Robustness and Sensitivity Metrics for Tuning the Extended Kalman Filter

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***Abstract*— In this paper, a robustness metric and a sensitivity metric have been defined, which can be used to determine a suitable combination of the filter tuning parameters of the extended Kalman filter. These metrics are related to the innovation covariance and their derivation necessitates a change of paradigm from the estimated states to the estimated measurements. The characteristics of these metrics have been inferred in detail and these have been used to predict the root-mean-squared error (RMSE) performances in a 2-D falling body problem. To do so, a general method has been proposed in this paper to obtain an initial choice of the filter tuning parameters based on the available literature. The RMSE performances are then obtained for a range of variation of the most critical tuning parameter, namely the filter process noise covariance. In general, the characteristics predicted from the metrics correlate significantly with the RMSE performances, and hence these can be used to obtain the desired tradeoff between robustness and sensitivity in various filter applications.**

***Index Terms*— Estimated measurements, extended Kalman filter (EKF), filter tuning, innovation covariance, robustness metric, sensitivity metric.**

I. Introduction

THE simple, yet most commonly used technique for the state estimation of nonlinear systems is the extended Kalman filter (EKF) [1], in which the covariance matrices consider the effects of the unmodeled dynamics and the measurement noises. However, the use of wrong *a priori* statistics of the noise affecting the process as well as the measurement in the design of the Kalman filter can lead to large (residual) estimation errors or in certain cases, to a divergence of the errors [2]. This limits the applicability of the filter since these statistics are either not available in most of the practical applications or else the accuracy of the available statistics is questionable.

To date, the system covariance matrices, particularly the process noise covariance matrix Q and the measurement noise covariance matrix R, that are used for filter tuning have to be supplied by the designer using some *ad hoc* procedures, thus posing a major challenge [3], [4]. In certain cases, it has been shown that adaptively tuning the filter parameters by altering Q or R suitably leads to improved robustness of the system to parameter variations by restricting the errors within proposed bounds [5], [6]. The method presented in [7] for estimating Q in a typical aerospace application, where R and the particular target tracking scenario are known, uses a difference vector between the theoretical residual covariance and the experimental one to provide a correction in Q using the sensitivity matrix.

In the EKF-based state estimation for both unconstrained and constrained systems, the Kalman filter gain is another critical parameter for the filter tuning [3], [8] and this is computed based on the estimated covariances. The Kalman gain calculation involves the use of the innovation covariance, where innovation is defined as the error between the actual measurement and the estimate of the same. An innovation-based adaptive estimation is used in [9], which introduces the calculated innovation covariance into the computation of the filter gain matrix directly. This innovation adaptive EKF algorithm is used for the real-time in-flight alignment of the position and orientation measurement system with a large initial heading error.

In [2], the maximum likelihood method and two correlation methods, one for obtaining the Kalman gain directly and the other for computing the filter covariance matrices Q or R adaptively, are stated, which also use the innovation covariance. Mehra [2] further relates the error between the actual covariance of the innovations and the theoretical innovation covariance, which arises due to an incorrect choice of a reduced R to an effective compensatory increase in the filter process noise covariance Q. Neto *et al.* [10] refers to this phenomenon as the QR-duality principle and uses it for a filter tuning procedure for tracking the various phases of a rocket.

Another utility of the innovations or the residuals is in ensuring the convergence of the filter theoretically [11]. The normalized innovation squared [11, p. 236], the normalized estimation error squared [11, p. 145] cost functions as well as the goodness of fit notions [11, pp. 164, 167] have often been used by researchers [11] to ensure theoretical convergence. One of the preliminary findings in this context was [12], which stated that if the mean and covariance of the innovations are not as expected, then this shows that the choice of any or all of the system matrices as well as the covariances is/are incorrect.

However, the major limitations of these tuning methods, including the adaptive techniques, are the need for a more complex algorithm [13] to obtain suitable Q or R without being able to relate or establish the robustness or sensitivity of the filter performances for the proposed choices. In this context, it is to be noted that a typical measure that is used as a benchmark for the comparison of the efficiency

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of various suboptimal filters is Cramer–Rao lower bound (CRLB). This provides the minimum variance of estimation error that is achievable by any unbiased filter but since it is necessary to know the true filter parameters in order to obtain these bounds and furthermore, it is computationally very complex, so its applicability to most of the problems is restricted [11].

Thus, there is a need to provide a method, which is offline, for determining *a priori* a possible range of these filter tuning parameters, particularly Q or R, that is supported by some theoretical insights into the expected robustness or sensitivity of the filter performances. This will enable the filter designer to choose one or more sets of these parameters for offline or online use in the particular application with its specific filter performance requirements. In this paper, the problem of selecting some suitable combinations of the filter tuning parameters of an EKF has been considered, which ensures a balanced root-mean-squared error (RMSE) performance in terms of robustness and sensitivity. To do so, two specific metrics have been defined, one for robustness and the other for sensitivity. Their combination can be used to provide an offline predictive method for obtaining the suitable set of filter tuning parameters. Some preliminary findings on this issue have been published in [14].

This paper is organized as follows. Section II contains the details of a target tracking problem, which has been considered to validate the efficacy of the proposed metrics. The shift in paradigm to estimated measurements and the role of innovations in EKF performance have been discussed in Section II-B. Section III contains the derivations of the robustness and sensitivity metrics and their characteristics as well as the interpretations of these metrics. Section IV contains the simulations and results, pertaining to the metrics as well as the obtained RMSE performances. A method for the initial choice of the filter tuning parameters has also been stated in Section IV-A. The conclusion is stated in Section V.

II. Problem Statement

Several researchers have applied various modifications of the KF in the practical applications of the general target tracking problem [15] and specifically for those of ballistic missiles and various maneuvering targets [16]–[18]. The particular problem considered in this paper is that of tracking a 2-D ballistic target and the process model for this the same, as discussed in [15]. However, the measurement model chosen differs from that in [15]. In this paper, the measurements considered are based on the realistic target–missile engagement scenario used by the aerospace community.

A. Truth Generation

1) Process Model: The target engagement scenario, the process model description, and the system parameter values are the same as in [15]. Here, it is assumed that the object enters the atmosphere in reentry phase under the presence of nonlinear air drag as well as gravity. The presence of drag as well as gravity makes the trajectory of the target a nonlinear one. The equivalent discrete-time target motion can be expressed using the state (1) [15] as

TABLE I

Process Parameters for Truth Generation

x_k	F_1
$\begin{bmatrix} x_{1k} \\ \dot{x}_{1k} \\ x_{2k} \\ \dot{x}_{2k} \end{bmatrix}$	$\begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$
G	Q_t
$\begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix}$	$\begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix}$

$$x_{k+1} = f(x_k) + Gu_k + w_k \tag{1}$$

where

$$f(x_k) = F_1 x_k + G f_{kk}(x_k).$$

Here, x_k is the state vector consisting of the positions of the ballistic object in the x and y directions, denoted as x_{1k} and x_{2k}, respectively, and their corresponding velocities. Of the other terms, $f(x_k)$ is a nonlinear state transition relation, $u_k = [0\ (-g)]^T$ is the input matrix with the associated structure matrix G while $w_k = N(0, \sqrt{Q_t})$ is the process noise. w_k is assumed to be a zero-mean white Gaussian noise with the fixed covariance matrix Q_t, which is dependent on T, the time interval between the radar measurements.

The nonlinear state transition $f(x_k)$ comprises of two parts as stated and in that, F_1 is the linear part of the state transition matrix, while $f_{kk}(x_k) = -(1/2\beta)\rho g v \left[\begin{smallmatrix} \dot{x}_{1k} \\ \dot{x}_{2k} \end{smallmatrix}\right]$ is the nonlinear state-dependent function associated with the common structure matrix G. Here, β is the ballistic coefficient, ρ is the air density, g is the acceleration due to gravity, whereas $v = (\dot{x}_{1k}^2 + \dot{x}_{2k}^2)^{1/2}$ is the velocity of the ballistic object. The parameters x_k, F_1, G, and Q_t are listed in Table I.

The initial value of the state vector x_0 is [232 km 2290cos (190°) ms^{-1} 88 km 2290sin (190°) ms^{-1}]^T. The acceleration due to gravity g is assumed to be constant at 9.81 ms^{-2}. Hence, u_k becomes time invariant, so it is henceforth referred to as the input matrix u. The ballistic target trajectory has been generated considering a constant target ballistic coefficient $\beta = 40\,000$ kgm^{-1}s^{-2} while the time interval between the radar measurements is considered to be $T = 2$ s.

A crucial parameter in this problem is the air density function ρ, which is considered to decay exponentially as $\rho = C_1 e^{-C_2 x_{2k}}$ with $C_1 = 1.227$, $C_2 = 1.093 \times 10^{-4}$, when $x_{2k} < 9144$ m; and $C_1 = 1.754$, $C_2 = 1.490 \times 10^{-4}$, when $x_{2k} \geq 9144$ m.

2) Measurement Model: The measurement equation is expressed as

$$y_k = h(x_k) + v_k \tag{2}$$

where $h(x_k)$ is the nonlinear function of the states and the measurement noise $v_k = N(0, \sqrt{R_t})$ is assumed to be a zero-mean white Gaussian noise with its corresponding covariance matrix R_t.

In the practical ballistic object scenario, usually, polar measurements are available and these depend on the choice of the sensor(s) and other practical constraints. Therefore, in this paper, the typical three radar measurements of the range rate $\dot{r}$, the elevation angle θ, and the sightline rate $\dot{\theta}$ have been considered. The corresponding measurement noise covariance matrix R_t is also considered to have values as typically obtained in realistic target–missile engagement scenarios.

The set of instantaneous measurements comprising of the range rate $\dot{r}_{mk}$, the elevation angle θ_{mk}, and the sightline rate $\dot{\theta}_{mk}$ has been generated by adding a white Gaussian noise $N(0, \sqrt{R_t})$ to the theoretical measurements $[\dot{r}_k\ \theta_k\ \dot{\theta}_k]^T$, which are obtained as follows:

$$\begin{bmatrix} \dot{r}_k \\ \theta_k \\ \dot{\theta}_k \end{bmatrix} = \begin{bmatrix} (x_{1k}\dot{x}_{1k} + x_{2k}\dot{x}_{2k}) / \left(\sqrt{x_{1k}^2 + x_{2k}^2}\right) \\ \tan^{-1}(x_{2k}/x_{1k}) \\ (x_{1k}\dot{x}_{2k} - x_{2k}\dot{x}_{1k}) / \left(x_{1k}^2 + x_{2k}^2\right) \end{bmatrix} \tag{3}$$

where

$$R_t = \begin{bmatrix} (15\ \mathrm{ms}^{-1})^2 & 0 & 0 \\ 0 & \left(\frac{0.1^\circ}{57.3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{0.01^\circ}{57.3}\mathrm{s}^{-1}\right)^2 \end{bmatrix}.$$

B. Shift in Filter Paradigm: Estimated Measurements

The typical objective of the EKF is to estimate the state vector x_k from the observations y_k in an optimal sense [8] (p. 145). The filter process and the measurement models have been considered to have a form similar to the corresponding truth model and the notations F_1, G, f_{kk}, and u used are the same as used in the truth. The number of states is denoted by n, whereas m is the number of measurements and p is the number of inputs.

Accordingly, the EKF algorithm as stated in [15] for this particular application is stated hereafter using the notations to be followed in this paper and the additional notions of the innovation q_k and its covariance S_k. These equations sequentially yield the *a priori* state estimate $\hat{x}_k^-$ and its error covariance P_k^-, the innovation q_k and its covariance S_k, the Kalman Gain K_k, and the *a posteriori* state estimate $\hat{x}_k^+$ and its error covariance P_k^+

$$\hat{x}_k^- = F_1\hat{x}_{k-1}^+ + Gf_{kk}\left(\hat{x}_{k-1}^+\right) + Gu \tag{4}$$
$$P_k^- = F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1} \tag{5}$$
$$q_k = y_k - h\left(\hat{x}_k^-\right) \tag{6}$$
$$S_k = H_kP_k^-H_k^T + R_k \tag{7}$$
$$K_k = P_k^-H_k^T\left(H_kP_k^-H_k^T + R_k\right)^{-1} = P_k^-H_k^TS_k^{-1} \tag{8}$$
$$x_k^+ = x_k^- + K_kq_k \tag{9}$$
$$\begin{aligned} P_k^+ &= (I - K_kH_k)P_k^-(I - K_kH_k)^T + K_kR_kK_k^T \\ &= (I - K_kH_k)\left[F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1}\right] \\ &\quad \times (I - K_kH_k)^T + K_kR_kK_k^T \end{aligned} \tag{10}$$

where

$$F_{k-1} = (F_1 + G\hat{f}_{k-1}) \tag{11}$$
$$\hat{f}_{k-1}(i, j) = \left[\frac{\partial f_{kk}(i)}{\partial \hat{x}_{k-1}^+(j)}\right], \quad i = [1:p], \quad j = [1:n] \tag{12}$$
$$H_k(i, j) = \left[\frac{\partial h_k(i)}{\partial \hat{x}_k^-(j)}\right], \quad i = [1:m], \quad j = [1:n]. \tag{13}$$

For this particular application, the *a priori* state update $\hat{x}_k^-$ and the innovation q_k rely on the relevant nonlinear terms and the *a priori* state estimation error covariance depends on the corresponding linearized input matrix. $\hat{f}_{k-1}$ denotes the Jacobian of the nonlinear process function $f_{kk}(x)$ evaluated for the available *a posteriori* estimate of the state vector $\hat{x}_{k-1}^+$ and so F_{k-1}, which is the linearized state transition matrix obtained at each time instant from the system matrix, is dependent on F_1, G and $\hat{f}_{k-1}$, as stated in (11). Similarly, the output matrix H_k in this case is obtained as the Jacobian of the nonlinear measurement function $h(x)$ evaluated for the available *a priori* estimate of the state vector $\hat{x}_k^-$. Thus, as is usual for the general EKF algorithm, in this application also, the linear matrices F_{k-1} and H_k are used in the evaluation of the covariances P_k^-, S_k, and P_k^+ and the Kalman gain K_k.

In accordance with the standard treatment in the existing literature on issues regarding the EKF [8, pp. 123–129], [11, p. 382], the stated algorithm focuses primarily on the estimated states and the errors thereof. From (9), however, it is observed that the correction in the state estimate of the EKF is in terms of the product of the innovation q_k and the Kalman gain K_k and this correction factor plays a major role in ensuring the optimized performance of the EKF. From (8), it is further observed that the Kalman gain K_k is inversely proportional to the innovation covariance S_k. These innovations are obtained in terms of the measured outputs, and hence the innovation covariance S_k depends on the *a priori* estimation error covariance of the measured outputs, namely $H_kP_k^-H_k^T$, and not just simply on the *a priori* estimation error covariance of the states P_k^-.

In this paper, the focus is shifted from the estimated states to the innovations in deriving suitable performance metrics for the EKF. This thus necessitates a shift in paradigm in the filtering approach from the estimated states to the estimated measurements.

III. Robustness and Sensitivity Metrics

The innovation q_k and/or its covariance S_k can thus be expected to provide a deterministic basis for the prediction of the filter tuning parameters. However, the innovations q_k are not suitable for predictions since these are random variables obtained in real time, and so the innovation error covariance S_k, as stated in (7), is considered. With this motivation, the innovation covariance S_k, the Kalman gain K_k, and the other filter covariances are manipulated algebraically to obtain the performance metrics for tuning the EKF.

A. Defining the Metrics

Let two terms A_k and B_k be defined as $A_k = H_kF_{k-1}P_{k-1}^+F_{k-1}^TH_k^T$ and $B_k = H_kQ_{k-1}H_k^T$. Then, pre- and postmultiplying (5) by H_k and H_k^T, respectively, yield

$$\begin{aligned} H_kP_k^-H_k^T &= H_k\left(F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1}\right)H_k^T \\ &= (A_k + B_k). \end{aligned} \tag{14}$$

Thus, the innovation covariance S_k in (7) can be expressed as

$$S_k = H_k\big(F_{k-1}P_{k-1}^{+}F_{k-1}^{T} + Q_{k-1}\big)H_k^T + R_k = (A_k + B_k + R_k). \tag{15}$$

Thereafter, premultiplying K_k in (8) by H_k and using (14) and (15) yields

$$H_kK_k = H_kP_k^{-}H_k^TS_k^{-1} = (A_k + B_k)(A_k + B_k + R_k)^{-1}. \tag{16}$$

Pre- and postmultiplying the *a posteriori* state estimation error covariance P_k^+, as stated in (10), by H_k and H_k^T, respectively, yields

$$\begin{aligned} &H_kP_k^+H_k^T \\ &= (I - H_kK_k)H_kP_k^-H_k^T(I - H_kK_k)^T + H_kK_kR_kK_k^TH_k^T \\ &= A_k + B_k - (A_k + B_k)(A_k + B_k + R_k)^{-1}(A_k + B_k). \end{aligned} \tag{17}$$

Thereafter, the resulting first term on the right is shifted to the left and then both sides of the result are premultiplied with $(A_k + B_k)^{-1}$ to obtain

$$\begin{aligned} &(A_k + B_k)^{-1}\big[H_k\big(P_k^+ - F_{k-1}P_{k-1}^+F_{k-1}^T\big)H_k^T\big] \\ &= (A_k + B_k)^{-1}B_k - (A_k+B_k + R_k)^{-1}(A_k + B_k + R_k - R_k) \\ &= (A_k + B_k)^{-1}B_k + (A_k + B_k + R_k)^{-1}R_k - I_m \end{aligned} \tag{18}$$

where I_m is an identity matrix of size m, which is the number of measurements.

Let $\mathrm{tr}(A)$ denotes the trace of the matrix A. Taking the trace of both sides of (18) and rearranging the terms, the two performance metrics J_{1k} and J_{2k} are defined as

$$\begin{aligned} J_{1k} &= \mathrm{tr}\big((A_k + B_k + R_k)^{-1}R_k\big), \\ J_{2k} &= \mathrm{tr}\big((A_k + B_k)^{-1}B_k\big), \\ \text{and} \quad J_{1k} + J_{2k} &= m - \mathrm{tr}(N_k) \end{aligned} \tag{19}$$

where $N_k = (A_k + B_k)^{-1}[H_k(F_{k-1}P_{k-1}^+F_{k-1}^T - P_k^+)H_k^T]$. It is observed from (19) that the value of $(J_{1k} + J_{2k})$ at any instant k deviates from the number of measurements m due to the contribution of the term $\mathrm{tr}(N_k)$ at that particular instant of time k.

For the evaluation of the overall filter performance, let the performance indices or metrics J_1, J_2 and a controlling parameter for the metrics, n_q, be defined in terms of the total horizon N as

$$J_1 = \frac{1}{N}\sum_{k=1}^{N} J_{1k} \tag{20}$$

$$J_2 = \frac{1}{N}\sum_{k=1}^{N} J_{2k} \tag{21}$$

$$\text{with} \quad n_q = \log\left\{\frac{1}{N}\sum_{k=1}^{N}\mathrm{tr}(B_k)\right\}.$$

B. Interpretation of the Metrics

The term A_k could be interpreted as the *a priori* state estimation error covariance in the measured output while the term B_k signifies the process noise covariance in the measured output. Therefore, the sum $(A_k + B_k)$ forms the state-dependent component of the innovation covariance S_k. Hence, of the three components in S_k, the two causative factors are the measurement noise covariance R_k and the projection of the process noise covariance Q_{k-1} onto the measurement, namely B_k, while the third component A_k, which is the corresponding projection of the *a priori* state estimation covariance $F_{k-1}P_{k-1}^+F_{k-1}^T$ onto the measurement, may be considered as the dependent factor.

Thus, while the performance metric $J_{2k} = \mathrm{tr}((A_k + B_k)^{-1}B_k)$ measures the effect of the process noise covariance Q_{k-1}, on the sum $(A_k + B_k)$; the performance metric $J_{1k} = \mathrm{tr}((A_k + B_k + R_k)^{-1}R_k)$ shows the effect of the other independent parameter, namely the measurement noise covariance R_k, on the total sum $(A_k + B_k + R_k) = S_k$. Thus, any change in J_{2k}, or J_2, is primarily driven by the causative factor B_k while any change in J_{1k}, and hence in J_1, may be primarily attributed to R_k.

Since any mismatch between the assumed process noise covariance Q_k and the covariance of the true process noise is due to the modeling error and/or noise, hence J_2 is termed the robustness metric. On the other hand, any mismatch between the assumed measurement noise covariance R_k and the covariance of the actual measurement noise directly affects the *a posteriori* state estimate. Since this is reflected in the value of J_{1k}, and hence in J_1, so J_1 is termed the sensitivity metric.

C. Characteristics of the Metrics

To obtain the characteristics of the metrics, it is needed to decide on an initial choice of the four filter tuning parameters, namely the initial state estimate x_0; the initial choice of the state estimation error covariance P_0; the process (model) noise covariance Q_k, and the measurement noise covariance R_k. The choices of the elements x_0 and P_0 are usually not very critical unless the initial estimation exceeds acceptable limits [19, pp. 151–155]. It is also known that a proper ratio of QR^{-1} affects the filter performance [10]. Of the two covariances Q_k and R_k, the choice of R_k is also noncritical and is guided by the sensor characteristics. Thus, among all the tuning parameters, the choice of the process noise covariance Q_k is considered to be the most critical [3] and complicated. This is so since all the model uncertainties and inaccuracies as well as the noises affecting the process have to be accounted for in Q_k.

In the view of these and since the independent term B_k appears explicitly in both J_1 and J_2, so changing B_k causes immediate changes in both the metrics J_1 and J_2. Hence, to obtain the characteristics of the robustness and sensitivity metrics, let the values of x_0, F_0, P_0^+, R_k, and a nominal Q_0, henceforth referred to as Q_{nom}, be fixed *a priori*. Thereafter, varying Q_0 suitably causes changes in the metrics J_1 and J_2 for a corresponding change in n_q. Then, the following observations can be made from (19), (20).

All comparisons of matrices in this paper have been stated in terms of the trace.

1) If the value of Q_0 is progressively increased such that B_k is much larger than R_k, then J_1 can be approximated by $1/N \sum_{k=1}^{N} \mathrm{tr}((A_k+B_k)^{-1}R_k)$, which is a very low value. Hence, for very large Q_0, J_1 is close to zero.
   However, when Q_0 is progressively reduced such that B_k is significantly smaller than R_k and A_k, then J_1 approximates $1/N \sum_{k=1}^{N} \mathrm{tr}((A_k+R_k)^{-1}R_k)$. Furthermore, if A_k is small, then J_1 approximates $1/N \sum_{k=1}^{N} \mathrm{tr}(I_m) = m$, but never exceeds it since $A_k = H_k F_{k-1} P_{k-1}^{+} F_{k-1}^{T} H_k^{T}$ is always nonnegative definite.
   Thus, J_1 is a bounded metric with values between zero and m.
2) Using a similar reasoning, it can be inferred that when B_k is significantly smaller than A_k, then J_2 approximates $1/N \sum_{k=1}^{N} \mathrm{tr}((A_k)^{-1}B_k)$ and for very small Q_0, J_2 approaches zero.
   In addition, as the value of Q_0 is progressively increased, then for a small A_k, J_2 approximates $1/N \sum_{k=1}^{N} \mathrm{tr}((B_k)^{-1}B_k) = 1/N \sum_{k=1}^{N} \mathrm{tr}(I_m) = m$. In this case also, the value of J_2 is always less than m because $(A_k + B_k) = (H_k F_{k-1} P_{k-1}^{+} F_{k-1}^{T} H_k^{T} + H_k Q_{k-1} H_k^{T})$ is always positive definite since A_k and B_k are quadratic forms of the nonnegative definite matrix P_{k-1}^{+} and the positive definite matrix Q_{k-1}, respectively. Hence, J_2 is also a bounded metric with values between zero and m.
3) Thus, robust filter RMSE performances may be expected for sufficiently large Q_0, and hence for sufficiently large values of n_q. In such cases, the robustness metric J_2 is expected to reach a steady maximum positive value, denoted as $J_{2\max}$, close to but less than m while the sensitivity metric J_1 is expected to reach a steady minimum positive value, denoted as $J_{1\min}$, close to zero.
   On the other hand, sensitive filter RMSE performances may be expected for sufficiently large R_k and hence, as inferred from the QR duality principle, for sufficiently small B_k. Therefore, these will be obtained when n_q is sufficiently small. In this case, the sensitivity metric J_1 is expected to attain a steady maximum positive value, denoted as $J_{1\max}$, close to but less than m while the robustness metric J_2 reaches a steady minimum positive value, denoted as $J_{2\min}$, close to zero.
4) If the EKF converges for a widely varying range of values of Q_0 about Q_{nom}, then as n_q changes from a small value to a large value, it is expected that the sensitivity metric J_1 decreases from $J_{1\max}$ to $J_{1\min}$, while the change for the robustness metric J_2 is the reverse from $J_{2\min}$ to $J_{2\max}$. As a result, there is a crossover of the two plots of the sensitivity metric J_1 versus n_q and the robustness metric J_2 versus n_q for some particular value of Q_0. This value is henceforth denoted as Q_{comp} and it shows the value of Q_0 for which $J_1 = J_2$ for the particular EKF with the stated choice of the other filter tuning parameters.
   In the cases when A_k attains a small value for increasing k, choices of $Q_0 \leq Q_{\mathrm{comp}}$ will usually yield more sensitive performances with improved steady-state responses since $J_1 > J_2$. Similarly, robust performances can be expected when $J_2 > J_1$, and hence for choices of $Q_0 \geq Q_{\mathrm{comp}}$.
5) The stability of the filter estimates depends on the accuracy of the filter model and also the actual disturbances in the system. These determine the actual values of P_k^{+} and hence A_k. It is observed that in most of the cases when A_k is small, P_k^{+} is also small. In cases when A_k is not small, filter performances will not be as desired. Then, filter tuning parameter choices for increased robustness might lead to larger steady-state errors while choices for increased sensitivity may lead to divergence of the filter estimates, particularly in response to some large sudden disturbances in the system.

IV. Simulation Results and Discussion

A. Initial Choice of Filter Parameters

In this paper, a general method is proposed for the initial choice of the filter tuning parameters, namely x_0, F_0, P_0^{+}, R_k, and a nominal Q_{nom} based on available literature. Thereafter, the use of the robustness and sensitivity metrics provides a tool to obtain a suitable choice of Q_0 for this combination.

The measurement noise covariance of the filter R is assumed to be known in terms of the information available from the sensors. Therefore, R is considered to be identical to the time-invariant R_t.

Of the other tuning parameters, the first two elements chosen are the initial state vector $\hat{x}_0 = \hat{x}_0^{+}$ and the initial choice of the state error covariance P_0^{+}. These are obtained using the two-point differencing method, as stated in [11, p. 247] and [15]. Then, to generalize the method of obtaining the filter tuning parameters, the adaptive initial state error covariance matrix $P_0 = P_{\mathrm{ad}}$ is calculated by setting Q_k identically equal to zero in an offline EKF for N number of measurement points in the horizon using the formulation, as stated in [4], as

$$P_{\mathrm{ad}} = \left[\frac{1}{N} \sum_{k=1}^{N} F_{k-1}^{T} H_k^{T} R^{-1} H_k F_{k-1} \right]^{-1} \tag{22}$$

where all the terms have their usual significance. This adaptive P_{ad} is then used as the initial state error covariance P_0^{+} in the simulations.

The third element to be chosen is the critical tuning parameter Q_k. As mentioned earlier, a nominal value of Q_0, namely Q_{nom}, has to be chosen. Using the adaptive $P_0^{+} = P_{\mathrm{ad}}$, Q_{nom} is chosen adaptively as Q_{ad} [4] based on the claim that it guarantees the convergence of the filter. For this, an initial guess of $Q = Q_{\mathrm{guess}}$ is required. In this problem, it is chosen to be Q_t, as stated in Table I. Then, Q_{nom} is obtained as

$$Q_{\mathrm{nom}} = Q_{\mathrm{ad}} = \frac{1}{N} \sum_{k=1}^{N} \left[Q_{\mathrm{guess}} - (P_k^{+} - P_k^{-}) \right]. \tag{23}$$

Since this formulation is based on the time average of $(P_k^{+} - P_k^{-})$ for $k = [1, N]$, hence $Q_k = Q_0$ is considered as a time-invariant filter process noise covariance matrix Q

TABLE II
FILTER TUNING PARAMETERS $\hat{x}_0^+$ AND P_0^+

$\hat{x}_0^+$	P_0^+
$10^5 \times \begin{bmatrix} 2.336 \\ -0.022 \\ 0.889 \\ -0.004 \end{bmatrix}$	$10^3 \times \begin{bmatrix} 99.39 & -0.56 & -22.38 & -1.73 \\ -0.56 & 0.23 & -0.01 & 0.04 \\ 22.38 & -0.01 & 9.15 & -0.58 \\ -1.73 & 0.04 & -0.58 & 0.09 \end{bmatrix}$

TABLE III
FILTER TUNING PARAMETERS Q_{nom} AND R

Q_{nom}	R
$10^2 \times \begin{bmatrix} 20.80 & 0.06 & -0.96 & -0.40 \\ 0.06 & 0.06 & -0.12 & 0.01 \\ -0.96 & -0.12 & 6.00 & 0.13 \\ -0.40 & 0.01 & 0.13 & 0.04 \end{bmatrix}$	$\begin{bmatrix} 15^2 & 0 & 0 \\ 0 & (\frac{0.1}{57.3})^2 & 0 \\ 0 & 0 & (\frac{0.01}{57.3})^2 \end{bmatrix}$

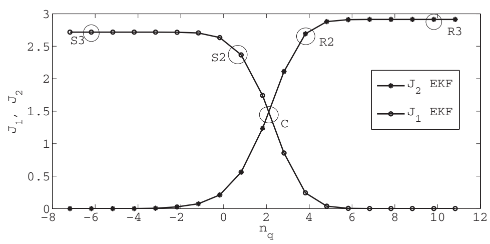

Fig. 1. Plots of J_1 and J_2 for different values of n_q.

in the present case. The values of $\hat{x}_0^+$ and P_0 are stated in Table II, whereas Q_{nom} and R are stated in Table III.

Thereafter, to obtain the changes in the robustness and sensitivity metrics defined in Section III-A and characterized in Section III-C, the value of $Q_0 = 10^p(Q_{\text{nom}})$ is varied continuously by varying the value of the integer p suitably. It may be observed that the method proposed for obtaining the initial choices of the filter tuning parameters is not binding on any designer. Neither does the efficacy of the metrics require the time invariance of the matrices Q_k and/or R_k. It is not even necessary to vary Q_0 in the manner specified or for p to be an integer. Instead any working choice of the filter parameters $\{x_0^+, P_0, Q_{\text{nom}}, R_0\}$ can be used by an engineer, and thereafter, any working method to vary Q_0 in a continuous manner can be adopted to obtain the J_1, J_2 versus n_q plots. However, in case no alternative choice exists or in absence of any other constraints, the method stated in this paper is simple, yet effective for obtaining a feasible initial choice of the filter tuning parameters.

B. Results

This particular application considers a Cartesian–polar model with the typical radar measurements of the range rate $\dot{r}$, the elevation angle θ, and the sightline rate $\dot{\theta}$. Using the nominal filter tuning parameters and thereafter varying the process noise covariance by changing the value of p systematically in the relation $Q = 10^p(Q_{\text{nom}})$, the sensitivity and robustness metrics J_1 and J_2 have been calculated offline. The plot of the metrics J_1 and J_2 for changing n_q has been shown in Fig. 1. The derived characteristics of the metrics listed in Section III-C have been validated for this problem and then these have been used to predict the natures of the filter performances, particularly in terms of their robustness or sensitivity.

1) Both J_1 and J_2 are bounded by the number of measurements $m = 3$ in the upper limit and zero in the lower limit for significantly high and low values of n_q. As n_q increases, the value of J_1 changes from $J_{1\max} = 2.72$ to zero while J_2 changes from zero to $J_{2\max} = 2.91$. The effectiveness of the choice of the EKF for obtaining reasonably robust and sensitive performances in this problem can be predicted by the closeness of $J_{1\max}$ and $J_{2\max}$, respectively, to the number of measurements $m = 3$.
2) The particular value of $Q_{\text{comp}} = 1.49$ in this problem and is marked as point C at $n_q = 2.11$. Thus, as predicted from the characteristics of the metrics, the points marked as $R2$ and $R3$ may be expected to yield RMSE performances in order of increasing robustness while the points marked $S2$ and $S3$ may be expected to yield progressively more sensitive performances in comparison with the point marked C.
3) It is expected that for the values of $n_q < (-2.18)$ where $J_1 = J_{1\max} = 2.72$, sensitive RMSE performances are obtained identical to that for $S3$. Similarly, robust performances identical to $R3$ are obtained for $n_q > 5.82$ when $J_2 = J_{2\max} = 2.91$. As also expected, $J_{1\min} = J_{2\min} = 0$ for high and low values of n_q, respectively.

To validate these predictions, the RMSE performances of the EKF have been obtained for 1000 Monte Carlo simulations using the different values of Q at the points $R2$, $R3$, C, $S2$, and $S3$ of Fig. 1. These RMSE plots as well as the theoretical best estimates as obtained from the CRLB are shown in Fig. 2. The predictions obtained using the metrics are correlated with the actual RMSE performances in Fig. 2 as well as the integral of timed square of error (ITSE) values for these choices, as shown in Fig. 5, and the observations are listed hereafter.

1) One particular feature of this problem is the large bias error in both the position estimates. This can be ascribed to the lack of the range measurement. However, for all choices of n_q, it is observed that a corrective action is initiated in the position estimates in midhorizon after which the error reduces to acceptably low values. Analyzing this phenomenon in terms of system observability, it is observed that although the observability matrix shows no loss or gain of rank throughout the horizon, yet the singular values of this matrix, particularly SV2 and SV3, increase significantly in midhorizon as is evident from Fig. 3. Analyzing this further in terms of state and parameter changes in the process, it is observed that due to the inherent nonlinearity in the system characteristics, the rate of change in position y reduces in the midhorizon and this is coupled to a corresponding increase in the air density ρ, as shown in Fig. 4. These changes in ρ and y cause a significant change in the system nonlinearity in this zone, which in turn becomes crucial for the filter performance in this application.

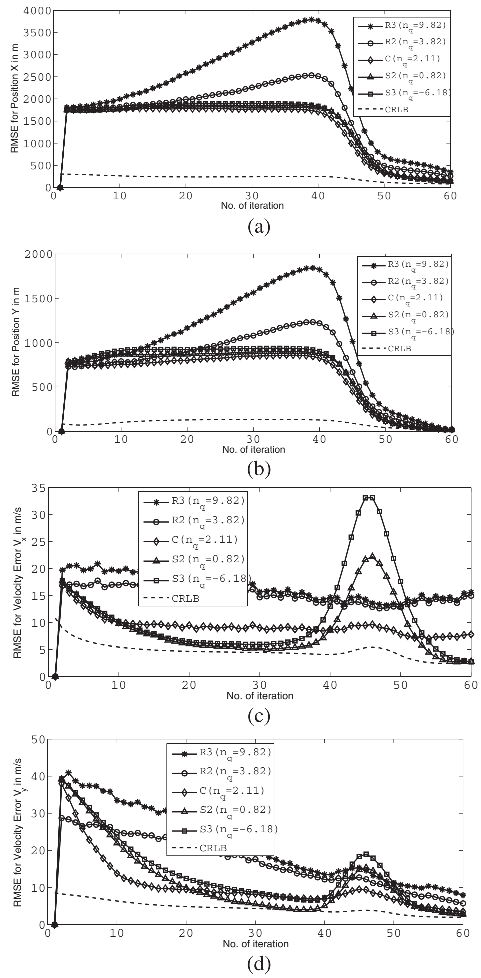

Fig. 2. RMSE plots of the states: a) position x, b) position y, c) velocity x and d) velocity y.

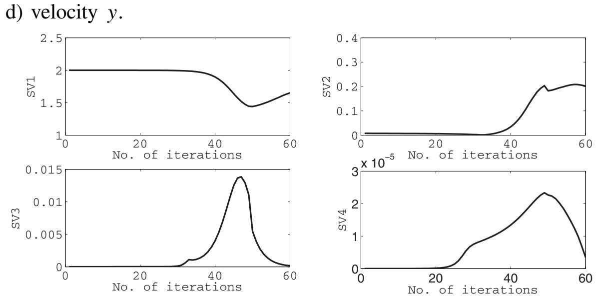

Fig. 3. Plots of singular values of the observability matrix.

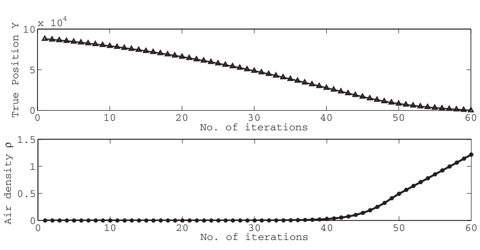

Fig. 4. Plots of true position y and ρ versus time.

2) It is observed in Fig. 2 that owing to the corrective action triggered by the system nonlinearity in midhorizon, the RMSEs of the position estimates are very close to the theoretically best CRLB in the end horizon while the RMSEs of the velocity estimates of one and/or other of the sensitive performances for C, $S2$, and/or $S3$ are close to the CRLB throughout the total time horizon. This validates the effectiveness of the choice of the particular formulation of the EKF, as stated in [15], for this problem.
3) It is further observed from the RMSE performances that as an adverse result of increasing robustness, the errors in all the states increase progressively from C to $R2$ and is maximum for $R3$ throughout the horizon. On the other hand, the benefit of increasing robustness is observed for these responses in terms of the induced correction in the position estimates as well as the inertness of the velocity estimates to the change in system nonlinearity in midhorizon.
4) In contrast, the increased sensitivity of the RMSE performances for C, $S2$, and $S3$ have an initial tendency of progressively reducing the errors for all the states, and hence these estimates have values close to the CRLB in midhorizon, which is particularly evident for the two velocity estimates. However, the adverse effect of increased sensitivity causes the velocity estimate errors to increase sharply from C to $S2$ and is maximum for $S3$ in midhorizon.
5) It has been independently verified that robust RMSE performances identical to $R3$ are obtained for $n_q > 5.82$ while sensitive RMSE performances identical to $S3$ are obtained for $n_q < (-2.18)$ as expected.
6) The tradeoff between sensitivity and robustness of the filter is clearly observed in the RMSE performances as Q is changed for $S3, S2, C, R2$, and $R3$, respectively. This tradeoff is observed quantitatively in the plots of the integral of the ITSEs for these points shown in Fig. 5. As predicted, the ITSE values are the minimum for all the estimated states for $Q = Q_{\text{comp}}$ at C.

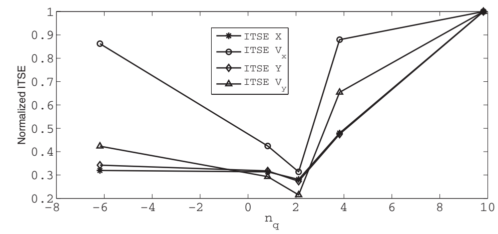

Fig. 5. Plots of ITSE for all states.

Therefore, given a nominal choice of the filter tuning parameters, the offline plot of the metrics can be used to obtain the valid range of Q for which the EKF exhibits sensitive to robust RMSE performances. Furthermore, it can be used to identify the particular value of Q_{comp} for which the best compromise in the RMSE performances in terms of sensitivity and robustness may be expected.

V. Conclusion

In this paper, a sensitivity metric J_1 and another robustness metric J_2 have been derived from the standard EKF algorithm and their various characteristics have been established from mathematical preliminaries. These metrics are useful in determining the balance between robustness and sensitivity of the RMSE performances of the EKF. The relation between these metrics and the innovation covariance as well as the effectiveness of the metrics in predicting the RMSE characteristics have been established. It has been shown that this necessitates the shift of paradigm in filtering from the estimated states to the estimated measurements.

It is known that the effectiveness of the EKF depends significantly on the selection of a suitable combination of the filter tuning parameters. With the available literature, a new general method has been proposed in this paper for obtaining an initial combination of the filter tuning parameters. Thereafter, only the filter process noise covariance Q has been varied to obtain the plots of the performance metrics, which can be done *a priori* and offline. These have been used to obtain the useful range of Q for which the RMSE performances vary from the most sensitive performance to the most robust performance as well the particular value of Q_{comp} for which the EKF can be expected to provide the judicious balance between robustness and sensitivity for the particular application. This offline method for obtaining the valid range of Q and Q_{comp} can be used to suitably adapt the value of Q online.

The application of these metrics to a 2-D falling body problem validates their efficacy in predicting the RMSE characteristics and also in predicting the limits and balance point for Q to obtain sensitive and robust performances in a particular application. The predicted characteristics of the performance metrics have been observed to correlate with the actual RMSE performances as well as the ITSE. Furthermore, it has been shown that the metric plots can be used to identify a suitable working range as well as the particular value Q_{comp} of the filter process noise covariances Q of the EKF for obtaining a desired tradeoff between robustness and sensitivity.

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REFERENCES

[1] R. Kalman, “A new approach to linear filtering and prediction problems,” *J. Basic Eng.*, vol. 82, no. 1, pp. 35–45, 1960.

[2] R. K. Mehra, “Approaches to adaptive filtering,” *IEEE Trans. Autom. Control*, vol. 17, no. 5, pp. 693–698, Oct. 1972.

[3] M. Ananthasayanam, “Kalman filter design by tuning its statistics or gains?” in *Proc. Int. Conf. Data Assimilation*, Jul. 2011, pp. 1–102.

[4] M. Ananthasayanam, A. Sarkar, and S. Vathsal, “Improved adaptive estimation of noise covariance in design of trajectory tracking filters,” in *Proc. AIAA Atmos. Flight Mech. Conf. Exhibit.*, Monterey, CA, USA, Aug. 2002, pp. 5–8.

[5] T. Lee, W. Ra, T. Yoon, and J. Park, “Robust Kalman filtering via Krein space estimation,” *IET Control Theory Appl.*, vol. 151, no. 1, pp. 59–63, 2004.

[6] Z. Ma and A. Ng, “Spacecraft attitude determination by adaptive Kalman filtering,” in *Proc. AIAA Guid., Navigat., Control Conf.*, Monterey, CA, USA, Aug. 2002, no. AIAA-02-4460.

[7] D. V. Stallard, “Process-noise-adaptive Kalman filters for tracking,” in *Proc. Guid. Control Conf.*, vol. 1. Aug. 1982, pp. 57–65.

[8] D. Simon, *Optimal State Estimation: Kalman, H_∞ and Nonlinear Approaches*. New York, NY, USA: Wiley, 2006.

[9] F. Jiangchen and Y. Sheng, “Study on innovation adaptive EKF for inflight alignment of airborne POS,” *IEEE Trans. Instrum. Meas.*, vol. 60, no. 4, pp. 1378–1388, Apr. 2011.

[10] J. Neto, J. Farid, and J. de Abreu, “QR-duality tuning of standard Kalman filter oriented to rocket velocity indirect measurement,” in *Proc. 12th Int. Conf. Comput. Model. Simul.*, 2010, pp. 74–79.

[11] Y. Bar Shalom, X. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. New York, NY, USA: Wiley, 2001.

[12] T. Kailath, “An innovations approach to least squares estimation. Part I: Linear filtering in additive white noise,” *IEEE Trans. Autom. Control*, vol. 13, no. 6, pp. 646–655, Dec. 1968.

[13] V. Bistrovs and A. Kluga, “Adaptive extended Kalman filter for aided inertial navigation system,” *Autom., Robot. Electron. Electr. Eng.*, vol. 6, no. 122, pp. 37–40, 2012.

[14] M. Saha, B. Goswami, and R. Ghosh, “Two novel metrics for determining the tuning parameters of the Kalman filter,” in *Proc. ACODS*, Bangalore, India, 2012, pp. 1–8.

[15] A. Farina, B. Ristic, and D. Benvenuti, “Tracking a ballistic target: Comparison of several nonlinear filters,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 38, no. 3, pp. 854–867, Jul. 2002.

[16] S. Rao, “Pseudo-linear estimator for bearings-only passive target tracking,” *IET Radar Sonar Navigat.*, vol. 148, no. 1, pp. 16–22, 2001.

[17] S. K. Rao, K. S. L. Murthy, and K. R. Rajeshwari, “Data fusion for underwater target tracking,” *IET Radar Sonar Navigat.*, vol. 4, no. 4, pp. 576–585, 2010.

[18] A. Sinha, T. Kirubarajan, and Y. Bar-Shalom, “Application of the Kalman-Levy filter for tracking maneuvering targets,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 3, pp. 1099–1107, Jul. 2007.

[19] P. Zarchan and H. Musoff, *Fundamentals of Kalman Filtering: A Practical Approach*. Washington, DC, USA: American Institute of Aeronautics and Astronautics, 2000.

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