Appendix A: Some Useful Mathematical Formulae

The symbols a, b, c, r and s represent real numbers, m and n are positive integers, and indices i and j are non-negative integers.

A.1 Binomial Formulae

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$

A.2 Power and Roots

- $a^0 = 1$
- $a^1 = a$
- $a^{-b} = \frac{1}{a^b}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^b a^c = a^{b+c}$
- $\frac{a^b}{a^c} = a^{b-c}$
- $a^c b^c = \left(ab\right)^c$
- $\frac{a^c}{b^c} = \left(\frac{a}{b}\right)^c$
- $\left(a^b\right)^c = a^{bc} = a^{cb} = \left(a^c\right)^b$

and the following are special cases of the above:

- $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$
- $a^{\frac{m}{n}} = \sqrt[n]{(a^m)} = \left(\sqrt[n]{a}\right)^m$
- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- $a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{(a^m)}} = \frac{1}{\left(\sqrt[n]{a}\right)^m}$

A.3 Logarithms

Definition: The logarithm of c to base a, written as $\log_a c$, is the real number b satisfying the equation $c = ab$, in which we assume that $c > 0$ and $a > 1$.

There are two special cases worth noting, namely $\log_a 1 = 0$, since $a^0 = 1$, and $\log_a a = 1$, since $a^1 = a$. From the definition, we immediately see that:

$a^{\log_a c} = c$ and $\log_a a^b = b$

and we can easily move from one base a to another a' using:

$\log_{a'} b = \log_{a'} a \times \log_a b$

Using this rule, we can simplify any big O notation that has a similar form to $O(\log_x n)$, where x is any positive integer, to become $O(\log n)$.

The key rules for logarithms are:

- $\log_a bc = \log_a b + \log_a c$
- $\log_a \frac{b}{c} = \log_a b - \log_a c$
- $\log_a (b^r) = r \log_a b$

and the following are special cases of those rules:

- $\log a^n = n \log a$
- $\log \sqrt[n]{a} = \frac{1}{n} \log a$

For large n we have the following approximation:

$\log n! = n \log n + O(n)$

A.4 Sums

We often find it useful to abbreviate a sum as follows:

$$S = \sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \ \dots \ + a_n$$

We can view this as an algorithm or program. Let `s` hold the sum at the end, and `a` be an array or a list holding the numbers we wish to add, that is $a[i] = a_i$. Then, to compute the sum:

```
s = 0
for i in range(n+1):
    s = s + a[i]
```

Note: that we have to use `range(n+1)` instead of `range(n)` because `range(n)` constructs a list of integers from 0 to $(n-1)$ inclusive.

The most common use of sums for our purposes is when investigating the time complexity of an algorithm or program. For that, we often have to count a variant of $(1 + 2 + \ \dots \ + n)$, so it is helpful to know that:

$$S = \sum_{i=0}^{n} i = 0 + 1 + 2 + \ \dots \ + n = \frac{n(n+1)}{2}$$

Proof:

Write out the sum twice but with the second in reverse as shown (we also omit the 0).

S	$=$	1	$+$	2	$+$	3	$+ \dots +$	n
S	$=$	n	$+$	$(n-1)$	$+$	$(n-2)$	$+ \dots +$	1

If we add both rows we get the sum of 1 to n, but twice. We notice that each pair adds up to $n+1$ and there are n pairs.

$2S$	$=$	$(n+1)$	$+$	$(n+1)$	$+$	$(n+1)$	$+ \dots +$	$(n+1)$
$2S$	$=$	$(n+1)$	$\times$	n				

Therefore, the sum of the rows is $2S = \big((n+1) \times n\big)$ which can be simplified to $S = \frac{n(n+1)}{2}$.