CHAPTER 4

Propagation of states and covariances

In this chapter, we will begin with our mathematical description of a dynamic system, and then derive the equations that govern the propagation of the state mean and covariance. The material presented in this chapter is fundamental to the state estimation algorithm (the Kalman filter) that we will derive in Chapter 5.

Section 4.1 covers discrete-time systems. Section 4.2 covers sampled-data systems, which are the most common types of systems found in the real world. In this type of system, the system dynamics are described by continuous-time differential equations, but the control and measurement signals are discrete time (e.g., control based on a digital computer and measurements obtained at discrete times). Section 4.3 covers continuous-time systems.

4.1 DISCRETE-TIME SYSTEMS

Suppose we have the following linear discrete-time system:

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} \tag{4.1}$$

where u_k is a known input and w_k is Gaussian zero-mean white noise with covariance Q_k. How does the mean of the state x_k change with time? If we take the expected value of both sides of Equation (4.1) we obtain

$$\begin{aligned} \bar{x}_k &= E(x_k) \\ &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \end{aligned} \tag{4.2}$$

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How does the covariance of x_k change with time? We can use Equations (4.1) and (4.2) to obtain

$$\begin{aligned} (x_k - \bar{x}_k)(\cdots)^T &= (F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} - \bar{x}_k)(\cdots)^T \\ &= [F_{k-1}(x_{k-1} - \bar{x}_{k-1}) + w_{k-1}][\cdots]^T \\ &= F_{k-1}(x_{k-1} - \bar{x}_{k-1})(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T + w_{k-1}w_{k-1}^T + \\ & \quad F_{k-1}(x_{k-1} - \bar{x}_{k-1})w_{k-1}^T + w_{k-1}(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T \end{aligned} \tag{4.3}$$

We therefore obtain the covariance of x_k as the expected value of the above expression. Since $(x_{k-1} - \bar{x}_{k-1})$ is uncorrelated with w_{k-1}, we obtain

$$\begin{aligned} P_k &= E\left[(x_k - \bar{x}_k)(\cdots)^T\right] \\ &= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1} \end{aligned} \tag{4.4}$$

This is called a discrete-time Lyapunov equation, or a Stein equation [Ste52]. We will see in the next chapter that Equations (4.2) and (4.4) are fundamental in the derivation of the Kalman filter.

It is interesting to consider the conditions under which the discrete-time Lyapunov equation has a steady-state solution. That is, suppose that $F_k = F$ is a constant, and $Q_k = Q$ is a constant. Then we have the following theorem, whose proof can be found in [Kai00, Appendix D].

Theorem 21 *Consider the equation $P = FPF^T + Q$ where F and Q are real matrices. Denote by $\lambda_i(F)$ the eigenvalues of the F matrix.*

1. *A unique solution P exists if and only if $\lambda_i(F)\lambda_j(F) \neq 1$ for all i, j. This unique solution is symmetric.*

2. *Note that the above condition includes the case of stable F, because if F is stable then all of its eigenvalues are less than one in magnitude, so $\lambda_i(F)\lambda_j(F) \neq 1$ for all i, j. Therefore, we see that if F is stable then the discrete-time Lyapunov equation has a solution P that is unique and symmetric. In this case, the solution can be written as*

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i \tag{4.5}$$

3. *If F is stable and Q is positive (semi)definite, then the unique solution P is symmetric and positive (semi)definite.*

4. *If F is stable, Q is positive semidefinite, and $(F, Q^{1/2})$ is controllable, then P is unique, symmetric, and positive definite. Note that $Q^{1/2}$, the square root of Q, is defined here as any matrix such that $Q^{1/2}(Q^{1/2})^T = Q$.*

Now let us look at the solution of the linear system of Equation (4.1):

$$x_k = F_{k,0}x_0 + \sum_{i=0}^{k-1} (F_{k,i+1}w_i + F_{k,i+1}G_i u_i) \tag{4.6}$$

The matrix $F_{k,i}$ is the state transition matrix of the system and is defined as

$$F_{k,i} = \begin{cases} F_{k-1}F_{k-2}\cdots F_i & k > i \\ I & k = i \\ 0 & k < i \end{cases} \tag{4.7}$$

Notice from Equation (4.6) that x_k is a linear combination of x_0, $\{w_i\}$, and $\{u_i\}$. If the input sequence $\{u_i\}$ is known, then it is a constant and can be considered to be a sequence of Gaussian random variables with zero covariance. If x_0 and $\{w_i\}$ are unknown but are Gaussian random variables, then x_k in Equation (4.6) is a linear combination of Gaussian random variables. Therefore, x_k is itself a Gaussian random variable (see Example 2.4). But we computed the mean and covariance of x_k in Equations (4.2) and (4.4). Therefore

$$x_k \sim N(\bar{x}_k, P_k) \tag{4.8}$$

This completely characterizes x_k in a statistical sense since a Gaussian random variable is completely characterized by its mean and covariance.

■ EXAMPLE 4.1

A linear system describing the population of a predator $x(1)$ and that of its prey $x(2)$ can be written as

$$\begin{aligned} x_{k+1}(1) &= x_k(1) - 0.8x_k(1) + 0.4x_k(2) + w_k(1) \\ x_{k+1}(2) &= x_k(2) - 0.4x_k(1) + u_k + w_k(2) \end{aligned} \tag{4.9}$$

In the first equation, we see that the predator population causes itself to decrease because of overcrowding, but the prey population causes the predator population to increase. In the second equation, we see that the prey population decreases due to the predator population and increases due to an external food supply u_k. The populations are also subject to random disturbances (with respective variances 1 and 2) due to environmental factors. This system can be written in state-space form as

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k \\ w_k &\sim (0, Q) \quad Q = \text{diag}(1, 2) \end{aligned} \tag{4.10}$$

Equations (4.2) and (4.4) describe how the mean and covariance of the populations change with time. Figure 4.1 depicts the two means and the two diagonal elements of the covariance matrix for the first few time steps when $u_k = 1$ and the initial conditions are set as $\bar{x}_0 = \begin{bmatrix} 10 & 20 \end{bmatrix}^T$ and $P_0 = \text{diag}(40, 40)$. It is seen that the mean and covariance eventually reach steady-state values given by

$$\begin{aligned} \bar{x} &= (I - F)^{-1}Gu \\ &= \begin{bmatrix} 2.5 & 5 \end{bmatrix}^T \\ P &\approx \begin{bmatrix} 2.88 & 3.08 \\ 3.08 & 7.96 \end{bmatrix} \end{aligned} \tag{4.11}$$

The steady-state value of P can also be found directly (i.e., without simulation) using control system software.[1] Note that since F for this example is stable and Q is positive definite, Theorem 21 guarantees that P has a unique positive definite steady-state solution.

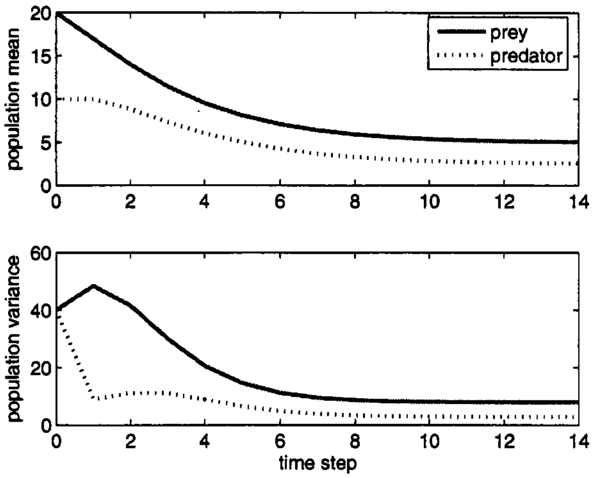

Figure 4.1 State means and variances for Example 4.1.

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In Equation (4.1), we showed the process noise directly entering the system dynamics. This is the convention that we use in this book. However, many times process noise is first multiplied by some matrix before it enters the system dynamics. That is,

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + L_{k-1}\tilde{w}_{k-1}, \qquad \tilde{w}_k \sim (0, \tilde{Q}_k) \tag{4.12}$$

How can we put this into the conventional form of Equation (4.1)? Notice that the rightmost term of Equation (4.12) has a covariance given by

$$\begin{aligned} E\left[(L_{k-1}\tilde{w}_{k-1})(L_{k-1}\tilde{w}_{k-1})^T\right] &= L_{k-1}E(\tilde{w}_{k-1}\tilde{w}_{k-1}^T)L_{k-1}^T \\ &= L_{k-1}\tilde{Q}_{k-1}L_{k-1}^T \end{aligned} \tag{4.13}$$

Therefore, Equation (4.12) is equivalent to the equation

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1}, \qquad w_k \sim (0, L_k Q_k L_k^T) \tag{4.14}$$

This idea is illustrated in Sections 7.3.1 and 7.3.2. The same type of transformation can be made with noisy measurement equations. That is, the measurement equation

$$y_k = H_k x_k + L_k \tilde{v}_k, \qquad \tilde{v}_k \sim (0, \tilde{R}_k) \tag{4.15}$$

is equivalent to the measurement equation

$$y_k = H_k x_k + v_k, \qquad v_k \sim (0, L_k \tilde{R}_k L_k^T) \tag{4.16}$$

[1] For example, we can use the MATLAB Control System Toolbox function DLYAP(F, Q).

4.2 SAMPLED-DATA SYSTEMS

Now we move on to sampled-data systems, which are the most frequently encountered systems in practice. A sampled-data system is a system whose dynamics are described by a continuous-time differential equation, but the input only changes at discrete time instants, because (for example) the input is generated by a digital computer. In addition, we are interested in estimating the state only at discrete time instants. We are interested in obtaining the mean and covariance of the state only at discrete time instants. The continuous-time dynamics are described as

$$\dot{x} = Ax + Bu + w \tag{4.17}$$

From Chapter 1 we know that the solution of $x(t)$ at some arbitrary time, say t_k, is given as

$$x(t_k) = e^{A(t_k - t_{k-1})}x(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}[B(\tau)u(\tau) + w(\tau)]\, d\tau \tag{4.18}$$

Now assume that $u(t) = u_k$ for $t \in [t_k, t_{k+1}]$; that is, the control $u(t)$ is piecewise constant.[2] If we make the definitions

$$\begin{aligned} \Delta t &= t_k - t_{k-1} \\ x_k &= x(t_k) \\ u_k &= u(t_k) \end{aligned} \tag{4.19}$$

then Equation (4.18) becomes

$$x_k = e^{A\Delta t}x_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)\, d\tau\, u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)\, d\tau \tag{4.20}$$

Now if we define F_k and G_k as

$$\begin{aligned} F_k &= e^{A\Delta t} \\ G_k &= \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B(\tau)\, d\tau \end{aligned} \tag{4.21}$$

then Equation (4.20) becomes

$$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)\, d\tau \tag{4.22}$$

$e^{A(t_k-\tau)}$ is the state transition matrix of the system from time τ to time t_k. Now take the mean of the above equation, remembering that $w(t)$ is zero-mean, to obtain

$$\begin{aligned} \bar{x}_k &= E(x_k) \\ &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \end{aligned} \tag{4.23}$$

[2]This assumes that a first-order hold is used for the control inputs. Other types of holds can be used in sampled data systems, but in this book we assume that first-order holds are used.

We can use the previous equations to obtain the covariance of the state as

$$
\begin{aligned}
P_k &= E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T] \\
&= E\left[\left(F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} w(\tau) d\tau - \bar{x}_k\right)\left(\cdots\right)^T\right] \\
&= F_{k-1}P_{k-1}F_{k-1}^T + E\left[\left(\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} w(\tau)\, d\tau\right)\left(\cdots\right)^T\right] \\
&= F_{k-1}P_{k-1}F_{k-1}^T + \int\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} E\left[w(\tau)w^T(\alpha)\right] e^{A^T(t_k-\alpha)}\, d\tau\, d\alpha
\end{aligned}
\tag{4.24}
$$

Now, if we assume that $w(t)$ is continuous-time white noise with a covariance of $Q_c(t)$, we see that

$$
E\left[w(\tau)w^T(\alpha)\right] = Q_c(\tau)\delta(\tau - \alpha) \tag{4.25}
$$

This means that we can use the sifting property of the impulse function (see Problem 4.10) to write Equation (4.24) as

$$
\begin{aligned}
P_k &= F_{k-1}P_{k-1}F_{k-1}^T + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)}\, d\tau \\
&= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1}
\end{aligned}
\tag{4.26}
$$

where Q_{k-1} is defined by the above equation; that is,

$$
Q_{k-1} = \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)}\, d\tau \tag{4.27}
$$

In general, it is difficult to calculate Q_{k-1}, but for small values of $(t_k - t_{k-1})$ we obtain

$$
\begin{aligned}
e^{A(t_k-\tau)} &\approx I \text{ for } \tau \in [t_{k-1}, t_k] \\
Q_{k-1} &\approx Q_c(t_k)\Delta t
\end{aligned}
\tag{4.28}
$$

■ EXAMPLE 4.2

Suppose we have a first-order, continuous-time dynamic system given by the equation

$$
\begin{aligned}
\dot{x} &= fx + w \\
E[w(t)w(t+\tau)] &= q_c\delta(\tau)
\end{aligned}
\tag{4.29}
$$

First-order equations can be used to describe many simple physical processes. For example, this equation describes the behavior of the current through a series RL circuit that is driven by a random voltage $w(t)$, where $f = -R/L$. Suppose we are interested in obtaining the mean and covariance of the state $x(t)$ every Δt time units; that is, $t_k - t_{k-1} = \Delta t$. For this simple scalar example, we can explicitly calculate Q_{k-1} in Equation (4.27) as

$$\begin{aligned}
Q_{k-1} &= \int_{t_{k-1}}^{t_k} \exp[f(t_k - \tau)] q_c \exp[f(t_k - \tau)]\, d\tau \\
&= \exp(2ft_k) q_c \int_{t_{k-1}}^{t_k} \exp(-2f\tau)\, d\tau \\
&= \exp(2ft_k) q_c \left[\frac{\exp(-2ft_{k-1}) - \exp(-2ft_k)}{2f} \right] \\
&= \frac{q_c}{2f} \left[\exp(2f(t_k - t_{k-1})) - 1 \right] \\
&= \frac{q_c}{2f} \left[\exp(2f\Delta t) - 1 \right]
\end{aligned} \tag{4.30}$$

For small values of Δt, we can expand the above equation in a Taylor series around $\Delta t = 0$ to obtain

$$\begin{aligned}
Q_{k-1} &= \frac{q_c}{2f} \left[\exp(2f\Delta t) - 1 \right] \\
&= \frac{q_c}{2f} \left[\left(1 + 2f\Delta t + \frac{(2f\Delta t)^2}{2!} + \cdots \right) - 1 \right] \\
&\approx \frac{q_c}{2f} \left[1 + 2f\Delta t - 1 \right] \\
&= q_c \Delta t
\end{aligned} \tag{4.31}$$

This matches Equation (4.28), which says that for small Δt we have $Q_{k-1} \approx q_c \Delta t$. The sampled mean of the state is computed from Equation (4.23) [noting that the control input in Equation (4.29) is zero] as

$$\begin{aligned}
\bar{x}_k &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \\
&= \exp\left[f(t_k - t_{k-1}) \right] \bar{x}_{k-1} + 0 \\
&= \exp(f\Delta t)\bar{x}_{k-1} \\
&= \exp(kf\Delta t)\bar{x}_0
\end{aligned} \tag{4.32}$$

We see that if $f > 0$ (i.e., the system is unstable) then the mean $\bar{x}_k$ will increase without bound (unless $\bar{x}_0 = 0$). However, if $f < 0$ (i.e., the system is stable) then the mean $\bar{x}_k$ will decay to zero regardless of the value of $\bar{x}_0$. The sampled covariance of the state is computed from Equation (4.26) as

$$\begin{aligned}
P_k &= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1} \\
&\approx (1 + 2f\Delta t)P_{k-1} + q_c\Delta t \\
P_k - P_{k-1} &= (2fP_{k-1} + q_c)\Delta t
\end{aligned} \tag{4.33}$$

From the above equation, we can see that P_k reaches steady state (i.e., $P_k - P_{k-1} = 0$) when $P_{k-1} = -q_c/2f$, assuming that $f < 0$. On the other hand, if $f \geq 0$ then $P_k - P_{k-1}$ will always be greater than 0, which means that $\lim_{k\to\infty} P_k = \infty$.

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4.3 CONTINUOUS-TIME SYSTEMS

In this section, we will look at how the mean and covariance of the state of a continuous-time linear system propagate. Consider the continuous-time system

$$\dot{x} = Ax + Bu + w \tag{4.34}$$

where $u(t)$ is a known control input and $w(t)$ is zero-mean white noise with a covariance of

$$E[w(t)w^T(\tau)] = Q_c\delta(t-\tau) \tag{4.35}$$

By taking the mean of Equation (4.34), we can obtain the following equation for the derivative of the mean of the state:

$$\dot{\bar{x}} = A\bar{x} + Bu \tag{4.36}$$

This equation shows how the mean of the state propagates with time. The linear equation that describes the propagation of the mean looks very much like the original state equation, Equation (4.34). We can also obtain Equation (4.36) by using the equation that describes the mean of a sampled-data system and taking the limit as $\Delta t = t_k - t_{k-1}$ goes to zero. Taking the mean of Equation (4.18) gives

$$\bar{x}_k = e^{A\Delta t}\bar{x}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)u(\tau)\,d\tau \tag{4.37}$$

The state transition matrix can be written as

$$\begin{aligned} F &= e^{A\Delta t} \\ &= I + A\Delta t + \frac{(A\Delta t)^2}{2!} + \cdots \end{aligned} \tag{4.38}$$

For small values of Δt, this can be approximated as

$$F \approx I + A\Delta t \tag{4.39}$$

With this substitution Equation (4.37) becomes

$$\bar{x}_k = (I + A\Delta t)\bar{x}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)u(\tau)\,d\tau \tag{4.40}$$

Subtracting $\bar{x}_{k-1}$ from both sides and dividing by Δt gives

$$\frac{\bar{x}_k - \bar{x}_{k-1}}{\Delta t} = A\bar{x}_{k-1} + \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)u(\tau)\,d\tau \tag{4.41}$$

Taking some limits as Δt goes to zero gives the following:

$$\begin{aligned} \lim_{\Delta t\to 0}\frac{\bar{x}_k - \bar{x}_{k-1}}{\Delta t} &= \dot{\bar{x}} \\ \lim_{\Delta t\to 0} e^{A(t_k-\tau)} &= I \text{ for } \tau \in [t_{k-1}, t_k] \end{aligned} \tag{4.42}$$

Making these substitutions in (4.41) gives

$$\dot{\bar{x}} = A\bar{x} + Bu \tag{4.43}$$

which is the same equation as the one we derived earlier in Equation (4.36) by a more direct method. Although the limiting argument that we used here was not necessary because we already had the mean equation in Equation (4.36), this method shows us how we can use limiting arguments (in general) to obtain continuous-time formulas.

Next we will use a limiting argument to derive the covariance of the state of a continuous-time system. Recall the equation for the covariance of a sampled data system from Equation (4.26):

$$P_k = F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1} \tag{4.44}$$

For small Δt we again approximate F_{k-1} as shown in Equation (4.39) and substitute into the above equation to obtain

$$\begin{aligned} P_k &\approx (I + A\Delta t)P_{k-1}(I + A\Delta t)^T + Q_{k-1} \\ &= P_{k-1} + AP_{k-1}\Delta t + P_{k-1}A^T\Delta t + AP_{k-1}A^T(\Delta t)^2 + Q_{k-1} \end{aligned} \tag{4.45}$$

Subtracting P_{k-1} from both sides and dividing by Δt gives

$$\frac{P_k - P_{k-1}}{\Delta t} = AP_{k-1} + P_{k-1}A^T + AP_{k-1}A^T\Delta t + \frac{Q_{k-1}}{\Delta t} \tag{4.46}$$

Recall from Equation (4.28) that for small Δt

$$Q_{k-1} \approx Q_c(t_k)\Delta t \tag{4.47}$$

This can be written as

$$\frac{Q_{k-1}}{\Delta t} \approx Q_c(t_k) \tag{4.48}$$

Therefore, taking the limit of Equation (4.46) as Δt goes to zero gives

$$\dot{P} = AP + PA^T + Q_c \tag{4.49}$$

This continuous-time Lyapunov equation, also sometimes called a Sylvester equation, gives us the equation for how the covariance of the state of a continuous-time system propagates with time.

It is interesting to consider the conditions under which the continuous-time Lyapunov equation has a steady-state solution. That is, suppose that $A(t) = A$ is a constant, and $Q_c(t) = Q_c$ is a constant. Then we have the following theorem, whose proof can be found in [Kai00, Appendix D].

Theorem 22 *Consider the equation $AP + PA^T + Q_c = 0$ where A and Q_c are real matrices. Denote by $\lambda_i(A)$ the eigenvalues of the A matrix.*

1. *A unique solution P exists if and only if $\lambda_i(A) + \lambda_j(A) \neq 0$ for all i, j. This unique solution is symmetric.*

2. *Note that the above condition includes the case of stable A, because if A is stable then all of its eigenvalues have real parts less than 0, so $\lambda_i(A)+\lambda_j(A) \neq 0$ for all i, j. Therefore, we see that if A is stable then the continuous-time Lyapunov equation has a solution P that is unique and symmetric. In this case, the solution can be written as*

$$P = \int_0^\infty e^{A^T\tau}Q_c e^{A\tau}\,d\tau \tag{4.50}$$

3. *If A is stable and Q_c is positive (semi)definite, then the unique solution P is symmetric and positive (semi)definite.*

4. *If A is stable, Q_c is positive semidefinite, and $\left[A, (Q_c^{1/2})^T\right]$ is controllable, then P is unique, symmetric, and positive definite. Note that $Q_c^{1/2}$, the square root of Q_c, is defined here as any matrix such that $Q_c^{1/2}(Q_c^{1/2})^T = Q_c$.*

■ EXAMPLE 4.3

Suppose we have the first-order, continuous-time dynamic system given by Equation (4.29):

$$\begin{aligned} \dot{x} &= fx + w \\ E[w(t)w(t+\tau)] &= q_c\delta(\tau) \end{aligned} \tag{4.51}$$

where $w(t)$ is zero-mean noise. The equation for the continuous-time propagation of the mean of the state is obtained from Equation (4.36):

$$\dot{\bar{x}} = f\bar{x} \tag{4.52}$$

Solving this equation for $\bar{x}(t)$ gives

$$\bar{x}(t) = \exp(ft)\bar{x}(0) \tag{4.53}$$

We see that the mean will increase without bound if $f > 0$ (i.e., if the system is unstable), but the mean will asymptotically tend to zero if $f < 0$ (i.e., if the system is stable). The equation for the continuous-time propagation of the covariance of the state is obtained from Equation (4.49):

$$\dot{P} = 2fP + q_c \tag{4.54}$$

Solving this equation for $P(t)$ gives

$$P(t) = \left(P(0) + \frac{q_c}{2f}\right)\exp(2ft) - \frac{q_c}{2f} \tag{4.55}$$

We see that the covariance will increase without bound if $f > 0$ (i.e., if the system is unstable), but the covariance will asymptotically tend to $-q_c/2f$ if $f < 0$ (i.e., if the system is stable). Compare these results with Example 4.2.

The steady-state value of P can also be computed using Equation (4.50). If we substitute f for A and q_c for Q_c in Equation (4.50), we obtain

$$\begin{aligned} P &= \int_0^\infty e^{2f\tau} q_c \, d\tau \\ &= \left. \frac{q_c}{2f} e^{2f\tau} \right|_0^\infty \end{aligned} \tag{4.56}$$

The integral converges for $f < 0$ (i.e., if the system is stable), in which case $P = -q_c/2f$.

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4.4 SUMMARY

In this chapter, we have derived equations for the propagation of the mean and covariance of the state of linear systems. For discrete-time systems, the mean and covariance are described by difference equations. Sampled-data systems are systems with continuous-time dynamics but control inputs that are constant between sample times. If the dynamics of a sampled-data system does not change between sample times, then the mean and covariance are described by difference equations, although the factors of the difference equations are more complicated than they are for discrete-time systems. For continuous-time systems, the mean and covariance are described by differential equations. These results will form part of the foundation for our Kalman filter derivation in Chapter 5.

The covariance equations that we studied in this chapter are named after Aleksandr Lyapunov, James Sylvester, and Philip Stein. Lyapunov was a Russian mathematician who lived from 1857 to 1918. He made important contributions in the areas of differential equations, system stability, and probability. Sylvester was an English mathematician and lawyer who lived from 1814 to 1897. He worked for a time in the United States as a professor at the University of Virginia and Johns Hopkins University. While at Johns Hopkins, he founded the *American Journal of Mathematics*, which was the first mathematical journal in the United States.

PROBLEMS

Written exercises

4.1 Prove that

$$\frac{d}{dt}\left(E[x]\right) = E\left[\frac{dx}{dt}\right]$$

4.2 Suppose that a dynamic scalar system is given as $x_{k+1} = fx_k + w_k$, where w_k is zero-mean white noise with variance q. Show that if the variance of x_k is σ^2 for all k, then it must be true that $f^2 = (\sigma^2 - q)/\sigma^2$.

4.3 Consider the system

$$\begin{aligned} x_k &= \begin{bmatrix} 1 & 1 \\ 0 & 1/2 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_{k-1} \\ w_k &\sim (0,1) \end{aligned}$$

where w_k is white noise.

a) Find all possible steady-state values of the mean of x_k.

b) Find all possible steady-state values of the covariance of x_k.

4.4 Consider the system of Example 1.2.

a) Discretize the system to find the single step state transition matrix F_k, the discrete-time input matrix G_k, and the multiple-step state transition matrix $F_{k,i}$.

b) Suppose the covariance of the initial state is $P_0 = \text{diag}(1,0)$, and zero-mean discrete-time white noise with a covariance of $Q = \text{diag}(1,0)$ is input to the discrete-time system. Find a closed-form solution for P_k.

4.5 Two chemical mixtures are poured into a tank. One has concentration c_1 and is poured at rate F_1, and the other has concentration c_2 and is poured at rate F_2. The tank has volume V, and its outflow is at concentration c and rate F. This is typical of many process control systems [Kwa72]. The linearized equation for this system can be written as

$$\dot{x} = \begin{bmatrix} -\frac{F_0}{2V_0} & 0 \\ 0 & -\frac{F_0}{V_0} \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ \frac{c_1 - c_0}{V_0} & \frac{c_2 - c_0}{V_0} \end{bmatrix} w$$

where F_0, V_0, and c_0 are the linearization points of F, V, and c. The state x consists of deviations from the steady-state values of V and c, and the noise input w consists of the deviations from the steady-state values of F_1 and F_2. Suppose that $F_0 = 2V_0$, $c_1 - c_0 = V_0$, and $c_2 - c_0 = 2V_0$. Suppose the noise input w has an identity covariance matrix.

a) Use Equation (4.27) to calculate Q_{k-1}.
b) Use Equation (4.28) to approximate Q_{k-1}.
c) Evaluate your answer to part (a) for small $(t_k - t_{k-1})$ to verify that it matches your answer to part (b).

4.6 Suppose that a certain sampled data system has the following state-transition matrix and approximate Q_{k-1} matrix [as calculated by Equation (4.28)]:

$$\begin{aligned} F_{k-1} &= \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \\ Q_{k-1} &= \begin{bmatrix} 2T & 3T \\ 3T & 5T \end{bmatrix} \end{aligned}$$

where $T = t_k - t_{k-1}$ is the discretization step size. Use Equation (4.26) to compute the steady-state covariance of the state as a function of T.

4.7 Consider the tank system described in Problem 4.5. Find closed-form solutions for the elements of the state covariance as functions of time.

4.8 Consider the system

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x_k + w_k \\ w_k &\sim (0, Q) \\ Q &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Use Equation (4.5) to find the steady-state covariance of the state vector.

4.9 The third condition of Theorem 21 gives a sufficient condition for the discrete-time Lyapunov equation to have a unique, symmetric, positive semidefinite solution. Since the condition is sufficient but not necessary, there may be cases that do not meet the criteria of the third condition that still have a unique, symmetric, positive semidefinite solution. Give an example of one such case with a nonzero solution.

4.10 Prove the sifting property of the continuous-time impulse function $\delta(t)$, which can be stated as

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)\,dt = f(\alpha)$$

Computer exercises

4.11 Write code for the propagation of the mean and variance of the state of Example 4.2. Use $m_0 = 1$, $P_0 = 2$, $f = -0.5$ and $q_c = 1$. Plot the mean and variance of x for 5 seconds. Repeat for $P_0 = 0$. Based on the plots, what does the steady-state value of the variance appear to be? What is the analytically determined steady-state value of the variance?

4.12 Consider the RLC circuit of Example 1.8 with $R = L = C = 1$. Suppose the applied voltage is continuous-time zero-mean white noise with a variance of 1. The initial capacitor voltage is a random variable with a mean of 1 and a variance of 1. The initial inductor current is a random variable (independent of the initial capacitor voltage) with a mean of 2 and a variance of 2. Write a program to propagate the mean and covariance of the state for five seconds. Plot the two elements of the mean of the state, and the three unique elements of the covariance. Based on the plots, what does the steady-state value of the covariance appear to be? What is the analytically determined steady-state value of the covariance? (Hint: The MATLAB function LYAP can be used to solve for the continuous-time algebraic Lyapunov equation.)

4.13 Consider the RLC circuit of Problem 1.18 with $R = 3$, $L = 1$, and $C = 0.5$. Suppose the applied voltage is continuous-time zero-mean white noise with a variance of 1. We can find the steady-state covariance of the state a couple of different ways.

- Use Equation (4.49).
- Discretize the system and use Equation (4.4) along with the MATLAB function DLYAP. In this case, the discrete-time white noise covariance Q is related to the continuous-time white noise covariance Q_c by the equation $Q = TQ_c$, where T is the discretization step size (see Section 8.1.1).

a) Analytically compute the continuous-time, steady-state covariance of the state.
b) Analytically compute the discretized steady-state covariance of the state in the limit as $T \to \infty$.
c) One way of measuring the distance between two matrices is by using the MATLAB function NORM to take the Frobenius norm of the difference between the matrices. Generate a plot showing the Frobenius norm of the difference between the continuous-time, steady-state covariance of the state, and the discretized steady-state covariance of the state for T between 0.01 and 1.