Approximating Text-to-Pattern Hamming Distances

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July 13, 2022

Problem

Inputs:

- Pattern string P and text string T with $|P| = m, |T| = n$.

Outputs:

- An $1 + \epsilon$ approximation of hamming distance between P and **any** size m substring of T.

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Goal: Running time $O(n \log^{1.5} n)$.

Approximate Hamming Distance $HD(X, Y)$

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Too slow since we have $\Theta(n)$ instances! Idea: sample many positions at one time.

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2. Independently uniformly sample $B_1, B_2, ..., B_c \subseteq [p]$ with $|B_i| = \log m$. Compute

$$x_i := h(\bigodot_{j \mod p \in B_i} X[j])$$

$$y_i = h(\bigodot_{j \mod p \in B_i} Y[j])$$

for each i. (Here $h(S)$ is the polynomial rolling hashing $\sum S[i] \cdot rand^i \mod m^c$.)

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3. Let d as the number of i that $x_i \neq y_i$. Return

$$f(d) = p \cdot \left(1 - \left(1 - \frac{d}{c}\right)^{\frac{1}{\log m}}\right)$$

Correctness

What to approximate

$$M := \{i \in [m] \mid X[i] \neq Y[i]\}$$

Instead we pick an appropriate parameter p and approximate

$$M' := \{b \in [p] \mid \bigodot_{i \mod p = b} X[i] \neq \bigodot_{i \mod p = b} Y[i]\}$$

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The following happens with large constant probability.

- With random prime p in $[ck \log m, 2ck \log m]$, we have $|M'| \approx |M|$.
- $f(d) \approx |M'|$.

$|M| \approx |M'|$

$$M' := \{b \in [p] \mid \bigodot_{i \bmod p = b} X[i] \neq \bigodot_{i \bmod p = b} Y[i]\}$$
$$= \{i \bmod p \mid i \in M\}$$

p is a random prime from $[\hat{p}, 2\hat{p}]$ where $\hat{p} = ck \log m$.
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Proof.

$$|M| - |M'| \leq \#\{i, j \in M \mid |i - j| \mod p = 0\}$$

$$\mathbb{E}\left[|M'| - |M|\right] \leq |M|^2 \cdot \frac{\log_{\hat{p}} m}{\hat{p} / \log \hat{p}} = O\left(|M|/c\right)$$

Markov's inequality: $\Pr[|M'| - |M| \geq \epsilon |M|] \leq O(1/\epsilon c)$

$\square$

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- Use the following lemma

Lemma

If the absolute derivative of $\log f(2^x)$ *is bounded by* c, *and* $(1-\epsilon)y < x < (1+\epsilon)y$, *then* $(1-2c\epsilon')f(y) < f(x) < (1+c\epsilon')f(y)$.

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Further facts: If k is too large, $E[d]/c$ will be too close to 0; if k is too small, $E[d]/c$ will be too close to 1: we can distinguish the cases where $k >> |M|$ or $k << |M|$.

text-pattern P, T hamming distance

Compute

$$h\left(\bigodot_{j \mod p \in B, j \in [m]} P[j]\right)$$

$$h\left(\bigodot_{j \mod p \in B, j \in [m]} T[i+j]\right)$$

For any i.

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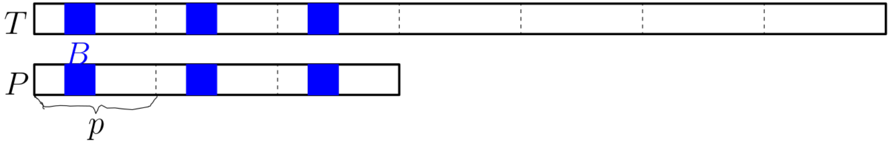

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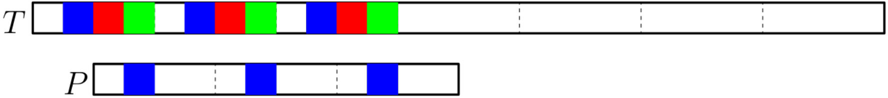

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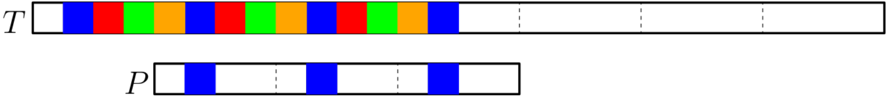

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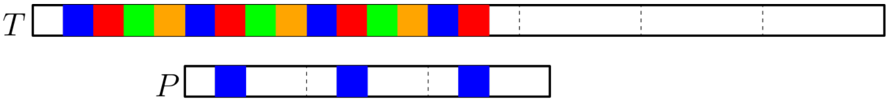

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Given a string S with $|S| = n/p$, computing $h(S')$ for any fix length substring S' of S cost $O(|S|) = O(n/p)$ running time.

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$O\left(\frac{n}{p} \cdot p \cdot |B|\right) = O(n \log n)$ for T.

$O\left(\frac{m}{p} \cdot |B|\right) = O(n/k)$ for P.

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$\sqrt{p}$ colors in T and $\sqrt{p}$ colors in P.

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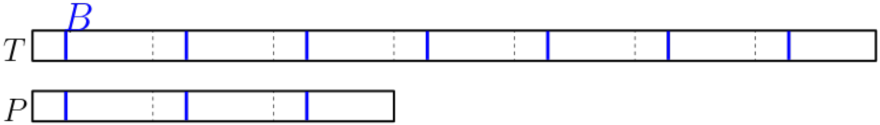

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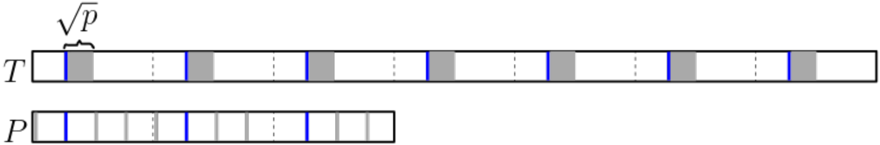

Colors $B + (1...\sqrt{p})$ in T and colors $B + (1...\sqrt{p}) \cdot \sqrt{p}$ in P.

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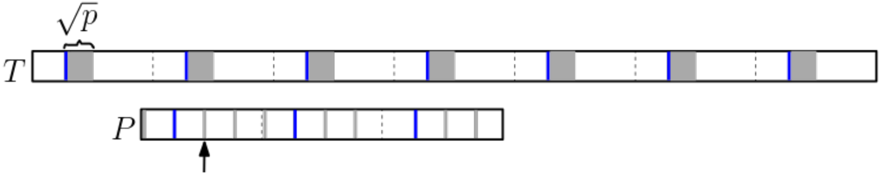

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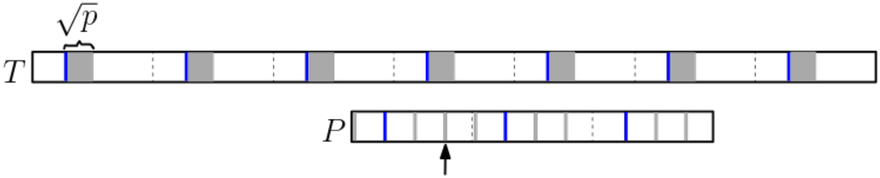

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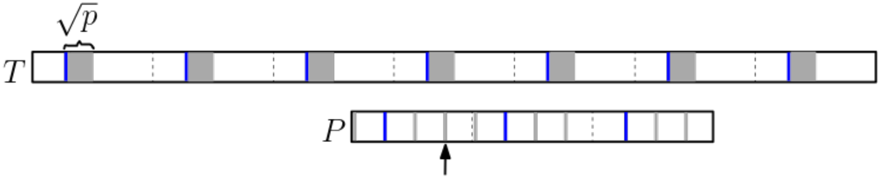

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$$O\left(\frac{m}{p} \cdot \sqrt{p} \cdot |B|\right) = O(n\sqrt{\log n}/\sqrt{k}) \text{ for } P.$$

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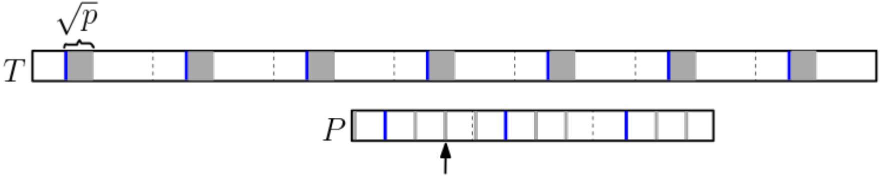

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Correctness: uniform random B (blue line) $\Rightarrow$ for any position of P, uniform random allignment.

Algorithm

For each $k = 2^i$ where $i \in [\log m]$ do:

1. Let p be a random prime from $\hat{p}, 2\hat{p}$ where $\hat{p} = \Theta(k \log m)$.
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3. Compute $h\left(\bigodot_{k+v\sqrt{p} \mod p \in B, k \in [m]} P[k]\right)$ for any sampled B and any $v \in [\sqrt{p}]$
   $O\left(\frac{m}{p} \cdot \frac{p}{k} \cdot \sqrt{p}\right) = O\left(\frac{m\sqrt{p}}{k}\right) = O\left(m\sqrt{\log m}/\sqrt{k}\right)$;
   $h\left(\bigodot_{k+i-u \mod p \in B, k \in [m]} T[i+k-1]\right)$ for any sampled B, any shift $u \in [\sqrt{p}]$ and for any i
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4. For any i, let the number of B that $h\left(\bigodot_{k+v_i\sqrt{n} \mod p\in B, k\in[m]} P[k]\right) = h\left(\bigodot_{k+i-u_i\sqrt{q} \mod p\in B, k\in[m]} T[i+k-1]\right)$ be d_i. Let $\tilde{d}_i = f^{-1}(d_i)$. If $\tilde{d}_i/c$ is not too closer to 0 or 1, then let $f(\tilde{d}_i)$ be the estimation for $HD(P, T[i, ..., i+m-1])$.$O(n)$

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For any i:

- Binary search for k:
  1. For any i, let the number of B that $h\left(\bigodot_{k+v_i\sqrt{n} \mod p \in B, k \in [m]} P[k]\right) = h\left(\bigodot_{k+i-u_i\sqrt{q} \mod p \in B, k \in [m]} T[i+k-1]\right)$ be d_i. Let $\tilde{d}_i = f^{-1}(d_i)$. If $\tilde{d}_i/c$ is not too closer to 0 or 1, then let $f(\tilde{d}_i)$ be the estimation for $HD(P, T[i, ..., i+m-1])$. $O(n)$

Correctness

Lemma

For any i **and any** k, *the algorithm returns a correct estimation of* $HD(P, T[i, ..., i+m-1])$, *or corretly determine whether* k *is too large or too small with large constant probability.*

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Total running time

$$O(\log n) \cdot \left(\sum_{k \in [logm]} O\left(\frac{m\sqrt{\log m}}{\sqrt{k}} \right) + \log\log n \cdot O(n) \right)$$

$$= O(n \log^{1.5} n)$$