VANISHING OF ℓ^2-BETTI NUMBERS OF LOCALLY COMPACT GROUPS AS AN INVARIANT OF COARSE EQUIVALENCE

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Abstract. We provide a short proof that the vanishing of ℓ^2-Betti numbers of unimodular locally compact second countable groups is an invariant of coarse equivalence.

1. Introduction

The insight that the vanishing of ℓ^2-Betti numbers provides a quasi-isometry invariant is due to Gromov [11, Chapter 8], and positive results around this insight have a long history. The most important contribution is by Pansu [16] whose work on asymptotic ℓ^p-cohomology includes the fact that the vanishing of ℓ^2-Betti numbers of discrete groups of type F_∞, is a quasi-isometry invariant.

There is a growing interest in the metric geometry of locally compact groups [2,3]. We thus think it is important to have the quasi-isometry and coarse invariance of the vanishing of ℓ^2-Betti numbers available in the greatest generality. Following Pansu's ideas and relying on more recent advances in the theory of ℓ^2-Betti numbers, we provide a proof of the following result.

Theorem 1. *Let G and H be unimodular locally compact second countable groups. If G and H are coarsely equivalent then the n-th ℓ^2-Betti number of G vanishes if and only the n-th ℓ^2-Betti number of H vanishes.*

Independently, Li [13] recently proved the coarse invariance for countable discrete groups, using different methods.

Every locally compact, second countable group G (hereafter abbreviated by **lcsc**) has a left-invariant proper continuous metric by a theorem of Struble [24]. As any two left-invariant proper continuous metrics on G are coarsely equivalent, every lcsc group has a well defined coarse geometry. Further, any coarse equivalence between compactly generated lcsc groups is a quasi-isometry with respect to word metrics of compact symmetric generating sets and vice versa. See [3, Chapter 4] for a systematic discussion of these notions.

To even state Theorem 1 in that generality, recent advances in the theory of ℓ^2-Betti numbers were necessary. ℓ^2-Betti numbers of discrete groups enjoy a long history but it was not until recently that ℓ^2-Betti numbers were defined for arbitrary unimodular lcsc groups by Petersen [17], and a systematic theory analogous to the discrete case emerged [12, 17, 18]. Earlier studies of ℓ^2-Betti numbers of locally compact groups in specific cases can be found in [4, 5, 9].

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Previous results on coarse invariance. Pansu [16] introduced asymptotic ℓ^p-cohomology and proved its invariance under quasi-isometries. If a group Γ is of type F_∞, then the ℓ^p-cohomology of Γ coincides with its asymptotic ℓ^p-cohomology [16, Théorème 1]. The geometric explanation for the appearance of the type F_∞ condition is that the finite-dimensional skeleta of the universal covering of a classifying space of finite type are uniformly contractible. As an immediate consequence of Pansu's result, the vanishing of ℓ^2-Betti numbers is a quasi-isometry invariant among discrete groups of type F_∞. The same arguments work for totally disconnected groups admitting a topological model of finite type [21].

Elek [6] investigated the relation between ℓ^p-cohomology of discrete groups and Roe's coarse cohomology and proved similar results. Another independent treatment is due to Fan [7]. Genton [10] elaborated upon Pansu's methods in the case of metric measure spaces.

Oguni [15] generalised the quasi-isometry invariance of the vanishing of ℓ^2-Betti numbers from discrete groups of type F_∞ to discrete groups whose cohomology with coefficients in the group von Neumann algebra satisfies a certain technical condition. A similar technical condition appears in the proof of quasi-isometry invariance of Novikov-Shubin invariants of amenable groups [23], and it is unclear how much this condition differs from the type F_∞-condition. Oguni's groupoid approach is inspired by [8, 23] and quite different from the approaches by Elek, Fan, and Pansu.

Li [13] recently proved Theorem 1 for discrete countable groups using groupoid techniques.

Structure of the paper. We review the necessary basics of ℓ^2-Betti numbers and continuous cohomology in Section 2. In Section 3 we define coarse ℓ^2-cohomology for lcsc groups and show that it is isomorphic to continuous cohomology. In Section 4 we conclude the proof of Theorem 1.

2. Continuous cohomology and ℓ^2-Betti numbers of lcsc groups

Let G be a unimodular lcsc group with Haar measure μ. Let X be a locally compact second countable space with Radon measure ν. Let E be a Fréchet space.

The space $C(X, E)$ of continuous functions from X to E becomes a Fréchet space when endowed with the topology of compact convergence. Let $L^2_{loc}(X, E)$ be the space of equivalence classes of measurable maps $f\colon X \to E$ up to ν-null sets such that $||f|_K||_E$ is square-integrable for every compact subset $K \subset X$. The L^2-norm of the function $||f|_K||_E$ defines a semi-norm p_K on $L^2_{loc}(X, E)$. The family of semi-norms p_K, $K \subset E$, turns $L^2_{loc}(X, E)$ into a Fréchet space.

We call a Fréchet space E with a continuous (i.e. $G \times E \to E$ is continuous) linear G-action a *G-module*. A continuous linear G-equivariant map between G-modules is a *homomorphism of G-modules*. If E is a G-module and G acts continuously and ν-preserving on X then $C(X, E)$ and $L^2_{loc}(X, E)$ become G-modules via $(g \cdot f)(x) = gf(g^{-1}x)$ for $x \in X$ and $g \in G$ [1, Proposition 3.1.1]. The usual homogeneous coboundary map

$$d^{n-1}f(g_0, ..., g_n) = \sum_{i=0}^{n} (-1)^i f(g_0, ..., \widehat{g_i}, ..., g_n) \tag{1}$$

defines cochain complexes $C(G^{*+1},E)$ and $L^2_{loc}(G^{*+1},E)$ of G-modules (cf. [1, Proposition 3.2.1]). Here we take the diagonal G-action on G^{*+1}. We recall the following definition.

Definition 2. The *(continuous) cohomology* of G in E is the cohomology

$$H^n(G,E)=H^n\big(C(G^{*+1},E)^G\big)$$

of the G-invariants of $C(G^{*+1},E)$. The *reduced (continuous) cohomology* $\underline{H}^*(G,E)$ is a quotient of $H^*(G,E)$ obtained by taking the quotient with the closure of $\operatorname{im} d^{*-1}$ instead of $\operatorname{im} d^{*-1}$.

We have an obvious inclusion

$$I^*: C(G^{*+1},E)\to L^2_{loc}(G^{*+1},E). \tag{2}$$

The maps I^* form a cochain map of G-modules. Taking a positive function $\chi\in C_c(G)$ there is a cochain map $R^*: L^2_{loc}(G^{*+1},E)\to C(G^{*+1},E)$ of G-modules

$$(R^nf)(g_0,...,g_n)=\int_{G^{n+1}}f(h_0,...,h_n)\chi(g_0^{-1}h_0)\cdot ...\cdot\chi(g_n^{-1}h_n)d\mu(h_0,...,h_n)$$

such that $I^*\circ R^*$ and $R^*\circ I^*$ are homotopic (as cochain maps of G-modules) to the identity [1, Proposition 4.8]. So we have the following useful fact:

Theorem 3. *The cochain map I^* in* (2) *induces isomorphisms in cohomology and in reduced cohomology.*

Next we turn to the case where the coefficient module $E=L^2(G)$ is the regular representation, relevant for the definition of ℓ^2-Betti numbers.

Let $L(G)$ be the *von Neumann algebra of G*; the Haar measure μ defines a semifinite trace tr_μ on $L(G)$. There are a natural left G-action and a natural right $L(G)$-action on $L^2(G)$, and the two actions commute. Hence also the G-actions on $C(G^{*+1},L^2(G))$ and $L^2_{loc}(G^{*+1},L^2(G))$ considered previously and the $L(G)$-actions induced from the right $L(G)$-action on $L^2(G)$ commute. So the (reduced and non-reduced) continuous cohomology of G in $L^2(G)$ is naturally a $L(G)$-module[1]. Obviously, the cochain map I^* above is compatible with the $L(G)$-module structures. The groups $H^*(G,L^2(G))$ are called the *(continuous) ℓ^2-cohomology* of G. Similarly for the reduced cohomology.

Petersen [17] extended Lück's dimension function from finite von Neumann algebras to semifinite von Neumann algebras. The dimension function $\dim_\mu$ with respect to (G,μ) is a non-trivial dimension for (algebraic) right $L(G)$-modules that is additive for short exact sequences of $L(G)$-modules. It scales as $\dim_{c\mu}=c^{-1}\dim_\mu$ for $c>0$. The fact that a $L(G)$-module has dimension zero can be expressed without referring to the trace: it is an algebraic fact. The following criterion was shown by the first author for finite von Neumann algebras [22, Theorem 2.4]; it was extended to the semifinite case by Petersen [17, Lemma B.27].

Theorem 4. *An $L(G)$-module M satisfies $\dim_\mu(M)=0$ if and only if for every $x\in M$ there is an increasing sequence (p_i) of projections in $L(G)$ with $\sup p_i=1$ such that $xp_i=0$ for every $i\in\mathbb{N}$.*

[1]When talking about $L(G)$-modules we mean the algebraic module structure and ignore topologies.

Definition 5. The n-th *ℓ^2-Betti number of* G is the $L(G)$-dimension of its reduced continuous cohomology with coefficients in $L^2(G)$, i.e.

$$\beta^n_{(2)}(G) := \dim_\mu \underline{H}^n(G, L^2(G)) \in [0, \infty].$$

Remark 6. Equivalently, the n-th ℓ^2-Betti number can be defined as the $L(G)$-dimension of the non-reduced cohomology $H^n(G, L^2(G))$. This is a non-trivial fact (see [12, Theorem A]). For discrete G, our definition coincides with Lück's definition in [14]. Again, this is non-trivial and shown in [19, Theorem 2.2].

The following lemma was observed in [17, Proposition 3.8]. Since it is a direct consequence of Theorem 4 we present the argument.

Lemma 7. $\beta^n_{(2)}(G) = 0 \Leftrightarrow \underline{H}^n(G, L^2(G)) = 0.$

Proof. Let $\beta^n_{(2)}(G) = 0$. Let $f\colon G^{n+1} \to L^2(G)$ be a cocycle representing a cohomology class $[f]$ in $\underline{H}^n(G, L^2(G))$. By Theorem 4 there is an increasing sequence of projections $p_j \in L(G)$ whose supremum is 1 such that each fp_j is a coboundary $d^{n-1}b_j$. It is clear that $fp_j = d^{n-1}b_j$ converges to f in the topology of $C(G^{n+1}, L^2(G))$, thus $[f] = 0$. □

3. Coarse equivalence and coarse ℓ^2-cohomology

Let G be a unimodular lcsc group. We fix a left-invariant proper continuous metric d on G. Let μ be a Haar measure on G. Let μ_n be the n-fold product measure of μ on G^n.

For every $R > 0$ and $n \in \mathbb{N}_0$ we consider the closed subset

$$G^n_R := \{(g_0, ..., g_{n-1}) \in G^n \mid d(g_i, g_j) \leq R \text{ for all } 0 \leq i, j \leq n-1\}$$

and a family of semi-norms for measurable maps $\alpha\colon G^{n+1} \to \mathbb{C}$ defined by

$$\|\alpha\|^2_R = \int_{G^{n+1}_R} |\alpha(g_0, ..., g_n)|^2 d\mu_{n+1} \in [0, \infty].$$

Let $CX^n_{(2)}(G)$ be the space of equivalence classes (up to μ_{n+1}-null sets) of measurable maps $\alpha\colon G^{n+1} \to \mathbb{C}$ such that $\|\alpha\|_R < \infty$ for every $R > 0$. The semi-norms $\|\_\|_R$, $R > 0$, turn $CX^n_{(2)}(G)$ into a Fréchet space. It is straightforward to verify that the homogeneous differential (1) yields a well-defined, continuous homomorphism $CX^n_{(2)}(G) \to CX^{n+1}_{(2)}(G)$ (cf. [10, Proposition 2.3.3]). Thus we obtain a cochain complex of Fréchet spaces.

Definition 8. The *coarse ℓ^2-cohomology of* G is defined as

$$HX^n_{(2)}(G) = H^n\big(CX^*_{(2)}(G)\big).$$

By taking the quotients by the closure of the differentials, one defines similarly the *reduced coarse ℓ^2-cohomology* $\underline{H}X^n_{(2)}(G)$.

Remark 9. The previous definition is the continuous analog of Elek's definition [6, Definition 1.3] in the discrete case (Elek gives credits to Roe [20]). It is very much related to Pansu's *asymptotic ℓ^2-cohomology* [16], which was considered in the generality of metric measure spaces by Genton [10]. The difference of our definition to the one in Genton [10] is as follows: $CX^*_{(2)}(G)$ is an inverse limit of spaces $L^2(G^{*+1}_R)$. Unlike us, Genton takes first the cohomology of $L^2(G^{*+1}_R)$ and then the inverse limit. Under some uniform contractibility assumptions the two definitions coincide but likely not in general.

Theorem 10. *Let G be a unimodular lcsc group. For every $n \geq 0$, there are isomorphisms of abelian groups*

$$H^n(G, L^2(G)) \cong HX^n_{(2)}(G),$$
$$\underline{H}^n(G, L^2(G)) \cong \underline{H}X^n_{(2)}(G).$$

Proof. We have the obvious embedding

$$L^2_{loc}(G^{n+1}, L^2(G)) \subset L^2_{loc}(G^{n+1}, L^2_{loc}(G))$$

and the exponential law (see [1, Lemme 1.4] for a proof but beware of the typo in the statement)

$$L^2_{loc}(G^{n+1}, L^2_{loc}(G)) \cong L^2_{loc}(G^{n+1} \times G).$$

Thus an element in $L^2_{loc}(G^{n+1}, L^2(G))^G$ is represented by a measurable complex function in $(n+2)$-variables. For $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$ we define μ_{n+2}-almost everywhere

$$F^n(\alpha)(x_0, \dots, x_n, x) = \alpha(x^{-1}x_0, \dots, x^{-1}x_n)(x).$$

The measurable function $F^n(\alpha)$ is invariant by translation in the $(n+2)$-th variable. By Fubini's theorem we may regard $F^n(\alpha)$ as a measurable function $E^n(\alpha)\colon G^{n+1} \to \mathbb{C}$ in the first $(n+1)$-variables. We may think of $E^n(\alpha)$ as an evaluation of α at e. Let $B(R)$ denote the R-ball around $e \in G$. Next we show that $\|E^n(\alpha)\|_R < \infty$ for every $R > 0$, thus $E^n(\alpha) \in CX^n_{(2)}(G)$.

Since $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$ we have

$$\begin{aligned}\infty &> \int_{B(2R)^{n+1}} \int_G |\alpha(x_0, x_1, ..., x_n)(x)|^2 d\mu d\mu_{n+1} \\ &= \int_{B(2R)^{n+1}} \int_G |\alpha(x, xx_0^{-1}x_1, ..., xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1}.\end{aligned}$$

The map

$$m\colon G^{n+2} \to G^{n+2}, (x_0, \dots, x_n, x) \mapsto (x, xx_0^{-1}x_1, \dots, xx_0^{-1}x_n, x_0)$$

is measure preserving. Further, we have

$$m\big(G^{n+1}_R \times B(R)\big) \subset B(2R)^{n+1} \times G.$$

This implies the first inequality below. The first equality follows from the fact that $(x_0, \dots, x_n, x) \mapsto (x^{-1}x_0, \dots, x^{-1}x_n, x)$ is a measure preserving measurable automorphism of $G^{n+1}_R \times B(R)$.

$$\begin{aligned}\int_{B(2R)^{n+1}} \int_G &|\alpha(x, xx_0^{-1}x_1, ..., xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1} \\ &\geq \int_{G^{n+1}_R} \int_{B(R)} |\alpha(x_0, \dots, x_n)(x)|^2 d\mu d\mu_{n+1} \\ &= \int_{G^{n+1}_R} \int_{B(R)} |\alpha(x^{-1}x_0, \dots, x^{-1}x_n)(x)|^2 d\mu d\mu_{n+1} \\ &= \mu(B(R)) \|E^n(\alpha)\|_R.\end{aligned}$$

Hence $\|E^n(\alpha)\|_R$ is finite for every $R > 0$. That $E^*\colon L^2_{loc}(G^{*+1}, L^2(G))^G \to CX^*_{(2)}(G)$ defines a cochain map is obvious. The above computation also implies that E^* is continuous with respect to the Fréchet topologies.

Given $\beta \in CX^n_{(2)}(G)$ we define

$$M^n(\beta)(g_0, \dots, g_n)(g) = \beta(g^{-1}g_0, \dots, g^{-1}g_n)$$

for μ_{n+2}-almost every $(g_0, \dots, g_n, g)$. The function $M^n(\beta)$ defines an element in $L^2_{loc}(G^{n+1}, L^2(G))^G$. The G-invariance of $M^n(\beta)$ is obvious. We have to show that $\|M^n(\beta)|_{B(R)^{n+1}}\|$ is square-integrable for every $R > 0$. This follows from the following computations which is based on the arguments above in reversed order.

$$\begin{aligned}\mu(B(R)) \int_{G^{n+1}_{2R}} |\beta(g_0, ..., g_n)|^2 d\mu_{n+1} &= \int_{G^{n+1}_{2R}} \int_{B(R)} |\beta(g_0, ..., g_n)|^2 d\mu d\mu_{n+1} \\ &\geq \int_{B(R)^{n+1}} \int_G |\beta(g^{-1}g_0, \dots, g^{-1}g_n)| d\mu d\mu_{n+1}\end{aligned}$$

Obviously, M^* is a chain map. Continuity follows from the previous computation. It is clear that M^* and E^* are mutual inverses. Using Theorem 3, this concludes the proof. □

4. Coarse invariance

We recall the notion of coarse equivalence. A map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is *coarse Lipschitz* if there is a non-decreasing function $a\colon [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} a(t) = \infty$ such that

$$d_Y(f(x), f(x')) \leq a(d(x, x'))$$

for all $x, x' \in X$. We say that two such maps f, g are *close* if

$$\sup_{x\in X} d_Y(f(x), g(x)) < \infty.$$

A coarse Lipschitz map $f\colon X \to Y$ is a *coarse equivalence* if there is a coarse Lipschitz map $g\colon Y \to X$ such that fg and gf are close to the identity. We say g is a *coarse inverse* of f.

Lemma 11. Coarsely equivalent lcsc groups are measurably coarse equivalent, i.e. if G and H are coarse equivalent lcsc groups then there are *measurable* coarse Lipschitz maps $f\colon G \to H$ and $g\colon H \to G$ such that fg and gf are close to the identity.

Proof. We choose left-invariant continuous proper metrics d_G and d_H on G and H, respectively. Let $f\colon G \to H$ be a coarse Lipschitz map with $d_H(f(x), f(x')) \leq a(d_G(x, x'))$. Let $t > 0$. We pick a countable measurable partition $\mathcal{U}$ of G whose elements have diameter $\leq t$ and choose an element $x_U \in U$ for every $U \in \mathcal{U}$. By setting $\tilde{f}(x) = f(x_U)$ for $x \in U$ we obtain a coarse Lipschitz map $\tilde{f}\colon G \to H$ which satisfies $d(\tilde{f}(x), \tilde{f}(x')) \leq a(d(x, x')+2t)$ and is close to f with $d(\tilde{f}(x), f(x)) \leq a(2t)$. Analogously, we construct a measurable coarse Lipschitz map $\tilde{g}$, constructed from a coarse Lipschitz map $g\colon H \to G$ which is a coarse inverse to f. It is obvious that $\tilde{g}$ is a coarse inverse to $\tilde{f}$. □

Theorem 12. *Coarsely equivalent unimodular lcsc groups have isomorphic reduced and non-reduced coarse ℓ^2-cohomology groups.*

Proof. Let G and H lcsc groups with Haar measures μ and ν, respectively. Let $f\colon G \to H$ be a coarse equivalence with coarse inverse g. Because of lemma 11 we can further assume that f and g are measurable. We define a map $\chi\colon G \times G \to \mathbb{R}$ by

$$\chi(x, y) = \frac{\mathbb{1}_{B_x(c)}(y)}{\mu(B(c))}$$

where we choose c such that $\mu(B(c)) \geq 1$. Then χ is a measurable function with $\chi(x, y) = \chi(y, x)$ and $\int_G \chi(x, y) d\mu(y) = 1$. We use the following notation:

$$\chi\colon G^{n+1} \times G^{n+1} \to \mathbb{R}, \quad \chi((x_0, ..., x_n), (y_0, ..., y_n)) = \chi(x_0, y_0) \cdot ... \cdot \chi(x_n, y_n).$$

Analogously, we define $\chi'\colon H^{n+1} \times H^{n+1} \to \mathbb{R}$ with some radius c'. Now we can define the maps $f^*\colon HX^*_{(2)}(H) \to HX^*_{(2)}(G)$ and $g^*\colon HX^*_{(2)}(G) \to HX^*_{(2)}(H)$ as follows where we use x_i for elements in G and y_i for elements of H:

$$f^*\alpha(x_0, ..., x_n) = \int_{H^{n+1}} \alpha(y_0, ..., y_n)\chi'\left((f(x_0), ..., f(x_n)), (y_0, ..., y_n)\right) d\nu_{n+1}$$
$$g^*\beta(y_0, ..., y_n) = \int_{G^{n+1}} \beta(x_0, ..., x_n)\chi\left((g(y_0), ..., g(y_n)), (x_0, ..., x_n)\right) d\mu_{n+1}.$$

The idea of averaging over a function like χ goes back to Pansu; it is necessary in our context since the maps f and g do not preserve the measure classes, in general.

First of all, we check that these are well-defined continuous cochain maps.

$$\begin{aligned}
\infty > \|\alpha\|^2_{a(R)+c'} &= \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \cdot \mathbb{1}_{H^n_{a(R)+c'}} d\nu_{n+1} \\
&\geq \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \int_{G^{n+1}_R} \chi'\left((f(x_0), ..., f(x_n)), (y_0, ..., y_n)\right) d\mu_{n+1} d\nu_{n+1} \\
&= \int_{G^{n+1}_R} \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \chi'\left((f(x_0), ..., f(x_n)), (y_0, ..., y_n)\right) d\nu_{n+1} d\mu_{n+1} \\
&\geq \int_{G^{n+1}_R} \left| \int_{H^{n+1}} \alpha(y_0, ..., y_n)\chi'\left((f(x_0), ..., f(x_n)), (y_0, ..., y_n)\right) d\nu_{n+1} \right|^2 d\mu_{n+1} \\
&= \int_{G^{n+1}_R} |f^n\alpha(x_0, ..., x_n)|^2 d\mu_{n+1} = \|f^n\alpha\|^2_R
\end{aligned}$$

It is a direct computation that $d^n \circ f^n = f^{n+1} \circ d^n$. It remains to show that there is a cochain homotopy $h\colon CX^*_{(2)}(H) \to CX^{*-1}_{(2)}(H)$ such that $\mathrm{Id} - g^*f^* = hd + dh$. We define $h^{n+1}_i\colon CX^{n+1}_{(2)}(H) \to CX^n_{(2)}(H)$ by

$$h^{n+1}_i \alpha(y_0, ..., y_n) = \int_{H^{n+1}} \alpha(\tilde{y}_0, ..., \tilde{y}_i, y_i, ..., y_n)\chi'((y_0, ..., y_n), (\tilde{y}_0, ..., \tilde{y}_n)) d\nu_{n+1}(\tilde{y})$$

and set

$$h^{n+1} = \sum_{i=0}^{n} (-1)^i h^{n+1}_i.$$

That h^* is well-defined is a similar consideration as to show that f^* and g^* are well-defined. Now let us denote the i-th term of the coboundary map by d^n_i, i.e.

$d_i^n \alpha(y_0, ..., y_{n+1}) = \alpha(y_0, ..., \widehat{y_i}, ..., y_{n+1})$. It is straightforward to verify that we have the following relations:

$$\begin{aligned} h_n^{n+1} \circ d_{n+1}^n &= g^n \circ f^n, \\ h_0^{n+1} \circ d_0^n &= \mathrm{Id}_{CX_{(2)}^n(H)}, \\ h_j^{n+1} \circ d_i^n &= d_i^{n-1} \circ h_{j-1}^n && \text{for } 1 \leq j \leq n \text{ and } i \leq j, \\ h_j^{n+1} \circ d_i^n &= d_{i-1}^{n-1} \circ h_j^n && \text{for } 1 \leq i \leq n \text{ and } i > j. \end{aligned}$$

We get $h^{n+1}d^n + d^{n-1}h^n = \mathrm{Id}_{CX_{(2)}^n(H)} - g^n \circ f^n$. The same construction applies to f^*g^* which completes the proof. □

Proof of Theorem 1. Let G and H be unimodular lcsc groups. Let G and H be coarsely equivalent. Then we have the following equivalences:

$$\begin{aligned} \beta_{(2)}^2(G) = 0 &\Leftrightarrow \underline{H}^n(G, L^2(G)) = 0 && \text{(Lemma 7)} \\ &\Leftrightarrow \underline{H}X_{(2)}^n(G) = 0 && \text{(Theorem 10)} \\ &\Leftrightarrow \underline{H}X_{(2)}^n(H) = 0 && \text{(Theorem 12)} \end{aligned}$$

Going the same steps backwards for the group H finishes the proof. □

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