteq g$: g extends f, $f = g|Dom(f)$.
   $Dom(f) \subseteq Dom(g)$ and $\forall x \in Dom(f)$, $f(x) = g(x)$.
4. $f \circ g$: composition of f and g. Domain
   $\{x : x \in Dom(g) \text{ and } g(x) \in Dom(f)\}$, value $f(g(x))$.
5. $f_\emptyset$: function defined nowhere. $Dom(f_\emptyset) = Ran(f_\emptyset) = \emptyset$.
   $f_\emptyset = g|\emptyset$ for any function g.

$\simeq$: similar-or-equal-to

Suppose $\alpha(\widetilde{x})$ and $\beta(\widetilde{x})$ are expressions involving $\widetilde{x} = (x_1, \cdots, x_n)$, then $\alpha(\widetilde{x}) \simeq \beta(\widetilde{x})$ means $\forall \widetilde{x}$, $\alpha(\widetilde{x})$ and $\beta(\widetilde{x})$ are either bother defined, or both undefined, and if defined they are equal.

- $f(x) \simeq g(x)$ is a kind of $f = g$.
- $f(x) \simeq y$ means $f(x)$ is defined and $f(x) = y$.

Partial and Total Function

- *n*-ary function: $f(\widetilde{x})$, $f(x_1, \cdots, x_n)$, $f : \mathbb{N}^n \to \mathbb{N}$.
- Partial function: $Dom(f)$ is not necessarily the whole $\mathbb{N}^n$.
  (In our class function means partial function)
- Total function: $Dom(f) = \mathbb{N}^n$.
- Zero function: $\mathbf{0} : \mathbb{N} \to \mathbb{N}$.

Relations

Relation

If A is a set, a property $M(x_1, \cdots, x_n)$ that holds for some n-tuple from A^n and does not hold for all other n-tuples from A^n is called an n-ary relation or predicate on A.

- Property $x < y$. $2 < 5$, $6 < 4$.
- f from $\mathbb{N}^n$ to $\mathbb{N}$ gives rise to predicate $M(\widetilde{x}, y)$ by: $M(x_1, \cdots, x_n, y)$ iff $f(x_1, \cdots, x_n) \simeq y$.

Equivalence Relation

- A binary relation R on A is called equivalence relation if

$$\left.\begin{array}{ll} \text{reflexivity} & \forall x \in A \;\; R(x,x) \\ \text{symmetry} & R(x,y) \Rightarrow R(y,x) \\ \text{transitivity} & R(x,y), R(y,z) \Rightarrow R(x,z) \end{array}\right\} \text{equivalence}$$

- A binary relation R on A is called a partial order if

$$\left.\begin{array}{ll} \text{irreflexivity} & \neg R(x,x) \\ \text{transitivity} & R(x,y), R(y,z) \Rightarrow R(x,z) \end{array}\right\} \text{partial order}$$

Example

	reflexive	symmetric	transitive
$<$	No	No	Yes
$\leq$	Yes	No	Yes
Parent of	No	No	No
$=$	Yes	Yes	Yes

Diagonal Method

The First Theorem

Theorem

The set of reals is uncountable.

The Second Theorem

Theorem

The power set of a set is always of greater cardinality than the set itself.

Hand Writing

- Small letters for elements and functions.
  - a, b, c for elements,
  - f, g for functions,
  - i, j, k for integer indices,
  - x, y, z for variables,
- Capital letters for sets. A, B, S. $A = \{a_1, \cdots, a_n\}$
- Bmall letters with tilde for vectors. $\widetilde{x}$, $\widetilde{y}$. $\widetilde{v} = \{v_1, \cdots, v_m\}$
- Bold capital letters for collections. $\mathbf{A}$, $\mathbf{B}$. $\mathbf{S} = \{S_1, \cdots, S_n\}$
- Blackboard bold capitals for domains (standard symbols). $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$.
- German script for collection of functions. $\mathscr{C}$, $\mathscr{S}$, $\mathscr{T}$.
- Greek letters for parameters or coefficients. α, β, γ.
- Double strike handwriting for bold letters.