FINANCE 601
Introduction to Finance

Lecture Notes

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Chapter 7

Options

A call option is a financial instrument that gives its holder the right, not the obligation, to purchase from its seller (writer) one unit of the underlying security, at a predetermined price, at or before an expiry date. In contrast, a put option gives its holder the right to sell to its seller instead. The buyer of the option is the party who pays for the right involved and, therefore, is the party who can make a choice (to exercise the option or not). The seller (writer) of the option is the party who receives the payment from the buyer and, therefore, is the party with no subsequent choice.

American and European options differ in that the latter can be exercised only at the expiry date. The 1973 work of Black and Scholes on the pricing of European call options of stocks has led to many innovations in the financial world. The most noticeable are the phenomenal growth of markets for trading various derivative securities and the corresponding research activities on creating and pricing such securities. Given its academic and practical significance, the Black-Scholes option pricing model has been part of the finance curriculum for decades now, even at the introductory level.

The derivation of the Black-Scholes model requires advanced mathematical tools including those for solving partial differential equations. The derivation of the corresponding binomial option pricing model is less complicated. For analytical tractability, the assumptions below are required for the derivations. Such assumptions are also required for the basic option properties to hold.

1. There are zero transaction costs.

2. Risk-free lending and borrowing are available at the same interest rate.

3. The underlying stock pays no dividend. (This assumption, if required, will be indicated explicitly.)

4. Arbitrage opportunities, if any, disappear very quickly.

The coverage in the main text of this chapter includes the following materials:

1. Illustrations of profits or losses from investments involving options for changes in the underlying stock price.

2. Basic option properties with explanations, and with numerical illustrations if necessary.

3. Put-call parity. The idea is that, for corresponding European call and put options on the same underlying stock that pays no dividends, the price of the call option and the price of the put option can be deduced directly from each other. The idea draws on a property that writing a call, buying the corresponding put, and owning the underlying stock at the same time is equivalent to owning a risk-free asset.

4. The Black-Scholes option pricing model: a formula for call options and a formula for put options, along with the underlying intuition.

5. The corresponding binomial option pricing model: a formula for call options and a formula for put options, along with the underlying intuition.

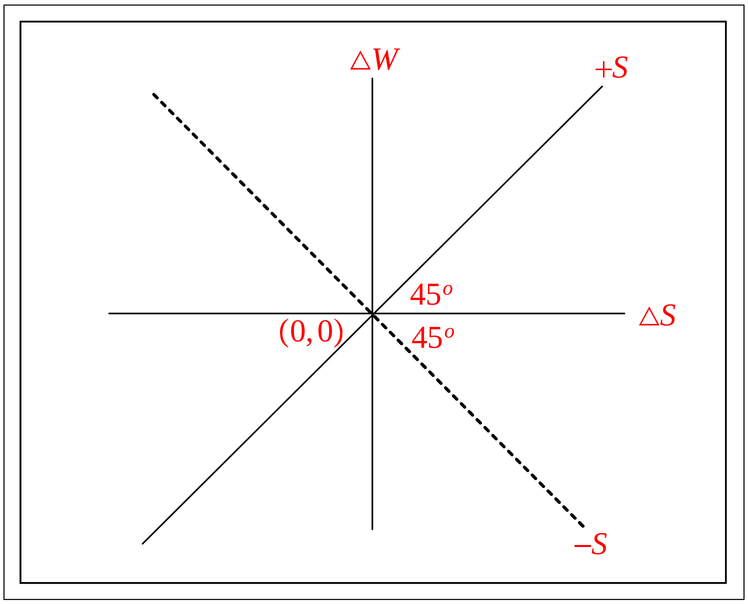
6. A practical application of options: the Chicago Board Options Exchange (CBOE) Volatility Index (VIX). The VIX is a well-known benchmark index to gauge the volatility of the U.S. equity market in the near future.

7.1 Gains or Losses for Changes in the Underlying Stock price

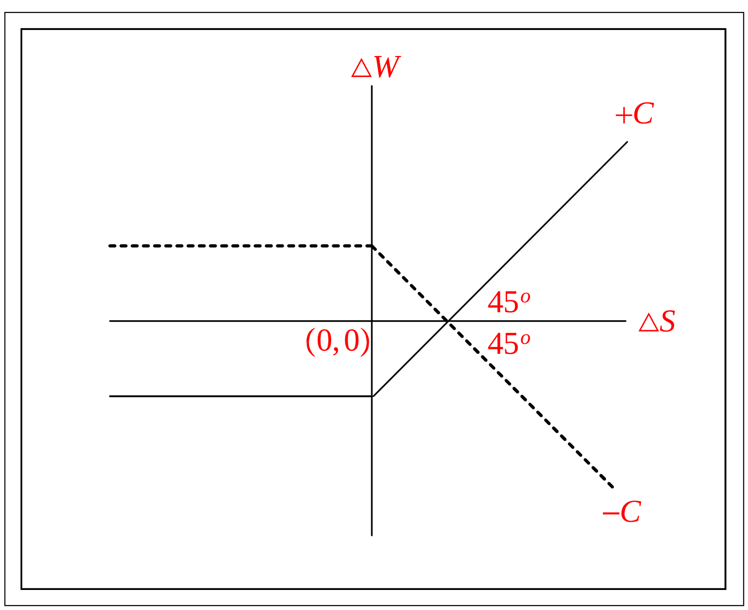
Let S be the price of the underlying stock and X (or E, which is also commonly used) be the exercise price of an option. The exercise price is also known as the strike price. If $S > X$, a call is *in the money*, but a put is *out of the money*. If $S < X$, a call is out of the money, but a put is in the money. If $S = X$, an option is *at the money*. Let

$$\begin{aligned} \triangle S &= \text{a change in the price of the underlying stock } (\textit{assuming the initial } S = X) \\ \text{and } \triangle W &= \text{the corresponding profit (if positive) or loss (if negative).} \end{aligned}$$

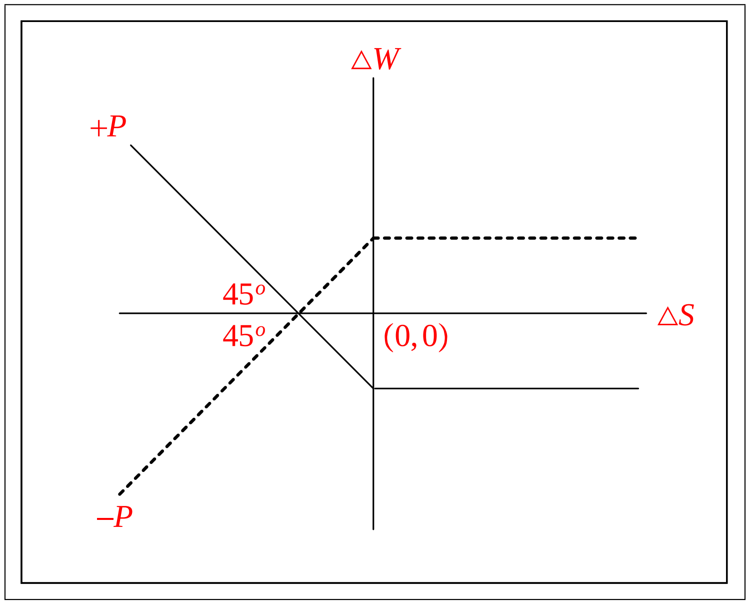
With $+S$ and $-S$ denoting the holder and the short seller of the stock, respectively, we have the following:



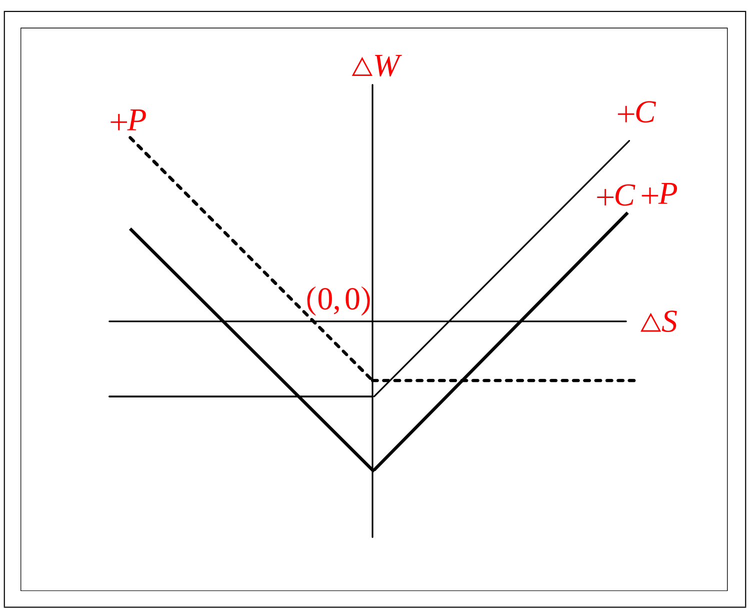
With $+C$ and $-C$ denoting the buyer and the writer of a call option on the stock, respectively, we have the following:



Likewise, with $+P$ and $-P$ denoting the buyer and the writer of a put option on the stock, respectively, we have the following:



Here is an example of combinations of options,; a straddle involves the purchase of a call option and a put option on the same underlying stock. With $+C+P$ denoting such a buyer, we have the following:



7.2 Some Basic Properties of Call Option Prices

1. The price of a call option cannot be higher than the price of the underlying stock.

$$C \leq S.$$

   - A call option gives the holder the right to buy a share of the stock at a predetermined price. Since the stock can also be purchased directly in the market, no rational investor would pay more than the share price for just the right to buy the stock.

2. The price of a call option cannot be negative.

$$C \geq 0.$$

   - An option is a right, not an obligation, for the holder. At the worst, it is worth nothing.

3. The price of a call option on a non-dividend paying stock cannot be lower than the difference between the stock price and the present value of the exercise price; that is,

$$C \geq S - Xe^{-rT}.$$

   - To see such a property, let us consider a European call option. Let r be the annual risk-free interest rate with continuous compounding. At this interest rate, each dollar today becomes
$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$
dollars after one year. Here, $e = 2.71828\ldots$
   - After T years (where T can be a fraction; for example, $T = \frac{1}{4}$ is three months), each dollar today becomes e^{rT} dollars. Since the present value of \$1 to be received or paid T years from now is e^{-rT} dollars, the corresponding present value of the exercise price of a European call is Xe^{-rT} dollars. The above statement about the price of a European call can be expressed as

$$C \geq S - Xe^{-rT}.$$

   The same statement applies to an American call as well. The reason is that an American call provides the holder with more opportunities than what a European call provides. Thus, the price of an American call cannot be lower than the price of an otherwise identical European call.

- Consider an example for a European call, where $S = \$100$, $X = \$60$, $r = 4\% = 0.04$, and $T = 0.5$ year. With $e^{-rT} = e^{-0.02} = 0.9802$, the difference between the stock price and the present value of the exercise price is

$$S - Xe^{-rT} = \$100 - \$60(0.9802) = \$100 - \$58.81 = \$41.19.$$

In this case, we expect the price of the option, C, to be no lower than \$41.19.

Suppose that $C = \$38$ which is less than \$41.19. Arbitrage opportunities exist. An arbitrageur would buy the call and short the stock to get an immediate cash inflow of

$$-\$38 + \$100 = \$62.$$

The \$62 would be invested risk-free until the expiry date of the call to become

$$\$62 \times e^{rT} = \$62 \times \frac{1}{0.9802} = \$63.25.$$

At the expiry date, should the stock price be greater than \$60, the arbitrageur would exercise the option for \$60, close the short position with the stock thus acquired, and make an arbitrage profit of

$$\$63.25 - \$60 = \$3.25.$$

Should the stock price be lower than \$60, the arbitrageur would discard the option, buy the stock in the market, and close out the short position for an even greater profit.

- The no-dividend assumption removes any potential dividend burden from the arbitrageur when holding the stock in a short position. The absence of arbitrage opportunities ensures that the condition

$$C \geq S - Xe^{-rT}$$

be satisfied.

4. The price of a call option at the expiry date is 0 or the difference between the underlying stock price and the exercise price, whichever is higher.

- At the expiry date, if $S \leq X$, no one would exercise the option. The value of the option is 0.
- At the expiry date, if $S > X$, the holder of the call is able to make $S - X$ dollars by exercising the option.

5. If two American call options differ only in their expiry dates, the one with a longer life cannot be worth less.
   - The holder of the longer-life American call has all the exercise opportunities that are available to the holder of the otherwise identical shorter-life American call, and more.
   - The same reason does not apply to European calls. However, it can be verified by an arbitrage proof that, if two European calls on a non-dividend paying stock differ only in their expiry dates, the one with a longer life cannot be worth less.

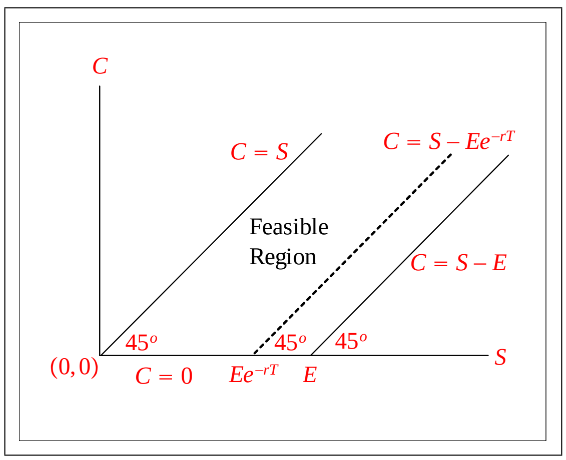
6. If two call options differ only in their exercise prices, then the one with a lower exercise price cannot be worth less.
   - If a call is exercised, the payoff is the gap between the underlying stock price and the exercise price. The lower the exercise price, the greater is the payoff.

7. An American call option on a non-dividend paying stock will not be exercised before the expiry date.
   - The idea is that, by exercising the option before the expiry date, the payoff is $S - X$. As the present value of X before the expiry date, Xe^{-rT}, is less than X, we have
   $$C \geq S - Xe^{-rT} > S - X \quad \text{(see property 3)}.$$
   If the holder of the call believes that the stock price will drop later, a better payoff can be achieved by selling the option for C dollars than by exercising it to get only $S - X$ dollars.
   In the example in property 3, $S = \$100$, $X = \$60$, $r = 4\%$, $T = 0.5$ year, and $C \geq \$41.19$. The payoff from exercising the option is $S - X = \$40$ and from selling the option is no less than $\$41.19$.
   - The price of an American call cannot be lower than the price of an otherwise identical European call. However, if there are no dividends to be paid over the life of the firm's options, an American call is worth exactly the same as an otherwise identical European call.

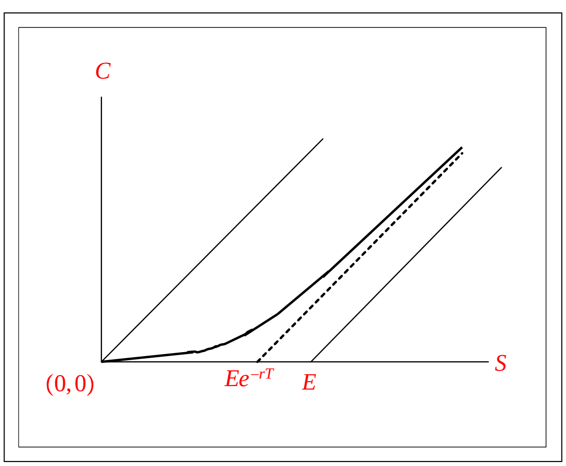
Arbitrage proofs for three of the above seven properties are provided in Appendix A of this chapter. Specifically, they include property 3, 5, and 6. For property 5, the proof is confined to European calls, as the same property is obvious for American calls. In an arbitrage proof, we use the alternative (opposite) conclusion to check whether arbitrage profits are available. The presence of arbitrage profits indicates that we can rule out the alternative conclusion. Students are strongly encouraged to read the appendices where arbitrage proofs are covered, given the usefulness of such proofs for verifying key results from various finance topics.

The value of a call option, for a given price S of the underlying stock, is always within the region as defined by the following:

$$\begin{array}{ll} C \leq S & \text{(see property 1);} \\ C \geq 0 & \text{(see property 2);} \\ C \geq S - Xe^{-rT} & \text{(see property 3).} \end{array}$$



In the above graph, the symbol E is used instead of X (as it is quite common to use E to represent the strike price of an option). Here is a sketch of C versus S:



To determine the solid curve requires the use of an option pricing model such as the Black-Scholes model. For such a purpose, the volatility of the underlying stock in terms of its standard deviation of returns is required.

Example 40 *Consider two European call options, labeled as #1 and #2, which have the same features except for the exercise price. Let us label their exercise prices as X_1 and X_2. Suppose that $S = \$40$, $r = 6\%$, $T = 0.5$ years, $X_1 = \$30$, and $X_2 = \$31$. Suppose also that the option prices are $C_1 = \$11$ and $C_2 = \$12$. Why is at least one of the two options mispriced? Illustrate also how arbitrage profits can be made from the mispricing?*

According to property 6, option #1 which has a lower exercise price cannot be worth less. However, since $C_1 = \$11$ and $C_2 = \$12$, this condition is violated. Thus, at least one of the two options is mispriced.

To make an arbitrage profit, you buy option #1 and write option #2. The cash inflow is $-\$11 + \$12 = +\$1$, and this amount is to be invested risk-free. At the expiry date, there are three possible outcomes. The stock price is: (a) below \$30, (b) between \$30 and \$31, and (c) above \$31.

(a) Neither option will be exercised. You will have $\$1 \times e^{rT} = \$e^{0.06\times 0.5} = \$1.03$ from the risk-free investment.

(b) Option #2 will not be exercised by its holder. You will exercise option #1 by paying \$30 for the stock whose market price is between \$30 and \$31. If you sell the stock thus acquired, the profit will be from \$0 to \$1, depending on its market price. With \$1.03 from your risk-free investment, the total profit will be from \$1.03 to \$2.03.

(c) You will exercise option #1 by paying \$30 for the stock. The holder of option #2 will exercise it by paying you \$31 for the same stock. Your profit, therefore, is $\$31 - \$30 = \$1$. With \$1.03 from your risk-free investment, the total profit will be \$2.03. ■

7.3 Some Basic Properties of Put Option Prices

1. The price of a put option cannot be higher than the exercise price. The price of a European put option cannot be higher than the present value of the exercise price.

That is,

$$\begin{array}{ll} P \leq X & \text{(for both American and European puts);} \\ P \leq Xe^{-rT} & \text{(for European puts only).} \end{array}$$

- The highest possible payoff from a put option is the exercise price X itself. This is the extreme case where $S = 0$.

  No rational investor would be willing to pay more than X dollars just for an opportunity to make X dollars or less.

- In the case of a European put, the highest possible payoff of X dollars is T years away (where T is a fraction). Its present value is Xe^{-rT}, which is less than X.

2. The price of a put option cannot be negative.

$$P \geq 0$$

- An option is a right, not an obligation. If the option becomes worthless, its price is 0.

3. The price of a put option on a non-dividend paying stock cannot be lower than the difference between the present value of the exercise price and the stock price.

$$P \geq Xe^{-rT} - S$$

- Consider a European put, where $S = \$50$, $X = \$55$, $r = 4\% = 0.04$, and $T = 0.5$ year. With $e^{-rT} = 0.9802$, the difference between the present value of the exercise price and the stock price is

$$\begin{aligned} Xe^{-rT} - S &= \$55(0.9802) - \$50 \\ &= \$53.91 - \$50 = \$3.91. \end{aligned}$$

  In this case, we would expect the price of the option, P, to be no lower than \$3.91.

  Suppose that $P = \$2$, which is less than \$3.91. An arbitrageur could borrow \$52 to buy both the stock and the put.

  At the expiry date, should the stock price be below \$55, the arbitrageur would exercise the option to sell the stock for \$55, repay the loan of

$$\$52e^{rT} = \$52 \times \frac{1}{0.9802} = \$53.05,$$

and make a profit of

$$\$55 - \$53.05 = \$1.95.$$

Should the stock price be greater than \$55, the arbitrageur would discard the option, sell the stock, and repay the loan to make a profit greater than \$1.95.

The absence of arbitrage opportunities ensures that the condition $P \geq Xe^{-rT} - S$ be satisfied.

4. The price of a put option at the expiry date is 0 or the difference between the exercise price and the underlying stock price, whichever is higher.

   - At the expiry date, if $S \geq X$, no one would exercise the option. The value of the option is 0.
   - At the expiry date, if $S < X$, the holder of the put is able to make $\$(X - S)$ by exercising the option.

5. If two American puts differ only in their expiry dates, the one with a longer life cannot be worth less.

   - The holder of the longer-life American put has all the exercise opportunities that are available to the holder of the otherwise identical shorter-life American put, and more.

6. If two put options differ only in their exercise prices, then the one with a higher exercise price cannot be worth less.

   - If a put option is exercised, the payoff is the gap between the exercise price and the underlying stock price. The higher the exercise price, the greater is the payoff.

7. It can be optimal to exercise an American put option on a non-dividend paying stock before the expiry date.

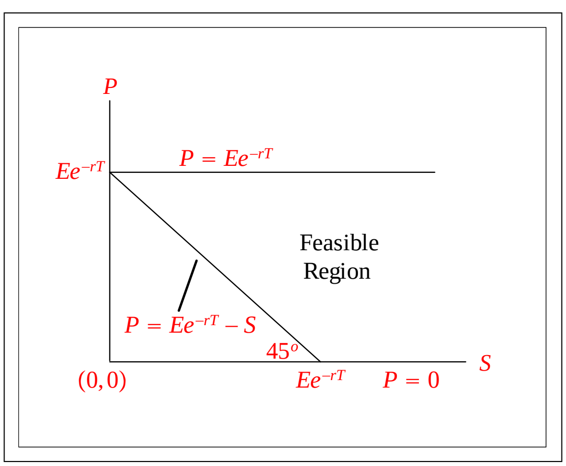
   - To illustrate, consider a scenario where an American put option is deep in the money with $X = \$100$ and $S = \$1$, on a date that is before the option's expiry. To exercise the option on that date, the payoff is \$99, which is nearly \$100. As the best payoff from exercising the option cannot exceed \$100, not much more can be gained by delaying the action. With the time value of money also considered, early exercise is appropriate here.

- The above illustration has considered a scenario where earlier exercise of an American put option is a rational decision. However, there are also scenarios where early exercise is suboptimal. Thus, we ought to be careful when stating this property; the use of the words "can be" here is to ensure that the statement does not preclude exercise of the option on the expiry date.
- In view of this property, the price of an American put option on a non-dividend paying stock cannot be lower than the price of an otherwise identical European put option.
- This property also leads to the result that, as early exercise is possible, the price of an American put option on a non-dividend paying stock cannot be lower than the difference between the exercise price and the stock price.
- This property is in sharp contrast with the corresponding property for an American call option on a non-dividend paying stock; in the latter case, early exercise is always suboptimal. Such a contrast has important implications when the Black-Scholes option pricing model is considered.

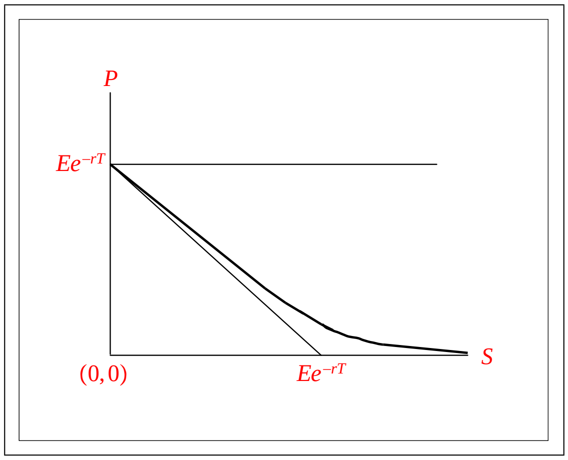
Arbitrage proofs for two of the above seven properties are also provided in Appendix A. Specifically, they include property 3 and 6. Some additional proofs for various properties of calls and puts are also included in the same appendix.

The value of a European put option, for a given price S of the underlying stock, is always within the region as defined by the following:

$$
\begin{array}{ll}
P \leq Xe^{-rT} & \text{(see property 1);} \\
P \geq Xe^{-rT} - S & \text{(see property 3);} \\
P \geq 0 & \text{(see property 2).}
\end{array}
$$



In the above graph, the symbol E is used instead of X. Here is a sketch of P versus S:



Again, to determine the solid curve requires the use of an option pricing model such as the Black-Scholes model. For such a purpose, the volatility of the underlying stock in terms of its standard deviation of returns is required.

7.3.1 Example: Put Options with Different Exercise Prices

Example 41 *Consider two European put options, labeled as #1 and #2, which have the same features except for the exercise price. Let us label their exercise prices as X_1 and X_2. Suppose that $S = \$20$, $r = 6\%$, $T = \frac{1}{3}$ years, $X_1 = \$30$, and $X_2 = \$29$. Suppose also that the option prices are: $P_1 = \$10$ and $P_2 = \$11$. Why is at least one of the two options mispriced? Illustrate also how arbitrage profits can be made from the mispricing?*

According to property 6, option #1 which has a higher exercise price cannot be worth less. However, with $P_1 = \$10$ and $P_2 = \$11$, this condition is violated. Thus, at least one of the two options is mispriced. To make an arbitrage profit, you buy option #1 and write option #2. The cash inflow is $-\$10 + \$11 = +\$1$, and this amount is to be invested risk-free.

At the expiry date, there are three possible outcomes. The stock price is: (a) below \$29, (b) between \$29 and \$30, and (c) above \$30.

(a) The holder of option #2 will exercise it by selling you the stock for \$29. You will exercise option #1 by selling the same stock for \$30. The profit from the two transactions will be

$\$30 - \$29 = \$1$. In addition, you will have

$$\$1 \times e^{rT} = \$e^{0.06/3} = \$1.02$$

from the risk-free investment. Thus, your total profit will be \$2.02.

(b) Option #2 will not be exercised by its holder. You will exercise option #1. That is, you will buy the stock at its market price (between \$29 and \$30) and sell it for \$30. The profit will be between \$0 and \$1. With \$1.02 from your risk-free investment, the total profit will be between \$1.02 and \$2.02.

(c) With neither option exercised, you will have \$1.02 from the risk-free investment.

7.4 Put-Call Parity

For European options on non-dividend paying stocks, there is a fixed relationship between the put and call prices for the same exercise price and the same maturity on an underlying stock.

From the value of a European call, the value of the corresponding put can be computed, and vice versa.

At time $t = 0$, you construct the portfolio: $+S + P - C$

That is, you buy 1 share of a stock for $\$S$, buy 1 put on the stock for $\$P$, and write 1 call on the stock to collect $\$C$ from the buyer.

Net investment $= \$(S + P - C)$.

The two options have the same expiry date and have the same exercise price X.

On the expiry date (at $t = T$), the underlying stock price is S_T.

	Value of Stock	+ Value of Put	− Value of Call	=	Value of Portfolio
If $S_T < X$:	S_T	$+\ (X - S_T)$	$-\ 0$	$=$	X
If $S_T > X$:	S_T	$+\ 0$	$-\ (S_T - X)$	$=$	X
If $S_T = X$:	S_T	$+\ 0$	$-\ 0$	$=$	$S_T = X$

If $S_T < X$: You will exercise the put and gain $\$(X - S_T)$. The holder of the call will not exercise it.

If $S_T > X$: You will not exercise the put. The holder of the call will exercise it.

If $S_T = X$: Both options will not be exercised.

Your investment is risk-free because, regardless of the underlying stock price S_T, the portfolio value at $t = T$ is the same $\$X$.

Let r be the annual risk-free interest rate with continuous compounding.

Since your investment $\$(S+P-C)$ is risk-free and becomes $\$X$ regardless of the underlying stock price, the following must hold:

$$\begin{aligned} \$(S+P-C)e^{rT} &= \$X; \\ S+P-C &= Xe^{-rT}. \end{aligned}$$

That is, the future value of $\$(S+P-C)$ is $\$X$; the present value of $\$X$ is $\$(S+P-C)$. Rearranging the terms,

$$C - P = S - Xe^{-rT}.$$

Put-call Parity is the name given to this relationship between put and call prices.

For a European call and a European put on the same underlying stock, with the same exercise price and the same expiry date, the difference in their values is the difference between the underlying stock price and the present value of the exercise price.

Note that e^{-rT} is the present value of \$1 to be received T years from now at the continuously compounded annual risk-free interest rate r. Here, T can be a fraction of a year. The present value factor e^{-rT} can also be written equivalently as $1/(1+r_{eff})^T$, where r_{eff} is the effective annual risk-free interest rate.

Writing a call + Buying the corresponding put + Owning the underlying stock

= Owning a risk-free asset

Writing a covered call

$$\begin{aligned} &= \text{Writing a call} + \text{Owning the underlying stock} \\ &= \text{Writing a put} + \text{Owning a risk-free asset} \end{aligned}$$

Example 42 *The price of a non-dividend paying stock is* \$31. *The continuously compounded annual risk-free interest rate is* 5%. *The price of a 6-month European call on the stock is* \$3 *and the price of a 6-month European put on the stock is* \$2.20. *The exercise price is* \$30. *Illustrate that we can make an arbitrage profit by buying the call, writing the put, and shorting the stock.*

According to put-call parity, we have

$$C - P = S - Xe^{-rT}.$$

In this example, we have

$$\begin{aligned} C - P &= \$3 - \$2.20 = \$0.80; \\ S - Xe^{-rT} &= \$31 - \$30e^{-0.05(0.5)} = \$31 - \$30 \times 0.9753 = \$1.74 \neq C - P. \end{aligned}$$

Since put-call parity is violated, an arbitrage profit is possible. The net cash inflow from buying the call, writing the put, and shorting the stock is

$$-\$3 + \$2.20 + \$31 = \$30.20.$$

We invest this amount risk-free until the expiry date of the options to obtain

$$\$30.20e^{0.05(0.5)} = \$30.20 \times 1.0253 = \$30.96.$$

If the stock price at the expiry date of the options is above \$30, we exercise the call (i.e., pay the \$30 exercise price to obtain the stock) but the holder of the put will not exercise it. We use the stock to close out the original short position. The net profit is $\$30.96 - \$30 = \$0.96$. If the stock price at the expiry date of the options is less than \$30, we discard the call but the holder of the put will exercise it for \$30. That is, we pay \$30 to the holder of the put for the stock. The stock will be used to close out the original short position. The net profit is still $\$30.96 - \$30 = \$0.96$. ■

7.5 The Black-Scholes Option Pricing Model

To derive the Black-Scholes option pricing model requires the use of advanced mathematical and statistical tools. As the analytical details are well beyond the scope of this course, the coverage below is confined to the resulting formula and the intuition involved. The Black-Scholes option pricing formula has two versions, with one version for call options and the other version for put options instead. These two versions are connected via put-call parity.

Consider a European call option on an underlying stock that pays no dividend in a market where the continuously compounded annual risk-free interest rate is r. The stock price is S, and the call option price is C. The option, which expires in T years (with T typically being a proportion), has an exercise price of X. The well-known Black-Scholes option pricing formula is

$$C = S\,\mathbf{N}(d_1) - X\,e^{-rT}\,\mathbf{N}(d_2), \tag{7.1}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + rT\right] + \frac{1}{2}\sigma\sqrt{T}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

A normal distribution that is standardized to have a zero mean and a unit standard deviation is called a standard normal distribution. In equation (7.1), besides the various symbols already defined, σ is the standard deviation of annual returns of the underlying stock, and $\mathbf{N}(\cdot)$ is the cumulative standard normal distribution. In the standard normal distribution function $f(z)$, $\mathbf{N}(d_1)$, for example, is the area under the symmetric bell-shaped curve from $z = -\infty$ to $z = d_1$; that is,

$$\mathbf{N}(d_1) = \int_{z=-\infty}^{d_1} f(z)dz.$$

Notice that $\mathbf{N}(-\infty) = 0$, $\mathbf{N}(0) = 0.5$, and $\mathbf{N}(\infty) = 1$.

- The Black-Scholes derivation of equation (7.1) starts with knowing that S changes over time according to a stochastic process and that, if S goes up (or down), so does C. Assume that it is permissible to buy or sell any proportion of a share of a stock and that there are no transaction costs. An investor who participates in the options market can achieve a risk-free hedge by writing a call and buying a matching proportion of a share of the underlying stock. In response to price changes in the market over time, the investor maintains the risk-free hedge by continuously adjusting the long and short positions (for the stock and the option, respectively) until the option expires.

- The maintenance of such a risk-free hedge leads to the *Black-Scholes partial differential equation* (BSPDE). Once the BSPDE has been reached, the next step is to reduce it to a one-dimensional heat equation for which a solution method is available. (Heat equations, which are well known in physical science and engineering fields, describe how temperature is distributed over space and time as heat spreads.) The end result is equation (7.1).

- There is another approach to reach equation (7.1). As introduced by Robert Merton, an alternative approach, which does not require the BSPDE to be solved and is known as *risk-neutral valuation*, takes advantage of the feature of the BSPDE that it does not contain any parameters affected by the attitudes towards risk of the participants in the options market. Should parameters such as the expected return of the underlying stock be present in the BSPDE as well, then its solution would not be adequate, from a market equilibrium standpoint, unless the issue of expected-return determination could also be addressed in the same model. Therefore, for the purpose of deriving the Black-Scholes option pricing model, we can simply consider a risk-neutral world, because the solution of the BSPDE is also applicable to such a world.

- Analytically, in a risk-neutral world, as investment risk is not a concern, the value of the call option is its expected value — which is the expected value of $\max(S - X, 0)$ — multiplied by a risk-free discount factor. By assuming a lognormal distribution of the the underlying stock price S, for which the probability density function is known, we can find the expected value of $\max(S - X, 0)$. The end result from risk-neutral valuation is the same as that from solving the BSPDE.

For a European put option on the same underlying stock, the put option price is

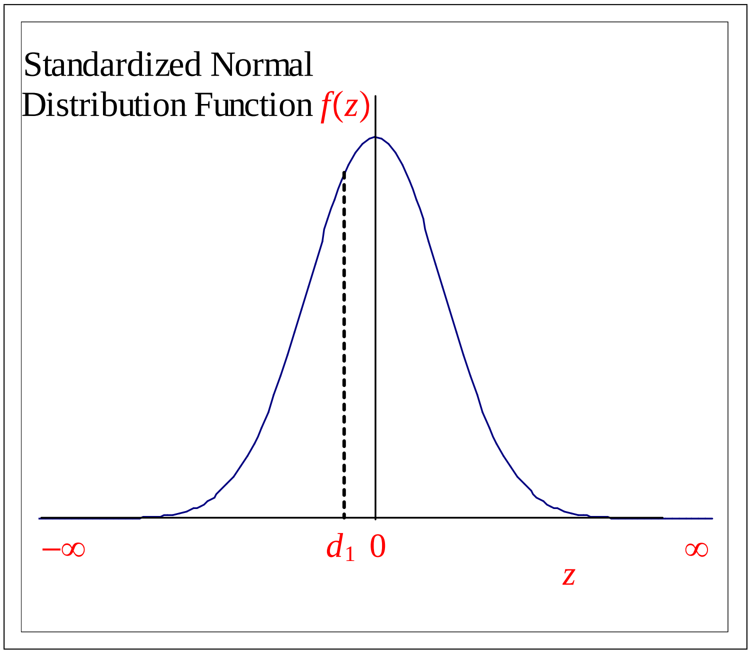
$$P = X\ e^{-rT}\,\mathbf{N}(-d_2) - S\ \mathbf{N}(-d_1). \tag{7.2}$$

Differencing the corresponding sides of equations (7.1) and (7.2) leads to

$$\begin{aligned} C - P &= S\ \mathbf{N}(d_1) - X\ e^{-rT}\,\mathbf{N}(d_2) - X\ e^{-rT}\,\mathbf{N}(-d_2) + S\ \mathbf{N}(-d_1) \\ &= S\ [\mathbf{N}(d_1) + \mathbf{N}(-d_1)] - X\ e^{-rT}\,[\mathbf{N}(d_2) + \mathbf{N}(-d_2)] \\ &= S - X\ e^{-rT}, \end{aligned}$$

which satisfies put-call parity. We have relied on the symmetry of the normal distribution to confirm that

$$\mathbf{N}(d_1) + \mathbf{N}(-d_1) = 1 \quad \text{and} \quad \mathbf{N}(d_2) + \mathbf{N}(-d_2) = 1.$$



For example, $\mathbf{N}(d_1)$ is the area enclosed by $f(z)$ and the z-axis from $z \to -\infty$ to $z = d_1$, and $\mathbf{N}(-d_1)$ is the enclosed area from $z \to -\infty$ to $z = -d_1$ or, equivalently, from $z = d_1$ to $z \to \infty$.

7.5.1 Impact of Each Underlying Parameter on Option Prices

The price of the call option, C, depends on S, X, T, r, and σ; so does the price of the put option, P. According to equation (7.1), we have the following:

$$\begin{array}{lll}
\partial C/\partial S > 0: & \text{if } S\uparrow & \text{then } C\uparrow \\
\partial C/\partial X < 0: & \text{if } X\uparrow & \text{then } C\downarrow \\
\partial C/\partial T > 0: & \text{if } T\uparrow & \text{then } C\uparrow \\
\partial C/\partial r > 0: & \text{if } r\uparrow & \text{then } C\uparrow \\
\partial C/\partial \sigma > 0: & \text{if } \sigma\uparrow & \text{then } C\uparrow
\end{array}$$

$$\begin{array}{lll}
\partial P/\partial S < 0: & \text{if } S \uparrow & \text{then } P \downarrow \\
\partial P/\partial X > 0: & \text{if } X \uparrow & \text{then } P \uparrow \\
\partial P/\partial T > 0: & \text{if } T \uparrow & \text{then } P \uparrow \\
\partial P/\partial r < 0: & \text{if } r \uparrow & \text{then } P \downarrow \\
\partial P/\partial \sigma > 0: & \text{if } \sigma \uparrow & \text{then } P \uparrow
\end{array}$$

The use of equations (7.1) and (7.2) enables us to confirm some previously established properties and to reveal some additional properties of European options. Both call and put prices increase with increasing time to expiry and with increasing volatility of the underlying stock. However, the impacts on call and put prices by changes in the underlying stock price, the exercise price, and the risk-free interest rate are opposite. The positive impact of the volatility of the return of the underlying stock on option prices is easily understandable. More volatile is the underlying stock return, more likely the option holder can gain from holding it.

7.5.2 American and European Call Options

Although the Black-Scholes option pricing formula in equation (7.1) has been derived for a European call option on a stock that pays no dividend, the same formula is applicable to the corresponding American call option as well. This is because, as property 7 of call options indicates, an American call option on a non-dividend paying stock will not be exercised before the expiry date. Such a property implies that the corresponding American and European call options must be worth exactly the same.

However, the same cannot be said about the corresponding American and European put options. As it can be worthwhile to exercise a deep-in-the-money American put option before its expiry date, we cannot justify the use of equation (7.2) for an American put option. It is important to recognize that equation (7.2) is intended for European put options only.

7.5.3 Numerical Illustrations

To illustrate the Black-Scholes option pricing model for either call or put options requires the use of numerical values from a standard normal distribution function. Areas under the standard normal distribution function $\int_0^{\mathbf{z}} f(z)dz$ are as follows:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

The above table was constructed by using the NORMSDIST worksheet function of Microsoft Excel^{TM}.

Example 43 *Using the standard normal distribution table, find* $\boldsymbol{N}(d_1)$ *for (1)* $d_1 = 1.25$, *(2)* $d_1 = -0.87$, *(3)* $d_1 = 2.124$, *and (4)* $d_1 = -1.367$.

(1) $d_1 = 1.25$: At the row for $z = 1.2$ and the column for $z = 0.05$ of the table, the value for $z = 1.25$ is 0.3944. This is the area under the bell-shaped curve from $z = 0$ to $z = 1.25$. Since the the area from $z = -\infty$ to $z = 0$ is 0.5000, we have

$$\mathbf{N}(d_1) = \mathbf{N}(1.25) = 0.5000 + 0.3944 = 0.8944.$$

(2) $d_1 = -0.87$: At the row for $z = 0.8$ and the column for $z = 0.07$ of the table, the value for $z = 0.87$ is 0.3078. This is the area under the bell-shaped curve from $z = 0$ to $z = 0.87$. Since the bell-shaped curve is symmetric, the area from $z = -0.87$ to $z = 0$ is also 0.3078. The area from $z = -\infty$ to $z = 0$ is 0.5000. Thus, the area from $z = -\infty$ to $z = -0.87$ is

$$\mathbf{N}(d_1) = \mathbf{N}(-0.87) = 0.5000 - 0.3078 = 0.1922.$$

(3) $d_1 = 2.124$: At the row for $z = 2.1$ and the column for $z = 0.02$ of the table, the value for $z = 2.12$ is 0.4830. At the row for $z = 2.1$ and the column for $z = 0.03$, the value for $z = 2.13$ is 0.4834.

The corresponding value for $z = 2.124$ can be obtained by linear interpolation as follows:

$$\begin{aligned}
\frac{2.124 - 2.12}{2.13 - 2.12} &= \frac{x - 0.4830}{0.4834 - 0.4830}; \\
\frac{0.004}{0.01} &= \frac{x - 0.4830}{0.0004}; \\
x &= 0.48316 \approx 0.4832.
\end{aligned}$$

This is the area under the bell-shaped curve from $z = 0$ to $z = 2.124$. Since the area from $z = -\infty$ to $z = 0$ is 0.5000, we have

$$\mathbf{N}(d_1) = \mathbf{N}(2.124) = 0.5000 + 0.4832 = 0.9832$$

(4) $d_1 = -1.367$: At the row for $z = 1.3$ and the column for $z = 0.06$ of the table, the value for $z = 1.36$ is 0.4131. At the row for $z = 1.3$ and the column for $z = 0.07$, the value for $z = 1.37$ is 0.4147.

The corresponding value for $z = 1.367$ can be obtained by linear interpolation as follows:

$$\begin{aligned}
\frac{1.367 - 1.36}{1.37 - 1.36} &= \frac{x - 0.4131}{0.4147 - 0.4131}; \\
\frac{0.007}{0.01} &= \frac{x - 0.4131}{0.0016}; \\
x &= 0.41422 \approx 0.4142.
\end{aligned}$$

This is the area under the bell-shaped curve from $z = 0$ to $z = 1.367$. Since the bell-shaped curve is symmetric, this is also the area from $z = -1.367$ to $z = 0$. Again, the area from $z = -\infty$ to $z = 0$ is 0.5000. Thus, we have

$$\mathbf{N}(d_1) = \mathbf{N}(-1.367) = 0.5000 - 0.4142 = 0.0858. \quad \blacksquare$$

Example 44 *(The Black-Scholes Option Pricing Model) Find the prices of a European call and a European put on a non-dividend paying stock with the following information: stock price* = \$21, *exercise price* = \$20; *expiry date = 6 months from now; continuously compounded annual risk-free interest rate* = 10%; *and stock volatility* = 20% *per annum.*

According to the Black-Scholes option pricing model:

$$C = S \cdot \mathbf{N}(d_1) - X \cdot e^{-rT} \cdot \mathbf{N}(d_2).$$

From $S = \$21$, $X = \$20$, $r = 0.10$, $T = 0.5$ year, and $\sigma = 0.20$, we have

$$\begin{aligned}
d_1 &= \frac{1}{0.20\sqrt{0.5}} \left\{ \ln\left(\frac{21}{20}\right) + \left[0.10 + \frac{(0.20)^2}{2}\right] 0.5 \right\} \\
&= \frac{1}{0.14142} \{0.04879 + [0.12]\, 0.5\} = 0.7693; \\
d_2 &= d_1 - \sigma\sqrt{T} \\
&= 0.7693 - 0.20\sqrt{0.5} = 0.6279.
\end{aligned}$$

From the standardized normal distribution table, the value for $z = 0.76$ is 0.2764 and the value for $z = 0.77$ is 0.2794. The value for $z = 0.7693$ can be found by linear interpolation:

$$\begin{aligned}
\frac{0.7693 - 0.76}{0.77 - 0.76} &= \frac{x - 0.2764}{0.2794 - 0.2764}; \\
x &= 0.2792.
\end{aligned}$$

Thus, we have

$$\mathbf{N}(d_1) = 0.5000 + 0.2792 = 0.7792.$$

From the same table, the value for $z = 0.62$ is 0.2324 and the value for $z = 0.63$ is 0.2357. The value for $z = 0.6279$ can also be found by linear interpolation:

$$\begin{aligned}
\frac{0.6279 - 0.62}{0.63 - 0.62} &= \frac{x - 0.2324}{0.2357 - 0.2324}; \\
x &= 0.2350.
\end{aligned}$$

Thus, we have

$$\mathbf{N}(d_2) = 0.5000 + 0.2350 = 0.7350.$$

The price of the call is

$$\begin{aligned}
C &= S \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2) \\
&= \$21(0.7792) - \$20(e^{-0.10 \times 0.5})(0.7350) \\
&= \$2.38.
\end{aligned}$$

Put-call parity gives us

$$C - P = S - Xe^{-rT}.$$

Accordingly, the price of the put is

$$\begin{aligned} P &= C - S + Xe^{-rT} \\ &= \$2.38 - \$21 + \$20(e^{-0.10\times 0.5}) \\ &= \$0.4046 \approx \$0.40. \quad \blacksquare \end{aligned}$$

- It is a good exercise to verify the above result for P by using equation (7.2) directly.

7.6 The Binomial Option Pricing Model for Call Options

John Cox, Stephen Ross, and Mark Rubinstein [in "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, (1979), 229-263] have derived a binomial option pricing model as a simple alternative to the Black-Scholes option pricing model. In a single-period setting — which is like a single toss of a biased coin in the coin-toss analogy — is able to capture the crucial idea of a risk-free hedge underlying the Black-Scholes derivation. In what follows, we use a European call option to illustrate the Cox-Ross-Rubinstein approach. After showing how a risk-free hedge can be formed and how the option value can be deduced in a single-period setting, we summarize the final result in a multi-period setting. The analytical detail of the entire task is covered in Appendix B of this chapter.

7.6.1 The Starting Point: A Single-period Setting

Let r_f be a one-period risk-free interest rate. Consider a call option that can be exercised at the end of the period with an exercise price X. Suppose that the underlying stock has a beginning-of-period price S and that its end-of-period price will be either uS or dS. Here, u and d are given multiplicative factors satisfying the condition of $u > 1 + r_f > 1 > d > 0$; they capture the up-down movements of S with probabilities q and $1 - q$, respectively. The magnitude of $u - d$ is a proxy for the volatility of the underlying stock returns.

Suppose also that the beginning-of-period option price is C. Let C_u and C_d be the two possible end-of-period option prices, corresponding to the up-down price movements of the underlying stock. As option prices can never be negative, we have

$$C_u = \max(0, uS - X) \tag{7.3}$$

and

$$C_d = \max(0, dS - X). \tag{7.4}$$

At the beginning of the period, an investor buys one share of the stock and writes (sells), as a hedge against any possible losses, m units of the call option that the stock underlies. The net investment is $S - mC$, which is positive.

A risk-free hedged portfolio requires

$$uS - mC_u = dS - mC_d,$$

from which the hedge ratio

$$m = \frac{(u-d)S}{C_u - C_d} \tag{7.5}$$

can be deduced. With r_f being the risk-free interest rate, we have

$$(1 + r_f)(S - mC) = uS - mC_u, \tag{7.6}$$

where the right hand side can also be $dS - mC_d$.

Combining equations (7.5) and (7.6) to eliminate m and to express C in terms of r_f, u, d, C_u, and C_d leads to

$$C = \frac{pC_u + (1-p)C_d}{1 + r_f} = \frac{p \max(0, uS - X) + (1-p) \max(0, dS - X)}{1 + r_f}, \tag{7.7}$$

where

$$p = \frac{(1 + r_f) - d}{u - d} \quad \text{and} \quad 1 - p = \frac{u - (1 + r_f)}{u - d}. \tag{7.8}$$

As long as

$$0 < d < 1 < 1 + r_f < u,$$

the conditions of

$$0 < p < 1 \quad \text{and} \quad 0 < 1 - p < 1$$

are assured. With p and $1-p$ taking the roles of the probabilities for C_u and C_d, respectively, the beginning-of-period option price can be viewed as the present value of the expected end-of-period option values.

- An alternative risk-free hedge, which also leads to equation (7.7), is provided in Appendix B. The analytical detail of the extension to a multi-period setting, which leads to end result of the model derivation, is also provided there.

The parameters p and $1-p$ are commonly called *risk-neutral probabilities*. The term originates from a special case where the investor involved is risk neutral. In a single-period setting, a risk-neutral investor is indifferent between investing risk-free a dollar amount equal to S to achieve $(1+r_f)S$ at the end of the period and investing in the stock instead for an expected end-of-period value of $quS+(1-q)dS$. It follows from

$$(1+r_f)S = quS + (1-q)dS$$

that

$$q = \frac{1+r_f-d}{u-d},$$

which is p. This is why p can be viewed as a risk-neutral probability.

Although p has the properties of a probability in a mathematical sense, it is not an actual probability of occurrence. This distinction is crucial, as the determination of the option price C does not require risk neutrality of investors. By interpreting p as a risk-neutral probability, we can express C simply as the expected value of potential outcomes discounted by a time-value factor, thus making the determination of C more intuitive. Indeed, the idea of risk-neutral probabilities, as introduced above, is widely used in pricing various derivative assets in practice.

7.6.2 An Extension to a Multi-Period Setting

In an N-period setting, let u and d be the multiplicative factors that capture the up-down movements each period of the underlying stock price, with probabilities q and $1-q$, respectively. Let r_f be the risk-free interest rate each period. Let also S and C be the corresponding beginning-of-period stock and option prices. To go from a single-period setting to a multi-period setting requires that the underlying parameters u, d, and r_f for each period be adjusted accordingly. The condition of

$$0 < d < 1 < 1 + r_f < u$$

must still hold.

As derived in Appendix B, the end result is

$$C = S\ B(n \geq a|N, p') - \frac{X}{(1+r_f)^N}\ B(n \geq a|N, p). \tag{7.9}$$

Here, a is the lowest integer n, in the range of 0 to N, satisfying the condition of

$$n > \frac{\ln[X/(d^N S)]}{\ln(u/d)}, \tag{7.10}$$

The two parameters,

$$p = \frac{1 + r_f - d}{u - d} \tag{7.11}$$

and

$$p' = p \cdot \frac{u}{1 + r_f}, \tag{7.12}$$

which satisfy the conditions of

$$0 < p < p' < 1,$$

take the roles of probabilities.

The two functions

$$B(n \geq a|N, p) = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

and

$$B(n \geq a|N, p') = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} (p')^n (1-p')^{N-n},$$

where $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 1$, and $0! = 1$, are complementary binomial distribution functions. As

$$B(n < a|N, p') + B(n \geq a|N, p') = 1$$

and

$$B(n < a|N, p) + B(n \geq a|N, p) = 1,$$

each pair of such functions fully covers all possible outcomes as characterized by the corresponding distribution.

7.6.3 An Illustration of a Complementary Binomial Distribution Function

To facilitate a comparison with the Black-Scholes formula, let us express equation (7.9) as

$$c = S\,\mathbf{B}(p') - \frac{X}{(1+r_f)^N}\,\mathbf{B}(p). \tag{7.13}$$

$$\mathbf{B}(p') = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!}(p')^n(1-p')^{N-n}$$

and

$$\mathbf{B}(p) = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}$$

for notational convenience. For given integers N and a, — where a, which depends on S, X, N, u, and d, is deduced from inequality (7.10) — each of the above two functions, denoted as $\mathbf{B}(p')$ and $\mathbf{B}(p)$, is a complementary binomial distribution function. They differ only in the probabilities involved. Each of the two probabilities, denoted as p and p' and defined by equations (7.11) and (7.12), respectively, depends on r_f, u, and d.

To illustrate intuitively what $\mathbf{B}(p)$ represents, consider $N = 10$ independent coin tosses. Suppose that the coin is biased and that the probabilities of getting heads and tails are p and $1 - p$, respectively. The **cumulative probability** of getting up to 4 heads is the sum of all probabilities of getting 0, 1, 2, 3, and 4 heads. The corresponding **complementary probability** is the sum of all remaining probabilities; that is, the sum of the probabilities of getting 5, 6, 7, 8, 9, and 10 heads.

What $\mathbf{B}(p)$ represents is a complementary probability. In N independent draws from a binomial distribution function where the probability of a successful draw is p, $\mathbf{B}(p)$ is the sum of all probabilities of getting $a, a+1, a+2, \ldots, N$ successful draws. The complementary probability $\mathbf{B}(p')$ can be illustrated in a similar manner.

The close similarity of equation (7.1) and equation (7.9) [or, equivalently, equation (7.13)], which are the Black-Scholes version and the Cox-Ross-Rubinstein version, respectively, is indeed remarkable. As $N \to \infty$, the two models converge analytically; however, their convergence still requires proper choices of the values of the underlying parameters. Given its analytical convenience, the binomial approach has been used extensively in valuation of various derivative assets.

7.7 The Binomial Option Pricing Models for Put Options

In a one-period setting, where the end-of-period stock price is either uS or dS, the corresponding end-of-period put price is either

$$P_u = \max(0, X - uS)$$

or

$$P_d = \max(0, X - dS).$$

With P being the beginning-of-period put price, if we buy 1 unit of the underlying stock and m units of the put option, the investment is $S + mP$. A risk-free investment requires

$$(1 + r_f)(S + mP) = uS + mP_u = dS + mP_d.$$

Eliminating m from above and express P in terms of the remaining parameters leads to

$$P = \frac{pP_u + (1-p)P_d}{1 + r_f} = \frac{p \max(0, X - uS) + (1-p) \max(0, X - dS)}{1 + r_f},$$

where p satisfies the condition of

$$0 < p = \frac{1 + r_f - d}{u - d} < 1.$$

To extend the above to a N-period setting requires the maintenance of a risk-free hedge at the beginning of each period. At the end of period 1, the stock price is uS or dS. At the end of period 2, the stock price is u^2S, udS, or d^2S. At the end of period 3, the stock price is u^3S, u^2dS, ud^2S, or d^3S, and so on. At the end of period N, the stock price is one of u^NS, $u^{N-1}dS$, $u^{N-2}d^2S$, $\ldots$, $ud^{N-1}S$, and d^NS, and the corresponding put option price is one of

$$\max(0, X - u^N d^{N-n} S), \text{ for } n = 0, 1, 2, \ldots, N.$$

As shown in Appendix C, the put option price at the beginning of period 1 is

$$P = \frac{X}{(1+r_f)^N} \; B(n < a|N, p) - S \; B(n < a|N, p'), \tag{7.14}$$

where the parameters p, p', and a have been defined in the preceding section.

- It is useful to confirm that equation (7.9) and equation (7.14) satisfy put-call parity, which is
$$C - P = S - \frac{X}{(1+r_f)^N}$$
in the current setting. As a cumulative distribution function and the corresponding complementary distribution function sum to 1, we have
$$B(n < a|N, p) + B(n \geq a|N, p) = 1$$
and
$$B(n < a|N, p') + B(n \geq a|N, p') = 1,$$
which are useful for confirming that put-call parity is satisfied.

7.8 Convergence of the Binomial Model to Black-Scholes

Although binomial and Black-Scholes models are differently formulated, they do share a common feature. In particular, both derivations require the formation of risk-free hedges. Not surprisingly, the two models are connected. The connection will become noticeable if the length of time before an option expires is divided into infinitely many short intervals for the binomial option pricing model.

The binomial model is a discrete-time model. In contrast, the Black-Scholes model is a continuous-time model. Intuitively, as the risk-free hedge that is formed at the start of each of the infinitely many periods can be viewed as continuously formed, the convergence of the two models can be expected. As the binomial option pricing model is much easier to follow, we can use it to reveal the economic intuition in the derivation of the Black-Scholes version, which is much more complicated.

To match the continuously compounded risk-free interest rate r in the Black-Scholes model, where the option expires in T years (or a proportion T of a year), we can set r_f in the binomial version according to

$$(1 + r_f)^N = \exp(rT).$$

There are different ways to set parameters u and d in an N-period setting to match the volatility parameter σ in the Black-Scholes model. One of them is

$$u = \frac{1}{d} = \exp\left(\sigma\sqrt{\frac{T}{N}}\right).$$

As $N \to \infty$, the binomial option pricing model will converge to the Black-Scholes version.

For an illustration, consider a European call option on a stock that pays no dividends. Suppose that $S = \$21$, $X = \$20$, $T = 0.25$ years, $r = 0.05$, and $\sigma = 0.20$. The computations below are by using Microsoft Excel's functions, including NORM.DIST for $\mathbf{N}(d_1)$ and $\mathbf{N}(d_2)$, as well as BINOM.DIST for $1 - B(n \geq a|N, p')$ and $1 - B(n \geq a|N, p)$. Using the Black-Scholes option pricing formula, we find

$$\begin{aligned}
\mathbf{N}(d_1) &= 0.74630, \\
\mathbf{N}(d_2) &= 0.71325, \\
\text{and } C &= \$1.58459.
\end{aligned}$$

We have the following results based on the Binomial option pricing model for call options:

N	10	100	1,000	10,000	100,000	1,000,000
u	1.03213	1.01005	1.00317	1.00100	1.00032	1.00010
d	0.96887	0.99005	0.99684	0.99900	0.99968	0.99990
$r_f \times 100$	0.1250782	0.0125008	0.0012500	0.0001250	0.0000125	0.0000013
a	5	48	493	4976	49923	499757
p	0.51187	0.50375	0.50119	0.50038	0.50012	0.50004
p'	0.52765	0.50875	0.50277	0.50087	0.50028	0.50009
$B(n \geq a\|N, p')$	0.44660	0.68268	0.72110	0.74054	0.74500	0.74537
$B(n \geq a\|N, p)$	0.40648	0.64615	0.68655	0.70712	0.71186	0.71226
C	1.34991	1.57371	1.58271	1.58450	1.58458	1.58459

Here, each column shows the computational results for each value of N. For a given T, which is 0.25 years in this example, as N increases, both u and d move towards 1 and all of r_f, p, and p' become smaller. Although $B(n \geq a|N, p')$ moves towards $\mathbf{N}(d_1)$ and $B(n \geq a|N, p)$

moves towards $\mathbf{N}(d_2)$ as N increases, a minor numerical difference still exists in each case for $N = 1,000,000$. However, the computed value of C for $N = 1,000$ already matches the Black-Scholes result very well for practical purposes.

7.9 A Practical Application of Options: Volatility Indices

The volatility index VIX was introduced to the finance world by Robert E. Whaley, as documented in the investment literature in 1993 (*Journal of Derivatives*, Volume 1, 1993, pp. 71-84). On September 23, 2003, the Chicago Board Options Exchange (CBOE) changed the VIX calculations. As a forward projection of volatility in real time, the VIX is intended to predict the volatility of the S&P 500 index during the next 30 calendar days. Although the VIX is not tradable in financial markets, its current and past values (since January 2, 1990) are available from various sources, such as Yahoo! Finance (for ^VIX) and Federal Reserve St. Louis (https://fred.stlouisfed.org/series/VIXCLS) among others.

The computation of the VIX is based on a portfolio of out-of-money options on S&P 500. Options for use in the computations must satisfy some specific conditions. As option prices are available in real time, so are the computed values of the VIX. Some basic information about the VIX can be found in a Wikipedia posting (https://en.wikipedia.org/wiki/VIX). There is also a 2019 CBOE White Paper, which illustrates the computation of the VIX (https://www.sfu.ca/~poitras/419_VIX. pdf).

The VIX is always positive. A value up to 12 indicates low volatility over the next 30 days. A value higher than 12 but lower than 20 is considered normal. A value that is 20 or higher indicates high volatility over the next 30 days. On October 24, 2008, which was during the US financial crisis, the intraday VIX reached an all-time high of 89.53. More recently, on March 12, 2020, which was during the early days of the Covid-19 pandemic, the VIX reached 75.47 at the close of the trading day.

Over time, CBOE has also introduced various other volatility indices. For example, there are extensions of the VIX for predicting the volatility of S&P 500 over different time intervals, such as 9 days, 3 months, 6 months, and one year. As of now, historical values of the 3-month volatility index are freely available in Yahoo! Finance (for ^VIX3M) and Federal Reserve St. Louis (https://fred.stlouisfed.org/series/VXVCLS).

Available CBOE volatility indices are typically for major stock indices, such as Dow Jones Industrial Average, NASDAQ Composite Index, S&P 500 Index, and Russell 2000 Index (for stocks of smaller companies), but not for the individual constituent stocks. However, daily closing values of volatility indices for some well-known stocks such as Amazon, Apple, Goldman Sachs, Google, and IBM are still freely available.

7.9.1 Preliminary Assessments of the Performance of the VIX

The availability of the above-mentioned CBOE volatility indices has provided equity investors with an additional investment tool. The attractiveness of such a tool is due to its forward-looking nature. The VIX, which predicts the volatility of S&P 500 in the month ahead, has emerged as a gauge of fear or stress in the stock market. Rightly or wrongly, it is perceived by many in the investment world as a *fear index*. As such, a high value of the VIX is also perceived by these investors as a signal of imminent market declines, thus affecting their trading activities. Such interpretations of the VIX are contrary to the well-established notion that expected return and volatility are in two different dimensions of an investment.

Even if the VIX is interpreted properly — only as a measure of expected volatility in the month ahead — its critics still question whether, in practice, it represents an improvement over simpler forecasting methods, given that many severe events in the future are unpredictable. The two graphs based on daily data from January 1990 to September 2009 near the end of the above-mentioned Wikipedia posting (https://en.wikipedia.org/wiki/VIX) can serve as an illustration of such a concern.

One of the two graphs shows the S&P standard deviation of daily returns next month (on the vertical axis) versus the VIX (on the horizontal axis), for which the correlation is 0.797. The other graph is where the variable on the horizontal axis is substituted by the currently measured S&P standard deviation of returns. The correlation is 0.769 instead. The comparison, therefore, is between the VIX and the currently measured S&P volatility as a predictor of the near future S&P volatility. Although no statistical tests are reported in the same Wikipedia posting, the observed increase of the correlation from 0.769 to 0.797 for using the VIX is deemed too small to make a meaningful difference.

This section is not intended to take a position as to whether the VIX and, by extension, any other CBOE volatility indices are superior in predicting future volatility, as compared to less sophisticated forecasting methods. Many future events, including catastrophes, are totally unpredictable regardless of what forecasting methods are used. Thus, the failure to foretell unpredictable future catastrophes should not be view as a weakness that is specific to the VIX and various other CBOE volatility indices alone.

However, as such indices have attracted so much attention of participants in the investment world, the availability of their real-time values might have been used unwisely by some investors in trading activities, which collectively could have consequential impacts on price movements and trading volumes of the stocks involved. It is such a practical concern that makes an exploration of potential causal relationships between the CBOE volatility indices and subsequent stock market activities interesting. Further, as there are times when the CBOE volatility indices outperform other forecasting methods and there are also times for the reverse to be true, an in-depth examination of these two situations could shed some light on the performance issue.

7.10 A Conceptual Perspective on Options and Corporate Finance

- We complete the main text of this chapter by offering a conceptual perspective, which applies the topic of options to the study of corporate finance, with attention to the roles of shareholders and bondholders in a firm.

At the conceptual level, a firm's equity can be viewed as a call option. To illustrate such an idea, let us consider a single-period setting, in which the firm is liquidated at the end of the period. If the firm's end-of-period value falls short of its debt obligation, each equity holder will receive nothing. Otherwise, the equity holders will receive what is left over after the firm meets its debt obligation. Thus, each equity holder's position is like the position of the holder of a call option. The end-of-period debt obligation is like the exercise price of the option.

In the same single-period setting, the position of the debt holders of the firm can be viewed as the owner of the firm who has written a call option with its exercise price being the required end-of-period debt obligation. If the firm's end-of-period value exceeds the debt obligation (the exercise price), then the option will be exercised, and the debt holders will receive the contractual debt payment. Otherwise, the option will not be exercised, and the debt holders will get whatever the firm is worth at the time.

Corporate debt in a single-period setting can also be viewed as a portfolio of two components. One component is a risk-free bond, with its end-of-period payment equal to the contractual debt obligation. The other component is a short position of a put option on the firm's assets, with the exercise price also equal to the same debt payment. Here, to hold a short position of the put option is to act as the writer of the option.

If the firm's assets are worth more than the contractual debt payment, the put option will not be exercised. If so, the holder of the portfolio receives the contractual debt payment. Otherwise, the option will be exercised; the writer of the put option will use the contractual debt payment from the risk-free bond (i.e., what is earned from the first component of the portfolio) to buy the firm's assets, which is worth less.

7.11 Appendix A: Arbitrage Proofs of Selected Option Properties

This appendix provides arbitrage proofs of some selected option properties. The assumptions as required for each proof are the same as those listed earlier. Any symbols, if undefined below, are also the same as those used earlier. A key feature of an arbitrage proof is to start with a proposition that is contrary to what we intend to prove. We then look for arbitrage profits, typically with transactions such as buying a potentially undervalued option, writing a potentially overvalued option, buying a potentially undervalued stock, or short selling a potentially overvalued stock. It is important to recognize that, in each proof, the exercise price X and the continuously compounded annual risk-free interest rate r are constants, but the option prices C and P, as well as the underlying stock price S do change over time.

7.11.1 Call Option Properties

(1) *For a European call option on a stock that pays no dividend, prove that*

$$C \geq S - Xe^{-rT}$$

Proof: Suppose that there is a European call option on a stock that pays no dividend, for which

$$C < S - Xe^{-rT} \tag{7.15}$$

instead. If so, we show below that an investor will earn an arbitrage profit by buying the call option and short selling the underlying stock (under the assumption of frictionless short sales).

The investor's immediate cash inflow is

$$S - C,$$

with S being the immediately available proceeds from short selling the stock and with C being the immediate payment for buying the call option. We have

$$S - C > Xe^{-rT} > 0,$$

given inequality (7.15) and given the fact that both X and e^{-rT} are positive. Thus, the investor can invest more than Xe^{-rT} risk-free until the option expires at time T (as measured in years). On the expiry date of the option, the investor will have

$$(S - C)e^{rT}$$

in cash, from the risk-free investment, which is greater than

$$(Xe^{-rT})e^{rT} = X.$$

On the expiry date of the option, if $S > X$, the investor will exercise the option to acquire the stock by paying X for it. The investor will then use the stock thus acquired to terminate the original short-sale arrangement. As the available cash from the risk-free investment is greater than X, the investor will still have some cash left after exercising the option. Thus, the investor will make an arbitrage profit.

On the expiry date of the option, if $S = X$ instead, there is no incentive for the investor to exercise the option. The investor can buy the stock in the market for a price that is equal to X, in order to terminate the original short-sale arrangement. The arbitrage profit will still be the same as that for the case of $S > X$.

On the expiry date of the option, if $S < X$ instead, the option will not be exercised. As the investor can acquire the stock in the market for a price lower than X (for the purpose of terminating the original short-sale arrangement), the arbitrage profit is even higher.

In all three cases, as an arbitrage profit will be made, inequality (7.15) cannot be valid. Therefore, we must have

$$C \geq S - Xe^{-rT}. \quad \blacksquare$$

(2) *Let C_1 and C_2 be the prices of two European call options on the same underlying stock that pays no dividend. The two options have identical features, except for the expiry dates. Let the corresponding times until expiry be T_1 and T_2 (as measured in years), with*

$$T_2 > T_1.$$

Prove that

$$C_2 \geq C_1.$$

Proof: Suppose that

$$C_2 < C_1$$

instead. If an investor buys the second option and writes the first option, the immediate cash inflow is

$$K = C_1 - C_2 > 0,$$

which can be invested risk-free. On the expiry date of the first option (at time T_1), the invested amount will become

$$Ke^{rT_1},$$

which is positive.

On the expiry date of the first option, if $S > X$, the first option will be exercised. The investor will incur a cash outflow of

$$S - X$$

for being the writer of the first option. Given that the remaining life of the second option at the time is $T_2 - T_1$, the investor can sell the second option for

$$C_2^{(T_1)} \geq S - Xe^{-r(T_2 - T_1)}.$$

Here, for notational clarity, we have used $C_2^{(T_1)}$, which is C_2 with a superscript (T_1), to indicate the value of the second option on the expiry date of the first option. As

$$T_2 - T_1 > 0,$$

we have

$$0 < e^{-r(T_2-T_1)} < 1$$

and thus

$$S - Xe^{-r(T_2-T_1)} > S - X.$$

That is, the investor's cash inflow from selling the second option exceeds the cash outflow for being the writer of the first option. Given also the dollar amount of Ke^{rT_1} from the risk-free investment, the investor's arbitrage profit is obvious.

On the expiry date of the first option, if $S \leq X$ instead, the first option will not be exercised. Given call option properties that

$$C \geq 0$$

and

$$C \geq S - Xe^{-rT},$$

we must have, in general,

$$C \geq \max(S - Xe^{-rT}, 0),$$

which is non-negative. At worst, the second option is worth nothing on the expiry date of the first option. Even so, the positive amount of Ke^{rT_1} from the risk-free investment still ensures the existence of an arbitrage profit.

To eliminate the existence of an arbitrage profit, we must rule out $C_2 < C_1$. That is, we must have

$$C_2 \geq C_1. \quad \blacksquare$$

(3) *Two European call options on the same underlying stock that pays no dividend have identical features, except for the exercise price. Let C_i and X_i be the value and the exercise price of option i, respectively, for $i = 1$ and 2. Prove that, if $X_1 < X_2$, we must have $C_1 \geq C_2$.*

Proof: Suppose that, for

$$X_1 < X_2,$$

we have

$$C_1 < C_2$$

instead, at the time when these two European call options will expire T years later. (For example, if the two options will expire in 3 months, we have $T = 1/4$.) If so, an investor buys option 1 and writes option 2. This is because either option 1 is considered to be undervalued or option 2 is considered to be overvalued, or both. The immediate cash inflow is

$$C_2 - C_1,$$

which is positive. The investor can invest this dollar amount risk-free, until the expiry date of the two options. The resulting value of this risk-free investment, on the expiry date of the two options, will be

$$(C_2 - C_1)e^{rT},$$

where r is the continuously compounded annual risk-free interest rate.

On the expiry date of the two options, we can have $S > X_2$, $X_2 \geq S > X_1$, or $S \leq X_1$. These three cases are now considered separately.

If $S > X_2$, which implies $S > X_1$, both options will be exercised. Being the holder of option 1 and the writer of option 2, the investor will receive the underlying stock by paying X_1 dollars to the writer of option 1 for it and will sell the same stock immediately to the holder of option 2 to receive X_2 dollars for it. The net cash inflow will be $X_2 - X_1$. As both $(C_2 - C_1)e^{rT}$ and $X_2 - X_1$ are positive, the investor will make a profit.

On the expiry date, if $X_2 \geq S > X_1$, only option 1 will be exercised. The investor will buy the underlying stock from the writer of option 1 for X_1 dollars and immediately sell it in the market for S dollars. The net cash inflow will be $S - X_1$. As both $(C_2 - C_1)e^{rT}$ and $S - X_1$ are positive, the investor will also make a profit.

On the expiry date, if $S \leq X_1$ instead, we also have $S < X_2$. Thus, neither option 1 nor option 2 will be exercised. As the investor still has $(C_2 - C_1)e^{rT}$, which is a positive dollar amount, from the risk-free investment, the investor will still make a profit.

To rule out any arbitrage profits, regardless of the price of the underlying stock on the expiry date of the two options, we must have $C_1 \geq C_2$ if $X_1 < X_2$. ■

More proofs of call option properties will be provided in Section 7.11.3.

7.11.2 Put Options Properties

(1) *For a European put option on a stock that pays no dividend, prove that*

$$P \geq Xe^{-rT} - S.$$

Proof: Suppose that there is a European put option on a stock that pays no dividend, for which

$$P < Xe^{-rT} - S \tag{7.16}$$

instead. If so, we show below that an investor will earn an arbitrage profit by buying both the put option and the underlying stock, entirely with borrowed money. A crucial point here is that the investor's immediate cash outlay is intended to be zero.

Given inequality (7.16), we have

$$P + S < Xe^{-rT}.$$

With the dollar amount of the loan being less than

$$Xe^{-rT},$$

the repayment of the loan on the expiry date of the option is less than

$$(Xe^{-rT})e^{rT} = X.$$

On the expiry date of the option, if $S < X$, the investor will exercise the option by selling the stock (that the investor has acquired at the beginning) to the writer of the option and will receive X for it. As the repayment of the loan is less than X, the investor will make an arbitrage profit.

On the expiry date of the option, if $S = X$ instead, there is no incentive for the investor to exercise the option. The investor will sell the stock in the market for a price that is equal to X. As the amount is more than what is required to repay the loan, the arbitrage profit will still be the same as that for the case of $S < X$.

On the expiry date of the option, if $S > X$ instead, the option will not be exercised. The investor will sell the stock in the market for a price that is greater than X. As the repayment for the loan requires an amount less than X, the arbitrage profit is even higher.

In all three cases, as an arbitrage profit will be made, inequality (7.16) cannot be valid. Therefore, we must have

$$P \geq Xe^{-rT} - S. \ \blacksquare$$

(2) *Two European put options on the same underlying stock that pays no dividend have identical features, except for the exercise price. Let P_i and X_i be the value and the exercise price of option i, respectively, for $i = 1$ and 2. Prove that, if $X_1 > X_2$, we must have $P_1 \geq P_2$.*

Proof: Suppose that, for

$$X_1 > X_2,$$

we have

$$P_1 < P_2$$

instead, at the time when these two European put options will expire T years later. If so, an investor buys option 1 and writes option 2. This is because either option 1 is considered to be undervalued or option 2 is considered to be overvalued, or both. The immediate cash inflow is

$$P_2 - P_1,$$

which is positive. The investor can invest this dollar amount risk-free, until the expiry date of the two options. The resulting value of this risk-free investment, on the expiry date of the two options, will be

$$(P_2 - P_1)e^{rT},$$

where r is the continuously compounded annual risk-free interest rate.

On the expiry date of the two options, we can have $S < X_2$, $X_2 \leq S < X_1$, or $S \geq X_1$. These three cases are now considered separately.

If $S < X_2$, which implies $S < X_1$, both options will be exercised. Being the holder of option 1 and the writer of option 2, the investor will receive the underlying stock by paying X_2 dollars to the holder of option 2 for it and will sell the same stock immediately to the writer of option 1 to receive X_1 dollars for it. The net cash inflow will be $X_1 - X_2$. As both $(P_2 - P_1)e^{rT}$ and $X_1 - X_2$ are positive, the investor will make a profit.

On the expiry date, if $X_2 \leq S < X_1$, only option 1 will be exercised. The investor will buy the underlying stock for S dollars and immediately sell it to the writer of option 1 for X_1 dollars. The net cash inflow will be $X_1 - S$. As both $(P_2 - P_1)e^{rT}$ and $X_1 - S$ are positive, the investor will also make a profit.

On the expiry date, if $S \geq X_1$ (which implies $S > X_2$) instead, neither option 1 nor option 2 will be exercised. As the investor still has $(P_2 - P_1)e^{rT}$, which is a positive dollar amount, from the risk-free investment, the investor will still make a profit.

To rule out any arbitrage profits, regardless of the price of the underlying stock on the expiry date of the two options, we must have $P_1 \geq P_2$ if $X_1 > X_2$. ■

7.11.3 More proofs of Option Properties

Here, we not only revisit some option properties that have already been covered, but also consider a few other simple ones. While analytical justifications, such as arbitrage proofs, are required for establishing some option properties, some others can also be proven by using familiar financial concepts. We continue with the same simplifying assumption of a frictionless capital market without transaction costs, where the underlying stock of each option pays no dividends. Further, risk-free lending or borrowing is available at a common interest rate. We also stay with the same notation, unless there is a need to make notational changes for better clarity.

Once again, it is important to recognize that X and r remain constant, T changes with the passage of time, and C, P, and S vary over time, due to volatility of price movements.

(1) *Prove that, for an American call option, the following condition must hold:*

$$C \geq S - X.$$

That is, the call option price cannot be less than the difference between the underlying stock price and the exercise price.

Proof: Suppose that, on the contrary,

$$C < S - X$$

instead. This is equivalent to

$$S - X - C > 0.$$

We can buy the option by paying C (in dollars) for it. This being an American option, we can exercise it any time, between the date of purchase and its expiry date. To earn an arbitrage profit, we exercise it immediately, by paying the writer X (in dollars) to acquire the underlying stock that is worth S (in dollars). We then sell the stock thus acquired in the market for S (in dollars). The net cash inflow is

$$S - X - C.$$

As the net cash inflow is positive, there is an arbitrage profit for us. To rule out the existence of any arbitrage profits, we must rule out

$$C < S - X.$$

Thus, the condition of

$$C \geq S - X$$

must hold for an American call option. ■

(2) *Prove that, for an American put option, the following condition must hold:*

$$P \geq X - S.$$

That is, the put option price cannot be less than the difference between the exercise price and the underlying stock price.

Proof: Suppose that, on the contrary,

$$P < X - S$$

instead. This is equivalent to

$$X - P - S > 0.$$

We can buy the option by paying P (in dollars) for it. At the same time, we also buy the underlying stock for S (in dollars). This option being an American option, we can exercise it any time, between the date of purchase and its expiry date. To earn an arbitrage profit, we immediately exercise the option, by selling the underlying stock to the writer of the option for X (in dollars). The net cash inflow is

$$X - P - S.$$

As the net cash inflow is positive, there is an arbitrage profit for us. To rule out the existence of any arbitrage profits, we must rule out

$$P < X - S.$$

Thus, the condition of

$$P \geq X - S.$$

must hold for an American put option. ■

(3) *Prove that, for an American call option on a stock that pays no dividend,*

$$C \geq S - Xe^{-rT}.$$

That is, the call option price cannot be less than the difference between the underlying stock price and the present value of the exercise price.

Proof: An American call option differs from a European call option on the same underlying stock, with the same exercise price and with the same expiry date, in that the former can be exercised any time until it expires. This is in contrast to the latter, which can only be exercised on the expiry date. That is, the American call option not only contains the same features as the corresponding European call option does, but also provides flexibility to option buyers as to when the option can be exercised. Thus, it cannot be worth less than the corresponding European call option to option buyers.

Let us label the European and American call option values as $C^{(E)}$ and $C^{(A)}$, respectively, for notational clarity. For the above reason, we must have

$$C^{(A)} \geq C^{(E)}.$$

It has been established earlier in this chapter that, for a European call option on a stock that pays no dividend,

$$C \geq S - Xe^{-rT}.$$

In the current notation, we can write

$$C^{(E)} \geq S - Xe^{-rT}.$$

As

$$C^{(A)} \geq C^{(E)},$$

we also have

$$C^{(A)} \geq S - Xe^{-rT},$$

thus confirming that, for an American call option on a stock that pays no dividend,

$$C \geq S - Xe^{-rT}. \quad \blacksquare$$

(4) *Prove that the prices of an American call option and the corresponding European call option, with the same exercise price and the same expiry date, on the same underlying stock that pays no dividend, are the same.*

Proof: As in (3), let us label the European and American call option values as $C^{(E)}$ and $C^{(A)}$, respectively, for notational clarity. In (3), we have established that

$$C^{(A)} \geq C^{(E)}.$$

We go further here, by establishing a tighter condition that

$$C^{(A)} = C^{(E)}.$$

The task here is an extension of the property that "(a)n American call option on a non-dividend paying stock will not be exercised before the expiry date."

To prove such a property, we have used the basic idea that, for a call option to be exercised, we must have $S - X > 0$; otherwise, exercising the option is not worthwhile. For an American call option to be exercised before its expiry date, the payoff is $S - X$. However, as the option can also be sold to others in the market for the price

$$C^{(A)} \geq S - Xe^{-rT},$$

which is greater than $S - X$, because the present value factor e^{-rT} (which is positive) is less than 1. Thus, the option ought not be exercises before its expiry date.

The holder of the option can either sell it to others in the market or hold on to it until the expiry date. As the option will not be exercised before its expiry date, its value must be the same as that of the corresponding European call option. Thus, we must have

$$C^{(A)} = C^{(E)}.$$

To establish the equality of the two option prices, we can also use an arbitrage proof. As we already have

$$C^{(A)} \geq C^{(E)}$$

from (3), all we have to do now is to rule out

$$C^{(A)} > C^{(E)},$$

the strict inequality.

For the arbitrage proof, suppose that

$$C^{(A)} > C^{(E)},$$

on the day when the two options have T years to go before maturity. Here, T is typically a proportion of a year. We can write the American call option and buy the corresponding European call option. The net cash inflow is

$$K = C^{(A)} - C^{(E)},$$

which is positive. The dollar amount K is invested risk-free to become Ke^{rT}, as measured at the time when the option expires.

As the American call option will not be exercised before its expiry date, we consider three situations on the expiry date here: $S > X$, $S = X$, and $S < X$. If $S > X$, both options will be exercised. Our net cash inflow will be zero, because the gain of $S - X$ from exercising the European call option as its holder is offset entirely by the loss of $X - S$ for being the writer of the American call option.

If $S \leq X$ instead, neither option will be exercised, and no cash flow will be involved. Given the risk-free income of Ke^{rT}, we have an arbitrage profit. To rule out the existence of any arbitrage profits, we must rule out

$$C^{(A)} > C^{(E)}.$$

Thus, we must have

$$C^{(A)} = C^{(E)}. \quad \blacksquare$$

(5) *Prove that, if two European call options on a non-dividend paying stock differ only in their expiry dates, then the one with a longer life cannot be worth less.*

Proof: The proof has been provided in this chapter. The proof there is an arbitrage proof. The alternative proof here is based directly on the result in (4).

Let us consider two European call options, which expire in T_1 and T_2 years (where T_1 and T_2 are proportions), with

$$T_2 > T_1.$$

Our task here is to prove that

$$C_2 \geq C_1,$$

where C_1 and C_2 are the corresponding European call option prices. For notational clarity, let use label them as $C_1^{(E)}$ and $C_2^{(E)}$, respectively. The labels for the corresponding American call options are $C_1^{(A)}$ and $C_2^{(A)}$.

As established in (4), we have

$$C_1^{(A)} = C_1^{(E)}$$

and

$$C_2^{(A)} = C_2^{(E)}.$$

This is because, for each of the two maturity dates, the corresponding American and European call options have the same price. Given these results, the comparison of the two European option prices can be treated as a comparison of the two corresponding American option prices.

The American call option that has a longer life not only has all the features of the American call option with a shorter life, but also has more time before maturity. Thus, the American call option cannot be worth less than the American call option with a shorter life. That is,

$$C_2^{(A)} \geq C_1^{(A)}.$$

Thus, we must have

$$C_2^{(E)} \geq C_1^{(E)}. \quad \blacksquare$$

7.11.4 Profits from Violations of Put-Call Parity: An Arbitrage Proof

Put-call parity states that, if a European call option and a European put option on the same underlying stock, which pays no dividend, have the same exercise price and the same expiry date, the condition

$$C - P = S - Xe^{-rT}$$

must hold in a frictionless market. Here, C is the call option price, P is the put option price, X is the exercise price, T is the number of years before the two options expire, and r is the continuously compounded annual risk-free interest rate. The task here is to establish analytically that, if this condition is violated, arbitrage profits are available. The materials involved are similar to arbitrage proofs of option properties considered earlier.

Proof: There are two possible ways for this condition to be violated; that is, for the two options, which will expire in T years, either

$$C - P > S - Xe^{-rT}$$

or

$$C - P < S - Xe^{-rT}.$$

(a) Let us start with the case where

$$C - P > S - Xe^{-rT}.$$

This inequality is equivalent to

$$P - C + S < Xe^{-rT}.$$

As

$$S \geq C \geq 0 \text{ and } P \geq 0,$$

we must have

$$0 \leq P - C + S.$$

An investor can buy the put option with P dollars, write the corresponding call option to receive C dollars, and buy the underlying stock with S dollars. The required cash is $P - C + S$ dollars. The investor can borrow less than Xe^{-rT} dollars, at a continuously compounded risk-free interest rate r, to make the investment of $P - C + S$ dollars entirely self-financed.

On the expiry date of the two options, which is T years afterwards, the repayment of the loan requires less than

$$(Xe^{-rT})e^{rT} = X$$

dollars. Let us label, on the expiry date of the two options, the stock price as S_T, the call option price as C_T, and the put option price as P_T. We can have $S_T > X$, $S_T = X$, or $S_T < X$.

If $S_T > X$, only the call option is exercised. With

$$C_T = S_T - X \text{ and } P_T = 0,$$

the investor's portfolio is worth

$$P_T - C_T + S_T = 0 - (S_T - X) + S_T = X$$

dollars.

If $S_T = X$, neither option is exercised. With

$$C_T = 0 \text{ and } P_T = 0,$$

the investor's portfolio is worth

$$P_T - C_T + S_T = 0 - 0 + S_T = S_T = X$$

dollars.

If $S_T < X$ instead, only the put option is exercised. With

$$C_T = 0 \text{ and } P_T = X - S_T,$$

the investor's portfolio is worth

$$P_T - C_T + S_T = (X - S_T) - 0 + S_T = X$$

dollars. As the loan repayment requires less than X dollars, the investor has an arbitrage profit in each of the three cases.

(b) Let us now consider the remaining case where

$$C - P < S - Xe^{-rT}.$$

This inequality is equivalent to

$$P - C + S > Xe^{-rT}.$$

As

$$Xe^{-rT} > 0,$$

we must have

$$P - C + S > 0.$$

An investor can write the put option to collect P dollars, buy the corresponding call option by paying C dollars, and short sell the underlying stock to collect S dollars. The net cash inflow is $P - C + S$ dollars. The investor can invest these $P - C + S$ dollars, which are more than Xe^{-rT} dollars, at a continuously compounded risk-free interest rate r, for T years, until the two options expire.

On the expiry date of the two options, which is T years afterwards, the risk-free investment will result in more than

$$(Xe^{-rT})e^{rT} = X$$

dollars. Let us label, on the expiry date of the two options, the stock price as S_T, the call option price as C_T, and the put option price as P_T. We can have $S_T > X$, $S_T = X$, or $S_T < X$.

If $S_T > X$, only the call option is exercised. The investor, being the holder of the call option, will buy the underlying stock from the writer of the call option, by paying X dollars for it, and immediately return the stock to its lender, so that the previous short-sale arrangement will be terminated. On the expiry date of the two options, the cash outflow from the investor's portfolio is X dollars.

If $S_T = X$, neither option is exercised. The investor will buy the underlying stock in the market for S_T dollars, which are equal to X dollars, so that the previous short-sale arrangement can be terminated.

If $S_T < X$ instead, only the put option is exercised. The writer, being the writer of the put option, will buy the underlying stock from the holder of the put option by paying X dollars for it. As the risk-free investment for T years will generate more than X dollars, the investor has an arbitrage profit in each of the three cases.

By ruling out (a) and (b), where arbitrage profits are available, we must have

$$C - P = S - Xe^{-rT}$$

instead. This is the put-call parity condition. While the approach to establish earlier is simpler and much more direct, the approach here is intended to illustrate analytically how arbitrage profits can be generated if the put-call parity condition is violated. ■

7.12 Appendix B: Derivation of the Binomial Model for Call Options

7.12.1 The Starting Point: A Single-period Setting

In the derivation in the main text, a risk-free hedge is achieved by buying one unit (share) of the underlying stock and writing m units of the call option. We now show that a risk-free investment is also achievable, by buying one unit of the call option and short selling h units (shares) of the stock on the option. Here, h can be a proportion of a share. The net investment is $C - hS$. A risk-free hedged portfolio requires

$$C_u - huS = C_d - hdS.$$

Thus, we have

$$C_u - C_d = h(u - d)S$$

and then

$$h = \frac{C_u - C_d}{(u - d)S}.$$

For the investment to be risk-free, we must have

$$(1 + r_f)(C - hS) = C_u - huS.$$

This equation can be written as

$$(1+r_f)C - (1+r_f)hS = C_u - huS$$

and then

$$C_u - (1+r_f)C = h[u - (1+r_f)]S.$$

Upon substituting the expression of h, we have

$$\begin{aligned} C_u - (1+r_f)C &= \frac{C_u - C_d}{(u-d)S}[u - (1+r_f)]S \\ &= \frac{(C_u - C_d)[u - (1+r_f)]}{u-d} \end{aligned}$$

and then

$$\begin{aligned} (1+r_f)C &= C_u - \frac{[u-(1+r_f)](C_u - C_d)}{u-d} \\ &= C_u\left[\frac{(1+r_f)-d}{u-d}\right] + C_d\left[\frac{u-(1+r_f)}{u-d}\right], \end{aligned}$$

which leads to equation (7.7) in the main text. That is,

$$\begin{aligned} C &= \frac{pC_u + (1-p)C_d}{1+r_f} \\ &= \frac{p\max(0, uS - X) + (1-p)\max(0, dS - X)}{1+r_f}, \end{aligned}$$

where

$$p = \frac{(1+r_f)-d}{u-d}$$

and

$$1 - p = \frac{u-(1+r_f)}{u-d}.$$

As long as

$$0 < d < 1 < 1 + r_f < u,$$

the conditions of $0 < p < 1$ and $0 < 1 - p < 1$ are assured.

The connection between the option price and the underlying stock price can be captured by the following diagram:

$$
\begin{array}{ccccccc}
 & & uS & & & & C_u = \max(0, uS - X) \\
 & \diagup & & & & \diagup & \\
S & & & & C & & \\
 & \diagdown & & & & \diagdown & \\
 & & dS & & & & C_d = \max(0, dS - X)
\end{array}
$$

7.12.2 Extension to a Multi-period Setting

Now, we propose that equation (7.7) be expressed as

$$C = \left[\sum\nolimits_{n=0}^{1} \binom{1}{n} p^n (1-p)^{1-n} \max\left(0, u^n d^{1-n} S - X\right) \right] \frac{1}{1+r_f}, \tag{7.17}$$

where

$$\binom{1}{0} = \frac{1!}{(1-0)!0!} = 1$$

and

$$\binom{1}{1} = \frac{1!}{(1-1)!1!} = 1.$$

The summation $\sum_{n=0}^{1}$ has two terms. As the term for $n = 0$ is

$$\binom{1}{0} p^0 (1-p)^{1-0} \max\left(0, u^0 d^{1-0} S - X\right) = (1-p) \max(0, dS - X)$$

and the term for $n = 1$ is

$$\binom{1}{1} p^1 (1-p)^{1-1} \max\left(0, u^1 d^{1-1} S - X\right) = p \max(0, uS - X),$$

the equivalence of equations (7.17) and (7.7) is confirmed. Although the use of a summation sign here may appear unnecessary at first glance, it actually facilitates the extension of the model to a multi-period setting.

In a multi-period setting, r_f is the risk-free interest rate for each period. In a two-period setting, where the up-down stock price movements are captured by the multiplicative factors u and d each period, the eventual stock price is one of u^2S, udS, and d^2S. The corresponding risk-neutral probabilities are p^2, $2p(1-p)$, and $(1-p)^2$, respectively. Likewise, in a three-period setting, the eventual stock price is one of u^3S, u^2dS, ud^2S, and d^3S, corresponding to risk-neutral probabilities p^3, $3p^2(1-p)$, $3p(1-p)^2$, and $(1-p)^3$, respectively. As the coefficients for the probability terms are from Pascal's triangle, we can easily extend the same idea to an N-period setting.

It is important to recognize that, in a multi-period setting, the risk-free hedge portfolio will have to be revised at the beginning of each period. Further, if a given time interval between the valuation date and the expiry date of the option is divided into increasingly more equal periods, the difference between the values of u and d involved — which captures the given volatility of the stock price movements each period — must be smaller. For the model to be practically relevant, the value of $u > 1$ must decrease and the value of $d < 1$ must increase when increasingly more periods are involved, thus resulting in a smaller difference between the values of these two multiplicative factors.

A Two-period Setting

In a two-period setting, the stock price at the end of the first period will be either uS or dS. The option price at the end of the first period will be either C_u or C_d. As there will still be one more period before the option expires, equations (7.3) and (7.4) in the main text, which are

$$\begin{aligned} C_u &= \max(0, uS - X) \\ \text{and } C_d &= \max(0, dS - X), \end{aligned}$$

respectively, are no longer applicable here. Let C_{uu}, C_{ud}, C_{du}, and C_{dd} be the option prices corresponding to stock prices of u^2S, udS, duS, and d^2S at the end of the second period. To find C_u in terms of C_{uu} and C_{ud}, we form a risk-free hedge portfolio at the end of the first period (which is also the start of the second period) when the stock price is uS.

To form a risk-free hedge, we can buy one unit of the call option and short sell h units (shares) of the stock on the option. Alternatively, as described in the main text, we can buy one unit (share) of the stock and write m units of the call option that the stock underlies instead. Both approaches will lead to the same end result. In the latter case, we have

$$u^2S - mC_{uu} = duS - mC_{ud},$$

which leads to

$$m = \frac{(u-d)uS}{C_{uu} - C_{ud}}.$$

This risk-free hedge portfolio ensures that

$$(1 + r_f)(uS - mC_u) = u^2 S - mC_{uu},$$

which can be written as

$$[u - (1 + r_f)]uS = m[C_{uu} - (1 + r_f)C_u] = \frac{(u-d)uS}{C_{uu} - C_{ud}}[C_{uu} - (1 + r_f)C_u].$$

Re-arranging the terms, we can write

$$C_{uu} - (1 + r_f)C_u = \frac{C_{uu} - C_{ud}}{u - d}[u - (1 + r_f)]$$

and then

$$C_u = \frac{1}{1 + r_f}\left\{\frac{C_{uu}[(1 + r_f) - d] + C_{ud}[u - (1 + r_f)]}{u - d}\right\},$$

which leads to

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{1 + r_f},$$

where p is as defined in equation (7.8). Notice that r_f here is the risk-free interest rate each period instead.

To find C_d in terms of C_{du} and C_{dd}, we form a risk-free hedge portfolio at the end of the first period when the stock price is dS. Using the same approach as described above, but with uS, C_u, C_{uu}, and C_{ud} in the algebraic expressions involved substituted by dS, C_d, C_{du}, and C_{dd}, respectively, we can reach

$$C_d = \frac{pC_{du} + (1-p)C_{dd}}{1 + r_f}.$$

As $C_{du} = C_{ud}$, we can write

$$\begin{aligned} C &= \frac{pC_u + (1-p)C_d}{1 + r_f} \\ &= \frac{p[pC_{uu} + (1-p)C_{ud}] + (1-p)[pC_{du} + (1-p)C_{dd}]}{(1 + r_f)^2} \\ &= \frac{p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}}{(1 + r_f)^2}. \end{aligned} \tag{7.18}$$

If the option expires at the end of the second period, as

$$C_{uu} = \max\left(0, u^2S - X\right),$$

$$C_{ud} = C_{du} = \max\left(0, udS - X\right),$$

and

$$C_{dd} = \max\left(0, d^2S - X\right),$$

the expression of C in equation (7.18) is equivalent to

$$C = \left[\sum_{n=0}^{2} \begin{pmatrix} 2 \\ n \end{pmatrix} p^n(1-p)^{2-n} \max\left(0, u^n d^{2-n} S - X\right)\right] \frac{1}{(1+r_f)^2}, \tag{7.19}$$

where

$$\begin{pmatrix} 2 \\ n \end{pmatrix} = \frac{2!}{(2-n)!n!},$$

for $n = 0$, 1, and 2. The summation $\sum_{n=0}^{2}$ has three terms, including $n = 0$, $n = 1$, and $n = 2$.

The term for $n = 0$ is

$$\begin{aligned}\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} p^0(1-p)^{2-0} \max\left(0, u^0 d^{2-0} S - X\right)\right] \frac{1}{(1+r_f)^2} &= \frac{(1-p)^2 \max\left(0, d^2S - X\right)}{(1+r_f)^2} \\ &= \frac{(1-p)^2 C_{dd}}{(1+r_f)^2}.\end{aligned}$$

The term for $n = 1$ is

$$\begin{aligned}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} p^1(1-p)^{2-1} \max\left(0, u^1 d^{2-1} S - X\right)\right] \frac{1}{(1+r_f)^2} &= \frac{2p(1-p) \max\left(0, udS - X\right)}{(1+r_f)^2} \\ &= \frac{2p(1-p) C_{ud}}{(1+r_f)^2}.\end{aligned}$$

The term for $n = 2$ is

$$\begin{aligned}\left[\begin{pmatrix} 2 \\ 2 \end{pmatrix} p^2(1-p)^{2-2} \max\left(0, u^2 d^{2-2} S - X\right)\right] \frac{1}{(1+r_f)^2} &= \frac{p^2 \max\left(0, u^2S - X\right)}{(1+r_f)^2} \\ &= \frac{p^2 C_{uu}}{(1+r_f)^2}.\end{aligned}$$

The connection between the option price and the underlying stock price can be captured by the following diagram:

$$
\begin{array}{cccccccccc}
 & & & & uuS & & & & & C_{uu} \\
 & & & / & & & & & / & = \max(0, u^2 S - X) \\
 & & uS & & & & & C_u & & \\
 & / & & \backslash & & & / & & \backslash & \\
S & & & & udS & C & & & & C_{ud} = C_{du} \\
 & \backslash & & / & = duS & & \backslash & & / & = \max(0, udS - X) \\
 & & dS & & & & & C_s & & \\
 & & & \backslash & & & & & \backslash & \\
 & & & & ddS & & & & & C_{dd} \\
 & & & & & & & & & = \max(0, d^2 S - X)
\end{array}
$$

An N-period Setting

In an N-period setting, we can extend equations (7.17) and (7.19) to

$$C = \left[\sum\nolimits_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} \max\left(0, u^n d^{N-n} S - X\right) \right] \frac{1}{(1+r_f)^N}, \tag{7.20}$$

where

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}, \tag{7.21}$$

with $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 1$, and $0! = 1$. The positive terms among cases of

$$\max\left(0, u^n d^{N-n} S - X\right)$$

all have

$$u^n d^{N-n} S > X.$$

We can establish the lowest n that ensures positive values of $\max\left(0, u^n d^{N-n} S - X\right)$ by solving this inequality or, equivalently,

$$(u/d)^n > \frac{X}{d^N S}$$

for n. The result is the lowest integer n, in the range of 0 to N, satisfying the condition of

$$n > \frac{\ln[X/(d^N S)]}{\ln(u/d)}.$$

Let us label this specific integer n as a. Then, equation (7.20) reduces to

$$C = \left[\sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \left(u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N}. \tag{7.22}$$

Let also

$$p' = \frac{pu}{1+r_f}.$$

As $u > 1+r_f$, we have $p' > p$. Our task now is to show that equation (7.22) can be written as

$$C = S\ B(n \geq a|N, p') - \frac{X}{(1+r_f)^N}\ B(n \geq a|N, p), \tag{7.23}$$

where

$$B(n \geq a|N, p) = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}.$$

This is the binomial option pricing model (for call options).

Before proceeding, notice that the functions $B(n \geq a|N, p')$ and $B(n \geq a|N, p)$, with each being a complementary binomial distribution function, differ only in the probabilities involved. The two functions are complementary distribution functions of their corresponding cumulative distribution functions, which are $B(n < a|N, p')$ and $B(n < a|N, p)$, respectively. As $B(n < a|N, p') + B(n \geq a|N, p') = 1$ and $B(n < a|N, p) + B(n \geq a|N, p) = 1$, each pair of such functions fully covers all possible outcomes as characterized by the corresponding distribution.

Further, under the condition of $1 + r_f < u$, which already ensures that $0 < p < 1$. Now, let us write

$$\begin{aligned} p' &= \frac{pu}{1+r_f} = \frac{1+r_f-d}{u-d} \cdot \frac{u}{1+r_f} \\ &= \frac{(1+r_f)u - ud}{(1+r_f)u - (1+r_f)d}. \end{aligned}$$

As $p' > p > 0$, the condition of $1 + r_f < u$ in the above equation ensures that $0 < p' < 1$. Thus, altogether, we have

$$0 < p < p' < 1.$$

This analytical result confirms that p' also has features of a probability.

To show that equation (7.22) leads to equation (7.23), let us first write equation (7.22) as

$$\begin{aligned} C &= \left[\sum_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \left(u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N} \\ &= K_1 - K_2, \end{aligned}$$

where

$$K_1 = \left[\sum_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} u^n d^{N-n}\right] \frac{S}{(1+r_f)^N}$$

and

$$K_2 = \left[\sum_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}\right] \frac{X}{(1+r_f)^N}.$$

For a binomial distribution function, $B(n < a|N, p)$ is a cumulative distribution function and $B(n \geq a|N, p)$ is the corresponding complementary distribution function. We can write

$$B(n < a|N, p) = \sum_{n=0}^{a-1} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

and

$$B(n \geq a|N, p) = \sum_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n},$$

with

$$\begin{aligned} B(n &< a|N, p) + B(n \geq a|N, p) \\ &= \sum_{n=0}^{a-1} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} + \sum_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \\ &= \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} = 1. \end{aligned}$$

Thus, we have directly

$$K_2 = \frac{X}{(1+r_f)^N} B(n \geq a|N, p).$$

To express K_1 in terms of a complementary distribution function, we first write it as

$$\begin{aligned} K_1 &= \left\{\sum_{n=a}^{N} \frac{N!}{(N-n)!n!} (up)^n [d(1-p)]^{N-n}\right\} \frac{S}{(1+r_f)^n (1+r_f)^{N-n}} \\ &= S \sum_{n=a}^{N} \frac{N!}{(N-n)!n!} \left(\frac{up}{1+r_f}\right)^n \left[\frac{d(1-p)}{1+r_f}\right]^{N-n}. \end{aligned}$$

As established in equation (7.8), we have

$$p = \frac{1 + r_f - d}{u - d}.$$

Letting

$$p' = \frac{pu}{1 + r_f},$$

we can write

$$p' = \frac{up}{1 + r_f} = \frac{u(1 + r_f - d)}{(1 + r_f)(u - d)}.$$

As

$$\begin{aligned} \frac{d(1-p)}{1+r_f} &= \frac{d}{1+r_f}\left(1 - \frac{1 + r_f - d}{u - d}\right) \\ &= \frac{d[u - d - (1 + r_f) + d)]}{(1 + r_f)(u - d)} \\ &= \frac{d[u - (1 + r_f)]}{(1 + r_f)(u - d)}, \end{aligned}$$

we have

$$\begin{aligned} p' + \frac{d(1-p)}{1+r_f} &= \frac{u(1 + r_f - d)}{(1 + r_f)(u - d)} + \frac{d[u - (1 + r_f)]}{(1 + r_f)(u - d)} \\ &= \frac{u(1 + r_f - d) + d[u - (1 + r_f)]}{(1 + r_f)(u - d)} \\ &= \frac{u(1 + r_f) - ud + du - d(1 + r_f)}{(1 + r_f)(u - d)} \\ &= \frac{(u - d)(1 + r_f)}{(1 + r_f)(u - d)} = 1 \end{aligned}$$

and thus

$$\frac{d(1-p)}{1+r_f} = 1 - p'.$$

As p' and $1 - p'$ have analytical features of probabilities, we can write

$$\begin{aligned} K_1 &= S \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} \left(\frac{up}{1+r_f}\right)^n \left[\frac{d(1-p)}{1+r_f}\right]^{N-n} \\ &= S \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} (p')^n (1 - p')^{N-n} \\ &= S\ B(n \geq a | N, p') \end{aligned}$$

and then

$$C = S\ B(n \geq a|N, p') - \frac{X}{(1 + r_f)^N}\ B(n \geq a|N, p),$$

which is equation (7.23).

$$C = \left[\sum\nolimits_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} p^n (1-p)^{N-n} \max\left(0, u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N}, \tag{7.24}$$

Then, equation (7.24) reduces to

$$C = \left[\sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \left(u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N}. \tag{7.25}$$

7.13 Appendix C: Derivation of the Binomial Model for Put Options

Given the close similarity in the derivations of the binomial option model for call options and for put options, only a sketch of the latter derivation is provided in this appendix. For a one-period case, let S and P be the beginning-of-period stock and put prices, respectively. With the end-of-period stock price being either uS or dS, the corresponding end-of-period put price is either

$$P_u = \max(0, X - uS)$$

or

$$P_d = \max(0, X - dS).$$

If we buy 1 unit of the underlying stock and m units of the put option, the investment is $S + mP$. A risk-free investment requires

$$(1 + r_f)(S + mP) = uS + mP_u = dS + mP_d.$$

Eliminating m from above and express P in terms of the remaining parameters leads to

$$P = \frac{pP_u + (1-p)P_d}{1 + r_f} = \frac{p \max(0, X - uS) + (1-p) \max(0, X - dS)}{1 + r_f},$$

where p satisfies the condition of

$$0 < p = \frac{1 + r_f - d}{u - d} < 1.$$

To extend the above to a two-period setting, with the stock price at the end of the second period being u^2S, udS, or d^2S, the corresponding put price is

$$P_{uu} = \max(0, X - u^2S),$$

$$P_{ud} = P_{du} = \max(0, X - udS),$$

or

$$P_{dd} = \max(0, X - d^2S).$$

Letting r_f be the risk-free interest rate each period instead, we form a risk-free hedge at the start of each period. We have

$$P = \frac{pP_u + (1-p)P_d}{1 + r_f},$$

where

$$P_u = \frac{pP_{uu} + (1-p)P_{ud}}{1 + r_f}$$

and

$$P_d = \frac{pP_{du} + (1-p)P_{dd}}{1 + r_f},$$

again with p satisfying the condition of

$$0 < p = \frac{1 + r_f - d}{u - d} < 1.$$

It is important to recognize that the r_f in the one-period setting considered earlier and the r_f in the two-period setting here are not the same. Upon substitutions of the terms involved, the analytical result here reduces to

$$\begin{aligned} P &= \frac{p^2P_{uu} + 2p(1-p)P_{ud} + (1-p)^2P_{dd}}{(1+r_f)^2} \\ &= \left[\sum\nolimits_{n=0}^{2} \binom{2}{n} p^n(1-p)^{2-n} \max(0, X - u^n d^{2-n} S)\right] \frac{1}{(1+r_f)^2}, \end{aligned}$$

where

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1, \ \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2, \text{ and } \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1.$$

Extension to an N-period setting requires that a risk-free hedge be formed at the start of each of the N periods. With r_f being the free-free interest rate each period, we can reach

$$P = \left[\sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} p^n (1-p)^{N-n} \max(0, X - u^n d^{N-n} S) \right] \frac{1}{(1+r_f)^N},$$

where

$$\begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{(N-n)!n!}.$$

To simplify $\max(0, X - u^n d^{N-n} S)$, let a be the smallest integer n in the range of 0 to N, satisfying the condition of

$$u^n d^{N-n} S - X > 0.$$

Then, $a-1$ must be the largest integer n to satisfy the condition of

$$X - u^n d^{N-n} S > 0.$$

Thus, we can write

$$\begin{aligned} P &= \left[\sum_{n=0}^{a-1} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} (X - u^n d^{N-n} S) \right] \frac{1}{(1+r_f)^N} \\ &= \frac{X}{(1+r_f)^N} B(n \le a-1 | N, p) - S\, B(n \le a-1 | N, p'), \end{aligned}$$

where

$$p = \frac{1 + r_f - d}{u - d}$$

and

$$p' = \frac{pu}{1 + r_f},$$

satisfying the condition of

$$0 < p < p' < 1.$$

Noting that

$$B(n \le a-1 | N, p) = B(n < a | N, p)$$

and

$$B(n \le a-1|N,p') = B(n < a|N,p'),$$

we can write

$$P = \frac{X}{(1+r_f)^N} B(n < a|N,p) - S\,B(n < a|N,p'),$$

which is equation (7.14) in the main text.

7.14 Exercises

1. (C50) Consider a European put on a non-dividend paying stock with a current stock price of \$37. The exercise price of the put, which will expire in 6 months, is \$40. The continuously compounded risk-free interest rate is 5% per annum, and the put is selling for \$1. Illustrate that the put is mispriced and that an arbitrage profit can be made by buying both the put and the underlying stock with borrowed money.

2. (C51) Consider a European call option with an exercise price of \$40 and a maturity date 6 month from now. The underlying stock price is \$28 and the standard deviation of the annual returns of the stock is $\sqrt{0.50}$ (which is about 70.7%). The continuously compounded risk-free interest rate is 6% per annum. What is the value of this European call if no dividend is expected from the stock for the next year.

3. (C52) Consider a European put option with an exercise price of \$20 and a maturity date 6 month from now. The underlying stock price is \$20 and the standard deviation of the annual returns of the stock is 60%. The continuously compounded risk-free interest rate is 8% per annum. What is the value of this European put if no dividend is expected from the stock for the 9 months.

4. (X19b) Both a European call option and a European put option with a common exercise price on the same underlying stock that pays no dividends will expire in 3 months. It is known that $X = \$20.00$, $r = 4.5\%$, and $\sigma = 20\%$. The current stock price is $S = \$19.75$. Based on such information, we have $d_1 = 0.0367$ and $d_2 = -0.0633$. Further, the areas under the standard normal distribution function $f(z)$, from $z = 0$ to $z = a$, for some selected values of a, are as follows:

a	0.03	0.04	0.05	0.06	0.07
$\int_{z=0}^{a} f(z)dz$	0.0120	0.0160	0.0199	0.0239	0.0279

Find the current price of the above European put option. The numerical result is intended to be to the nearest of $1/1,000$ of a dollar.

5. (X23b) Consider a European call option and a European put option on the same stock that pays no dividends. Both options expire in 3 months. Currently, the stock price is \$52, the exercise price of each option is \$50, and the continuously compounded risk-free annual interest rate is 4%. In a frictionless market with no transaction costs, if the call option is selling at \$3.100 and the put option is selling at \$0.585, can any arbitrage profits be made? Explain clearly by considering all possible scenarios of the underlying stock price on the expiry date of the two options.

6. (X23a) Consider a European put option on a stock that pays no dividends. The option expires in 3 months. Currently, the stock price is \$38, the exercise price of the option is \$40, and the continuously compounded risk-free annual interest rate is 5%. In a frictionless market with no transaction costs, if the option is selling at \$1.45, can any arbitrage profits be made? Explain clearly by considering all possible scenarios of the underlying stock price on the expiry date of the option.

7. (X16a23c) For the binomial option pricing model in an N-period setting, the present value of the exercise price is $X/(1+r_f)^N$. Here, X is the exercise price and r_f is the risk-free interest rate each period. The present value of the exercise price in the Black-Scholes option pricing model is $X\exp(-rT)$ instead. Here, r is the continuously compounded risk-free annual interest rate and T is the number of years (which is usually a proportion of a year) before the option expires. To establish the correspondence between the two models, the time interval as covered by the T years is divided into N periods.

   (a) Given that r is 6.5% per annum and that a European call option expires in 2 months, find the corresponding r_f, the risk-free interest rate per period for $N = 10$.

   (b) For the binomial option pricing model in an N-period setting, suppose that the underlying stock price at the beginning of period n is S_n and that it becomes either uS_n or dS_n at the end of the corresponding period. Here, u and d are given multiplicative factors satisfying the condition of $u > 1 + r_f > 1 > d > 0$. It is known that $u = 1/d = 1.00160$ corresponding to $N = 10,000$, for a European call option that expires in 3 months. Infer from the convergence of the binomial version to the Black-Scholes option pricing model at $N = 10,000$ the standard deviation of annual returns of the underlying stock.

7.15 Highlights of This Chapter

- A call option is a financial instrument that gives its holder the right to purchase from its seller (writer) one unit of the underlying security, at a predetermined price, at or before an expiry date.

- A put option gives its holder the right to sell to its seller instead.

- The buyer of the option is the party who pays for the right involved and, therefore, is the party who can make a choice (to exercise the option or not).

- The seller (writer) of the option is the party who receives the payment from the buyer and, therefore, is the party with no subsequent choice.

- American and European options differ in that the latter can be exercised only at the expiry date.

- Let S be the price of the underlying stock and X be the exercise price of an option. The exercise price is also known as the strike price. If $S > X$, a call is *in the money*, but a put is *out of the money*. If $S < X$, a call is out of money, but a put is in the money. If $S = X$, an option is *at the money*.

- The following assumptions are typically required for establishing basic option properties: (1) The absence of transaction costs. (2) Risk-free lending and borrowing at the same interest rate. (3) No dividends from the underlying stock. (4) The absence of arbitrage opportunities.

- The price of a call option cannot be higher than the price of the underlying stock.

- The price of a call option cannot be negative.

- The price of a call option on a non-dividend paying stock cannot be lower than the difference between the stock price and the present value of the exercise price. With C being the call option price, S being the underlying stock price, X being the exercise price, r being the continuously compounded annual risk-free interest rate, and T being the proportion of a year before the call option expires, this property can be stated analytically as
$$C \geq S - Xe^{-rT}.$$

- The price of a call option at the expiry date is 0 or the difference between the underlying stock price and the exercise price, whichever is higher.

- If two American call options differ only in their expiry dates, the one with a longer life cannot be worth less. If two European call options on a non-dividend paying stock differ only in their expiry dates, the one with a longer life cannot be worth less. As the latter property is not as obvious, a proof such as an arbitrage proof is needed for its confirmation. An arbitrage proof is by starting with the contrary, in order to see whether arbitrage profits exist.

- If two call options differ only in their exercise prices, then the one with a lower exercise price cannot be worth less.

- An American call option on a non-dividend paying stock will not be exercised before the expiry date. An implication of this property is that, if there are no dividends to be paid over the life of a firm's options, an American call option is worth exactly the same as an otherwise identical European call option.

- The price of a put option cannot be higher than the exercise price. The price of a European put option cannot be higher than the present value of the exercise price.

- The price of a put option cannot be negative.

- The price of a put option on a non-dividend paying stock cannot be lower than the difference between the present value of the exercise price and the stock price. With P being the put option price, S being the underlying stock price, X being the exercise price, r being the continuously compounded annual risk-free interest rate, and T being the proportion of a year before the put option expires, this property can be stated analytically as
$$P \geq Xe^{-rT} - S.$$

- The price of a put option at the expiry date is 0 or the difference between the exercise price and the underlying stock price, whichever is higher.

- If two American puts differ only in their expiry dates, the one with a longer life cannot be worth less.

- If two put options differ only in their exercise prices, then the one with a higher exercise price cannot be worth less.

- It can be optimal to exercise an American put option on a non-dividend paying stock before the expiry date. Given this property, the price of an American put option on a non-dividend paying stock cannot be lower than the price of an otherwise identical

European put option. Further, since early exercise is possible, the price of an American put option on a non-dividend paying stock cannot be lower than the difference between the exercise price and the stock price.

- When we use arbitrage proofs to verify an option property, we must recognize that the exercise price X and the continuously compounded annual risk-free interest rates r are constants, but the remaining symbols, including option prices C and P, the underlying stock price S, and the time to expiry T are not constants.

- For the corresponding call and put options on a stock that pays no dividends, *put-call parity* is the property that the difference between the call price and the put price is the same as the difference between the price of the underlying stock and the present value of the exercise price. By using the same symbols as defined earlier, put-call parity can be stated analytically as

$$C - P = S - Xe^{-rT}.$$

- Put-call parity can be established by noting that simultaneously writing a call, buying the corresponding put, and owing the underlying stock is equivalent to owing a risk-free security.

- Writing a *covered call* is equivalent to simultaneously writing a call and owning the underlying stock.

- Writing a covered call is also equivalent to simultaneously writing a put and owning a risk-free asset.

- The original derivation of the Black-Scholes option pricing model for a call option starts with knowing that S changes over time according to a stochastic process and that, if S goes up (or down), so does C. An investor can achieve a risk-free hedge by writing a call option and buying a matching proportion of a share of the underlying stock. In response to price changes in the market over time, the investor maintains a risk-free hedge by continuously adjusting the long and short positions (for the stock and the option, respectively) until the option expires. The maintenance of a risk-free hedge is crucial for establishing a partial differential equation, which leads to the Black-Scholes option pricing formula.

- The Black-Scholes option pricing formula for European and American call options on a stock that pays no dividends is

$$C = S\,\mathbf{N}(d_1) - X\,e^{-rT}\,\mathbf{N}(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + rT\right] + \frac{1}{2}\sigma\sqrt{T} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

Besides the symbols that have already been defined, σ is the standard deviation of annual returns of the underlying stock and $\mathbf{N}(\cdot)$ is the cumulative standard normal distribution. In the standard normal distribution function $f(z)$, $\mathbf{N}(d_1)$, for example, is the area under the symmetric bell-shaped curve from $z = -\infty$ to $z = d_1$; that is,

$$\mathbf{N}(d_1) = \int_{z=-\infty}^{d_1} f(z)dz.$$

Notice that $\mathbf{N}(-\infty) = 0$, $\mathbf{N}(0) = 0.5$, and $\mathbf{N}(\infty) = 1$.

- The Black-Scholes option pricing formula for a European put options on a stock that pays no dividends is

$$P = X\ e^{-rT}\,\mathbf{N}(-d_2) - S\,\mathbf{N}(-d_1).$$

- The Black-Scholes formulas allow us to establish the following signs

$$\partial C/\partial S > 0;\quad \partial C/\partial X < 0;\quad \partial C/\partial T > 0;\quad \partial C/\partial r > 0;\quad \partial C/\partial \sigma > 0;$$

$$\partial P/\partial S < 0;\quad \partial P/\partial X > 0;\quad \partial P/\partial T > 0;\quad \partial P/\partial r < 0;\quad \text{and}\quad \partial P/\partial \sigma > 0.$$

These results confirm some previously established properties and reveal some additional properties of European options. Both call and put prices increase with increasing time to expiry and with increasing volatility of the underlying stock. However, the impacts on call and put prices by changes in the underlying stock price, the exercise price, and the risk-free interest rate are opposite. The positive impact of the volatility of the return of the underlying stock on option prices is easily understandable. More volatile is the underlying stock return, more likely the option holder can gain from holding it.

- In a multi-period setting, the binomial option pricing model as developed by Cox, Ross, and Rubinstein also uses a risk-free hedge at the beginning of each period.

- The binomial option pricing model for a call option — implicitly, a European call option on a stock that pays no dividends — is

$$C = S\ B(n \geq a|N, p') - \frac{X}{(1+r_f)^N}\ B(n \geq a|N, p).$$

Besides the symbols that have already been defined, (1) N is the number of periods considered, (2) r_f is the risk-free interest rate each period, (3) a is the smallest integer that is greater than

$$\frac{\ln[X/(d^N S)]}{\ln(u/d)},$$

where u and d are multiplicative factors of stock prices each period (to capture stock price movements each period) under the condition of $0 < d < 1 < 1 + r_f < u$, (4) p and p' are parameters defined as

$$p = \frac{1 + r_f - d}{u - d} \quad \text{and} \quad p' = \frac{pu}{1 + r_f},$$

respectively, where $0 < p < p' < 1$, and (5)

$$B(n \geq a|N, p) = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

and

$$B(n \geq a|N, p') = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} (p')^n (1-p')^{N-n}$$

are complementary binomial distribution functions, where $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 1$, and $0! = 1$.

- The binomial option pricing model for a put option — implicitly, a European put option on a stock that pays no dividends — is

$$P = \frac{X}{(1 + r_f)^N} B(n < a|N, p) - S\, B(n < a|N, p'),$$

where $B(n < a|N, p)$ and $B(n < a|N, p')$ are cumulative binomial distribution functions.

- As a cumulative distribution function and the corresponding complementary distribution function sum to 1, we have

$$B(n < a|N, p) + B(n \geq a|N, p) = 1$$

and

$$B(n < a|N, p') + B(n \geq a|N, p') = 1,$$

- In an N-period setting, put-call parity can be stated as

$$C - P = S - \frac{X}{(1 + r_f)^N}.$$

- To match the continuously compounded risk-free interest rate r in the Black-Scholes model, where the option expires in T years (or a proportion T of a year), we can set r_f in the binomial version according to

$$(1 + r_f)^N = \exp(rT),$$

which allows r_f to be determined from

$$1 + r_f = \exp\left(\frac{rT}{N}\right).$$

- There are different ways to set parameters u and d in an N-period setting to match the volatility parameter σ in the Black-Scholes model. One of them is

$$u = \frac{1}{d} = \exp\left(\sigma\sqrt{\frac{T}{N}}\right),$$

as provided by Cox, Ross, and Rubinstein.

- As $N \to \infty$, the binomial option pricing model will converge to the Black-Scholes version.

- In the Black-Scholes version, the continuously compounded risk-free interest rate r and the option expires in T years. In the binomial case, the risk-free interest rate r_f is over each of the N time intervals.

- The binomial option pricing model uses the multiplicative up-down factors u and d for the underlying stock price over each of the N time steps to capture the volatility parameter σ in the Black-Scholes model.

- The popular volatility index known as the VIX is the first forward-looking volatility index provided by the Chicago Board Options Exchange (CBOE). The index is for predicting the volatility of the S&P 500 index during the next 30 calendar days. The computation of the VIX is based on a portfolio of out-of-money options on S&P 500. Options for use in the computations must satisfy some specific conditions. As option prices are available in real time, so are the computed values of the VIX.

- Over time, CBOE has also introduced various other volatility indices. For example, there are extensions of the VIX for predicting the volatility of S&P 500 over different time intervals, such as 9 days, 3 months, 6 months, and one year. Currently, available CBOE volatility indices are typically for major stock indices, such as Dow Jones Industrial Average, NASDAQ Composite Index, S&P 500 Index, and Russell 2000 Index (for stocks of smaller companies), but not for the individual constituent stocks. However, daily closing values of volatility indices for some well-known stocks such as Amazon, Apple, Goldman Sachs, Google, and IBM are still available.

- The availability of various CBOE volatility indices has provided equity investors with an additional investment tool. The attractiveness of such a tool is due to its forward-looking nature. In particular, the VIX, which predicts the volatility of S&P 500 in the month ahead, has emerged as a gauge of fear or stress in the stock market. Rightly or wrongly, it is perceived by many in the investment world as a *fear index*. As such, a high value of the VIX is also perceived by these investors as a signal of imminent market declines, thus affecting their trading activities. Such interpretations of the VIX are contrary to the well-established notion that expected return and volatility are in two different dimensions of an investment.

- Even if the volatility indices are interpreted properly, their critics still question whether, in practice, they represent improvements over the use of simpler forecasting methods, given that many severe events in the future are unpredictable. As such indices have attracted so much attention in the investment world, the availability of their real-time values might have been used unwisely by some investors in trading activities, which collectively could have consequential impacts on price movements and trading volumes of the stocks involved. It is such a practical concern that makes an exploration of potential causal relationships between the CBOE volatility indices and subsequent stock market activities interesting.

- Consider a firm with two sources of funds, which are equity and debt. In a single-period setting, if the firm's end-of-period value falls short of its debt obligation, each equity holder will receive nothing. Otherwise, equity holders will receive what is left over after the firm meets its debt obligation. Thus, each equity holder's position is like the position of the holder of a call option. The end-of-period debt obligation is like the exercise price of the option. The position of the debt holders of the firm can be viewed as the owner of the firm who has written a call option with its exercise price being the required end-of-period debt obligation.

- In a single-period setting, corporate debt can be viewed as a portfolio consisting of a risk-free bond and a short position of a put option on the firm's assets (by writing the put option), with the exercise price equal to the same debt payment. If the firm's assets are worth more than the contractual debt payment, the put option will not be exercised. If so, the holder of the portfolio receives the contractual debt payment. Otherwise, the option will be exercised; the writer of the put option will use the contractual debt payment from the risk-free bond to buy the firm's assets, which is worth less.