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투자론

- R과 Excel을 통한 금융데이터 분석 -

6주차
회귀 분석 및 APT and Multifactor Model

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Unit 02

APT and Multifactor Model

Overview

- Arbitrage Pricing Theory
- Factor Model
- Multi-factor Model

Arbitrage Pricing Theory

- The APT is an approach to determine risk-return tradeoffs based on the law of one price and no arbitrage
- It is a multi-factor model

Arbitrage Pricing Theory

- **The APT is derived from no arbitrage arguments and a statistical model for returns. This contrasts with the CAPM, an equilibrium model**
  - To get the APT, we do not have to assume that everyone is optimizing like the CAPM.
  - This makes the APT a much more “palatable” theory
- **Unlike the CAPM, we need very few assumptions to get the APT**

APT Assumptions

- **Before specifying the APT, it is useful to list the assumptions necessary**
  - All securities follow a factor structure with finite expected values and variances
  - Some investors can trade in well-diversified portfolios
  - There are no taxes & transaction costs
- **That is it. We have considerably fewer assumptions than the CAPM**

APT Assumptions

- What makes the APT work?
- Ultimately, the APT will deliver a multi-factor security market line (SML)

A Factor Model of Returns

- The APT requires us to put some structure on security prices, so we start by assuming a statistical model drives returns. Basically, this is just an extension of the market model that we studied earlier

- Assume that all risky security returns are driven by K factors:

$$r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + \cdots$$

$$= E(r_i) + \sum_{k=1}^{K} \beta_{ik}F_k + \epsilon_i,$$

where, $E(\epsilon_i) = E(F_k) = 0, Cov(\epsilon_i, F_k) = Cov(\epsilon_i, \epsilon_j) = 0$

A Factor Model of Returns

- The F's are factors

- Assets have different sensitivities to each factor through the β's
  - Assets have different sensitivities to each factor through the β's. The β's are commonly called factor loadings

- The residuals (the ϵ's) will be firm-specific.

Interpretation

$$r_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} F_k + \epsilon_i$$

- Think of the factors (the F's) as representing new information about macroeconomic conditions such as
  - Interest rates, industrial production, inflation, volatility, etc
- For each of the factors, $E(F_k)$=0. This simply means that instead of defining F_k directly as economic growth, we define it as the surprise in economic growth over what was expected

Interpretation

$$r_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} F_k + \epsilon_i$$

- Assuming $Cov(\epsilon_i, \epsilon_j) = 0$ implies that ϵ_i captures the non-systematic, idiosyncratic, or residual risk for security i.

Diversified Portfolios

- The key to the APT is that some investor can construct well-diversified portfolios.

- Definition. In the context of the APT, a diversified portfolio is a portfolio that carries no idiosyncratic risk:

$$r_p = E(r_p) + \sum_{k=1}^{K} \beta_{pk} F_k$$

Diversified Portfolios

- So, a diversified portfolio is defined relative to a specific factor model.

- This contrasts with the notion of a diversified portfolio in the context of the CAPM as a diversified portfolio will have the smallest variance for a given level of expected return.

The Underpinnings of the APT - No Arbitrage Pricing

- By assuming that diversified portfolios can be constructed by some investor in the economy, the APT uses simple no arbitrage arguments to derive a multi-factor security market line (SML)
- The key to the APT is that if the number of risky securities in the economy is much larger than the number of systematic factors, then simple no arbitrage restrictions will impose a great deal of structure on expected returns
- With many risky securities, it is possible to construct many different portfolios that generate the same exposure to the systematic factors
- This generates a tight structure on expected returns

Reminder — Arbitrage Opportunity

What is an arbitrage opportunity?

An arbitrage opportunity is a self-financing trading strategy that

1. costs nothing or has a negative price today,
2. never pays a negative cash flow in the future,
3. and pays a strictly positive cash flow in the future with a positive probability if it costs nothing today.

   ⇒ "Money-for-Nothing"

Note: Arbitrage opportunities are riskless [like T-Bonds].

But they don't cost anything today [unlike T-Bonds] ···

Reminder — The Law of One Price

In a frictionless market, the absence of arbitrage opportunities implies that two portfolios with the same set of cash flows must have the same price.

- **This is the key to the APT. Also, all derivative pricing is based on this idea. The general approach to derivative pricing is:**

  1. Determine the cash flows of the derivative security.
  2. Find a portfolio with a known price with the same set of cash flows.

APT — Risk Premiums

- Again, holding portfolios with a large number of assets will diversify away firm-specific risk
- The systematic risk through the common factors cannot be avoided
- So, diversified portfolios will be exposed to the systematic risk from the factors
- By trading the diversified portfolios in a setting with no arbitrage, the risk premium of each factor can be uncovered
- This gives us a multifactor risk-return model.

APT's multiple factor security market line

- By imposing the Law of One Price, the APT's implications is a cross-sectional restriction on expected risky security returns:

$$E(r_i) = r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + \cdots + \beta_{ik}\lambda_K$$

- So, we now have a multifactor model of risk and return or security market line (SML)
  - The λ's are factor risk premia as they tell you how much extra return you receive by holding a portfolio exposed to the factors
  - Portfolios with no risk (for all k ,β_{ik}=0) just return the diskless rate r_f
- The APT SML holds exactly for well-diversified portfolios and approximately for individual stocks with idiosyncratic risk

Single Factor APT Example

- Suppose all risky stock returns are driven by one economic factor which proxies for economic innovation EI:

$r_i = E(r_i) + \beta_i F_{EI} + \epsilon_i$, where $E(F_{EI}) = 0$ which simply means that we have normalized factor

- Now we build two well-diversified portfolios:

One portfolio H loads on high levels of economic innovation, while the other portfolio L loads on low levels of economic innovation

$$r_H = E(r_H) + \beta_H F_{EI}, r_L = E(r_L) + \beta_L F_{EI}$$

Using no arbitrage, we will now arrive at a restriction on expected returns on all risky securities

Single Factor APT Example

Arbitrage Restriction

- Consider a portfolio p of the two well-diversified portfolios with portfolio weights w_H and $1-w_H$:

$$r_p = (w_H E(r_H) + (1 - w_H)E(r_L) + (w_H\beta_H + (1 - w_H)\beta_L)F_{EI}$$

- Suppose we set the weights such that $w_H\beta_H + (1 - w_H)\beta_L = 0$ implying $w_H = -{}^{\beta_L}/_{\beta_H - \beta_L}$. Then,

$$r_p = w_H E(r_H) + (1 - w_H)E(r_L) = -{}^{\beta_L}/_{\beta_H - \beta_L} E(r_H) + \left(1 + {}^{\beta_L}/_{\beta_H - \beta_L}\right) E(r_L)$$
$$= -\frac{\beta_L}{\beta_H - \beta_L}E(r_H) + \frac{\beta_L}{\beta_H - \beta_L}E(r_L)$$

- This portfolio has no exposure to the economic innovation factor, so by no arbitrage it should have the same return, r_f, as the riskless bond.

Single Factor APT Example

Implication

- Imposing no arbitrage leads to a restriction on the expected returns of the H and L portfolios.
- Setting the last line on the previous slide to r_f and rearranging gives:

$$-\frac{\beta_L}{\beta_H-\beta_L}E(r_H)+\frac{\beta_H}{\beta_H-\beta_L}E(r_L)=r_f$$

$$\rightarrow -\beta_L E(r_H)+\beta_H E(r_L)=r_f(\beta_H-\beta_L)$$

$$\rightarrow \beta_H\big(E(r_L)-r_f\big)=\beta_L(E(r_H)-r_f)$$

$$\rightarrow \frac{E(r_L)-r_f}{\beta_L}=\frac{E(r_H)-r_f}{\beta_H}=\lambda_{EI}$$

Single Factor APT Example

Implication

$$-\frac{\beta_L}{\beta_H-\beta_L}E(r_H)+\frac{\beta_H}{\beta_H-\beta_L}E(r_L)=r_f$$

$$\rightarrow -\beta_L E(r_H)+\beta_H E(r_L)=r_f(\beta_H-\beta_L)$$

$$\rightarrow \beta_H\left(E(r_L)-r_f\right)=\beta_L(E(r_H)-r_f)$$

$$\rightarrow \frac{E(r_L)-r_f}{\beta_L}=\frac{E(r_H)-r_f}{\beta_H}=\lambda_{EI}$$

- The last line implies that for every well-diversified portfolio, its risk premium divided by its factor loading equals the same constant λ_{EI}
- Rearranging, we arrive at the SML under the APT $E(r_i)=r_f+\beta_i\lambda_{EI}$ which holds exactly for well-diversified portfolios and approximately for individual securities (we will not be showing this)

Example — Determining the Security Market Line

- The intuition from the previous equation showed that in a single factor model, we only need two well-diversified portfolios to "price" all other assets
- To further shore up the intuition of the APT, consider a setting where all asset returns can be explained by two factors, economic innovation (EI), and monetary policy (MP)
- Now, how many well-diversified portfolios do I need to price all assets?
- In other words, how many well-diversified portfolios do I need to recover SML assuming the APT is true?
  - How many factor risk premia do I need to find the SML?
  - What about the riskless rate?

Multi-Factor Model Equation

Two-Factor Model

- The multifactor APT is similar to the one-factor case:

$$r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + \epsilon_i$$

- Track with diversified factor portfolios:
  - β=1 for one of the factors and 0 for all other factors
  - The factor portfolios track a particular source of macroeconomic risk, but are uncorrelated with other sources of risk

Example — Data

- An analysts gives you the following information on 3 well-diversified funds:

Fund	$E(r_i)$	$\beta_{i,EI}$	$\beta_{i,MP}$
BRI	14%	1.5	0.0
SAC	14%	2.0	-1.0
APL	5%	0.0	0.0

- Assuming that the two factors EI and MP explain asset returns, what is the SML implied by this fund?

- What is the expected return according to the APT on the funds SOR with $\beta_{SOR,EI} = 1$ and $\beta_{SOR,MP}$=0.5?

Derivation of Multi-Factor APT

(w arning: technical)

- One factor APT can be easily extended to multi-factor APT:

$$E(r_i) = r_f + \beta_{i1}\lambda_1 + \cdots + \beta_{iN}\lambda_N$$

- Proof

Idea:

1. Create a well-diversified portfolio with zero cost (arbitrage portfolio)
2. Note that an arbitrage portfolio should have no exposure to systematic risk factors

Multi-Factor APT

Proof

$$r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + \cdots + \beta_{iK}F_K + \epsilon_i$$

- Conditions for an arbitrage portfolio,

$$r_p = \sum_{i=1}^{N} w_i r_i = \sum_{i=1}^{N} w_i E(r_i) + \sum_{j=1}^{K}\sum_{i=1}^{N} w_i \beta_{ij} F_j + \sum_{i=1}^{N} w_i \epsilon_i$$

When the economy is already in equilibrium, creating a portfolio with zero cost permits no changes in wealth, hence we expect zero rate of return. Zero rate of return means the arbitrage portfolio has no exposures to any systematic risk factors. Thus,

1. $\sum_{i=1}^{N} w_i = 0$) zero cost)
2. $\sum_{i=1} w_i\beta_{ij} = 0, \forall j = 1, \cdots, K$) no exposure to risk factors)

Multi-Factor APT

Proof

- Since the arbitrage portfolio is well-diversified, it has no exposure to idiosyncratic risk; $\sum_{i=1}^{N} w_i \epsilon_i \approx 0$.

- Zero expected return implies $r_p = \sum_{i=1}^{N} w_i E(r_i) = 0$.
  If this is violated, then an arbitrage opportunity emerges. Why?

Multi-Factor APT

Proof

- Summary of 3 key conditions for an arbitrage portfolio in the world of multi-factor model:

1. $$\sum_{i=1}^{N} w_i = 0$$

2. $$\sum_{i=1}^{N} w_i \beta_{ij} = 0, \forall j = 1, \cdots, K$$

3. $$r_p = \sum_{i=1}^{N} w_i E(r_i) = 0$$

Multi-Factor APT

Proof

- Multiply an arbitrary constant number λ_0 to the first equation, and multiply $\lambda_j, j = 1, \cdots, K$ for each of the second equation to obtain:

$$\begin{gathered} \lambda_0(w_1 + \cdots + w_N) = 0 \\ \lambda_1(2_1\beta_{11} + \cdots + w_N\beta_{N1}) = 0 \\ \vdots \\ \lambda_K(w_1\beta_{1K} + \cdots + w_N\beta_{NK}) = 0 \end{gathered}$$

- By summing them up, we have

$$w_1\left[\lambda_0 + \sum_{j=1}^{K}\lambda_j\beta_{1j}\right] + w_2\left[\lambda_0 + \sum_{j=1}^{K}\lambda_j\beta_{2j}\right] + \cdots + w_N\left[\lambda_0 + \sum_{j=1}^{K}\lambda_j\beta_{Nj}\right] = 0$$

Multi-Factor APT

Proof

- Now we compare this equation with $r_p = \sum_{i=1}^{N} w_i E(r_i) = 0$.

- For the equality of two equations with arbitrary investment weights $w_i, i = 1, \cdots, N$ (of course, $\sum_{i=1}^{N} w_i = 0$), each coefficients on w_i on both sides must be the same. Thus, we obtain

$$\begin{gathered} E(r_1) = \lambda_0 + \lambda_1\beta_{11} + \cdots + \lambda_K\beta_{1K} \\ E(r_2) = \lambda_0 + \lambda_1\beta_{21} + \cdots + \lambda_K\beta_{2K} \\ \vdots \\ E(r_N) = \lambda_0 + \lambda_1\beta_{N1} + \cdots + \lambda_K\beta_{NK} \\ \Rightarrow E(r_i) = \lambda_0 + \sum_{j=1}^{K} \lambda_j\beta_{ij} \end{gathered}$$

Multi-Factor APT

Proof

- Where there is a riskless asset in the multi−factor APT world, then the riskless asset must have zero sensitivities on all risk factors such that

$$Er_f = \lambda_0 + \sum_{j=1}^{K} \lambda_j \beta_{f,j} = \lambda_0$$

Rewriting the previous APT equation yields

$$E(r_i) = r_f + \lambda_1 \beta_{i1} + \cdots + \lambda_K \beta_{iK}!!!$$

APT and CAPM

CAPM is a special case of APT

- Take an asset l whose sensitivity with respect to j-th factor is 1 $(\beta_{lj} = 1)$ and zero with regard to all other factors. Let it's expected return be δ_j.
  Then, $\delta_j = E(r_l) = r_f + \lambda_j \Rightarrow$ APT implies that the risk premium λ_j is $\delta_j - r_f$.
  Putting this back into the equation to obtain

$$E(r_l) = r_f + \beta_{lj}[\delta_j - r_f]$$

- By extending the same arguments to all factors j=1,⋯,K, we obtain

$$E(r_i) = r_f + \sum_{j=1}^{K} \lambda_j \beta_{ij} = r_f + \sum_{j=1}^{K} [\delta_j - r_f]\beta_{ij}$$

APT and CAPM

CAPM is a special case of APT

- Now we assume that all the securities' returns can be explained by one factor (= market factor). Then,

$$E(r_i) = r_f + [\delta_M - r_f]\beta_{iM}$$

where δ_M is the expected rate of return of an asset whose sensitivity is 1 respect to the market factor and zero with regard to all the other factors. In this market, we do not have any other factors. So, δ_M becomes the expected rate of return of the market portfolio, $E(r_M)$. Also, since β_{iM} is the asset i's sensitivity with respect to the market portfolio, it is the asset i's beta (β_i). Therefore,

$$E(r_i) = r_f + [E(r_M) - r_f]\beta_i. \Leftarrow \text{ CAPM = the special case of APT}$$

APT and CAPM

Summary of Comparison

APT	CAPM
Assumes a well-diversified portfolio, but residual risk is still a factor	Model is based on an inherently unobservable "market" portfolio
Does not assume investors are mean-variance optimizers	Rests on mean-variance efficiency. The actions of many small investors restore CAPM equilibrium.
Uses an observable market index	
Reveals arbitrage opportunities	

Fama-French Three-Factor Model

$$r_{it} - r_f = \alpha_i + \beta_{iM}(r_{Mt} - r_f) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \epsilon_{it}$$

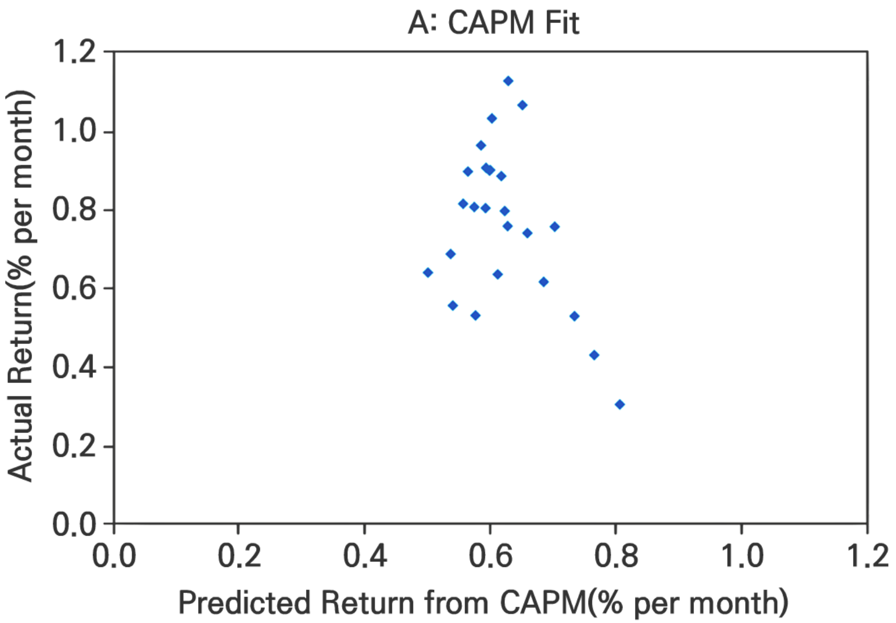
- SMB =
- HML =
- Are these firm characteristics correlated with actual systematic risk factors?

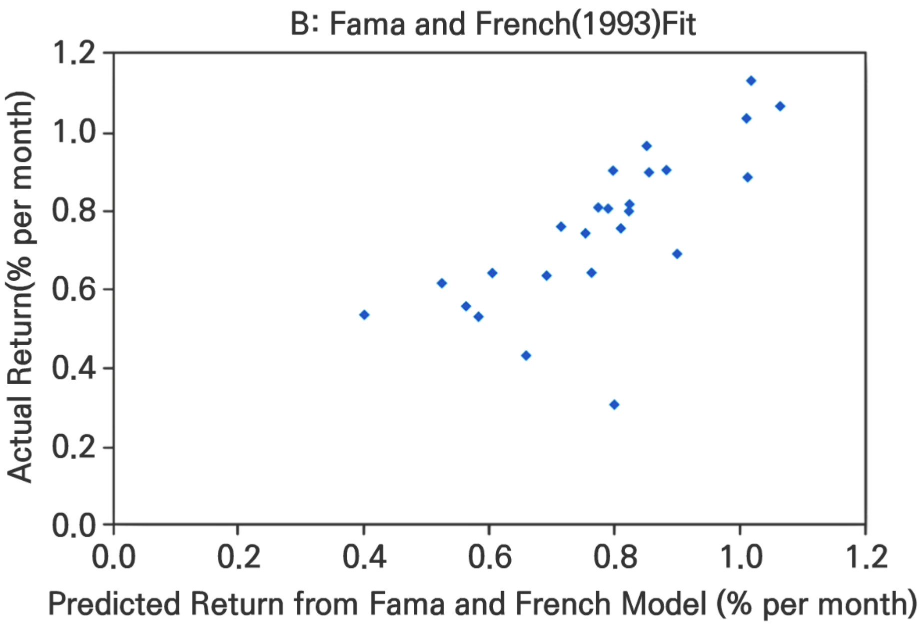
Creation of Fama-French Factors

		Small Cap	Large Cap
		Small	Big
High B/M	Value	Small Value	Big Value
	Neutral	Small Neutral	Large Neutral
Low B/M	Growth	Small Growth	Large Growth

- SMB = (1/3)x(Small Value + Small Neutral + Small Growth)
  − (1/3)x(Big Value + Big Neutral + Big Growth)
- HML = (1/2)x(Small Value + Big Value) − (1/2)x(Small Growth + Big Growth)

Goodness of Fit: CAPM vs FF 3-Factor





출처 : Bodie, Kane, Markus, Investments 10th

Example Problem 1

- Suppose there two independent economic factors, F, and F2. The risk-free rate is 6%, and all stocks have independent firm-specific components with a standard deviation of 45%. Portfolio A and B are both well-diversified with the following properties:

Portfolio	Beta on F_1	Beta on F_2	Expected Return
A	1.5	2.0	31%
B	2.2	-0.2	27%

What is the expected return-beta relationship in this economy?

Example Problem 2

- Suppose that the market can be described by the following three sources of systematic risk with associated risk premiums.

Factor	Risk Premium
Industrial Production *(I)*	6%
Interest rates *(R)*	2
Consumer confidence *(C)*	4

The return on a particular stock is generated according to the following equation:

$$r = 15\% + 1.0I + 0.5R + 0.75C + e$$

Find the equilibrium rate of return on this stock using the APT. The T-bill ate is 6%. Is the stock over- or underpriced? Explain.