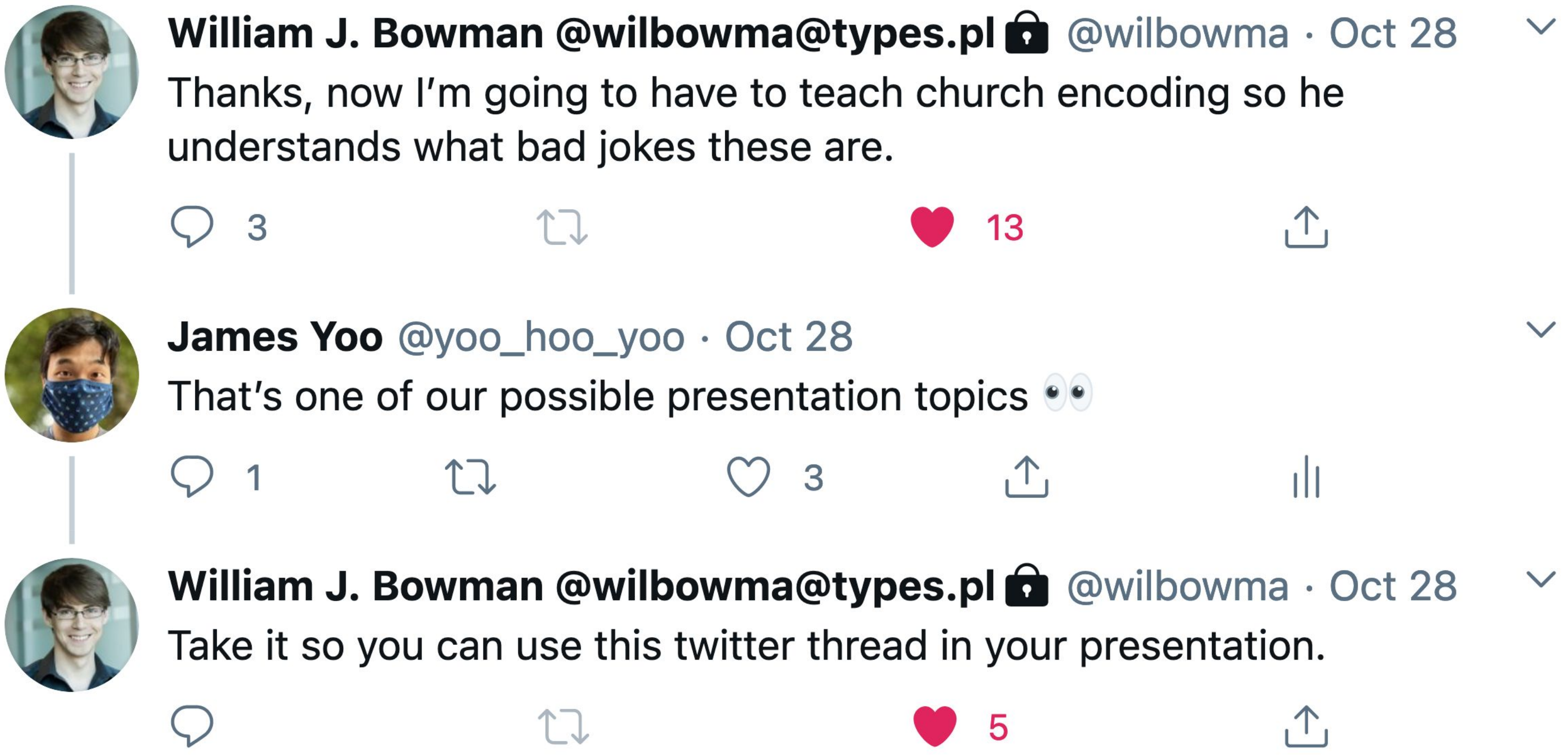
Church Encoding

Alison Li, James Yoo



Twitter, 2020 (colourized)

William J. Bowman @wilbowma@types.pl 🔒 @wi[redacted]

Thanks, now I'm going to have to teach church [redacted] understands what bad jokes these are.

3

James Yoo @yoo_hoo_yoo[redacted]

That's one of our possi[redacted] topics 👀

1 [redacted] 3

William [redacted] ilbowma@types.pl 🔒 @wilbowma · Oct 28

Ta[redacted] use this twitter thread in your presentation.

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YOU ASKED FOR IT

Twitter, 2020 (colourized)

Overview of Presentation

- Motivation
- Background
- Definitions
- Activity
- Summary

Motivation

“ using lambda, you can encode boolean expressions like *true*, *false*, and *if*, and you can encode **numbers** and **arithmetic**. – Garcia, 2013

$$x$$

$$\lambda x.t$$

$$t\ t$$

all you need to express

any computation whatsoever

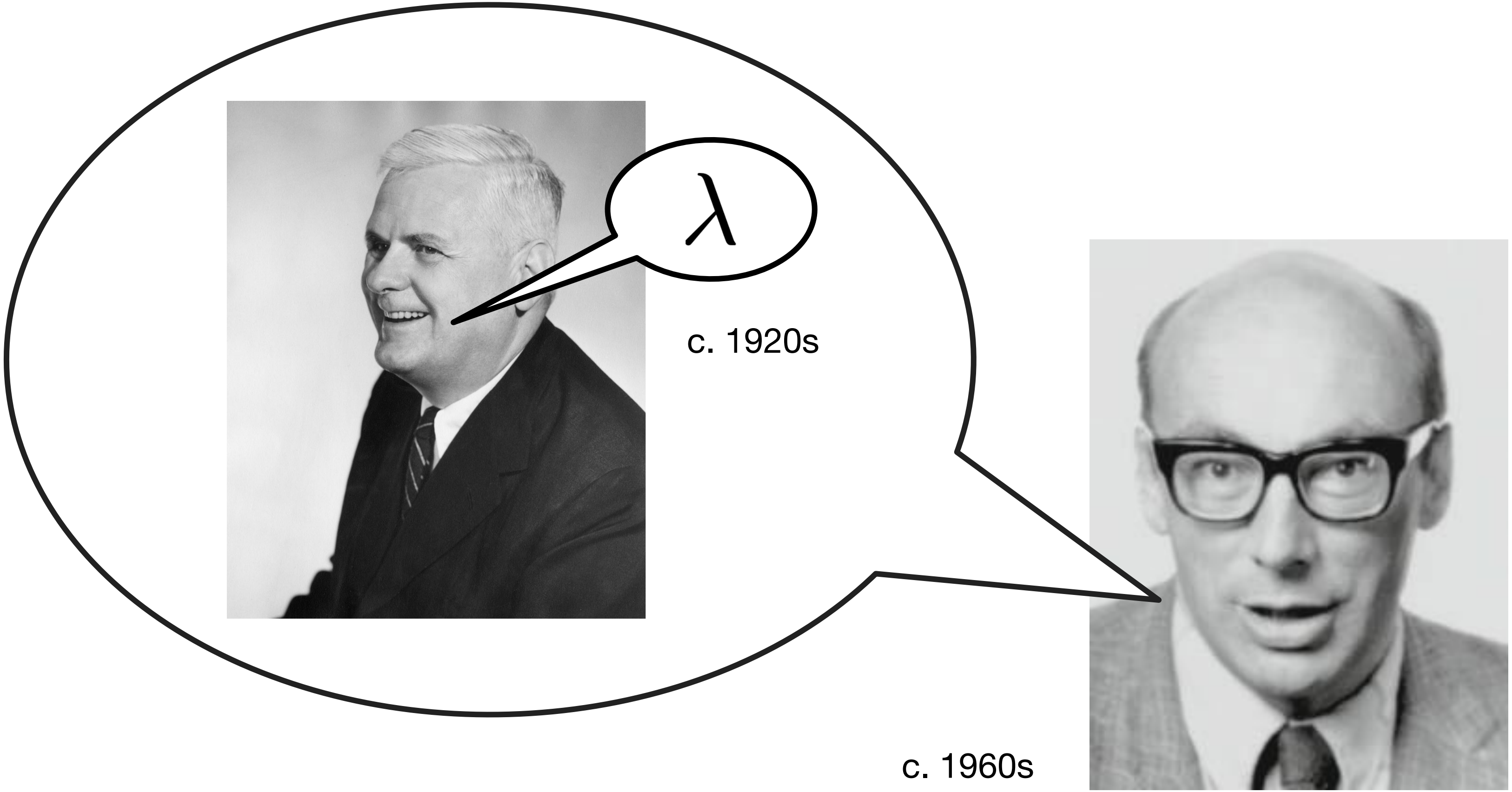
$$n \equiv \lambda f . \lambda . x . f^{\circ n} x$$

$\lambda f.\ \lambda x.\ f(f(f(f(fx))))$

Background

c. 1920s

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(fx))))))$



$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(fx)))))))$

$$\underbrace{\overbrace{(\lambda x.t_1)}^{\text{function } \textit{definition}}t_2}_{\text{function } \textit{application}}$$

λ-calculus

$$x \in \textsc{Var}, \quad t \in \textsc{Term}, \quad v \in \textsc{Value}$$
$$\begin{array}{lll} t & ::= & x \mid t\ t \mid \lambda x.t \\ v & ::= & \lambda x.t \end{array}$$

$\boxed{[t/x] : \textsc{Term} \to \textsc{Term}}$ **Substitution**

$$\begin{aligned} [t/x]x &= t \\ [t/x]y &= y \quad \text{if } y \neq x \\ [t/x]t_1\ t_2 &= ([t/x]t_1)\ ([t/x]t_2) \\ [t/x]\lambda x_0.t_0 &= \lambda x_0.[t/x]t_0 \end{aligned}$$

$\boxed{\longrightarrow\ \subseteq \textsc{Term} \times \textsc{Term}}$ **Single-step Reduction**

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}\ \text{(e-app1)} \qquad \frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'}\ \text{(e-app2)} \qquad \frac{}{(\lambda x.\ t_{12})v \longrightarrow [v/x]t_{12}}\ \text{(e-appabs)}$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(fx)))))))))$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \text{ (e-app1)}$$

$$\frac{t_1 \longrightarrow \lambda x.t}{t_1\ t_2 \longrightarrow (\lambda x.t) t_2}$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(fx))))))))))$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'}\ \text{(e-app2)}$$

$$\frac{t_2 \longrightarrow \boxed{x'}}{(\lambda x.t)t_2 \longrightarrow (\lambda x.t)\boxed{x'}}$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(f(f(fx))))))))))))$

$$\frac{}{(\lambda x.\, t_{12})v \longrightarrow [v/x]t_{12}} \quad \text{(e-appabs)}$$

$$(\lambda x.t)x' \longrightarrow [x'/x]t$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(f(f(fx))))))))))))$

Definitions

Zero

$$0 \equiv \lambda f.\underbrace{\lambda x.\overbrace{x}^{\text{body of \textit{inner} lambda}}}_{\text{body of \textit{outer} lambda}}$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(f(f(f(fx)))))))))))))$

One

$$1 \equiv \lambda f.\lambda x.\, fx$$

$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(f(f(f(f(fx))))))))))))))$

Two

$$2 \equiv \lambda f.\lambda x.\boxed{f(fx)}$$

$$\lambda f.\ \lambda x.\ f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(fx)))))))))))))))$$

$$
\begin{aligned}
0 &\equiv \lambda f.\lambda x.x \\
1 &\equiv \lambda f.\lambda x.f\ x \\
2 &\equiv \lambda f.\lambda x.f(f\ x) \\
3 &\equiv \lambda f.\lambda x.f(f(f\ x)) \\
4 &\equiv \lambda f.\lambda x.f(f(f(f\ x))) \\
5 &\equiv \lambda f.\lambda x.f(f(f(f(f\ x)))) \\
6 &\equiv \lambda f.\lambda x.f(f(f(f(f(f\ x))))) \\
&\dots \\
420 &\equiv \lambda f.\lambda x.f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(f(\\
&\dots
\end{aligned}
$$

$$n \equiv \lambda f.\lambda.x.\overbrace{f^{\circ n} x}^{\text{apply } f \text{ n times to } x}$$

$\lambda f.\ \lambda x.\ f^{\circ 19}\ x$

Successor $succ(n) = n + 1$

$$succ \equiv \lambda n.\ \lambda f.\ \lambda x.\ \underbrace{f\ \overbrace{(n\ f\ x)}^{\text{apply } f \text{ n times to } x}}_{\text{apply } f \text{ to the result}}$$

$\lambda f.\ \lambda x.\ f^{\circ 20}\ x$

succ(0) = 1

$$\overbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}^{\textit{succ}}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{\textit{Church encoding of 0}}$$

$\lambda f.\ \lambda x.\ f^{\circ 21}\ x$

succ(0) = 1

$$\underbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}_{\text{succ}}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{\text{Church encoding of 0}}$$

$\lambda f.\ \lambda x.\ f^{\circ 22}\ x$

succ(0) = 1

$$\underbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}^{\textit{succ}}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{\textit{Church encoding of 0}}$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ x)\ f\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 23}\ x$

succ(0) = 1

$$\underbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}^{\textit{succ}}\ \underbrace{(\lambda f.\ \lambda x.\ x)}^{\textit{Church encoding of 0}}$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ x)\ f\ x)$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda x.\ x)\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 24}\ x$

succ(0) = 1

$$\underbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}_{\textit{succ}}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{\textit{Church encoding of 0}}$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ x)\ f\ x)$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda x.\ x)\ x)$$

$$\lambda f.\ \lambda x.\ f(x)$$

$\lambda f.\ \lambda x.\ f^{\circ 25}\ x$

succ(0) = 1

$$\underbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}_{\textit{succ}}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{\textit{Church encoding of 0}}$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ x)\ f\ x)$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda x.\ x)\ x)$$

$$\lambda f.\ \lambda x.\ f(x)$$

$$\lambda f.\ \lambda x.\ \boxed{f\ x}$$

succ(0) = 1

$$\overbrace{(\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))}^{\textit{succ}}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{\textit{Church encoding of 0}}$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ x)\ f\ x)$$

$$\lambda f.\ \lambda x.\ f\ ((\lambda x.\ x)\ x)$$

$$\lambda f.\ \lambda x.\ f(x)$$

$$\underbrace{\lambda f.\ \lambda x.\ f\ x}_{\textit{Church encoding of 1}}$$

$\lambda f.\ \lambda x.\ f^{\circ 27}\ x$

Addition $m + n$

$$plus \equiv \lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ \underbrace{m\ f\ \overbrace{(n\ f\ x)}^{\text{apply } f \text{ n times to } x}}_{\text{apply } f \text{ m times to the result}}$$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}_{\textit{plus}}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{c_2}$$

c_n denotes the Church encoding of n

$\lambda f.\ \lambda x.\ f^{\circ 29}\ x$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}_{plus}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{c_2}$$

$\lambda f.\ \lambda x.\ f^{\circ 30}\ x$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}_{plus}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$\lambda f.\ \lambda x.\ f^{\circ 31}\ x$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}_{plus}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 32}\ x$

plus(1, 2) = 3

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}^{plus}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}^{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda x.\ f\ (f\ x))\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 33}\ x$

plus(1, 2) = 3

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}^{plus}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}^{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda x.\ f\ (f\ x))\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ (f\ x))$$

$\lambda f.\ \lambda x.\ f^{\circ 34}\ x$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}_{plus}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda x.\ f\ (f\ x))\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.\ f\ x)\ (f\ (f\ x))$$

$\lambda f.\ \lambda x.\ f^{\circ 35}\ x$

plus(1, 2) = 3

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}^{plus}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}^{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda x.\ f\ (f\ x))\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.\ f\ x)\ (f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ f\ (f\ (f\ x))$$

plus(1, 2) = 3

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x))}^{plus}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}^{c_2}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda x.\ f\ (f\ x))\ x)$$

$$\lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ (f\ x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.\ f\ x)\ (f\ (f\ x))$$

$$\underbrace{\lambda f.\ \lambda x.\ f\ (f\ (f\ x))}_{c_3}$$

Multiplication $m * n$

$$mult \equiv \lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (n\ f)\ x$$

mult(2, 0) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{\textit{mult}}\overbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\overbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

mult(2, 0) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}_{\textit{mult}}\underbrace{(\lambda f.\lambda x.\ f(fx))}_{c_2}\underbrace{(\lambda f.\ \lambda x.x)}_{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

mult(2, 0) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{mult}\overbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\overbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$\lambda f.\ \lambda x.\ f^{\circ 41}\ x$

mult(2, 0) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{mult}\overbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\overbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (\lambda x.x)\ x)$$

mult(2, 0) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{mult}\overbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\overbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (\lambda x.x)\ x)$$

$$(\lambda f.\ \lambda x.\lambda x.\ (\lambda x.x)((\lambda x.x)x))$$

$\lambda f.\ \lambda x.\ f^{\circ 43}\ x$

mult(2, 0) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{mult}\underbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\underbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (\lambda x.x)\ x)$$

$$(\lambda f.\ \lambda x.\lambda x.\ (\lambda x.x)((\lambda x.x)x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.x)\ x$$

$\lambda f.\ \lambda x.\ f^{\circ 44}\ x$

mult(2, 0) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{\textit{mult}}\underbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\underbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (\lambda x.x)\ x)$$

$$(\lambda f.\ \lambda x.\lambda x.\ (\lambda x.x)((\lambda x.x)x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.x)\ x$$

$$\lambda f.\ \lambda x.\ x$$

$\lambda f.\ \lambda x.\ f^{\circ 45}\ x$

mult(2, 0) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ m\ (nf)\ x)}^{mult}\overbrace{(\lambda f.\lambda x.\ f(fx))}^{c_2}\overbrace{(\lambda f.\ \lambda x.x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (nf)\ x)(\lambda f.\ \lambda x.x)$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ ((\lambda f.\ \lambda x.x)f)\ x$$

$$(\lambda f.\ \lambda x.\ (\lambda f.\lambda x.\ f(fx))\ (\lambda x.x)\ x)$$

$$(\lambda f.\ \lambda x.\lambda x.\ (\lambda x.x)((\lambda x.x)x))$$

$$\lambda f.\ \lambda x.\ (\lambda x.x)\ x$$

$$\underbrace{\lambda f.\ \lambda x.\ x}_{c_0}$$

$\lambda f.\ \lambda x.\ f^{\circ 46}\ x$

Exponentiation m^n

$$exp \equiv \lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ \underbrace{(n\ m)}_{\text{m applied to itself n times}}\ f\ x$$

$\lambda f.\ \lambda x.\ f^{\circ 47}\ x$

exp(1, 0) = 1^0 = 1

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}^{exp}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{c_0}$$

$\lambda f.\ \lambda x.\ f^{\circ 48}\ x$

exp(1, 0) = 1^0 = 1

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}_{exp}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{c_0}$$

$\lambda f.\ \lambda x.\ f^{\circ 49}\ x$

exp(1, 0) = 1^0 = 1

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}^{exp}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (n\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x)\ (\lambda f.\ \lambda x.\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 50}\ x$

exp(1, 0) = 1^0 = 1

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}_{exp}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (n\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x)\ (\lambda f.\ \lambda x.\ x)$$

$$\lambda f.\ \lambda x.\ ((\lambda f.\ \lambda x.\ x)\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x$$

exp(1, 0) = 1^0 = 1

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}^{exp}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (n\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x)\ (\lambda f.\ \lambda x.\ x)$$

$$\lambda f.\ \lambda x.\ ((\lambda f.\ \lambda x.\ x)\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x$$

$$\lambda f.\ \lambda x.\ (\lambda x.\ x)\ f\ x$$

exp(1, 0) = 1^0 = 1

$$\underbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}_{exp}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\ \underbrace{(\lambda f.\ \lambda x.\ x)}_{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (n\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x)\ (\lambda f.\ \lambda x.\ x)$$

$$\lambda f.\ \lambda x.\ ((\lambda f.\ \lambda x.\ x)\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x$$

$$\lambda f.\ \lambda x.\ (\lambda x.\ x)\ f\ x$$

$$\lambda f.\ \lambda x.\ \boxed{f\ x}$$

$\lambda f.\ \lambda x.\ f^{\circ 53}\ x$

exp(1, 0) = 1⁰ = 1

$$\overbrace{(\lambda m.\ \lambda n.\ \lambda f.\ \lambda x.\ (n\ m)\ f\ x)}^{exp}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\ \overbrace{(\lambda f.\ \lambda x.\ x)}^{c_0}$$

$$(\lambda n.\ \lambda f.\ \lambda x.\ (n\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x)\ (\lambda f.\ \lambda x.\ x)$$

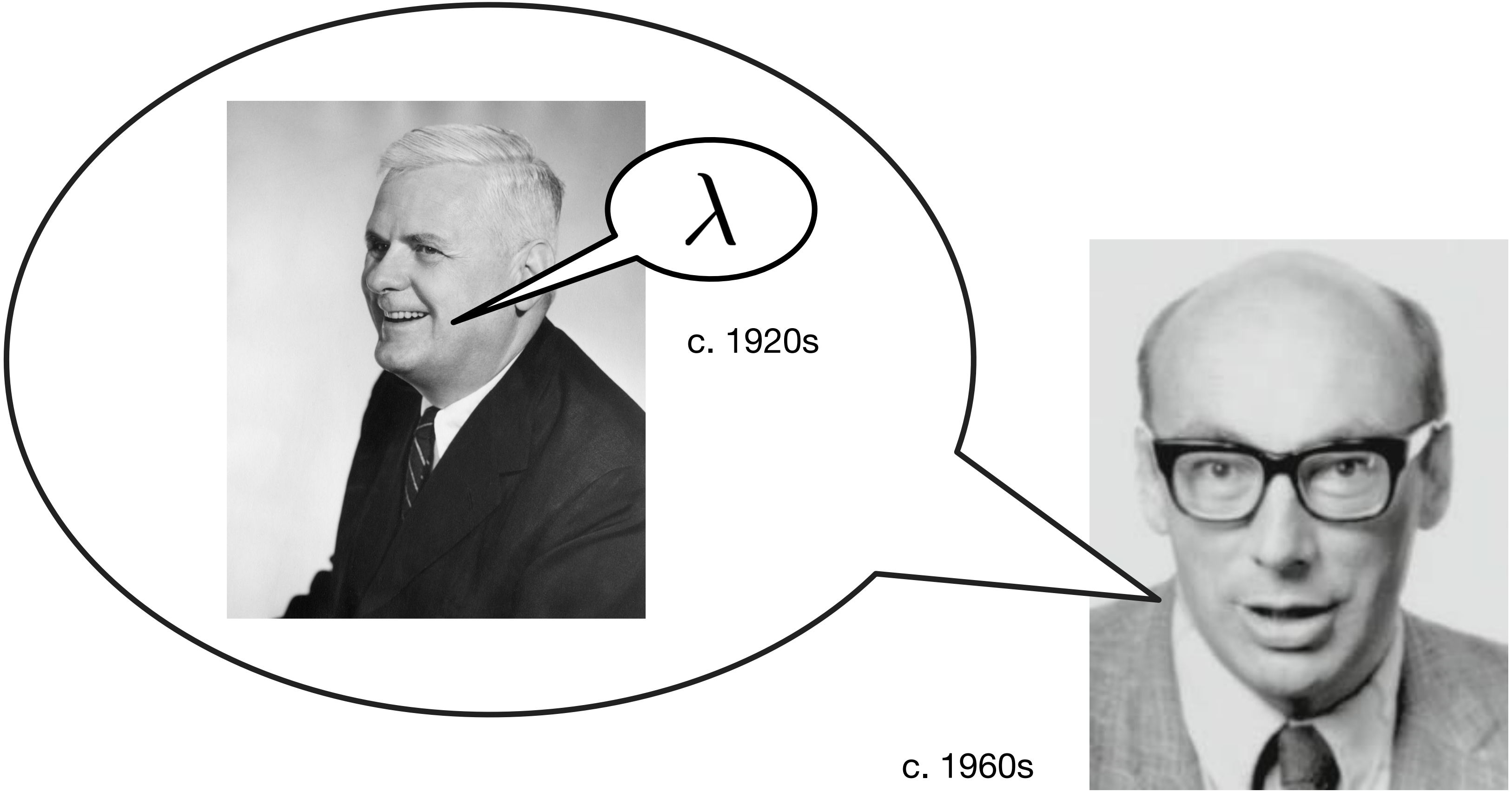
$$\lambda f.\ \lambda x.\ ((\lambda f.\ \lambda x.\ x)\ (\lambda f.\ \lambda x.\ f\ x))\ f\ x$$

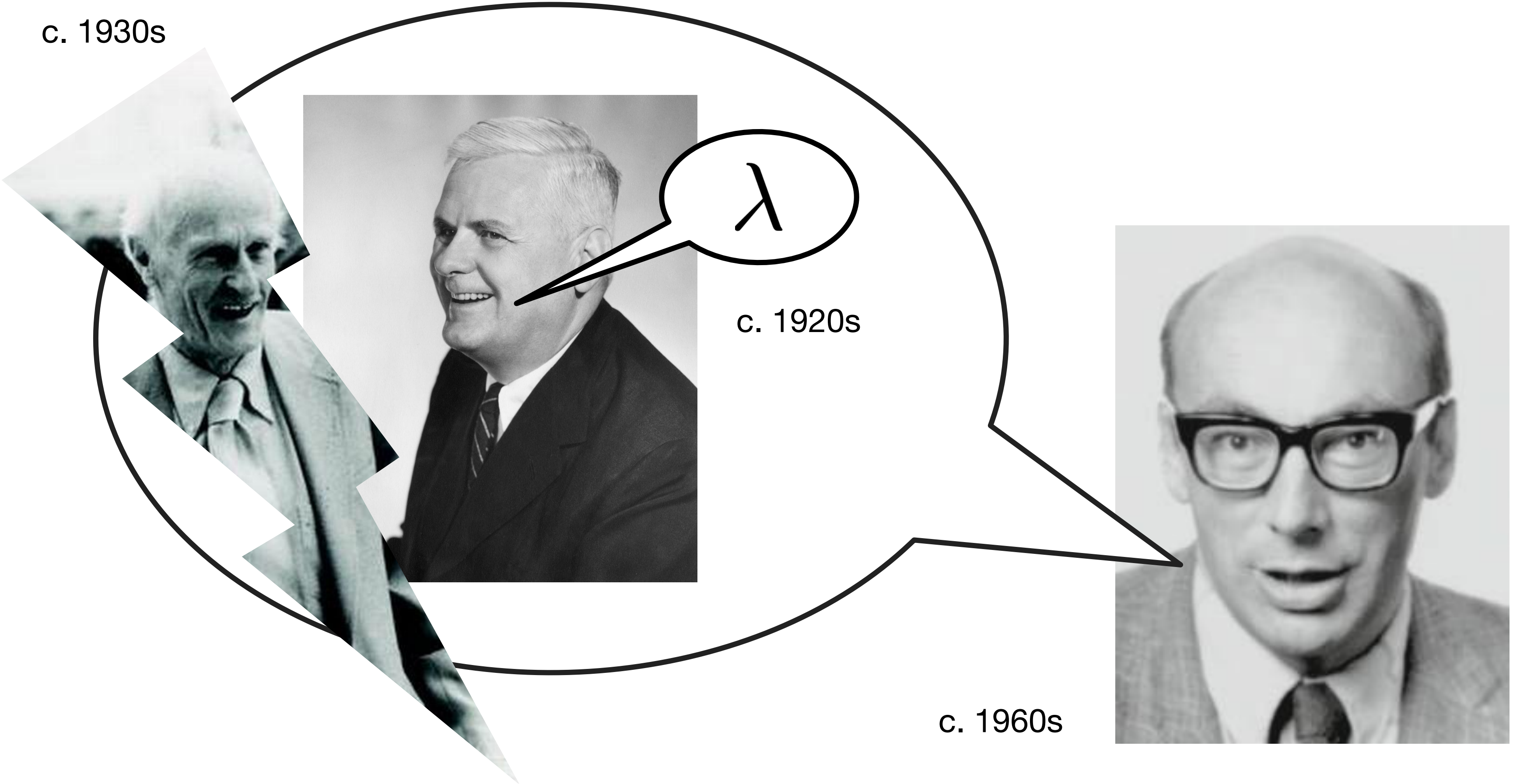
$$\lambda f.\ \lambda x.\ (\lambda x.\ x)\ f\ x$$

$$\underbrace{\lambda f.\ \lambda x.\ f\ x}_{c_1}$$

What about predecessor and subtraction?

🤔


λ
c. 1920s
c. 1960s


c. 1930s
λ
c. 1920s
c. 1960s

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$

$$\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)(\lambda u.\ u)$$

$\lambda f.\ \lambda x.\ f^{\circ 57}\ x$

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$
$$\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)(\lambda u.\ u)$$

$\lambda f.\ \lambda x.\ f^{\circ 58}\ x$

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$
$$\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)(\lambda u.\ u)$$

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$

$$\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ \underbrace{(\lambda u.\ x)}_{\text{constant function}}(\lambda u.\ u)$$

constant function

(takes argument *u* and ignores it, returning *x*)

$\lambda f.\ \lambda x.\ f^{\circ 60}\ x$

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$

$$\lambda n.\ \lambda f.\ \lambda x.\ n\ \underbrace{(\lambda g.\ \lambda h.\ h\ (g\ f))}\ (\lambda u.\ x)(\lambda u.\ u)$$

<name> gets
applied *n* times

$\lambda f.\ \lambda x.\ f^{\circ 61}\ x$

Predecessor

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$$

$$pred \equiv$$

$$\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x) \underbrace{(\lambda u.\ u)}_{\text{identity function}}$$

$\lambda f.\ \lambda x.\ f^{\circ 62}\ x$

pred(1) = 0

$$\underbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}_{pred}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}$$

$\lambda f.\ \lambda x.\ f^{\circ 63}\ x$

pred(1) = 0

$$\overbrace{\left(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\right)}^{\textit{pred}}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$\lambda f.\ \lambda x.\ \left(\lambda f.\ \lambda x.\ f\ x\right)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$\lambda f.\ \lambda x.\ f^{\circ 64}\ x$

pred(1) = 0

$$\overbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}^{pred}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$\lambda f.\ \lambda x.\ \Big(\lambda f.\ \lambda x.\ f\ x\Big)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\Big)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$\lambda f.\ \lambda x.\ f^{\circ 65}\ x$

pred(1) = 0

$$\overbrace{\left(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\right)}^{pred}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$\lambda f.\ \lambda x.\ \left(\lambda f.\ \lambda x.\ f\ x\right)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \left(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\right)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \left((\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\right)\ (\lambda u.\ u)$$

pred(1) = 0

$$\underbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}_{pred}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}$$

$$\lambda f.\ \lambda x.\ \Big(\lambda f.\ \lambda x.\ f\ x\Big)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\Big)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big((\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda h.\ h\ ((\lambda u.\ x)\ f)\Big)\ (\lambda u.\ u)$$

pred(1) = 0

$$\underbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}_{pred}\ \underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}$$

$$\lambda f.\ \lambda x.\ \Big(\lambda f.\ \lambda x.\ f\ x\Big)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\Big)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big((\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda h.\ h\ ((\lambda u.\ x)\ f)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ (\lambda u.\ u)\ ((\lambda u.\ x)\ f))$$

pred(1) = 0

$$\overbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}^{\textit{pred}}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$\lambda f.\ \lambda x.\ \Big(\lambda f.\ \lambda x.\ f\ x\Big)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\Big)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big((\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda h.\ h\ ((\lambda u.\ x)\ f)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ (\lambda u.\ u)\ ((\lambda u.\ x)\ f))$$

$$\lambda f.\ \lambda x.\ (\lambda u.\ u)\ (x)$$

pred(1) = 0

$$\overbrace{\Big(\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ x)\ (\lambda u.\ u)\Big)}^{pred}\ \overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$\lambda f.\ \lambda x.\ \Big(\lambda f.\ \lambda x.\ f\ x\Big)\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda x.\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ x\Big)\ (\lambda u.\ x)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big((\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ \Big(\lambda h.\ h\ ((\lambda u.\ x)\ f)\Big)\ (\lambda u.\ u)$$

$$\lambda f.\ \lambda x.\ (\lambda u.\ u)\ ((\lambda u.\ x)\ f))$$

$$\lambda f.\ \lambda x.\ (\lambda u.\ u)\ (x)$$

$$\underbrace{\lambda f.\ \lambda x.\ x}\ \ c_0$$

☕

$\lambda f.\ \lambda x.\ f^{\circ 70}\ x$

Subtraction $m - n$

$$minus \equiv \lambda m.\ \lambda n.\ n\ \underbrace{pred}_{\lambda n.\ \lambda f.\ \lambda x.\ n\ (\lambda g.\ \lambda h.\ h\ (g\ f))\ (\lambda u.\ x)(\lambda u.\ u)}\ m$$

$\lambda f.\ \lambda x.\ f^{\circ 71}\ x$

minus(1, 1) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}^{minus}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$\lambda f.\ \lambda x.\ f^{\circ 72}\ x$

minus(1, 1) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}^{minus}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$(\lambda n.\ n\ pred\ (\lambda f.\ \lambda x.\ f\ x))(\lambda f.\ \lambda x.\ f\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 73}\ x$

minus(1, 1) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}_{minus}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}_{c_1}$$

$$(\lambda n.\ n\ pred\ (\lambda f.\ \lambda x.\ f\ x))(\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda f.\ \lambda x.\ f\ x)\ pred\ (\lambda f.\ \lambda x.\ f\ x)$$

minus(1, 1) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}^{minus}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$(\lambda n.\ n\ pred\ (\lambda f.\ \lambda x.\ f\ x))(\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda f.\ \lambda x.\ f\ x)\ pred\ (\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda x.\ pred\ x)(\lambda f.\ \lambda x.\ f\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 75}\ x$

minus(1, 1) = 0

$$\underbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}^{minus}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\underbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$(\lambda n.\ n\ pred\ (\lambda f.\ \lambda x.\ f\ x))(\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda f.\ \lambda x.\ f\ x)\ pred\ (\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda x.\ pred\ x)(\lambda f.\ \lambda x.\ f\ x)$$

$$pred\ (\lambda f.\ \lambda x.\ f\ x)$$

$\lambda f.\ \lambda x.\ f^{\circ 76}\ x$

minus(1, 1) = 0

$$\overbrace{(\lambda m.\ \lambda n.\ n\ pred\ m)}^{minus}\overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}\overbrace{(\lambda f.\ \lambda x.\ f\ x)}^{c_1}$$

$$(\lambda n.\ n\ pred\ (\lambda f.\ \lambda x.\ f\ x))(\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda f.\ \lambda x.\ f\ x)\ pred\ (\lambda f.\ \lambda x.\ f\ x)$$

$$(\lambda x.\ pred\ x)(\lambda f.\ \lambda x.\ f\ x)$$

$$pred\ (\lambda f.\ \lambda x.\ f\ x)$$

$$\underbrace{\lambda f.\ \lambda x.\ x}_{c_0}$$

👁👄
👁

$\lambda f.\ \lambda x.\ f^{\circ 77}\ x$

👆
your face on Zoom

It’s your turn! ✨

your face on Zoom 👉

Activity

Breakout Room 2: Electric Boogaloo

Summary

So what’s the big deal?

$$\lambda x.t$$

$$x$$

$$t\ t$$

We’re no strangers to lambdas

You know the vars, and so do I

A strong abstraction’s what I’m thinking of

You wouldn’t get this from any other guy

References

References

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