투자론

- *R*과 Excel을 통한 금융데이터 분석 -

5주차
포트폴리오 이론 3

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Unit 01

Single Index Model

Overview

- Advantages of a single factor model
- Risk decomposition: systematic vs firm-specific
- Single index model and its estimation

Background

Input list of the Markowitz Model

- When analyzing 50 stocks for optimal portfolio choice, inputs are following
  - $n = 50$ estimates of expected returns $E(r_i)$'s
  - $n = 50$ estimates of variances, Var(r_i)'s
  - $(n^2 - n)/2 = (50^2 - 50)/2 = 1{,}225$ estimates of covariances, Cov$(r_i, r_j), i \neq j$

This is a formidable task even with high computing power
And imagine 4,000 stocks, and more…

Background

Input list of the Markowitz Model

– To reduce the number of inputs, we hypothetically decompose asset return into the expected and unexpected as follows

$r_i = E(r_i)$ + unanticipated surprise

– Unexpected component of the stock return can be attributed to 1 purely firm-specific, or 2. unexpected changes of economic conditions that affect the broad economy

- We decompose the sources of return uncertainty into uncertainty about the economy as a whole, captured by a systematic market factor (call it, m), and uncertainty about the firm, which is captured by a firm-specific random variable (call it, e_i)

A Single Factor Model

Advantages

- Reduces the number of inputs for diversification
- Easier for security analysts to specialize

Model

$$r_i = E(r_i) + \beta_i m + e_i$$

β_i = sensitivity, m = market factor, e_i = firm-specific

Single Index Model

We need to specify m— it should be observable and estimable
(we can estimate the mean, the volatility, and individual sensitivity to it)

Regression equation

$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$, $\beta_i \equiv \frac{Cov(R_i, R_M)}{Var(R_M)}$, sensitivity

Expected return−beta relationship

$E(R_i) = \alpha_i + \beta_i E(R_M), since E(e_i) = 0$ (what does this mean?)

We also assume that $Cov(e_i, e_j) = 0$, for all i , j.

Single Index Model

We need to specify m— it should be observable and estimable

(we can estimate the mean, the volatility, and individual sensitivity to it)

- **Variance = Systematic Risk + Firm Specific Risk**

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma(e_i)^2$$

- **Covariance = Product of Betas Market Risk**

$$Cov(r_i, r_j) = Cov(\beta_i m + e_i \ , \beta_j m + e_j) = \beta_i \beta_j \sigma_m^2$$

- **Correlation**

$$Corr(r_i, r_j) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} = \frac{\beta_i \sigma_M^2 \beta_j \sigma_M^2}{\sigma_i \sigma_M \sigma_j \sigma_M}$$

$$= Corr(r_i, r_M) \times Corr(r_j, r_M)$$

Diversification

Variance of the equally-weighted portfolio of firm-specific components

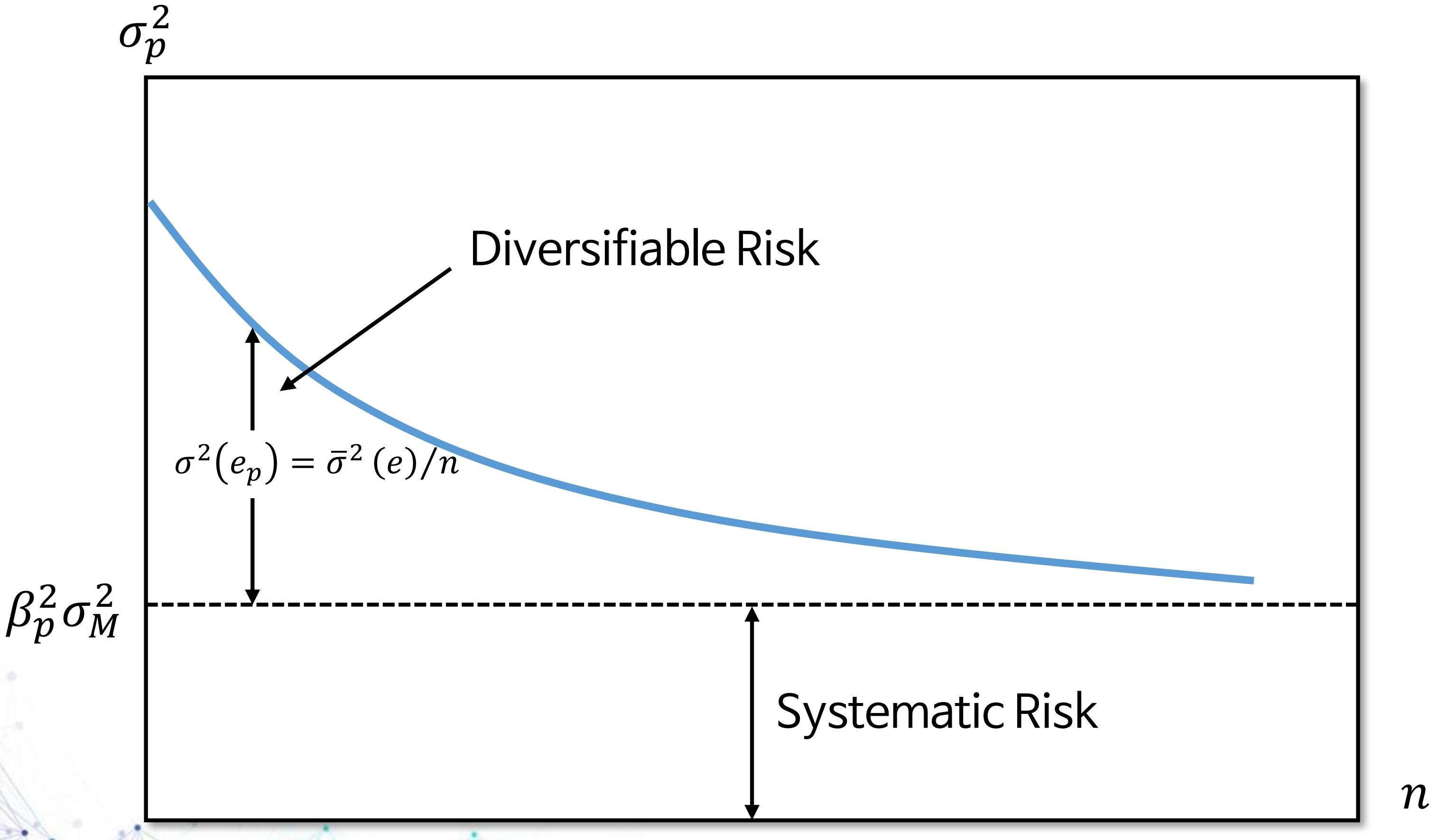
$$r_p = \sum_{\mathrm{i}=1}^{n} w_{\mathrm{i}} r_{\mathrm{i}} = \frac{1}{n} r_1 + \cdots + \frac{1}{n} r_n$$

$$\sigma(e_p)^2 = \sum_{\mathrm{i}=1}^{n} \left(\frac{1}{n}\right)^2 \sigma(e_i)^2 = \frac{1}{n} \bar{\sigma}^2(\mathrm{e})$$

– When gets large, $\sigma^2(e_P) \to 0$:
firm-specific risk gets diversified away!

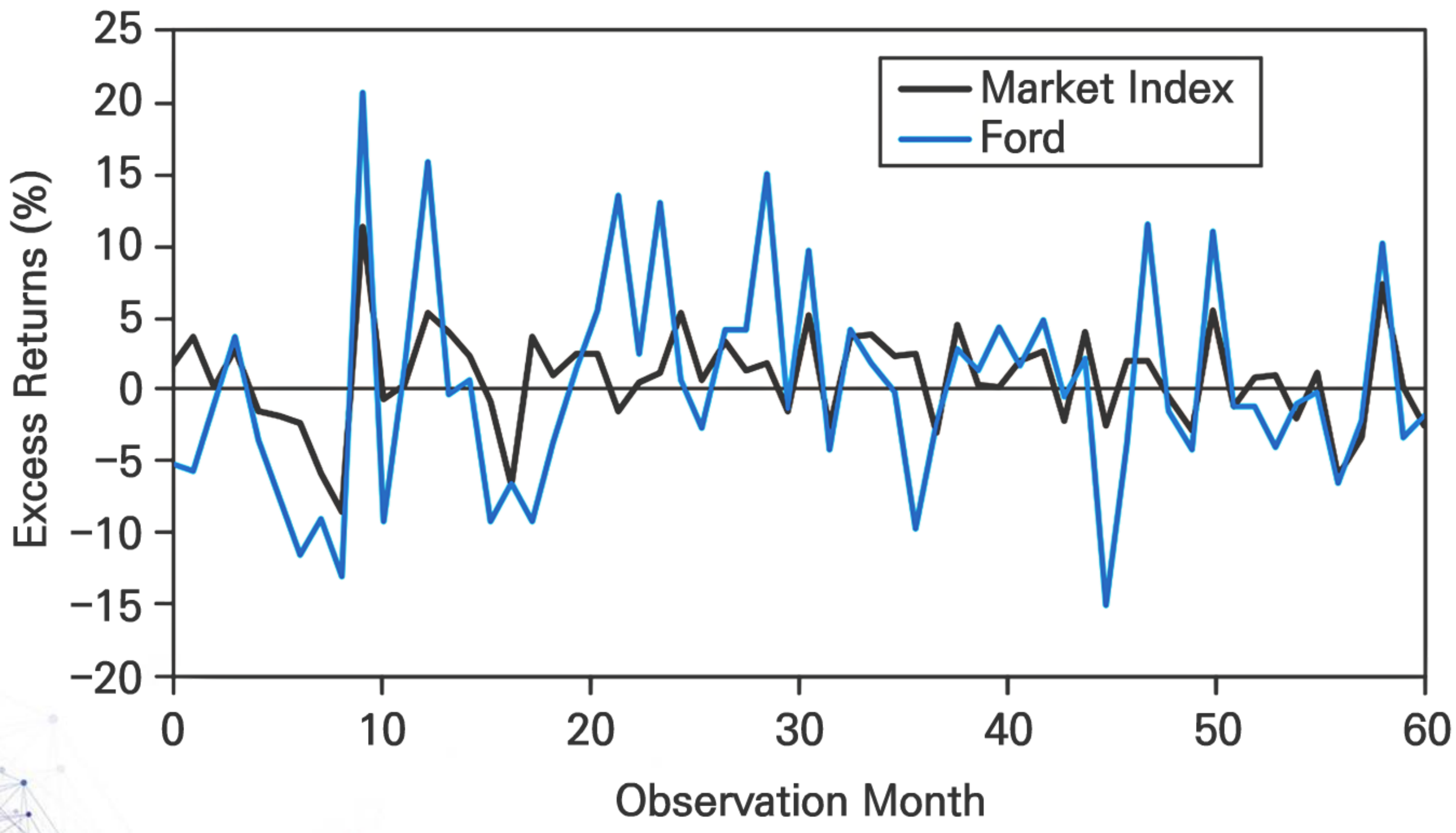
Diversification

Figure : The variance of an equally weighted portfolio with risk coefficient β_p in the single-factor economy as a function of the number of firms included in the portfolio



Estimation of the Single Index Model – Ford Case

Figure: Excess returns on Ford and market index



Estimation of the Single Index Model – Ford Case

Figure: Scatter diagram for Ford against the market index and Ford's security characteristic line

Regression Equation

- $\boldsymbol{R_i(t) = \alpha_i + \beta_i R_{S\&P500}(t) + e_i(t)}$
  - $R_i(t) \rightarrow$ excess return of security i
  - $\alpha_i \rightarrow$ expected excess return when the market excess return is zero
  - $\beta_i \rightarrow$ sensitivity of security i's return to changes in the return of the market
  - $R_{S\&P500}(t) \rightarrow$ expected excess return of the market
  - $e_i(t) \rightarrow$ zero-mean, firm-specific surprise in security i's return in month t (the residual)

Regression Equation

Excel output: Regression statistics for Ford's SCL

Regression Statistics				
Multiple R	0.6280			
R-square	0.3943			
Adjusted R-square	0.3839			
Standard error	0.0577			
Observations	60			
	Coefficients	Standard Error	t-Stat	p-Value
Intercept	−0.0098	0.0077	−1.2767	0.2068
Market index	1.3258	0.2157	6.1451	0.0000

Explained Variation of the Return by the Model

$$R^2 = \frac{\beta^2 \sigma_M^2}{\sigma^2}$$

Total Variation of the Return Data

In a simple linear model, $R^2 = \rho_{i,M}^2$

Interpretation

- Correlation of Ford with the S&P 500 is 0.6280
- The model explains about 38% of the variation in Ford
- Ford's alpha is −0.98% per month, but not statistically significant
- Ford's beta is 1.3258, but the 95% confidence interval is [0.90, 1.75]

Exercise Problem 1

Suppose that the index model for stocks A and B is estimated from excess returns with the following results:

$$r_A = 3\% + 0.7r_M + e_A,$$
$$r_B = -2\% + 1.2r_M + e_B,$$
$$\sigma_M = 20\%, R_A^2 = 0.2, R_B^2 = 0.12.$$

1) What is the standard deviation of each stock?
2) Break down the variance of each stock into its systematic and firm-specific components.
3) What are the covariance and the correlation coefficient between the two stocks?
4) What is the covariance between each stock and the market index?
5) For portfolio P with investment proportions of .60 in A and 40 in B, rework Problems 1), 2), and 4).
6) Rework Problem 5) for portfolio Q with investment proportions of .50 in P .30 in the market index, and .20 in T-bills