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투자론

- R과 Excel을 통한 금융데이터 분석 -

3주차
포트폴리오 이론 1

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Unit 01

Optimal Portfolio 2

Overview

- Optimal Risky Portfolio: Part 2
- Diversification

General Case

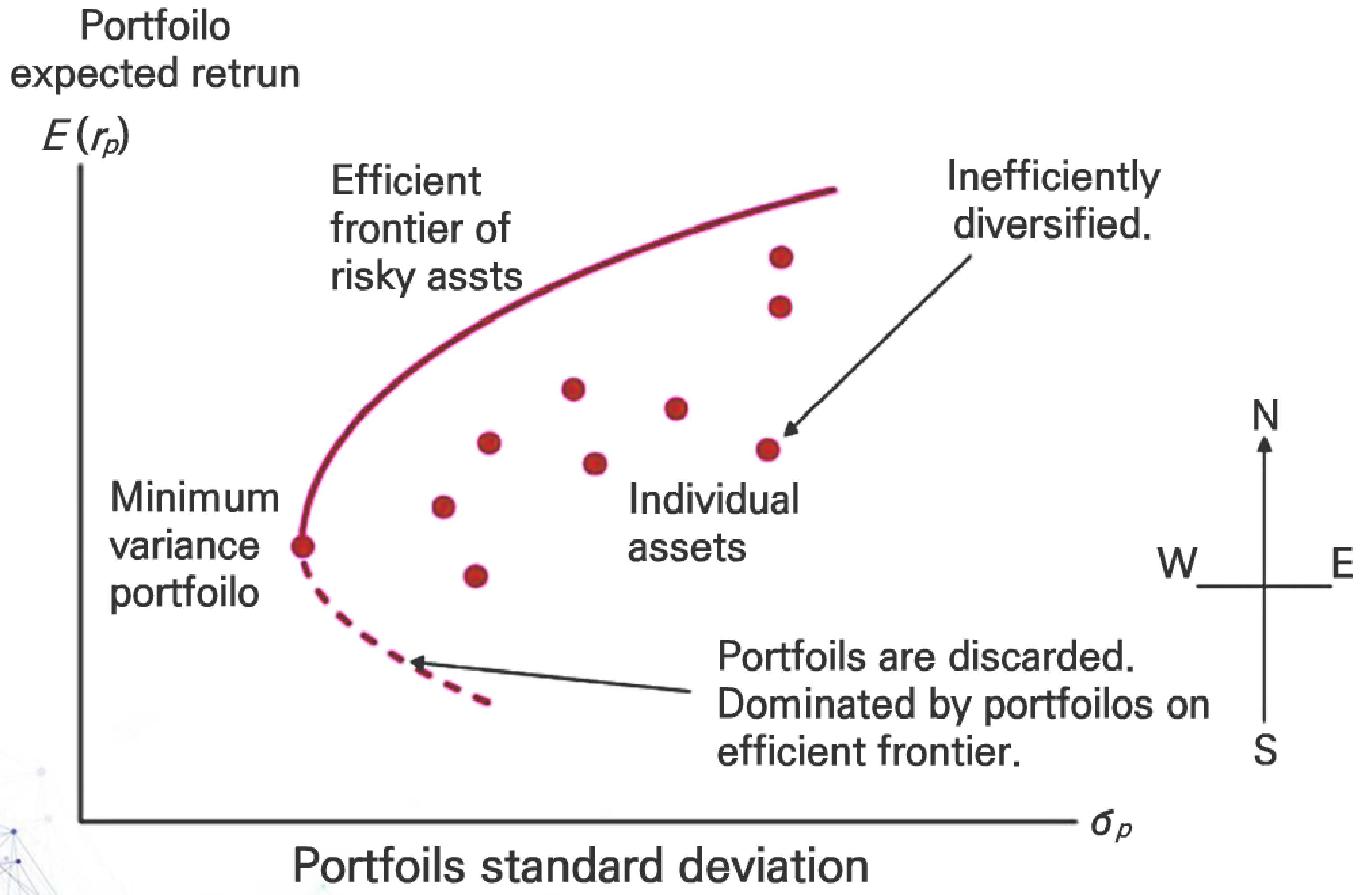
- When there are many assets, the shape of variance critically depend on correlation coefficients among assets.

- The goal of an investor is to maximize expected return of his/her portfolio at the given level of risk (volatility or variance) of the portfolio. This is equivalent to minimize the volatility of the portfolio at the given (or target) expected return of his/her portfolio.

$$\min_{(w_1,\cdots,w_N)} \sigma_p^2 \ \ w.r.t.\ \ E(r_p) = \mu_p, \sum_{i=1}^{N} w_i = 1$$

$$\Leftrightarrow \min_{(w_1,\cdots,w_N)} \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i>j}^{N} w_i w_j \rho_{i,j}\sigma_i\sigma_j.\ w.r.t.\ \ E(r_p) = \mu_p, \sum_{i=1}^{N} w_i = 1$$

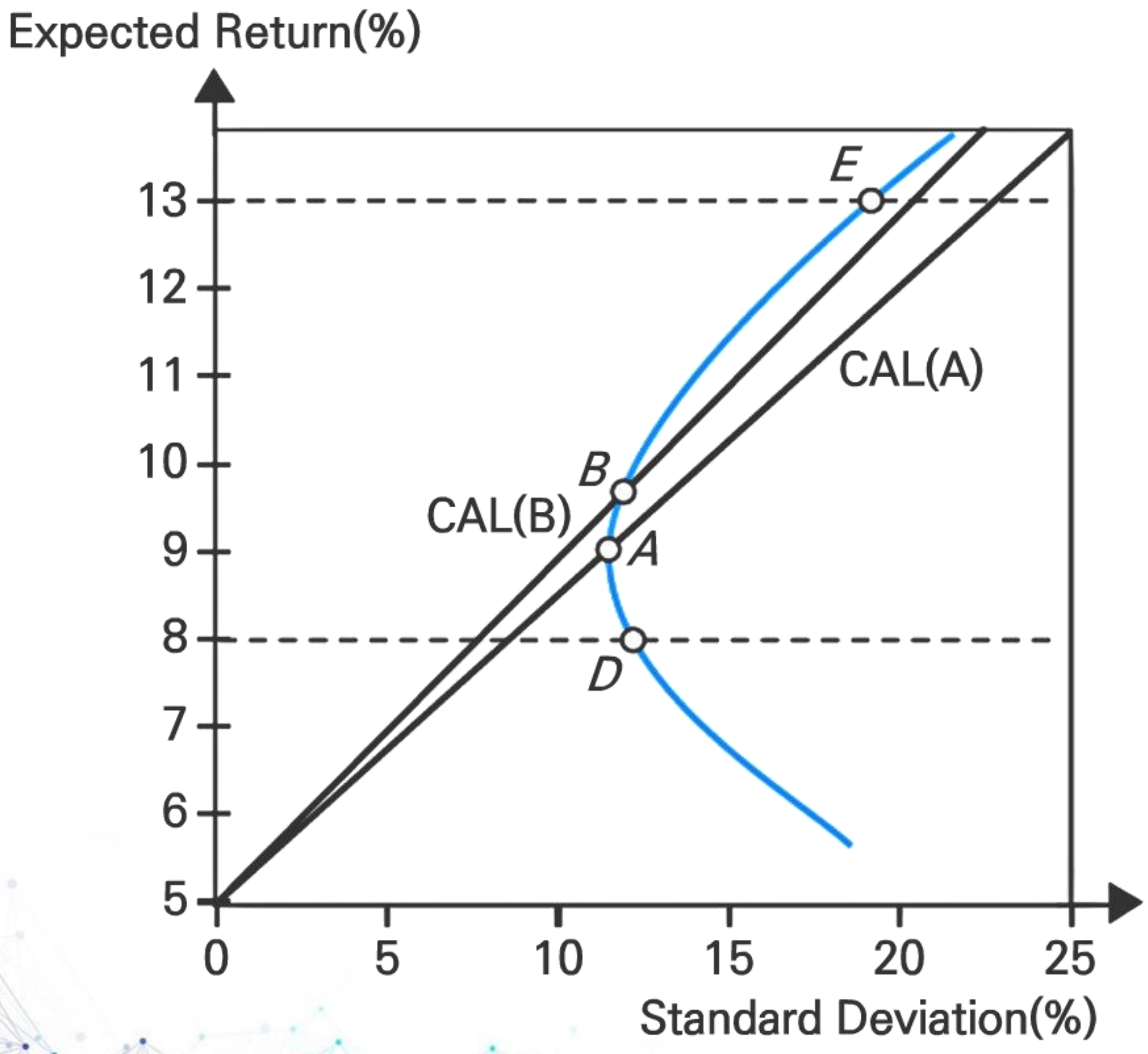
Efficient friontier of risky assets



Two Risky Assets

- Assume that we already solved the problem of minimization of the variance of one's portfolio
- Then, we have a mean–variance frontier with two risky assets
- What is the next? Goal = maximizing the performance of the portfolio, which is equivalent to maximizing the slope of the CAL. (feasibility, indifference curve, and etc.)

Two Risky Assets



Two Risky Assets

$$\max_{w_i's} S_p \left(= \frac{E(r_p) - r_f}{\sigma_p} \right) \quad subject\ \ to \sum_{i=1}^{N} w_i = 1$$

In the case of two risky assets (r_D and r_E), the solution is

$$w_D = \frac{E(r_D - r_f)\sigma_E^2 - E(r_E - r_f)\sigma_{D,E}}{E(r_D - r_f)\sigma_E^2 + E(r_E - r_f)\sigma_D^2 - [E(r_D - r_f) + E(r_E - r_f)]\sigma_{D,E}},$$

$$w_E = 1 - w_D$$

Example

From the previous example

- $E(r_D) = 8\%, E(r_E) = 13\%, \sigma_D = 12\%, \sigma_E = 20\%, \sigma_{D,E} = 72, \rho_{D,E} = 0.3,$

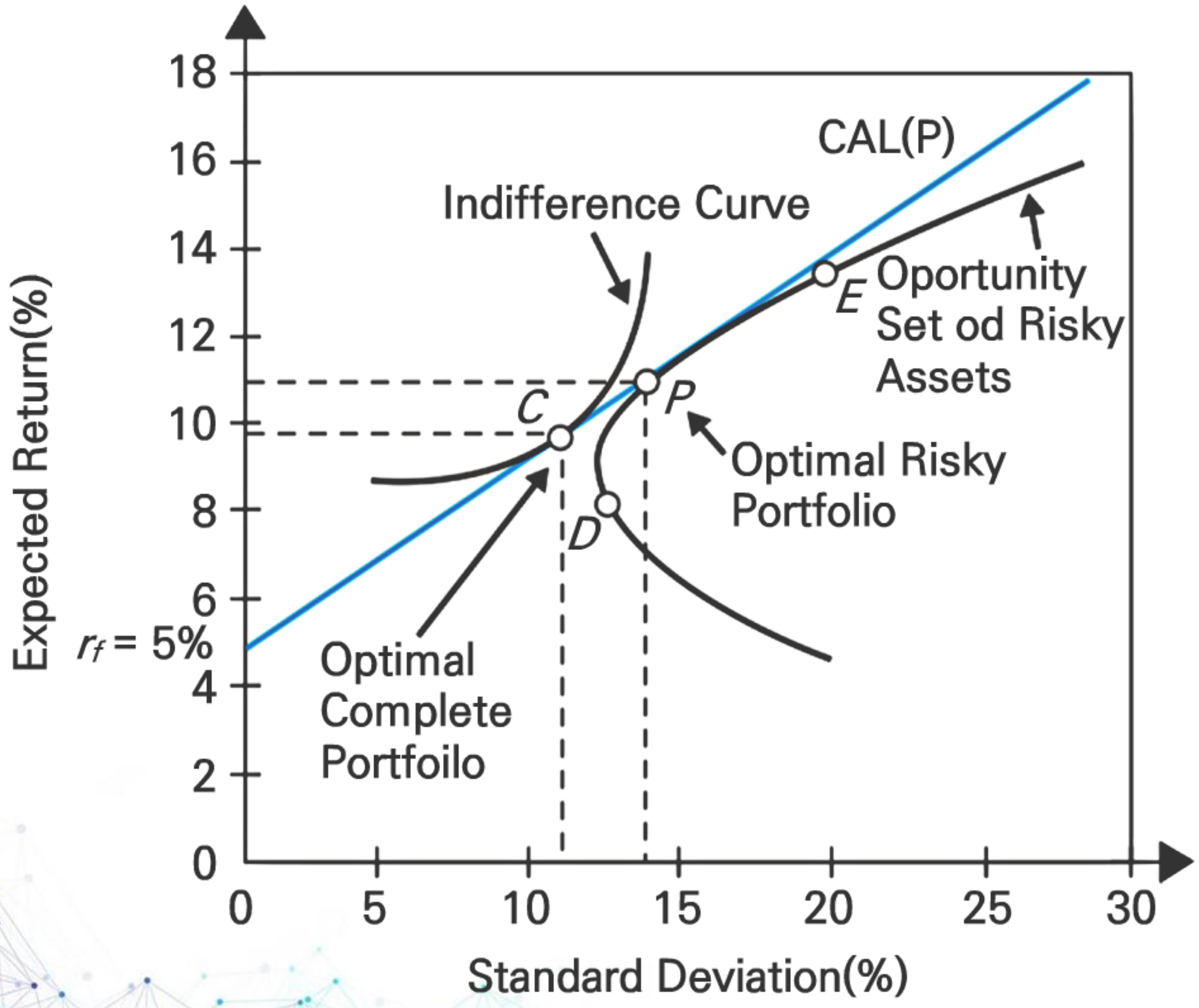
- $w_D = \dfrac{(8-5)400 - (13-5)72}{(8-5)400 + (13-5)144 - (8-5+13-5)72} = 0.4$
- $w_E = 1 - w_D = 1 - 0.4 = 0.6$
- $E(r_p) = (.4 \times 8) + (.6 \times 13) = 11\%$
- $\sigma_p = \sqrt{(.4^2 \times 144) + (.6^2 \times 400) + (2 \times 0.4 \times 0.6 \times 72)} = 14.2\%$
- Sharpe ratio = $S_p = \dfrac{11-5}{14.2} = 0.42$

Example

- An investor with risk aversion parameter A=4 would take a position in portfolio P of

$$y = \frac{E(r_p) - r_f}{A\sigma_p^2} = \frac{0.11 - 0.05}{4 \times 0.142^2} = 0.7439$$

Example


18
16
14
12
10
8
6
4
2
0
Expected Return(%)
r_f = 5%
0
5
10
15
20
25
30
Standard Deviation(%)
CAL(P)
Indifference Curve
E
Oportunity Set od Risky Assets
C
P
Optimal Risky Portfolio
D
Optimal Complete Portfoilo

Markowitz Portfolio Optimization Model

Security selection

- Determine the risk-return opportunities available.
- All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations.

Markowitz Portfolio Optimization Model

- Search for the CAL with the highest reward-to-variability ratio
- Everyone invests in P, regardless of their degree of risk aversion
  - More risk averse investors put less in P
  - Less risk averse investors put more in P

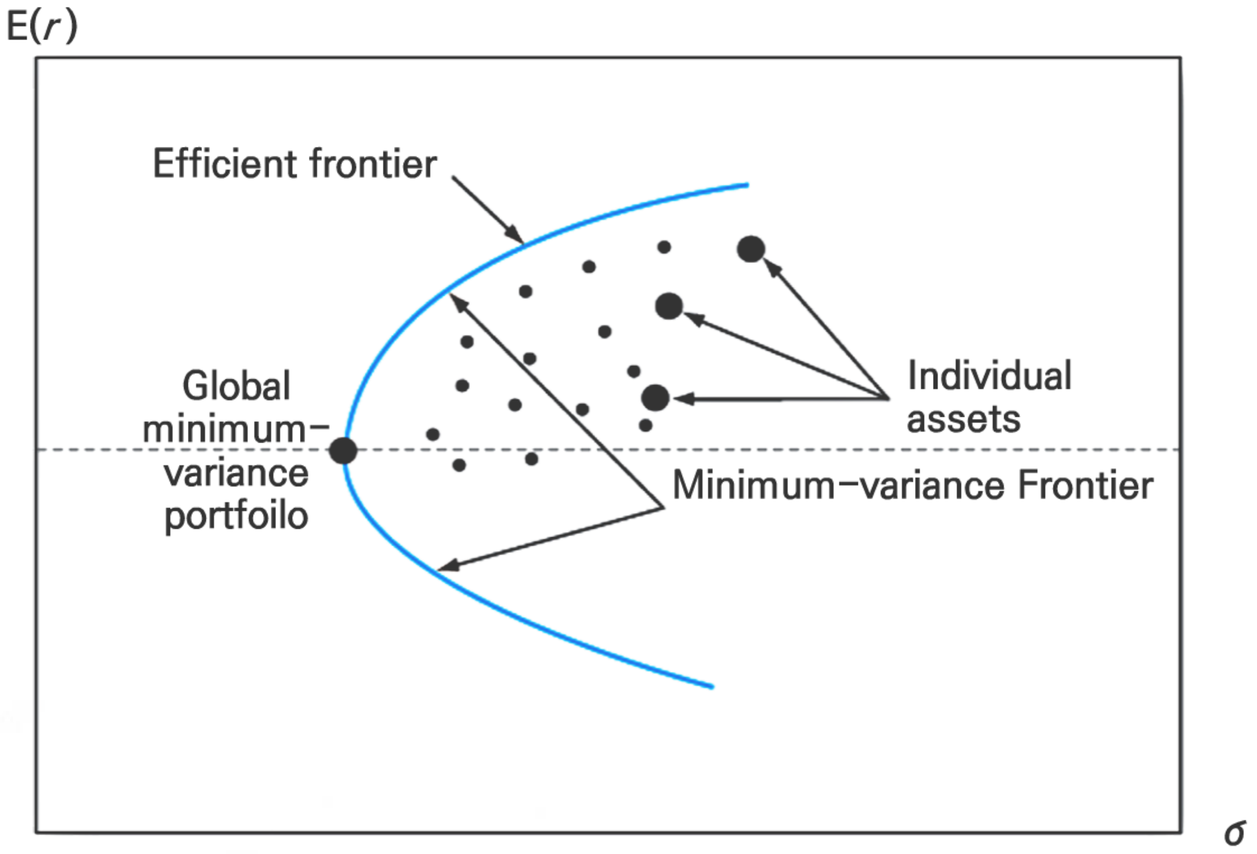
Markowitz Portfolio Optimization Model

Capital Allocation and the Separation Property

- Portfolio choice problem may be separated into two independent tasks
  - Determination of the optimal risky portfolio is purely technical
  - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference (risk attitude)

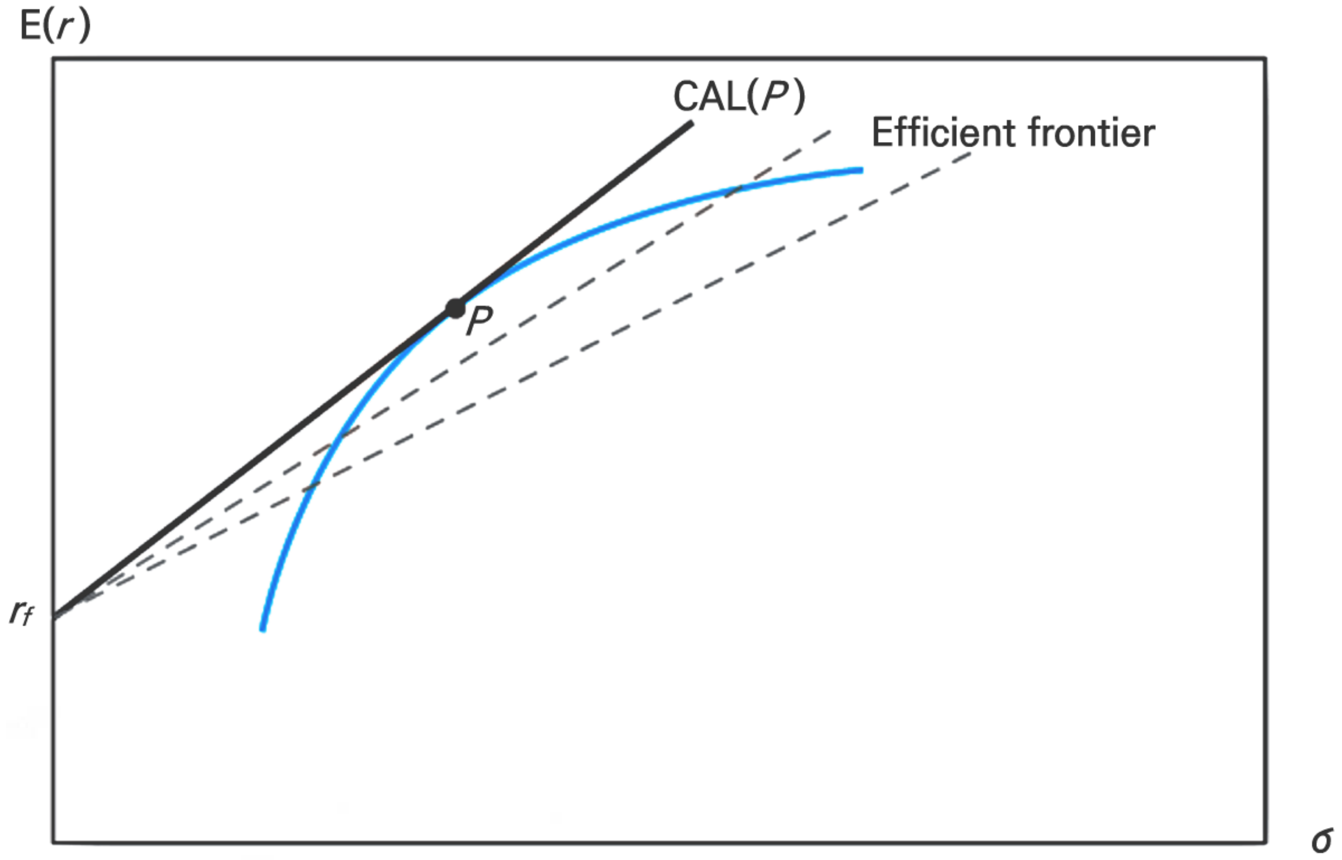
Summary of Mean Variance Method

1. The Minimum-Variance Frontier of Risky Assets



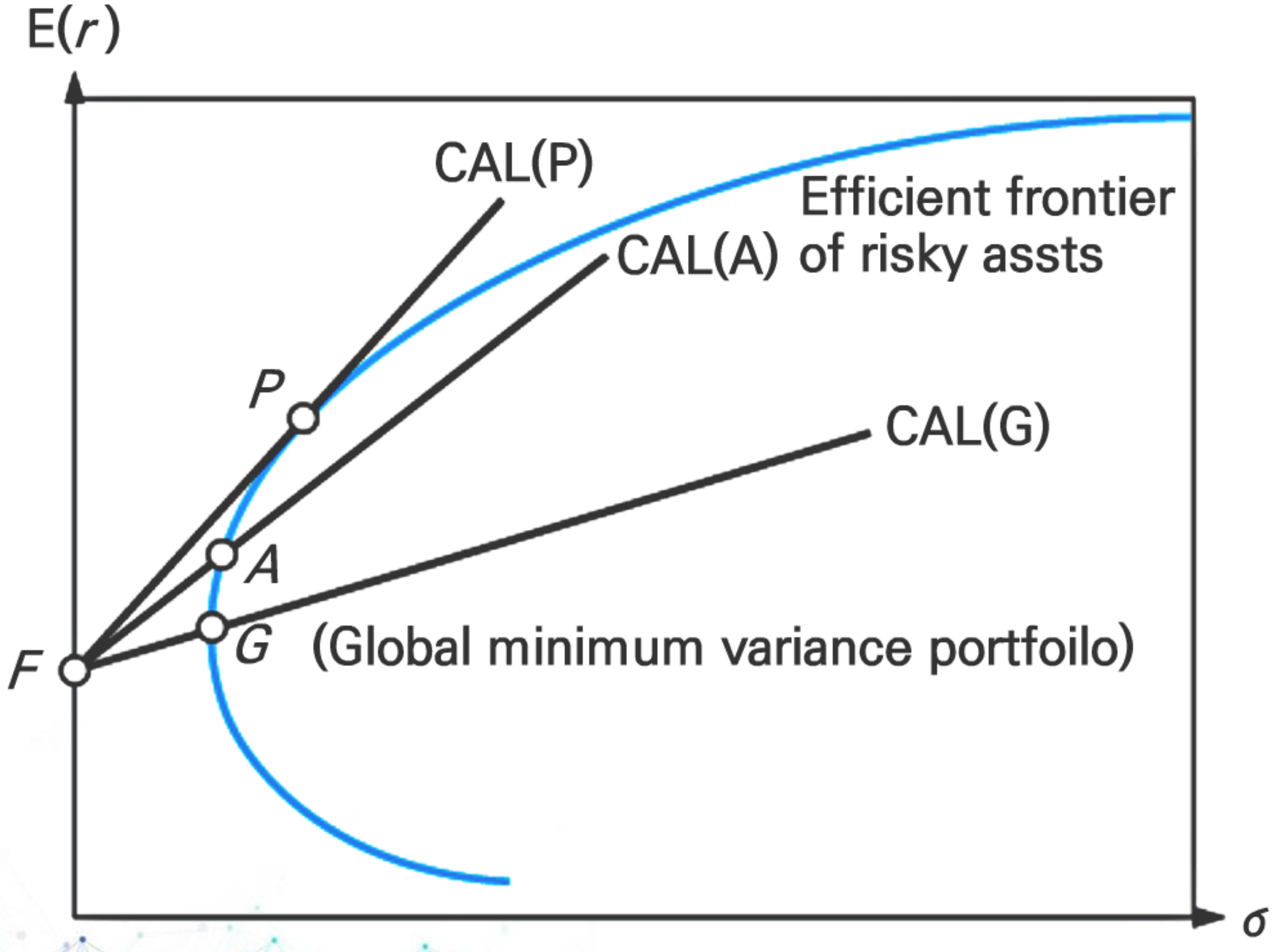
Summary of Mean Variance Method

2. The Efficient Frontier of Risky Assets with the Optimal CAL



Summary of Mean Variance Method

3. Optimal Capital Allocation Line + Utility Maximization



Diversification

The power of diversification

- Remember that we first minimized the variance of the portfolio (at the target rate of expected return)

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j}$$

- We can then express portfolio variance as

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov\left(r_i, r_j\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sigma_i^2 + \sum_{j=1, i\neq j}^{n}\sum_{i=1}^{n}\frac{1}{n^2}Cov\left(r_i, r_j\right),$$

when $w_i = \dfrac{1}{n}$ for each asset

Diversification

The power of diversification

- If we define the average variance and average covariance of the securities as

$$\overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{N}\sigma_i^2, \quad cov = \frac{1}{n(n-1)}\sum_{j=1, j\neq i}^{N}\sum_{i=1}^{N}\sigma_{i,j}, \text{ then, } \sigma_p^2 = \frac{1}{n}\,\overline{\sigma}^2 + \frac{n-1}{n}\,\overline{Cov}$$

- Portfolio variance can be driven to zero if the average covariance is zero
- EX: same variance σ^2, same correlation coefficient ρ → covariance $\rho\sigma^2$

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 \text{ goes to } \rho\sigma^2 \text{ which is the average covariance}$$

The irreducible risk of a diversifiable portfolio depends on the covariance on the covariance of the returns

Risk Reduction of Equally Weighted Portfolios

		$\rho=0$		$\rho=0.40$	
Universe Size n	Portfolio Weights W=1/n(%)	Standard Deviation(%)	Reduction in σ	Standard Deviation(%)	Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.03	0.70
6	16.67	20.41		36.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

Excersise Problem 1

Consider two risky assets A, B, and one riskless asset C. Return distributions for assets are as follows: $E(r_A)$ = 8%, $SD(r_A)$ = 12%, $E(r_B)$ = 13%, $SD(r_B)$ = 20%, r_f = 5%, and the correlation coefficient between two risky assets is $\rho_{A,B}$ = 0.3.

1. Draw efficient investment opportunity set using assets A and B when investment weights, $w_i \geq 0$.

2. What is the tangent portfolio when we use all three assets? Compute the expected return and the standard deviation of the tangent portfolio.

3. Now let your utility function be $U = E(r) - 0.005$ x A x σ_r^2, where A = 5.
   What is your optimal complete portfolio, r_C?

4. Briefly describe portfolio separation theorem based on above questions.

Excersise Problem 2

Consider many stocks that are traded on the market. Assume that the return distributions are the same for all stocks, and their expected returns and standard deviations are given by $E(r) = 15\%$, $SD(r) = 30\%$. Correlation coefficients among all stocks are the same and are given by 0.5.

1. When using an equal weighted portfolio using 25 stocks, compute the expected return and the standard deviation.

2. What is the ultimate systematic risk of the above portfolio?

3. Assume that the correlations are zero among all stocks. What is the ultimate systematic risk of the portfolio?