Solution for Problem Set 7

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1 Problem 1

(a) We do `MAKESET` for n times and each of them cost $\Theta(1)$. Notice during each `UNION`, we append the list of length 1 to the longer list, so it also cost $\Theta(1)$. Hence the total running time is $n + n - 1 = \Theta(n)$.

(b) Notice every node of the trees will be visited only once while we do $FIND$ by path compression. Hence the total time is $O(n)$.

2 Problem 2

(a) We mantain an array X to support $LOOKUP$ in $O(1)$ time. Every time we do $BLACKEN(i)$, we will set $X[i] \leftarrow 1$. However, we need to do more. Treating the continuous 'black' elements in X as a set in the form of rooted-tree, all these sets make up a disjoint-set. Moreover, the leader of each set should store the index of the first 'white' element after the 'black' interval'. So when we do $BLACKEN(i)$,

(1) If $X[i-1]$ and $X[i+1]$ are 'white', we just need to do $MakeSet(X[i])$ to create a set containing only $X[i]$ whose leader is also $X[i]$, and the 'white index' it stores is $i + 1$.

(2) Otherwise, $X[i]$ is adjancent to one or two 'black intervals', we have to $UNION$ them. And the 'white index' of the leader of the new set should be the 'white index' of the right-most set among them in X.

Here we use union-by-height to implement $UNION$, so the amortized time for $BLACKEN$ is $O(logn)$. To implement $NEXTWHITE(i)$, if $X[i]$ is 'white', return i directly. Otherwise, we can first call $Find(X[i])$ to find the leader and return its 'white index'. However, this will still cost $O(logn)$. Maybe more operations are needed in $BLACKEN(i)$, but I haven't make it yet.

(b) I haven't work out it yet. Maybe we can use union-by-rank and path-compression to implement $UNION$ in the data structure of part (a). Then the amortized time for $NEXTWHITE$ is $O(log * n)$, however, the worst-case time for $BLACKEN$ is not $O(n)$. Awful!

3 Problem 3

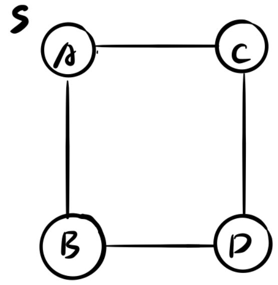
Suppose the adjacency matrix is A and the i^{th} vertex satisfying the property, then the matrix row $A[i][1, \cdots, |V|]$ must be all 0 and the column $A[1, \cdots, |V|][i]$ must be all 1 except the element $A[i][i]$.

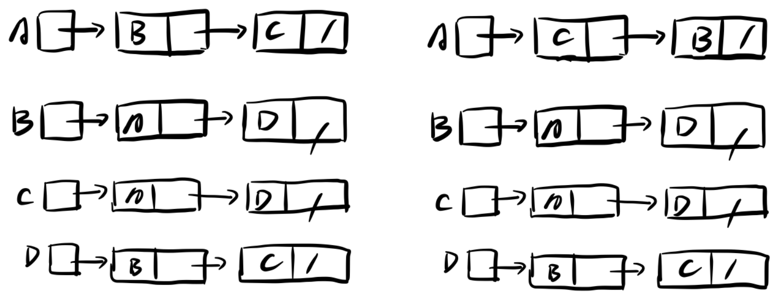
Suppose we start at the left upper corner $A[1][1]$ and then we start moving. If the current element is 0 we move right. Otherwise we will go down. At last we may finish at $A[k][|V|]$ or $A[|V|][k]$, then check if the k^{th} element satisfies the property. If it does, we find the vertex we want. Otherwise the vertex we want does not exist.

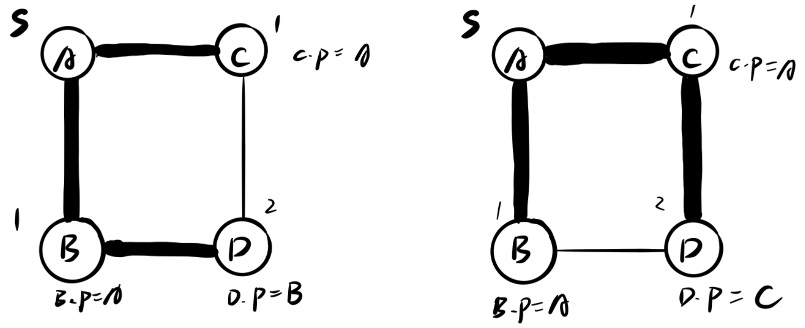
4 Problem 4

(a) Suppose the source vertex is s, then for every vertex u reachable from s, the distance $u.d$ assigned to it must be the distance of the shortest path(Theorem 22.5, CLRS). Therefore, once the source vertex is fixed, the distance for every vertex is fixed as well, which is independent of the order in the agjacency list.

See from the following example

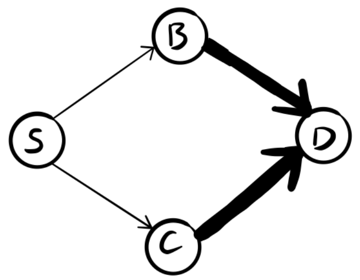






Suppose the source vertex is A, for adjacency lists of different ordering, the breadth-first tree is different since $D.p = B$ in the left tree while $D.p = C$ in the right one.

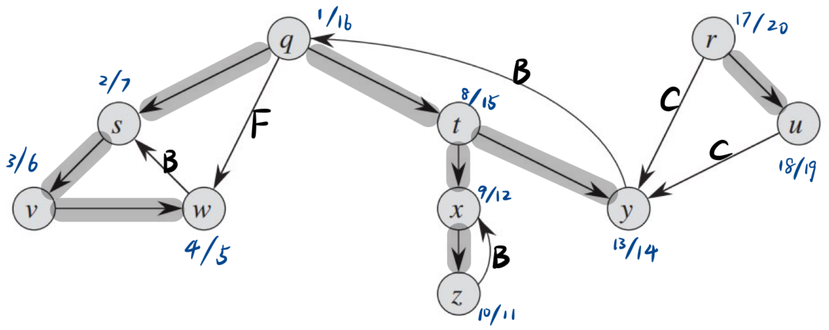
(b) The graph is shown below



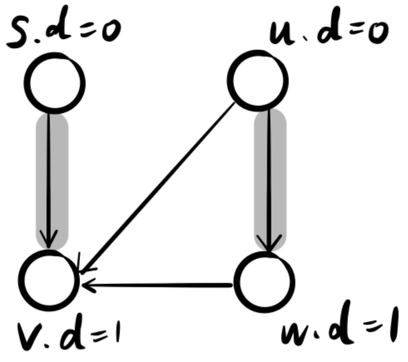
No matter B precedes C or C precedes B in the adjacency list $Adj[s]$, one of the two bold edges will not be produced.

5 Problem 5

(a) The discovery and finishing time is marked near each vertex and the tree edges are bold, back edges are B, forward edges are F, cross edges are C.



(b) A counterexample is given in the folowing



Assume the DFS procedures the vertices in alphabetical order, we can find that $u.d = 0 < v.d = 1$, and graph G contains a path from u to v. However, v is not a descendant of u since the edge (u, v) is a cross edge.

6 Problem 6

Algorithm 1 DFS(G,s)

```
$time = time + 1$
$s.d = time$
$s.color = GRAY$
for (each edge $(s, v)$ in E) do
    if $v.color == BLACK$ then
        if $v.d > s.d$ then
            print $(s, v)$ is forward edge
        else
            print $(s, v)$ is cross edge
        end if
    else if $v.color == GRAY$ then
        print $(s, v)$ is back edge
    else
        $DFS(G, v)$
    end if
end for
$s.color = BLACK$
$time = time + 1$
$s.f = time$
```

7 Problem 7

(a) Inspired by the concept of implicit graph, we can build a new graph to solve the problem.

Suppose the origin directed graph is $G = (V, E)$, we make three copies V_0, V_1, V_2 of the vertices. For each edge $(u, v) \in E$ in the origin graph, we create new edges (u_i, v_j) in the new graph where u_i is the corresponding vertex of u in V_i, v_j is the corresponding vertex of v in V_j and $j = (i + 1)\ mod\ 3$ $(i = 0, 1, 2)$. i.e. each edge (u, v) has three corresponding edges (u_0, v_1), (u_1, v_2), (u_2, v_0) in the new graph.

Then we comes to the problem. Given two vertices s^0 and s^k in the origin graph G and a path connected them of length k, assume all vertices along the path is s^0, $s^1 \cdots, s^{k-1}$, s^k, and the corresponding vertices in the new graph is s^0_m, $s^1_{(m+1)\ mod\ 3}$, $\cdots$, s^k_n. Obviously, $n = (m + k)\ mod\ 3$. If k is divisible by 3, then $n = m$. Samely, we can prove the inverse proposition is true as well.

Deduce from the above proposition, if there exists a path from s_m to t_m,(e.g. s_0 and t_0) in the new graph, which can be implement by BFS, then there must exist a walk from s to t whose length is divisible by 3 in the origin graph, and the algorithm will return `true`.