Exam I : Next week

$$f(x) = ax^2 + bx + c = 0$$

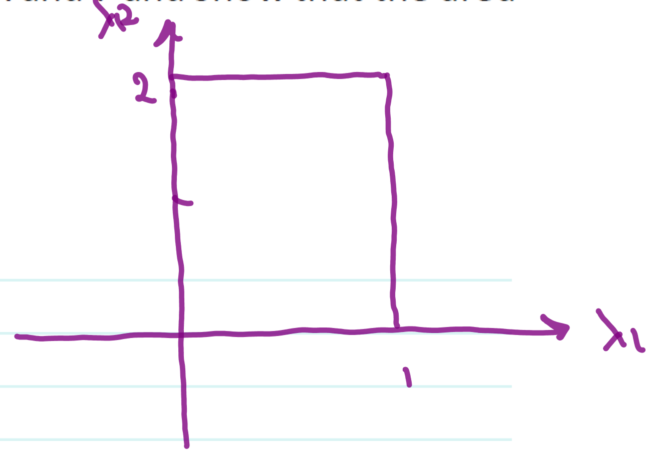
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Final Term project

3. Consider the following joint density function for the random variable X and Y uniformly distributed as:

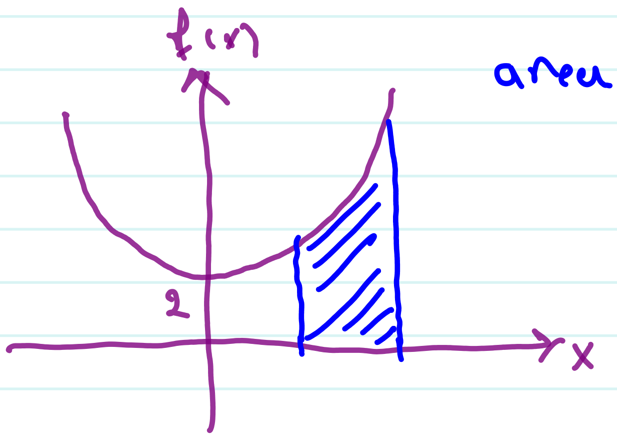
$$f_{X,Y}(\lambda_1,\lambda_2)=\begin{cases} c\lambda_1^2+\dfrac{\lambda_1\lambda_2}{3} & ;\ 0<\lambda_1<1 \text{ and } 0<\lambda_2<2 \\ 0 & ;\ \quad Else \end{cases}$$

a. Find the constant c.
b. Find the marginal density $f_X(\lambda_1)$ and $f_Y(\lambda_2)$.
c. Graph the marginal density function for random variable X and Y and show that the area under each curve is unity.
d. Find $P(\lambda_1+\lambda_2>1)$
e. Are random variable X and Y independent?



$f(x) = x^2 + 2$

area $f(x)$ $1 < x < 2$



area $= \int_1^2 f(x)\,dx$

independency check

$$f_{x,y}(\lambda_1, \lambda_2) \stackrel{?}{=} f_x(\lambda_1) * f_y(\lambda_2)$$

$$c\lambda_1^2 + \frac{\lambda_1 \lambda_2}{3} \stackrel{?}{=} \qquad \forall \lambda_1, \lambda_2$$

$$f_{xy}(\lambda_1, \lambda_2) = \begin{cases} \frac{1}{\lambda_1 \lambda_2} & : \\ 0 & : \end{cases}$$

$$\frac{1}{\lambda_1 \lambda_2} = \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2}$$

x, y are independent

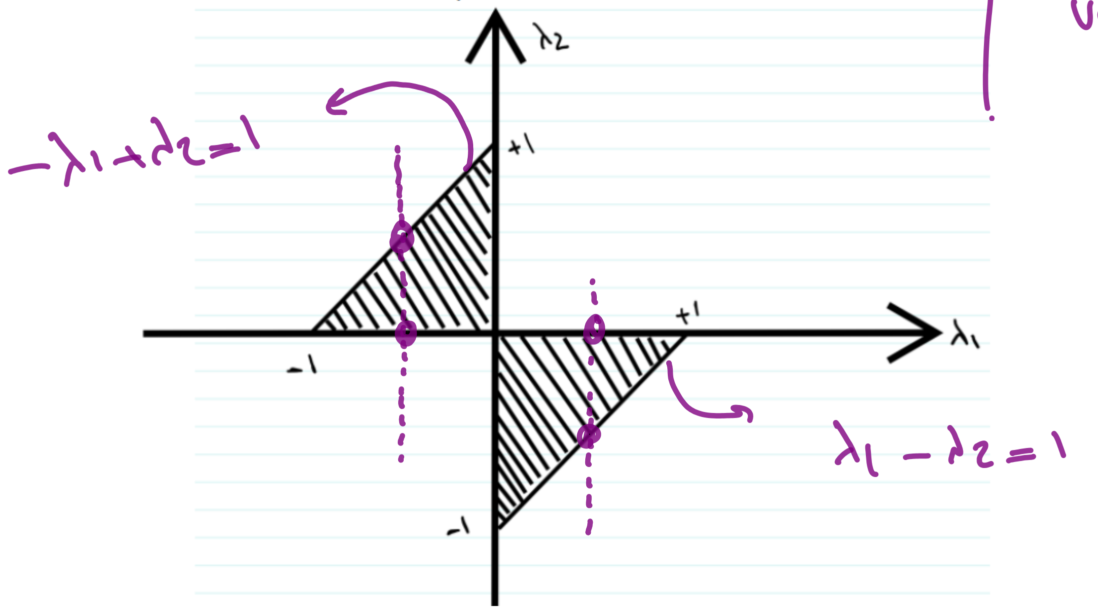
$$f_x(\lambda_1) = \frac{1}{\lambda_1}$$

$$f_y(\lambda_2) = \frac{1}{\lambda_2}$$

1. Consider the following joint density function for the random variable X and Y with the following distribution:

$$f_{X,Y}(\lambda_1,\lambda_2) = \begin{cases} c \; ; & -1 < \lambda_1 < 1 \; and \; -1 < \lambda_2 < 1 \\ 0 \quad ; & Else \end{cases}$$

a. Find the constant c.
b. Find the marginal density $f_X(\lambda_1)$ and $f_Y(\lambda_2)$
c. Graph the marginal density function for random variable X and Y and show that the area under each curve is unity.
d. What is E[X]? what is E[Y]?
e. Find and graph $f_{X|Y}(\lambda_1|\lambda_2)$. What is $f_{X|Y}(\lambda_1|\lambda_2)$ when λ_2 = 0.5.
f. Find and graph $f_{Y|X}(\lambda_2|\lambda_1)$. What is $f_{Y|X}(\lambda_2|\lambda_1)$. when λ_1 = 0.5.
g. Are random variable X and Y independent?



Alternative:

volume = area * height

$1 = (\frac{1}{2}+\frac{1}{2}) * c$

$\boxed{c = 1}$

$$\int_{-\infty}^{\infty}\int f_{x,y}(\lambda_1,\lambda_2)\, d\lambda_2\, d\lambda_1 = 1$$

$$\int_{-1}^{0}\int_{0}^{1+\lambda_1} c \; d\lambda_2 \; d\lambda_1 \quad + \quad \int_{0}^{1}\int_{\lambda_1-1}^{0} c \, d\lambda_2 \; d\lambda_1 = 1$$

$$\int_{-1}^{0} c(1+\lambda_1)\, d\lambda_1 \quad + \quad \int_{0}^{1} c(1-\lambda_1)\, d\lambda_1 = 1$$

$$c\left(\lambda_1 + \frac{\lambda_1^2}{2}\right)\Big|_{-1}^{0} \quad + \quad c\left(\lambda_1 - \frac{\lambda_1^2}{2}\right)\Big|_{0}^{1} = 1$$

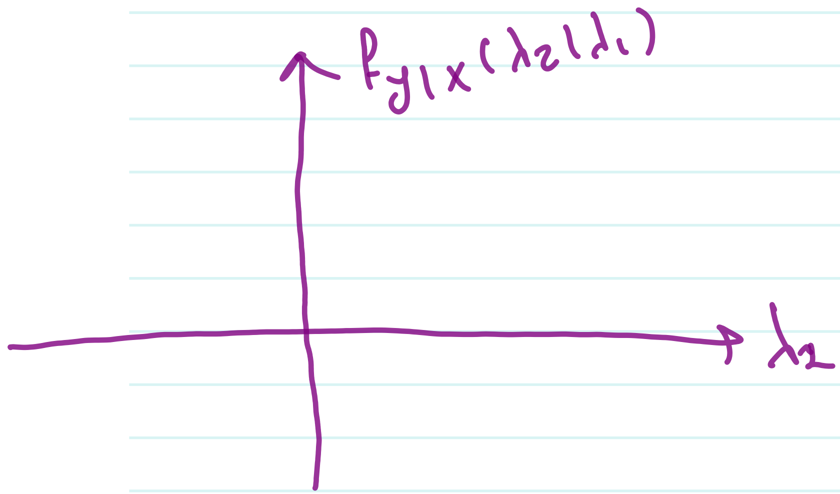
$$c\Big(0 - \overbrace{\left(-1+\tfrac{1}{2}\right)}^{-1/2}\Big) \quad + \quad \underbrace{c\left(1-\tfrac{1}{2}\right)} = 1$$

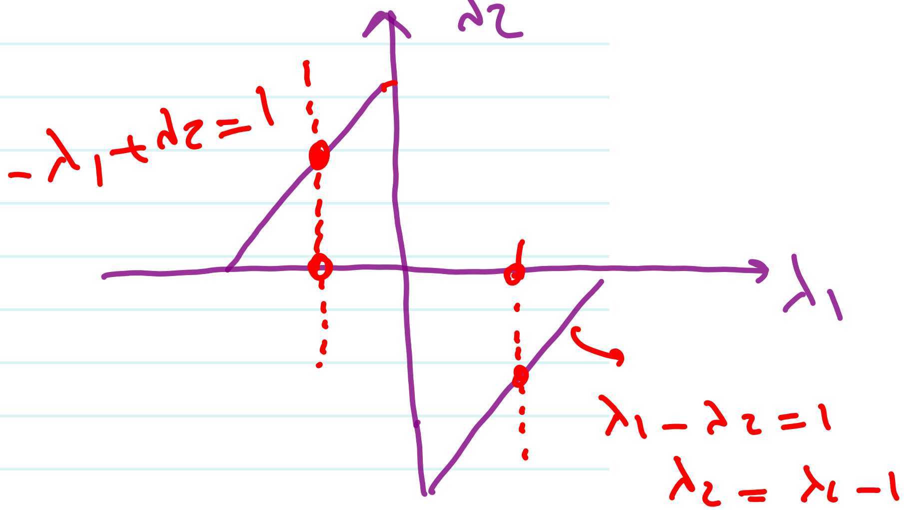
$$\frac{c}{2} \quad + \quad \frac{c}{2} = 1$$

$$\boxed{c = 1}$$

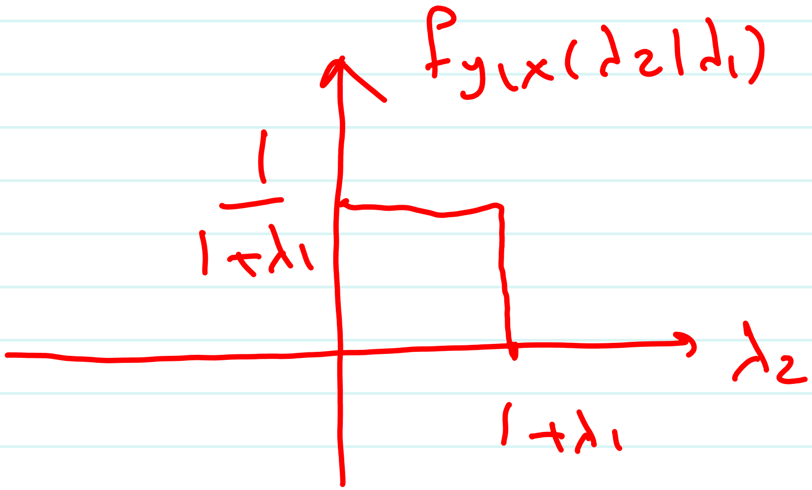
$$f_X(\lambda_1) = \begin{cases} 1+\lambda_1 & : \quad -1<\lambda_1<0 \\ 1-\lambda_1 & : \quad 0<\lambda_1<1 \\ 0 & : \quad \text{Else} \end{cases}$$

$$\underline{f_{Y|X}(\lambda_2|\lambda_1)} = \frac{f_{XY}(\lambda_1,\lambda_2)}{f_X(\lambda_1)} = \begin{cases} \dfrac{1}{1+\lambda_1} & -1<\lambda_1<0 \\ \dfrac{1}{1-\lambda_1} & 0<\lambda_1<1 \\ 0 & : \quad \text{Else} \end{cases}$$



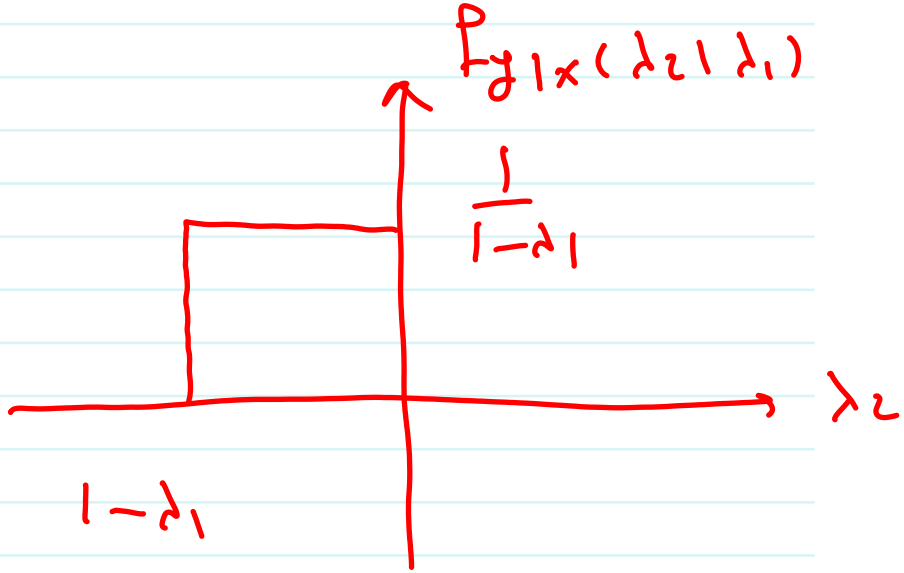


Case I : $\boxed{-1 < \lambda_1 < 0}$

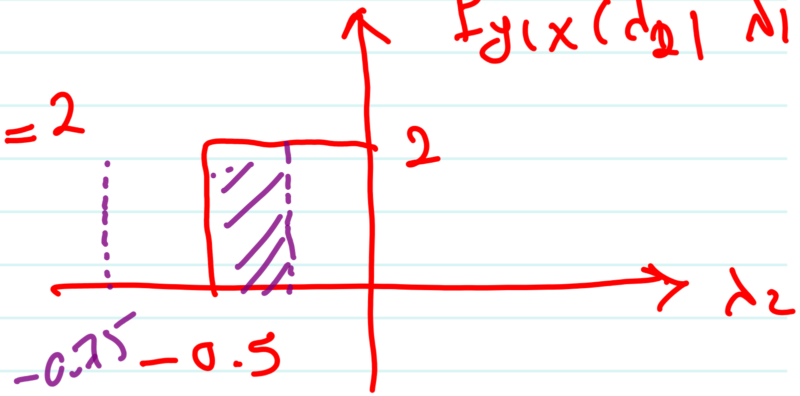


Case II: $\boxed{0 < \lambda_1 < 1}$

$$f_{y|x}(\lambda_2|\lambda_1) = \frac{1}{1-\lambda_1}$$



$f_{y|x}(\lambda_2 | \lambda_1 = 0.5) = \frac{1}{\frac{1}{2}} = 2$



$Pr(-0.5 < \lambda_2 < -0.75 \mid \lambda_1 = 0.5) = 0$

$Pr(-0.5 < \lambda_2 < -0.25 \mid \lambda_1 = 0.5) = 50\%$

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p + \epsilon$$

$$\begin{cases} y_1 = \beta_0 + \beta_1 x_{11} + \cdots + \beta_p x_{1p} + \epsilon_1 \\ y_2 = \beta_0 + \beta_1 x_{21} + \cdots + \beta_p x_{2p} + \epsilon_2 \\ \vdots \\ y_n = \beta_0 + \beta_1 x_{n1} + \cdots + \beta_p x_{np} + \epsilon_n \end{cases}$$

β_i are unknown

$$\underline{Y} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{n \times 1} \qquad \underline{X} = \underbrace{\begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & \vdots & \ddots & \vdots \\ \vdots & & & \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix}}_{n \times (p+1)} \qquad \underline{\beta} = \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}}_{(p+1) \times 1} \qquad \epsilon = \underbrace{\begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}}_{n \times 1}$$

y

$$\boxed{\underset{n\times 1}{Y} = \underset{n\times(p+1)}{X}\underset{(p+1)\times 1}{B} + \underset{n\times 1}{\epsilon}}$$

$$\hat{Y} = XB$$

Find $\underline{\beta}$: $SSE = e_1^2 + \cdots + e_n^2$

$$\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \qquad SSE = \underset{1\times n}{\underline{e}^T} \cdot \underset{n\times 1}{\underline{e}}$$

$$= (Y - \hat{Y})^T (Y - \hat{Y})$$

$$= (Y - XB)^T (Y - XB)$$

$$= (Y^T - B^T X^T)(Y - XB)$$

$$(AB)^T = B^T \cdot A^T$$

$Y^T X B$

$$\underbrace{\widetilde{SSE}}= \underbrace{Y^T \cdot Y} - \underbrace{Y^T X B} - \underbrace{B^T X^T Y} + B^T \overbrace{X^T X}^{A} B$$

$$\frac{\partial SSE}{\partial \beta} = 0 - X^T Y - X^T Y + 2 X^T X \beta = 0$$

$$0 = -2X^T Y + 2X^T X \beta$$

$$\underbrace{X^T X} \beta = X^T Y$$

$$\boxed{\beta^*_{LSE} = (X^T X)^{-1} X^T Y}$$

$(P+1)\times 1$

Normal Eq.

Least square estimator.

$$f(\underline{x}) = h_1 x_1 + h_2 x_2 + \cdots + h_n x_n$$

$$= \underline{h}^T \underline{x}$$

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} = h_1 \\ \frac{\partial f}{\partial x_2} = h_2 \\ \vdots \\ \frac{\partial f}{\partial x_n} = h_n \end{bmatrix}$$

$$\frac{\partial f}{\partial \underline{x}} = \underline{h}$$

$$f(\underset{-}{x}) = x^T \underline{A} x$$

$$\frac{\partial f}{\partial x} = 2Ax$$

erval for this estimate.

Bill($) (x)	Tip($) (y)	$\hat{y}$	e	e^2
1	2	2.8	-0.8	0.64
2	4	3.4	0.6	0.36
3	5	4	1	1
4	4	4.6	-0.6	0.36
5	5	5.2	-0.2	0.04

$\boxed{SSE = 2.4}$

$$\hat{y} = B_0 + B_1 x$$

$$\hat{y} = 2.2 + 0.6x$$

$$Y = XB \rightarrow B^* = (X^T X)^{-1} X^T Y$$

$$Y = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{matrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{matrix}$$

$$\beta^* = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

$$\beta^* = \left(\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}^{-1} \right) \begin{bmatrix} 20 \\ 66 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

$$= \frac{\begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} 20 \\ 66 \end{bmatrix}}{5 * 55 - 15 * 15}$$

$$= \frac{\begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 66 \end{bmatrix}}{\frac{55 - 45}{10}} = \underbrace{\begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 6.6 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 0.6 \end{bmatrix}$$

$\beta_0 = 2.2$

$\beta_1 = 0.6$

$$\boxed{\hat{y} = 2.2 + 0.6x}$$

$$\hat{y}_i \pm 1.96\, \hat{\sigma}_e \sqrt{1 + x^{*}(X^T X)^{-1} x^{*}}$$

$$\sigma_e^2 = \frac{2.4}{n-p-1} = \frac{2.4}{5-1-1} = \boxed{0.8}$$

$$\hat{y}_1 : \sqrt{1 + [1 \quad 1] \underbrace{\begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix}}_{10} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \sqrt{1 + \frac{6}{10}} = \sqrt{\frac{16}{10}}$$

$$\hat{y}_2 \quad \sqrt{1 + [1 \quad 2] \begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

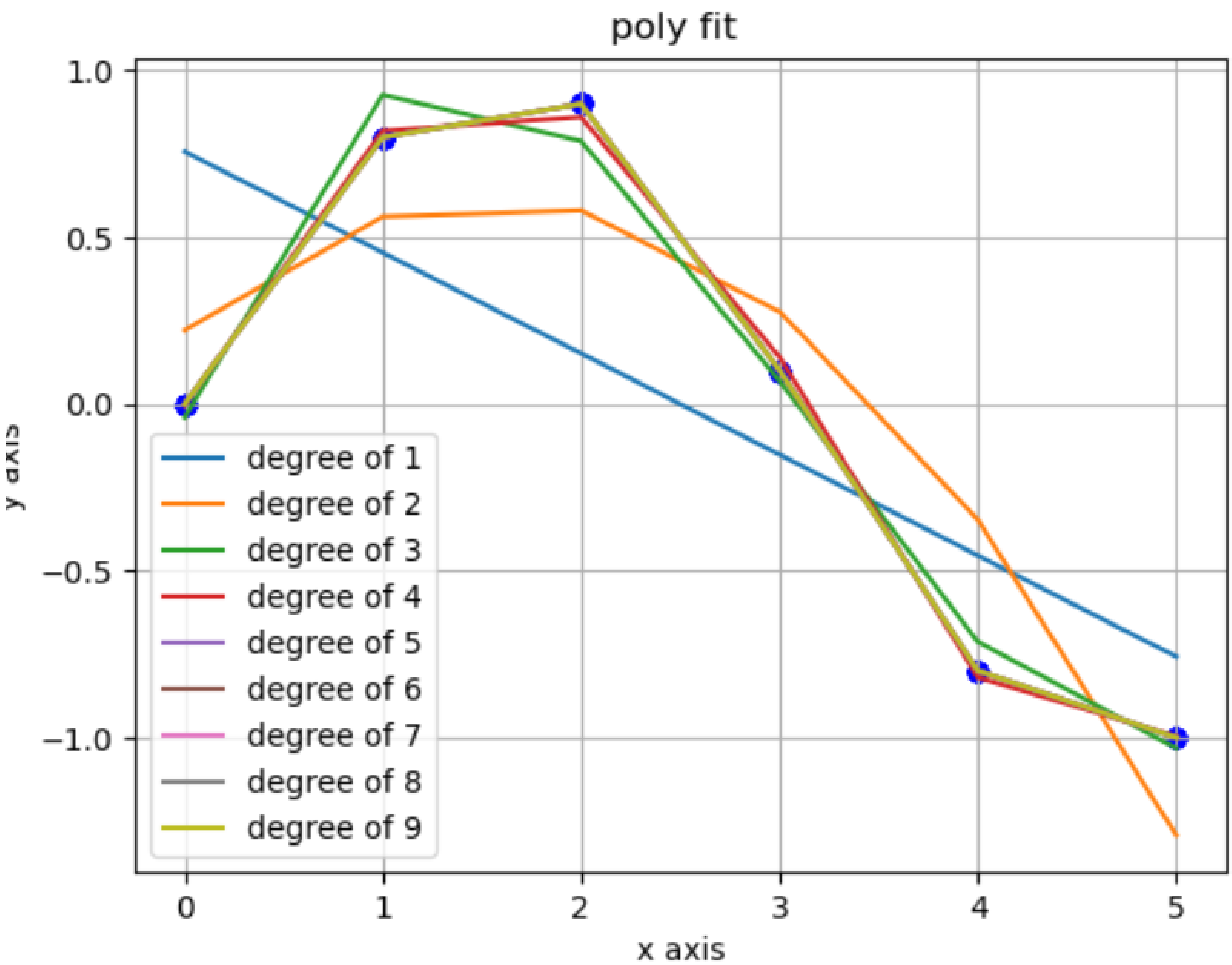
$\hat{y}$, interval

$$2.8 \pm 1.96 \sqrt{\frac{8}{10}} \sqrt{\frac{16}{10}} =$$

$$2.8 \pm 1.96 \sqrt{\frac{\overset{2\times 4}{8 \times 16}}{100}} = 2.8 \pm \frac{1.96 \, \overset{4}{\cancel{8}}\sqrt{2}}{\cancel{10} \, 5}$$

$$\boxed{2.8 \pm \overbrace{1.96 \frac{4\sqrt{2}}{5}}}$$ 0.2

95% $\left\{ \begin{array}{c} 3 \\ 2.6 \end{array} \right\}$ 2.8


poly fit
1.0
0.5
0.0
−0.5
−1.0
y axis
0
1
2
3
4
5
x axis
degree of 1
degree of 2
degree of 3
degree of 4
degree of 5
degree of 6
degree of 7
degree of 8
degree of 9