ℓ^2-BETTI NUMBERS OF DISCRETE AND NON-DISCRETE GROUPS

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ABSTRACT. In this survey we explain the definition of ℓ^2-Betti numbers of locally compact groups – both discrete and non-discrete. Specific topics include proportionality principle, Lück's dimension function, Petersen's definition for locally compact groups, some concrete examples.

1. Introduction

This short survey addresses readers who, motivated by geometric group theory, seek after a brief overview of the (algebraic) definitions of ℓ^2-Betti numbers of discrete and non-discrete locally compact groups. I limit myself to the group case, mostly ignoring the theory of ℓ^2-Betti numbers of equivariant spaces.

ℓ^2-Betti numbers share many formal properties with ordinary Betti numbers, like Künneth and Euler-Poincare formulas. But the powerful *proportionality principle* is a distinctive feature of ℓ^2-Betti numbers. It also was a motivation to generalize the theory of ℓ^2-Betti numbers to locally compact groups.

In its easiest form, the proportionality principle states that the ℓ^2-Betti number in degree p, which is henceforth denoted as $\beta_p(\_)$, of a discrete group Γ and a subgroup $\Lambda < \Gamma$ of finite index satisfy the relation

$$\beta_p(\Lambda) = [\Gamma : \Lambda]\, \beta_p(\Gamma). \tag{1.1}$$

The next instance of the proportionality principle involves lattices Γ and Λ in a semisimple Lie group G endowed with Haar measure μ. It says that their ℓ^2-Betti numbers scale according to their covolume:

$$\beta_p(\Gamma)\mu(\Lambda\backslash G) = \beta_p(\Lambda)\mu(\Gamma\backslash G) \tag{1.2}$$

The proof of (1.2) becomes easy when we use the original, analytic definition of ℓ^2-Betti numbers by Atiyah [1]. According to this definition, the p-th ℓ^2-Betti numbers of Γ, denoted by $\beta_p(\Gamma)$, is given in terms of the heat kernel of the Laplace operator on p-forms on the symmetric space $X = G/K$:

$$\beta_p(\Gamma) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta^p}(x,x))\, \mathrm{dvol}\,.$$

Here $\mathcal{F} \subset X$ is a measurable fundamental domain for the Γ-action on X and $e^{-t\Delta^p}(x,x) : \mathrm{Alt}^p(T_xX) \to \mathrm{Alt}^p(T_xX)$ is the integral kernel – called *heat kernel* – of the bounded operator $e^{-t\Delta^p}$ obtained from the unbounded Laplace operator Δ^p on $L^2\Omega^p(X)$ by spectral calculus. Since G acts transitively on X by isometries, it is clear that the integrand in the above formula is constant in x. So there is a

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constant $c > 0$ only depending on G such that $\beta_p(\Gamma) = c\,\mathrm{vol}(\Gamma\backslash X)$, from which one deduces (1.2).

Gaboriau's theory of ℓ^2-Betti numbers of measured equivalence relations [9] greatly generalized (1.2) to the setting of measure equivalence.

Definition 1.1. Let Γ and Λ be countable discrete groups. If there is a Lebesgue measure space (Ω, ν) on which Γ and Λ act in a commuting, measure preserving way such that both the Γ- and the Λ-action admit measurable fundamental domains of finite ν-measure, we call Γ and Λ *measure equivalent*. We say that (Ω, ν) is a *measure coupling* of Γ and Λ.

Theorem 1.2 (Gaboriau)**.** *If (Ω, ν) is a measure coupling of Γ and Λ, then*

$$\beta_p(\Gamma)\nu(\Lambda\backslash\Omega) = \beta_p(\Lambda)\nu(\Gamma\backslash\Omega).$$

If Γ and Λ are lattices in the same locally compact G, then G endowed with its Haar measure is a measure coupling for the actions described by $\gamma \cdot g := \gamma g$ and $\lambda \cdot g := g\lambda^{-1}$ for $g \in G$ and $\gamma \in \Gamma$, $\lambda \in \Lambda$. So Gaboriau's theorem generalizes (1.2) to lattices in arbitrary locally compact groups.

Motivated by (1.1) and (1.2), Petersen introduced ℓ^2-Betti numbers $\beta_*(G, \mu)$ of a second countable, locally compact unimodular group G endowed with Haar measure μ. Earlier, Dymara [6] and Davis-Dymara-Januszkiewicz-Okun [7] introduced ℓ^2-Betti numbers for groups with a BN-pair (the passage between these two definitions is part of a forthcoming work of the author with Henrik Petersen and Andreas Thom). The work of Davis, Dymara, Januszkiewicz, Okun provides a rich source of computations.

If G is discrete, we always take μ to be the counting measure. If a locally compact group possesses a lattice, it is unimodular. Petersen [21] (if $\Gamma < G$ is cocompact or G is totally disconnected) and then Kyed-Petersen-Vaes [11] (in general) showed the following generalization of (1.1).

Theorem 1.3 (Petersen, Kyed-Petersen-Vaes)**.** *Let Γ be a lattice in a second countable, locally compact group G with Haar measure μ. Then*

$$\beta_p(\Gamma) = \mu(\Gamma\backslash G)\,\beta_p(G, \mu).$$

In Section 2 we explain the definition of continuous cohomology. And we explain how to compute it for discrete and totally disconnected groups by geometric models. In Section 3 we start with the definition of the von Neumann algebra of a unimodular locally compact group and its semifinite trace. The goal is to understand Lück's dimension function for modules over von-Neumann algebras. In Section 4 we define ℓ^2-Betti numbers, comment on their quasi-isometry invariance, and present some computations.

2. Continuous cohomology

Throughout, let G be a second countable, locally compact group, and let E be topological vector space with a continuous G-action, i.e. the action map $G \times E \to E$, $(g, e) \mapsto ge$, is continuous. A topological vector space with such a G-action is called a *G-module*. A *G-morphism* of G-modules is a continuous, linear, G-equivariant map.

2.1. **Definition via bar resolution.** Let $C(G^{n+1},E)$ be the vector space of continuous maps from G^{n+1} to E. The group G acts from the left on $C(G^{n+1},E)$ via

$$(g\cdot f)(g_0,\dots,g_n) = gf(g^{-1}g_0,\dots,g^{-1}g_n).$$

If G is discrete, the continuity requirement is void, and $C(G^{n+1},E)$ is the vector space of all maps $G^{n+1}\to E$. The fixed set $C(G^{n+1},E)^G$ is just the set of continuous equivariant maps. The *homogeneous bar resolution* is the chain complex

$$C(G,E)\xrightarrow{d^0} C(G^2,E)\xrightarrow{d^1} C(G^3,E)\to\dots$$

with differential

$$d^n(f)(g_0,\dots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\dots,\hat{g}_i,\dots,g_{n+1}).$$

The chain groups $C(G^{n+1},E)$ are endowed with the compact-open topology turning them into G-modules. This is a non-trivial topology even for discrete G, where it coincides with pointwise convergence.

Since the differentials are G-equivariant we obtain a chain complex $C(G^{*+1},E)^G$ by restricting to the equivariant maps.

Definition 2.1. The cohomology of $C(G^{*+1},E)^G$ is called the *continuous cohomology* of G in the G-module E and denoted by $H^*(G,E)$.

The differentials are continuous but usually have non-closed image, which leads to a non-Hausdorff quotient topology on the continuous cohomology. Hence it is natural to consider the following.

Definition 2.2. The *reduced continuous cohomology* $\bar{H}^*(G,E)$ of G in E is defined as the quotient $\ker(d^n)/\operatorname{clos}(\operatorname{im}(d^{n-1}))$ of $H^n(G,E)$, where we take the quotient by the closure of the image of the differential.

In homological algebra it is common to compute derived functors, such as group cohomology, by arbitrary injective resolutions. The specific definition of continuous cohomology by the homogeneous bar resolution, which is nothing else than usual group cohomology if G is discrete, is the quickest definition in the topological setting. But there is also an approach in the sense of homological algebra, commonly referred to as *relative homological algebra*.

We call an injective G-morphism of G-modules *admissible* if it admits a linear, continuous (not necessarily G-equivariant) inverse.

Definition 2.3. A G-module E is *relatively injective* if for any admissible injective G-morphism $j:U\to V$ and a G-morphism $f:U\to E$ there is a G-morphism $\bar{f}:V\to E$ such that $\bar{f}\circ j = f$.

Example 2.4. Let E be a G-module. Then $C(G^{n+1},E)$ is relatively injective. Let $j:U\to V$ be an admissible G-morphism. Let $s:V\to U$ be a linear continuous map with $s\circ j=\mathrm{id}_U$. Given a G-morphism $f:U\to C(G^{n+1},E)$, the G-morphism

$$(\bar{f})(v)(g_0,\dots g_n) = f\big(g_0 s(g_0^{-1}v)\big)(g_0,\dots,g_n)$$

satisfies $\bar{f}\circ j = f$. Similarly, if K is a compact subgroup, then $C((G/K)^{n+1},E)$ is relatively injective. Here the extension is given by

$$(\bar{f})(v)([g_0],\dots[g_n]) = \int_K f\big(g_0 k s(k^{-1}g_0^{-1}v)\big)([g_0],\dots,[g_n])d\mu(k),$$

where μ is the Haar measure normalized with $\mu(K)=1$.

As the analog of the fundamental lemma of homological algebra we have:

Theorem 2.5. *Let E be a G-module. Let $0\to E\to E^0\to E^1\to\dots$ be a resolution of E by relatively injective G-modules E^i. Then the cohomology of $(E^*)^G$ is (topologically) isomorphic to the continuous cohomology of G in E.*

2.2. **Injective resolutions to compute continuous cohomology.** The homogeneous bar resolution is useful for proving general properties of continuous cohomology. But other injective resolutions coming from geometry are better suited for computations.

Definition 2.6. Let G be totally disconnected (e.g. discrete). Let X be a cellular complex on which G acts cellularly and continuously. We require that for each open cell $e\subset X$ and each $g\in G$ with $ge\cap e\neq\emptyset$ multiplication by g is the identity on e. We say that X is a *geometric model* of G if X is contractible, its G-stabilizer are open and compact, and the G-action on the n-skeleton $X^{(n)}$ is cocompact for every $n\in\mathbb{N}$.

For simplicial G-actions the requirement on open cells can always be achieved by passage to the barycentric subdivision. A cellular complex with cellular G-action that satisfies the above requirement on open cells and whose stabilizers are open is a *G-CW-complex* in the sense of [26, II.1.]. This means that the n-skeleton $X^{(n)}$ is built from $X^{(n-1)}$ by attaching G-orbits of n-cells according to pushouts of G-spaces of the form:

$$\begin{tikzcd}
\coprod_{U\in\mathcal{F}_n} G/U\times S^{n-1} \arrow[r] \arrow[d, hook] & X^{(n-1)} \arrow[d, hook] \\
\coprod_{U\in\mathcal{F}_n} G/U\times D^{n} \arrow[r] & X^{(n)}
\end{tikzcd}$$

Here $\mathcal{F}_n$ is a set of representatives of conjugacy classes of stabilizers of n-cells. We require that each $X^{(n)}$ is cocompact. So $\mathcal{F}_n$ is finite. Each coset space G/U is discrete. Let us fix a choice of pushouts, which is not part of the data of a cellular complex and corresponds to an equivariant choice of orientations for the cells. The horizontal maps induces an isomorphism in relative homology by excision. Let $C_*^{\mathrm{cw}}(X)$ be the cellular chain complex with $\mathbb{C}$-coefficients: We obtain isomorphisms of discrete G-modules:

(2.1)
$$\bigoplus_{U\in\mathcal{F}_n}\mathbb{C}[G/U]\cong H_n\Big(\coprod_{U\in\mathcal{F}_n} G/U\times(D^n,S^{n-1})\Big)\xrightarrow{\cong} H_n(X^{(n)},X^{(n-1)})\overset{\mathrm{def}}{=}C_n^{\mathrm{cw}}(X)$$

The G-action $(gf)(x)=gf(g^{-1}x)$ turns

$$\hom_{\mathbb{C}}(C_n^{\mathrm{cw}}(X),E)\cong\prod_{U\in\mathcal{F}_n}C(G/U,E)=\bigoplus_{U\in\mathcal{F}_n}C(G/U,E)$$

into a G-module. By Example 2.4 it is relatively injective. Further, the contractibility of X implies that $\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X),E)$ is a resolution of E. The next statement follows from Theorem 2.5.

Theorem 2.7. *Let X be a geometric model for a totally disconnected group G. Then*

$$H^n(G,E) \cong H^n(\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X),E)^G).$$

Not every discrete or totally disconnected group has a geometric model. For discrete groups having a geometric model means that the group satisfies the finiteness condition F_∞. But by attaching enough equivariant cells to increase connectivity one can show [14, 1.2]:

Theorem 2.8. *For every totally disconnected G there is a contractible G-CW-complex whose stabilizers are open and compact.*

More is true but not needed here: Every totally disconnected group has a classifying space for the family of compact-open subgroups.

What about the opposite case of a connected Lie group G? There we lack geometric models to compute continuous cohomology, but we can use the infinitesimal structure. The van-Est isomorphism relates continuous cohomology and Lie algebra cohomology. We won't discuss it here and instead refer to [10].

2.3. **Definition and geometric interpretation of ℓ^2-cohomology.** Let G be a second countable, locally compact group. For convenience, we require that the the Haar measure μ is unimodular, i.e. invariant under left and right translations. Unimodularity becomes a necessary assumption in Section 3.

We consider continuous cohomology in the G-module $L^2(G)$ which consists of measurable square-integrable functions on G modulo null sets. A left G-action on $L^2(G)$ is given by $(gf)(h) = f(g^{-1}h)$.

The G-module $L^2(G)$ also carries the right G-action $(fg)(h) = f(hg^{-1})$, which commutes with the left action. In Section 2 we ignore this right action. It becomes important in Sections 3 and 4 when we define ℓ^2-Betti numbers.

Definition 2.9. We call $H^*(G,L^2(G))$ and $\bar{H}^*(G,L^2(G))$ the *ℓ^2-cohomology* and the *reduced ℓ^2-cohomology* of G, respectively.

Assume that G possesses a geometric model X. By Theorem 2.7 the ℓ^2-cohomology of G can be expressed as the cohomology of the G-invariants of the chain complex $\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X),L^2(G))$. In view of (2.1), this is a chain complexes of Hilbert spaces

$$\hom_{\mathbb{C}}(C_n^{\mathrm{cw}}(X),L^2(G))^G \cong \bigoplus_{U\in\mathcal{F}_n} C(G/U,L^2(G))^G \cong \bigoplus_{U\in\mathcal{F}_n} L^2(G)^U \tag{2.2}$$

with bounded differentials.

Let us rewrite this chain complex in a way so that the group G does not occur anymore: As a (non-equivariant) cellular complex the n-th cellular chain group $C_n^{\mathrm{cw}}(X)$ comes with a preferred basis B_n, given by n-cells, which is unique up to permutation and signs. We define the subvector space

$$\ell^2 C^n_{\mathrm{cw}}(X) := \bigl\{ f : C_n^{\mathrm{cw}}(X) \to \mathbb{C} \mid \sum_{e\in B_n} |f(e)|^2 < \infty \bigr\} \subset \hom_{\mathbb{C}}(C_n^{\mathrm{cw}}(X),\mathbb{C})$$

of ℓ^2-cochains in the cellular cochains; it has the structure of a Hilbert space with Hilbert basis $\{f_e \mid e \in B_n\}$ where $f_e(e') = 1$ for $e' = e$ and $f_e(e') = 0$ for $e' \in B_n \backslash \{e\}$. For general cellular complexes the differentials in the cellular cochain complex are not bounded as operators, but in the presence of a cocompact group action on skeleta they are.

Definition 2.10. The *(reduced) ℓ^2-cohomology* of X is defined as the (reduced) cohomology of $\ell^2 C^*_{\mathrm{cw}}(X)$ and denoted by $\ell^2 H^*(X)$ or $\ell^2 \bar{H}^*(X)$, respectively.

Proposition 2.11. *The ℓ^2-cohomology of G and the ℓ^2-cohomology of a geometric model of G are isomorphic. Similarly for the reduced cohomology.*

Proof. The maps

$$\hom_{\mathbb{C}}(C_n^{\mathrm{cw}}(X), L^2(G))^G \to \ell^2 C^n_{\mathrm{cw}}(X).$$

that take f to the ℓ^2-cochain that assigns to $e \in B_n$ the essential value of the essentially constant function $f(e)|_U$, where $U < G$ is the open stabilizer of e, form a chain isomorphism. The claim follows now from Theorem 2.7. $\square$

2.4. **Reduced ℓ^2-cohomology and harmonic cochains.**

Definition 2.12. Let $W^0 \xrightarrow{d^0} W^1 \xrightarrow{d^1} W^2 \dots$ be a cochain complex of Hilbert spaces such that the differentials are bounded operators. The *Laplace-operator* in degree n is the bounded operator $\Delta^n = (d^n)^* \circ d^n + d^{n-1} \circ (d^{n-1})^* : W^n \to W^n$. Here $*$ means the adjoint operator. A cochain $c \in W^n$ is *harmonic* if $\Delta^n(c) = 0$.

Proposition 2.13. *Every harmonic cochain is a cocycle, and inclusion induces a topological isomorphism* $\ker(\Delta^n) \xrightarrow{\cong} \bar{H}^n(W^*) = \ker(d^n)/\operatorname{clos}(\operatorname{im}(d^{n-1}))$.

Proof. Since Δ^n is a positive operator, we have $c \in \ker(\Delta^n)$ if and only if

$$\langle \Delta^n(c), c \rangle = \|d^n c\|^2 + \|(d^{n-1})^*(c)\|^2 = 0.$$

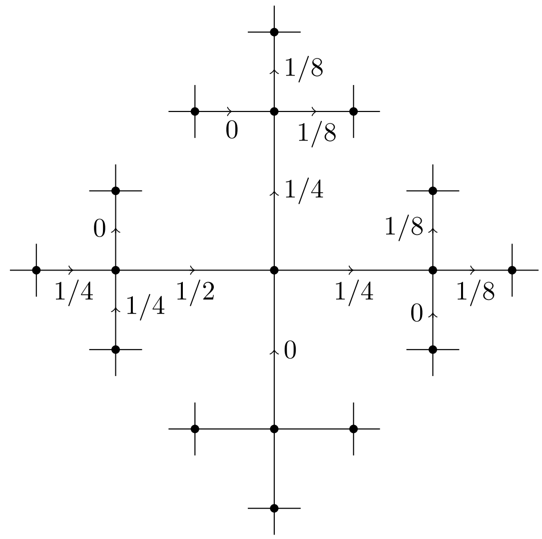
Hence $\ker(\Delta^n) = \ker(d^n) \cap \ker((d^{n-1})^*)$. The second statement follows from

$$\begin{aligned} W^p &= (\ker(d^n) \cap \operatorname{im}(d^{n-1})^\perp) \oplus \operatorname{clos}(\operatorname{im}(d^{n-1})) \oplus \ker(d^n)^\perp \\ &= (\ker(d^n) \cap \ker((d^{n-1})^*)) \oplus \operatorname{clos}(\operatorname{im}(d^{n-1})) \oplus \ker(d^n)^\perp \\ &= \ker(\Delta^n) \oplus \operatorname{clos}(\operatorname{im}(d^{n-1})) \oplus \ker(d^n)^\perp. \end{aligned}$$
$\square$

As a consequence of Propositions 2.11 and 2.13, we obtain:

Corollary 2.14. *Let X be a geometric model of G. The space of harmonic n-cochains of $\ell^2 C^*_{\mathrm{cw}}(X)$ is isomorphic to $\bar{H}^n(G, L^2(G))$.*

Example 2.15. The 4-regular tree is a geometric model of the free group F of rank two. Let $d : \ell^2 C^0_{\mathrm{cw}}(X) \to \ell^2 C^1_{\mathrm{cw}}(X)$ be the differential. Since there are no 2-cells, the first Laplace operator Δ^1 is just dd^*. A 1-cochain is harmonic if and only if it is in the kernel of d^*. We choose a basis B of $C_1^{\mathrm{cw}}(X)$ by orienting 1-cells in the following way:



For $e \in B$ let $e(-)$ be the starting point and $e(+)$ the end point of e. Then:

$$d(f)(e) = f(e(+)) - f(e(-))$$
$$d^*(g)(v) = \sum_{e(+)=v} g(e) - \sum_{e(-)=v} g(e)$$

In the second equation the sums run over all edges whose starting or end point is v. The picture indicates a non-vanishing $c \in \ell^2 C^1_{\mathrm{cw}}(X)$ with $d^*(c) = 0$, thus $\Delta^1(c) = 0$. Therefore $\bar{H}^1(F, \ell^2(F)) \neq 0$.

3. von-Neumann algebras, trace and dimension

Throughout, we fix a second countable, locally compact group G with left invariant Haar measure μ. We require that G is unimodular, that is, μ is also right invariant.

3.1. **The von Neumann algebra of a locally compact group.** The continuous functions with compact support $C_c(G)$ form a $\mathbb{C}$-algebra with involution through the convolution product

$$(f * g)(s) = \int_G f(t) g(t^{-1}s) d\mu(t)$$

and the involution $f^*(s) = \overline{f(s^{-1})}$. Taking convolution with $f \in C_c(G)$ is still defined for a function $\phi \in L^2(G)$. It follows from the integral Minkowski inequality that

$$\|f * \phi\|_2 \leq \|f\|_1 \|\phi\|_2,$$

where $\|\_\|_p$ denotes the L^p-Norm. We obtain a $*$-homomorphism into the algebra of bounded operators on $L^2(G)$:

$$\lambda : C_c(G) \to \mathcal{B}(L^2(G)),\ \lambda(f)(\phi) = f * \phi$$

Similarly, we obtain a $*$-anti-homomorphism

$$\rho : C_c(G) \to \mathcal{B}(L^2(G)),\ \rho(f)(\phi) = \phi * f.$$

Definition 3.1. A *von-Neumann algebra* is a subalgebra of the bounded operators of a Hilbert space that is closed in the weak operator topology and closed under taking adjoints. The *von-Neumann algebra $L(G)$ of the group G* is the von-Neumann algebra defined as the weak closure of $\mathrm{im}(\lambda)$.

The weak closure $R(G)$ of $\mathrm{im}(\rho)$ in $\mathcal{B}(L^2(G))$ is the commutant of $L(G)$ inside $\mathcal{B}(L^2(G))$.

Remark 3.2. Consider the operator $u_g \in \mathcal{B}(L^2(G))$ such that

$$u_g(\phi)(s) = \phi(g^{-1}s).$$

Similarly, we define $r_g(\phi)(s) = \phi(sg)$. We claim that $u_g \in L(G)$. If G is discrete, then the Kronecker function δ_g on G is continuous, so $\delta_g \in C_c(G)$, and we have $u_g := \lambda(\delta_g) \in L(G)$. If G is not discrete, we can choose a sequence $f_n \in C_c(G)$ of positive functions with $\|f_n\|_1 = 1$ whose supports tend to g. Then $(\lambda(f_n))_{n\in\mathbb{N}}$ converges strongly, thus weakly, to u_g, implying $u_g \in L(G)$. One can also show that $L(G)$ is the weak closure of the span of $\{u_g \mid g \in G\}$. Similary it follows that $r_g \in R(G)$

Remark 3.3. Let $j : L^2(G) \to L^2(G)$ be the conjugate linear isometry with

$$j(\phi)(s) = \phi^*(s) = \overline{\phi(s^{-1})}.$$

Then

$$J : L(G) \to R(G),\ J(T) = j \circ T^* \circ j$$

is a $*$-anti-isomorphism, that is, $J(T^*) = J(T)^*$ and $J(T \circ S) = J(S) \circ J(T)$. Furthermore, we have $J(\lambda_g) = r_g$ and $J(\lambda(f)) = J(\rho(f))$ for $f \in C_c(G)$.

Definition 3.4 ($L(G)$-module structures)**.** The Hilbert space $L^2(G)$ is naturally a left $L(G)$-module via $T \cdot \phi = T(\phi)$ for $T \in L(G)$ and $\phi \in L^2(G)$. The left $L(G)$-module structure restricts to the left G-module structure via $g \cdot \phi := u_g(\phi)$. The Hilbert space $L^2(G)$ becomes a right $L(G)$-module by the anti-isomorphism J, explicitly $\phi \cdot T := J(T)(\phi)$. In the sequel, $L^2(G)$ will be regarded as a bimodule endowed with the left G-module structure and the right $L(G)$-module structure.

3.2. **Trace.** We explain the semifinite trace on the von Neumann algebra of a locally compact group. This a bit technical, but indispensable for the dimension theory. We start by discussing traces on an arbitrary von-Neumann algebra $\mathcal{A}$.

Let $\mathcal{A}_+$ be the subset of positive operators in $\mathcal{A}$. For $S, T \in \mathcal{A}_+$ one defines $S \leq T$ by $T - S \in \mathcal{A}_+$; it is a partial order on $\mathcal{A}_+$. Every bounded totally ordered subset of $\mathcal{A}_+$ has a supremum in $\mathcal{A}_+$.

Definition 3.5. A *trace* on a von Neumann algebra $\mathcal{A}$ is a function $\tau : \mathcal{A}_+ \to [0, \infty]$ such that

(1) $\tau(S) + \tau(T) = \tau(S + T)$ for $S, T \in \mathcal{A}_+$;
(2) $\tau(\lambda S) = \lambda\tau(S)$ for $S \in \mathcal{A}_+$ and $\lambda \geq 0$;
(3) $\tau(SS^*) = \tau(S^*S)$ for $S \in \mathcal{A}$.

Let $\mathcal{A}_+^\tau = \{S \in \mathcal{A}_+ \mid \tau(S) < \infty\}$. A trace τ is *faithful* if $\tau(T) > 0$ for every $T \in \mathcal{A}_+\backslash\{0\}$. It is *finite* if $\mathcal{A}_+^\tau = \mathcal{A}_+$. It is *semifinite* if $\mathcal{A}_+^\tau$ is weakly dense in $\mathcal{A}_+$. It is *normal* if the supremum of traces of a bounded totally ordered subset in $\mathcal{A}_+$ is the trace of the supremum.

If $\mathcal{A} \subset \mathcal{B}(H)$ is a von-Neumann algebra with trace τ, then the $n \times n$-matrices $M_n(\mathcal{A}) \subset \mathcal{B}(H^n)$ are a von-Neumann algebra with trace

$$(\tau \otimes \mathrm{id}_n)(S) := \tau(S_{11}) \ldots + \tau(S_{nn}).$$

Remark 3.6. Let $\mathcal{A}^\tau$ be the linear span of $\mathcal{A}^\tau_+$. It is clear that τ extends linearly to $\mathcal{A}^\tau$. If τ is finite, we have $\mathcal{A}^\tau = \mathcal{A}$, so τ is defined on all of $\mathcal{A}$. Further, $\mathcal{A}^\tau$ is always an ideal in $\mathcal{A}$ [25, p. 318]. The *trace property*

$$\tau(ST) = \tau(TS) \text{ for all } S \in \mathcal{A}^\tau \text{ and } T \in \mathcal{A}.$$

holds true. Its deduction from the third property in Definition 3.5 takes a few lines and uses polarization identities. See [25, Lemma 2.16 on p. 318].

After this general discussion we turn again to the von-Neumann algebra of G. We call $\phi \in L^2(G)$ *left bounded* if there is a bounded operator, denoted by λ_ϕ on $L^2(G)$ such that $\lambda_\phi(f) = \phi * f$ for every $f \in C_c(G)$. Of course, every element $f \in C_c(G)$ is left bounded and $\lambda_f = \lambda(f)$. Define for an element $S^*S \in L(G)_+$ (every positive operator can be written like this):

$$\tau_{(G,\mu)}(S^*S) = \begin{cases} \|\phi\|_2^2 & \text{if there is a left bounded } \phi \in L^2(G) \text{ with } \lambda_\phi = S; \\ \infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Notation 3.7. The Haar measure is only unique up to scaling. Hence we keep μ in the notation $\tau_{(G,\mu)}$. If $G = \Gamma$ is discrete, we always take the counting measure as Haar measure and simply write τ_Γ.

See [19, 7.2.7 Theorem] for a proof of the following fact.

Theorem 3.8. *$\tau_{(G,\mu)}$ is a faithful normal semifinite trace on $L(G)$.*

Let us try to obtain a better understanding of Defintion (3.1). To this end, we first consider the case that $G = \Gamma$ is a discrete group. Then $\delta_e \in C_c(\Gamma) = \mathbb{C}[\Gamma] \subset L^2(\Gamma)$. For every $S \in L(G)$ it is $S(\delta_e) \in L^2(\Gamma)$ and $S = \lambda_{S(\delta_e)}$. This implies that every element in $L(\Gamma)_+$ has finite τ_Γ-trace. Hence τ_Γ is finite. Further, from

$$\tau_\Gamma(S^*S) = \|S(\delta_e)\|_2^2 = \langle S(\delta_e), S(\delta_e)\rangle_{L^2(\Gamma)} = \langle S^*S(\delta_e), \delta_e\rangle_{L^2(\Gamma)}$$

we conclude and record:

Remark 3.9 (Trace for discrete groups)**.** If $G = \Gamma$ is discrete, then τ_Γ is finite and thus everywhere defined. For every $T \in L(\Gamma)$ we have $\tau_\Gamma(T) = \langle T(\delta_e), \delta_e\rangle_{L^2(\Gamma)}$.

Remark 3.10 (Trace for totally disconnected groups)**.** Let G be totally disconnected. Then we have a decreasing neighborhood basis (K_n) by open-compact subgroups. Then $\frac{1}{\mu(K_n)}\lambda(\chi_{K_n}) \in L(G)$ is a projection (see Example 3.21). Let $S \in L(G)$. Then $\phi_n := S(\chi_{K_n}) \in L^2(G)$ is a left bounded element such that $\lambda_{\phi_n} = S \circ \lambda(\chi_{K_n})$. From that and (3.1) it is easy to see that

$$\tau_{(G,\mu)}(S^*S) = \lim_{n\to\infty} \frac{1}{\mu(K_n)^2} \cdot \|S(\chi_{K_n})\|_2^2 \in [0,\infty]$$

Now back to general G:

Remark 3.11 (Trace for arbitrary groups)**.** Because of $\lambda_f = \lambda(f)$ for every $f \in C_c(G)$ we obtain that $\tau_{(G,\mu)}(\lambda(f)^*\lambda(f)) = \|f\|_2^2$. One quickly verifies that the latter is just evaluation at the unit element: $\|f\|_2^2 = (f^*f)(e)$. It turns out that $C_c(G) \subset L(G)^{\tau_{(G,\mu)}}$, and the trace $\tau_{(G,\mu)}$ is evaluation at $e \in G$ on $C_c(G)$.

3.3. **Dimension.** We explain first Lück's dimension function over a von-Neumann algebra endowed with a finite trace. We refer to [12,13] for proofs. Lück's work was a major advance in creating a general and algebraic theory of ℓ^2-Betti numbers. An alternative algebraic approach was developed by Farber [8]. After that we discuss Petersen's generalization to von-Neumann algebras with semifinite traces [20].

3.3.1. *Finite traces.* Let τ be a finite normal faithful trace on a von-Neumann algebra $\mathcal{A}$. According to Remark 3.6, τ is a functional on $\mathcal{A}$ which satisfies the trace property $\tau(ST) = \tau(TS)$ for $S, T \in \mathcal{A}$. We start by explaining the dimension for finitely generated projective right $\mathcal{A}$-modules. Let P be such a module. This means that P is isomorphic to the image of left multiplication $l_M : \mathcal{A}^n \to \mathcal{A}^n$ with an idempotent matrix $M \in M_n(\mathcal{A})$. In general the sum $\sum_{i=1}^n M_{ii}$ of the diagonal entries of M depends not only on P but on the specific choice of M. However, Hattori and Stallings observed that its image in the quotient $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ by the additive subgroup generated by all commutators is independent of the choices [4, Chapter IX]. This is true for an arbitrary ring $\mathcal{A}$. The trace τ is still defined on the quotient (see Remark 3.6). Therefore one defines:

Definition 3.12. The *Hattori-Stallings rank* $\mathrm{hs}(P) \in \mathcal{A}/[\mathcal{A},\mathcal{A}]$ of a finitely generated projective right $\mathcal{A}$-module P is defined as the image of $\sum_{i=1}^n M_{ii}$ in $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ for any idempotent matrix $M \in M_n(\mathcal{A})$ with $P \cong \mathrm{im}(l_M)$. The *dimension* of P is defined as

$$\dim_\tau(P) = \tau(\mathrm{hs}(P)) = (\tau \otimes \mathrm{id}_n)(M) \in [0, \infty).$$

Henceforth $\mathcal{A}$-modules are understood to be right $\mathcal{A}$-modules.

Remark 3.13. Let $\ell^2(\mathcal{A}, \tau)$ be the GNS-construction of $\mathcal{A}$ with respect to τ, that is the completion of the pre-Hilbert space $\mathcal{A}$ with inner product $\langle S, T\rangle_\tau = \tau(ST^*)$. By an observation of Kaplansky [13, Lemma 6.23 on p. 248], every finitely generated projective $\mathcal{A}$-module can be described by a projection matrix M, that is a matrix M with $M^2 = M$ *and* $M^* = M$. Then M yields a right Hilbert $\mathcal{A}$-submodule of $\ell^2(\mathcal{A}, \tau)^n$, namely the image of left multiplication $\ell^2(\mathcal{A}, \tau)^n \to \ell^2(\mathcal{A}, \tau)^n$ with A. And $\dim_\tau(M)$ coincides with the von-Neumann dimension of this Hilbert $\mathcal{A}$-submodule. See [13, Chapter 1] for more information on Hilbert $\mathcal{A}$-modules. The classic reference is [16].

The idea of how to extend $\dim_\tau$ to arbitrary modules is almost naive; the difficulty lies in showing its properties.

Definition 3.14. Let M be an arbitrary $\mathcal{A}$-module. Its *dimension* is defined as

$$\dim_\tau(M) = \sup\{\dim_\tau(P) \mid P \subset M \text{ fin. gen. projective submodule}\} \in [0, \infty].$$

First of all, using the same notation $\dim_\tau$ as before requires a justification. But indeed, the new definition coincides with the old one on finitely generated projective modules. And so $\dim_\tau(\mathcal{A}) = \tau(1_\mathcal{A})$ which we usually normalize to be 1. The following two properties are important and, in the end, implied by the additivity and normality of τ.

Theorem 3.15 (Additivity)**.** *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence of* $\mathcal{A}$*-modules, then* $\dim_\tau(M_2) = \dim_\tau(M_1) + \dim_\tau(M_3)$*. Here* $\infty + a = a + \infty = \infty$ *for* $a \in [0, \infty]$ *is understood.*

Theorem 3.16 (Normality)**.** *Let M be an $\mathcal{A}$-module, and let M be the union of an increasing sequence of $\mathcal{A}$-submodules M_i. Then* $\dim_\tau(M) = \sup_{i\in\mathbb{N}} \dim_\tau(M_i)$

Example 3.17. The only drawback of $\dim_\tau$ – in comparision to the dimension of vector spaces – is that $\dim_\tau(N) = 0$ does not, in general, imply $N = 0$. Here is an example. Take a standard probability space (X, ν). Then the ν-integral is a finite trace τ on $\mathcal{A} = L^\infty(X)$. Let $X = \bigcup_{i=1}^n X_i$ be an increasing union of measurable sets such that $\nu(X_i) < 1$ for every $i \in \mathbb{N}$. Each characteristic function χ_{X_i} is a projection in $\mathcal{A}$ with trace $\nu(X_i)$. And each $\chi_{X_i}\mathcal{A} = L^\infty(X_i)$ is a finitely generated projective module of dimension $\tau(\chi_{X_i}) = \nu(X_i)$. Let $M \subset \mathcal{A}$ be the increasing union of submodules $\chi_{X_i}\mathcal{A}$. Then $\mathcal{A}/M \neq 0$ but $\dim_\tau(\mathcal{A}/M) = \dim_\tau(\mathcal{A}) - \dim_\tau(M) = 0$ by additivity and normality.

3.3.2. *Semifinite traces.* Let $\mathcal{A}$ be a von-Neumann algebra with a semifinite trace τ. The definition of the dimension function only needs small modifications, and the proof of being well-defined and other properties run basically like the one in the finite case [21, Appendix B].

We say that a projective $\mathcal{A}$-module P is *τ-finite* if P is finitely generated and if one, then any, representing projection matrix $M \in M_n(\mathcal{A})$, $P \cong \operatorname{im}(l_M : \mathcal{A}^n \to \mathcal{A}^n)$, satisfies $(\tau\otimes\mathrm{id}_n)(M) < \infty$. The *dimension* $\dim_\tau(P) \in [0, \infty)$ of a τ-finite projective $\mathcal{A}$-module $P \cong \operatorname{im}(l_M)$ is defined as $(\tau \otimes \mathrm{id}_n)(M)$.

Definition 3.18. Let M be an arbitrary $\mathcal{A}$-module. Its *dimension* is defined as

$$\dim_\tau(M) = \sup\{\dim_\tau(P) \mid P \subset M \ \tau\text{-finite projective submodule}\} \in [0, \infty].$$

Similarly as before, the notation is consistent with the one for τ-finite projective modules, and additivity and normality hold true.

Remark 3.19. While the situation in the semifinite case is quite similar to the finite case, there is one important difference: If τ is not finite, then $\dim_\tau(\mathcal{A}) = \tau(1_\mathcal{A}) = \infty$.

Notation 3.20. We write $\dim_{(G,\mu)}$ instead of $\dim_{\tau_{(G,\mu)}}$ and $\dim_\Gamma$ instead of $\dim_{\tau_\Gamma}$.

Example 3.21. Let $K < G$ be an open-compact subgroup. Then the characteristic function χ_K is continuous and $P = \lambda(\frac{1}{\mu(K)}\chi_K)$ is a projection in $L(G)$. According to the remark at the end of Subsection 3.2, we have $\tau_{(G,\mu)}(P) = 1/\mu(K)$, thus the dimension of the projective $L(G)$-module $P \cdot L(G)$ is $1/\mu(K)$. The $L(G)$-module $P \cdot L^2(G)$ is not projective, but, by an argument involving rank completion, one can show [21, B.25 Proposition] that

$$\dim_{(G,\mu)}(P \cdot L^2(G)) = \dim_{(G,\mu)}(P \cdot L(G)) = \frac{1}{\mu(K)}. \tag{3.2}$$

One easily verifies that $P \cdot L^2(G)$ consists of all left K-invariant functions, so

$$P \cdot L^2(G) = L^2(G)^K.$$

4. ℓ^2-Betti numbers of groups

Throughout, G denotes a locally compact unimodular group.

4.1. **Definition.** In the definition of ℓ^2-cohomology we only used the left G-module structure. The right $L(G)$-module structure survives the process of taking G-invariants of the bar resolution, and so $H^*(G, L^2(G))$ inherits a right $L(G)$-module structure from the one of $L^2(G)$. The following definition is due to Petersen for second countable, unimodular, locally compact G. It is modelled after and coincides with Lück's definition for discrete G.

Definition 4.1. The ℓ^2-Betti numbers of G with Haar measure μ are defined as

$$\beta_p(G,\mu) := \dim_{(G,\mu)}\bigl(H^p(G, L^2(G))\bigr) \in [0,\infty].$$

Remark 4.2. In Lück's book [13], where only the case of discrete G is discussed, the ℓ^2-Betti numbers are defined as $\dim_G\bigl(H_p(G, L(G))\bigr)$. By [22, Theorem 2.2] Lück's definition coincides with the one above.

The homological algebra in Section 2 can be carried through such that the additional right $L(G)$-module structure is respected. In particular, we obtain (see Theorem 2.7):

Theorem 4.3. *Let G be totally disconnected and X a geometric model of G. Then*

$$\beta_p(G,\mu) = \dim_{(G,\mu)}\bigl(H^p\bigl(\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X), L^2(G))^G\bigr)\bigr).$$

For totally disconnected groups we can compute the ℓ^2-Betti numbers also through the reduced cohomology [21]:

Theorem 4.4. *Let G be totally disconnected. Then*

$$\beta_p(G,\mu) = \dim_{(G,\mu)}\bigl(\bar{H}^p(G, L^2(G))\bigr)$$

If X is a geometric model of G, then

$$\beta_p(G,\mu) = \dim_{(G,\mu)}\bigl(\bar{H}^p\bigl(\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X), L^2(G))^G\bigr)\bigr).$$

Many properties of ℓ^2-Betti numbers of discrete groups possess analogues for locally compact groups. We refer to [21] for more information and discuss only the Euler-Poincare formula:

Let X be a cocompact proper G-CW complex. Let $K_1, \dots, K_n < G$ be the stabilizers of G-orbits of p-cells of X. The *weighted number of equivariant p-cells* of X is then defined as

$$c_p(X; G, \mu) = \mu(K_1)^{-1} + \dots + \mu(K_n)^{-1}.$$

Definition 4.5. The *equivariant Euler characteristic* of X is defined as

$$\chi(X; G, \mu) := \sum_{p\geq 0}(-1)^p c_p(X; G).$$

Theorem 4.6 (Euler-Poincare formula)**.** *Let G be totally disconnected, and let X be a cocompact geometric model of G. Then*

$$\sum_{p\geq 0}(-1)^p\,\beta_p(G,\mu) = \chi(X; G, \mu).$$

Proof. Let $C^* := \hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X), L^2(G))^G$, and let H^* be the cohomology of C^*. Let Z^p be the cocycles in C^p and B^p be the coboundaries in C^p. Note that $c_p(X; G, \mu) = \dim_{(G,\mu)}(C^p)$. We have exact sequences $0 \to Z^p \to C^p \to B^{p+1} \to 0$

and $0 \to B^p \to Z^p \to H^p \to 0$ of $L(G)$-modules. By additivity of $\dim_{(G,\mu)}$ we conclude that

$$\begin{aligned}\chi(X; G, \mu) &= \sum_p (-1)^p \dim_{(G,\mu)}(C^p)\\ &= \sum_p (-1)^p (\dim_{(G,\mu)}(Z^p) + \dim_{(G,\mu)}(B^{p+1}))\\ &= \sum_p (-1)^p (\dim_{(G,\mu)}(B^p) + \dim_{(G,\mu)}(H^p) + \dim_{(G,\mu)}(B^{p+1}))\\ &= \sum_p (-1)^p \beta_p(G, \mu).\end{aligned}$$ □

Remark 4.7. It turns out that

$$\beta_p(G,\mu) > 0 \Leftrightarrow \bar{H}^p\big(\hom_{\mathbb{C}}(C_*^{\mathrm{cw}}(X), L^2(G))^G\big) \neq 0.$$

This is in general false for the non-reduced continuous cohomology. So constructing non-vanishing harmonic cocycles is a way to show non-vanishing of ℓ^2-Betti numbers (see Subsection 2.4). Example 2.15 shows that the first ℓ^2-Betti number of a non-abelian free group is strictly positive.

4.2. **Quasi-isometry invariance.** Let G be a compactly generated totally disconnected unimodular group, and let X be a cocompact (simplicial) geometric model of G. We endow G with the word metric of a compact symmetric generating set. The space X with the simplicial path metric is quasi-isometric to G. By Proposition 2.11 and Remark 4.7 we obtain that

$$\beta_p(G) > 0 \Leftrightarrow \ell^2\bar{H}^p(X) \neq 0.$$

Let H be another compactly generated totally disconnected group, and let Y be a cocompact geometric model of H. Now assume that G and H are quasi-isometric. Hence X and Y are quasi-isometric. By the connect-the-dots technique (see e.g. [3, Proposition A.1]) there is a Lipschitz homotopy equivalence $f\colon X \to Y$. Pansu [18] proves that from that we obtain an isomorphism

$$\ell^2\bar{H}^p(X) \cong \ell^2\bar{H}^p(Y)$$

in all degrees p. This sketches the proof of the following theorem.

Theorem 4.8. *The vanishing of the p-th ℓ^2-Betti number is a quasi-isometry invariant among totally disconnected unimodular groups having a cocompact geometric model.*

The above proof follows Pansu's proof of the corresponding result [18] for discrete groups. In fact, his proof requires only the existence of a geometric model (and not a cocompact one). Building on ideas in [23, 24], Oguni further relaxed the hypothesis on having a geometric model [17]. This should hold true in the totally disconnected situation as well.

The phenomenon that group homological invariants can be viewed as coarse-geometric invariants is not unique to the theory of ℓ^2-cohomology or ℓ^2-Betti numbers, of course. For instance, $H^*(\Gamma, \ell^\infty(\Gamma))$ is isomorphic to the uniformly finite homology by Block and Weinberger [2]. But unlike for $H^*(\Gamma, \ell^\infty(\Gamma))$ the Hilbert-space structure allows to numerically measure the size of the groups $H^*(\Gamma, \ell^2(\Gamma))$, which are in general huge and unwieldy as abelian groups.

4.3. **Examples and computations.** Computations of ℓ^2-Betti numbers of groups are rare, especially the ones where there is a non-zero ℓ^2-Betti number in some degree. But sometimes the computation, at least the non-vanishing result, follows quite formally. We present two such cases.

Example 4.9. Let $\Gamma = F_2$ be the free group of rank 2. A geometric model is the 4-regular tree T. By the explicit coycle construction in Example 2.15 we already know that $\beta_1(\Gamma) \neq 0$. But since $\beta_0(\Gamma) = 0$ (since Γ is infinite) and $\beta_p(\Gamma) = 0$ for $p > 1$, this also follows from the Euler-Poincare formula:

$$\beta_1(\Gamma) = -\chi(T; \Gamma) = -\chi(S^1 \vee S^1) = 1$$

The only relevant information for the previous example was the number of equivariant cells in the geometric model. The same technique helps in the next example (cf. [21, 5.29]).

Example 4.10. Let $G = SL_3(\mathbb{Q}_p)$. We show $\beta_2(G, \mu) \neq 0$ as easily as possible and then apply this to the deficiency of lattices in G. As geometric model, we take the Bruhat-Tits building X of G, which is 2-dimensional. By the fact that there are no 3-cells and by additivity of dimension, we obtain that

$$\begin{aligned}\beta_2(G, \mu) &= \dim_{(G,\mu)}(\operatorname{coker}(d^1)) \\ &\geq \dim_{(G,\mu)}\bigl(\hom_{\mathbb{C}}(C_2^{\mathrm{cw}}(X), L^2(G))^G\bigr) - \dim_{(G,\mu)}\bigl(\hom_{\mathbb{C}}(C_1^{\mathrm{cw}}(X), L^2(G))^G\bigr)\end{aligned}$$

In dimension 2 there is only one equivariant cell with stabilizer B, the Iwahori subgroup of G. Hence

$$\hom_{\mathbb{C}}(C_2^{\mathrm{cw}}(X), L^2(G))^G \cong \operatorname{map}(G/B, L^2(G))^G \cong L^2(G)^B$$

and with Example 3.21 it follows that $\dim_{(G,\mu)}\bigl(\hom_{\mathbb{C}}(C_2^{\mathrm{cw}}(X), L^2(G))^G\bigr) = 1/\mu(B)$. We normalize μ such that $\mu(B) = 1$. There are three equivariant 1-cells corresponding to the 1-dimensional faces of the 2-dimensional fundamental chamber. The stabilizer of each splits into $p+1$ many cosets of B. Therefore the μ-measure of each stabilizer is $(p+1)$. Similarly as above, this yields

$$\dim_{(G,\mu)}\bigl(\hom_{\mathbb{C}}(C_1^{\mathrm{cw}}(X), L^2(G))^G\bigr) = \frac{3}{p+1}.$$

Let $p \geq 3$. Then we obtain that $\beta_2(G, \mu) \geq 1 - \frac{3}{p+1} > 0$. Let us consider a lattice $\Gamma < G$. By Theorem 1.3,

$$\beta_2(\Gamma) = \mu(\Gamma\backslash G)\, \beta_p(G, \mu) \geq (1 - \frac{3}{p+1})\mu(\Gamma\backslash G).$$

Let R be a finite presentation of Γ, and let g be the number of generators and r be the number of relations in R. Let $X(R)$ be the universal covering of the presentation complex of R. One can regard $X(R)$ as the 2-skeleton of a geometric model Y from which we can compute the ℓ^2-Betti numbers of Γ. By the Euler-Poincare formula,

$$g - r = 1 - \chi(X(R); \Gamma) \leq 1 - \beta_0(\Gamma) + \beta_1(\Gamma) - \beta_2(\Gamma) \leq 1 - (1 - \frac{3}{p+1})\mu(\Gamma\backslash G).$$

We also used that Γ has property (T) which implies that $\beta_1(\Gamma) = 0$. Hence the deficiency of Γ, which is defined as the maximal value $g - r$ over all finite presentations, is bounded from above by $1 - (1 - 3/(p+1))\mu(\Gamma\backslash G)$.

Remark 4.11. The computation of ℓ^2-Betti numbers of locally compact groups reduces to the case of totally disconnected groups. Let G be a (second countable, unimodular) locally compact group. If its amenable radical K, its largest normal amenable (closed) subgroup, is non-compact, then $\beta_p(G,\mu) = 0$ for all $p \geq 0$ by [11, Theorem C], which generalizes a result of Cheeger and Gromov for discrete groups [5]. So let us assume that K is compact. Endowing G/K with the push-forward ν of μ, one obtains that $\beta_p(G/K,\nu) = \beta_p(G,\mu)$ [21, Theorem 3.14]. So we may and will assume that the amenable radical of G is trivial. Upon replacing G by a subgroup of finite index, G splits then as a product of a centerfree non-compact semisimple Lie group H and a totally disconnected group D. This is an observation of Burger and Monod [15, Theorem 11.3.4], based on the positive solution of Hilbert's 5th problem. Since H possesses lattices, one can use to Borel's computations of ℓ^2-Betti numbers of such lattices [13, Chapter 5] and Theorem 1.3 to obtain a computation for H. A Künneth formula [21, Theorem 6.7] then yields the ℓ^2-Betti numbers of $G = H \times D$ provided one is able to compute the ℓ^2-Betti numbers of D.

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