Euler-Lagrange formulation for dynamics of an n-link manipulator

In the Euler-Lagrange dynamics formulation, the dynamics of an n-link manipulator are written as:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad , i = 1, \ldots, n \tag{1}$$

where the Lagrangian $\mathcal{L}$ is defined as $\mathcal{L} = K - P$ with K being the kinetic energy of the system and P being the potential energy of the system. τ_i is the force/torque corresponding to the i^{th} joint of the manipulator.

Given an n-link manipulator, the kinetic energy of the manipulator can be written as:

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} \tag{2}$$

with $D(q)$ defined as

$$D(q) = \sum_{i=1}^{n} \left\{ m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i^0 I_i (R_i^0)^T J_{\omega_i} \right\}. \tag{3}$$

Here, m_i denotes the mass of the i^{th} link, I_i denotes the inertia matrix in the link-fixed frame with its origin at the center of mass of the link, J_{v_i} denotes the velocity Jacobian for the center of mass of link i, and J_{ω_i} denotes the angular velocity Jacobian for link i, i.e., the velocity (written relative to frame 0) of the center of mass of link i is written as $v_i^{(0)} = J_{v_i}(q)\dot{q}$ and the angular velocity (written relative to frame 0) of link i is written as $\omega_i^{(0)} = J_{\omega_i}(q)\dot{q}$. Note that $D(q)$ as defined in equation (3) is a symmetric matrix.

Inertia matrix: Note that since I_i is the inertia matrix written relative to the link-fixed frame, $R_i^0 I_i (R_i^0)^T$ is the inertia matrix written relative to an inertial frame (with the origin of the frame at the center of mass of the link). The inertia matrix is typically a constant matrix when written in the link-fixed frame. The inertia matrix I_i is a 3×3 symmetric matrix whose elements can be found by a volume integration, i.e.,

$$I_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{4}$$

where $I_{xx} = \int\int\int (y^2 + z^2)\rho(x,y,z)dx\,dy\,dz$, $I_{yy} = \int\int\int (x^2 + z^2)\rho(x,y,z)dx\,dy\,dz$, $I_{zz} = \int\int\int (x^2 + y^2)\rho(x,y,z)dx\,dy\,dz$, $I_{xy} = I_{yx} = -\int\int\int xy\rho(x,y,z)dx\,dy\,dz$, $I_{xz} = I_{zx} = -\int\int\int xz\rho(x,y,z)dx\,dy\,dz$, and $I_{yz} = I_{zy} = -\int\int\int yz\rho(x,y,z)dx\,dy\,dz$. $\rho(x,y,z)$ denotes the mass density of the rigid body at the position (x,y,z). The integrals in the expressions for I_{xx}, I_{xy}, etc., are computed over the entire volume of the rigid body.

The potential energy of the n-link manipulator can be written as

$$P = \sum_{i=1}^{n} m_i g^T r_{ci} \tag{5}$$

where g is the acceleration due to gravity (written relative to frame 0) and r_{ci} is the position of the center of mass of link i (again, written relative to frame 0).

If the kinetic energy and potential energy functions that were found as in equations (2) and (5) are algebraically simple, then it is easy to simply substitute $\mathcal{L} = K - P$ into the Euler-Lagrange equation (1) to find the dynamics equations. Alternatively, a more formal procedure is to use the Christoffel symbols defined below.

From the matrix $D(q)$ that was found in equation (3), the Christoffel symbols c_{ijk} are found as:

$$c_{ijk} = \frac{1}{2}\left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \tag{6}$$

where expressions such as d_{ij} denote the $(i,j)^{th}$ element, etc., of the matrix $D(q)$. The Christoffel symbols need to be found for all i,j,k in $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,n\}$, $k \in \{1,\ldots,n\}$. In writing the Christoffel symbols, we can use the property that $c_{ijk} = c_{jik}$ to reduce the number of Christoffel symbols that need to be explicitly calculated by around a half.

From the potential energy (5), define the functions

$$g_k(q) = \frac{\partial P}{\partial q_k} \quad k = 1,\ldots,n. \tag{7}$$

The Euler-Lagrange dynamics equations can be written as:

$$\sum_{i=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k \quad k = 1,\ldots,n. \tag{8}$$

From the Christoffel symbols, define a matrix $C(q, \dot{q})$ to be the $n \times n$ matrix that has its $(k, j)^{th}$ element to be

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i. \tag{9}$$

Then, the Euler-Lagrange dynamics equations from (8) can be written in a matrix form as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{10}$$

where $g(q) = [g_1(q), \ldots, g_n(q)]^T$ and $\tau = [\tau_1, \ldots, \tau_n]^T$.

Some properties of the D and C matrices:

- The $D(q)$ matrix is symmetric and positive-definite.
- The matrix $N(q, \dot{q})$ defined as $N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q})$ is skew symmetric, i.e., $[N(q, \dot{q})]^T = -N(q, \dot{q})$.

Derivation of equation (8): Denoting the $(i, j)^{th}$ element of the matrix $D(q)$ by d_{ij}, the kinetic energy of the manipulator is seen from equation (2) to be of the form

$$K = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j. \tag{11}$$

The potential energy P depends only on q and does not depend on $\dot{q}$. Hence, we see that for any k in $1, \ldots, n$:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j. \tag{12}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} &= \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\left\{\frac{d}{dt}d_{kj}(q)\right\}\dot{q}_j &\quad (13)\\
&= \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j. &\quad (14)
\end{aligned}$$

Also, note that

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}. \tag{15}$$

Hence, the Euler-Lagrange equation (1) can be written as:

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \tag{16}$$

By interchanging the dummy variables of summation, we can write
$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_j\dot{q}_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j$.
Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j. \tag{17}$$

Therefore, from (16), we get

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \tag{18}$$

Hence, from the definition of the Christoffel symbols from (6), we get the dynamics equations shown in equation (8).