WHO I WORKED WITH FOR THE ASSIGNMENT.

1) I attended zoom call with the Proffessor on Monday.

2) This assignment was challenging, and I worked with few people.

* Aditya
* Jyothi
* Pavan.

3) READ a lot on the internet and refered the class notes a lot of times

Question 13.

I didn't how to write the equation. ~~I DO KN~~

I DON'T KNOW THE ANSWER.

Question 1.

(i) If D is countable then D is $P(\Sigma^*)$ countable.

Proof: Since D is countable we know there exists a onto function

$$f : N \xrightarrow{\text{onto}} D$$

Thus, $D = \{ f(n), n \in N \}$.

We know that D is C-countable if there exists $A \in C$ and.

$D \subseteq \{ A_n \mid n \in N \}$, and we can now say,

$A \in P(\Sigma^*)$. Now to define A, we can write

$A = \{ \langle n, y \rangle \mid n \in N, y \in f(n) \}$, and when we take the n^{th} slice,

$A_n = \{ y \in \Sigma^* \mid \langle n, y \rangle \in A \} = f_n \ldots$ Eqn ①.

We further strengthen eqn ① by saying, since we defined A for all $\langle n, y \rangle$, where $n \in N$ and $y \in f(n) \}$.

Therefore D is now, $D = \{ f(n) \mid n \in N \}$, that shows that

$D = \{ A_n \mid n \in N \}$ and we can conclude saying

D is $P(\Sigma^*)$ countable.

(ii) If D is $P(\Sigma^*)$ countable, then D is countable.

Since D is $P(\Sigma^*)$ countable, we can say that there exists.

$A \in P(\Sigma^*)$ such that $D \subseteq \{ A_n \mid n \in N \}$ Derived from Property 2.

Lets takes b as a language of D. We define a function.

$h : N \xrightarrow{\text{onto}} D$, then for all $n \in N$, then

$$h(n) = \begin{cases} A_n & \text{if } g(n) \in D \\ b & \text{otherwise} \end{cases}$$

Since g is an onto function, as the range is D and the domain N, so D is countable

9 (b) DEC is not DEC countable.

Let us assume that DEC is DEC countable.

From Property 2, we can say that,

$$DEC \subseteq \{A_n \mid n \in N\} \quad \& \quad DEC \subseteq \{A_n \mid n \in \Sigma^*\}$$

By diagnolization, we construct D,

$D \in DEC$, where $D \{x \mid \langle x, x\rangle \notin A\}$. which reasons out as $x \in D$, and $\langle x, x \rangle \notin A$, $\Rightarrow x \notin A$, which further proves.

$D \in DEC$ (as $x \in D$) but $D \notin \{A_n \mid n \in \Sigma^*)$ $(x \notin A_n]$.

$\therefore$ $D \notin \{A_n \mid n \in N\} \Rightarrow D \in (DEC \setminus \{A_n \mid n \in N\})$ which is a clear contradiction of the statement. Therefore we can conclude saying DEC is not DEC countable.

Question 14 .

To Prove : A partial function $f : \subseteq \Sigma^* \to \Sigma^*$ is computable if and only if its graph

$$G_f = \{ \langle x, f(x) \rangle \mid x \in \text{dom } f \} \text{ is c.e}$$

Proof:

We call $f : \subseteq \Sigma^* \to \Sigma^*$ a partial function when domain f.

A partial function is computable if there exists a TM such that $x \in$ domain (f) and $f(x) = y$.

Lets take a TM with an input $\langle x, w \rangle$, $f : \subseteq \Sigma^* \to \Sigma^*$.

We know TM on input x will be accepted and $f(x) = w$. otherwise it may reject or else loop forever. ... Eqn ①

~~Assume~~ Assume $A = L(M)$ for some enumerator M.

We construct an M that compute $f(x)$

> M starts N to enumerate all pair ~~$\langle x, y \rangle$~~ $\langle x, f(x) \rangle \in A$.

> If $(x, f(x))$ appears for some y, the M returns some y, else M loops.

Hence we conclude that computing a function is the same as accepting its graph. Hence we conclude that G_f is ce.

Problem 11,

Ans Consider the set $A : P(\Sigma^*)$ of all languages. We then define countability as

$$CTBL = \{ E \subseteq P(\Sigma^*) \mid E \text{ is countable} \} \quad \ldots \text{ Eqn } ①$$

Lemma 7 and Lemma 8 tell us that CTBL is an ideal on $P(\Sigma^*)$.

To prove: CTBL is an δ-ideal on $P(\Sigma^*)$.

We know ~~on a set~~ a σ-ideal on a set X is an ideal on X with the following closure under countable unions:

$$E_n \in I \text{ for all } n \in N, \text{ then } \bigcup_{n=0}^{\infty} E_n \in I$$

From Eqn ①,

We can break it down as, that countable sets are subsets of $P(\Sigma^*)$, and also we know by countability, we know we have $g : N \xrightarrow{\text{onto}} E$. From Lemma 8 we can ~~too~~ know, if A and B are countable, the $A \cup B$ is countable.

Based on Lemma 8, we can conclude saying that subset E_i, which is also countable infinite union.

Thus, because $P(\Sigma^*)$ is infinite and contains all the languages, we can conclude saying CTBL is an σ-ideal on $P(\Sigma^*)$ as.

E_1 is countable, therefore $E_1 \cup E_2$ also countable, hence if $E_1, E_2 \ldots, E_m$ is countable the $(E_1, E_2 \ldots E_m) \cup (E_{m+1})$ is also countable.

12. To prove all the inclusions in the infinite diagram.

Given:

① $\Sigma_1^0 = CE$

To prove:

② $\Pi_n^0 = co\,\Sigma_n^0 = co\,CE$

① $\Delta_n^0 \subseteq \Pi_n^0$ & $\Delta_n^0 \subseteq \Sigma_n^0$.

③ $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

② $\Pi_n^0 \subseteq \Delta_{n+1}^0$

③ $\Sigma_n^0 \subseteq \Delta_{n+1}^0$

④ $\Sigma_{n+1}^0 = \exists \Pi_n^0$

⑤ $\Pi_1^0 \overset{(2)}{=} co\,\Sigma_1^0 \overset{(1)}{=} co\,CE \overset{col\,30}{=} \forall DEC.$

⑥ $\Delta_1^0 \overset{(3)}{=} \Sigma_1^0 \cap \Pi_1^0 \overset{(1,5)}{=} CE \cap co\,CE \overset{lemma\,5}{=} DEC.$

(7) $\Sigma_2^0 \overset{(4)}{=} \exists \Pi_1^0 \overset{5}{=} \exists co CE \overset{col\,30}{=} \exists\forall DEC$

(8) $\Pi_2^0 \overset{(2)}{=} co\,\Sigma_2^0 \overset{7}{=} co\,\exists\forall DEC = \forall\exists DEC.$

(9) $\Pi_{n+1}^0 = \forall \Sigma_n^0.$

To prove $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

1) $\Delta_1 = \Sigma_1^0 \cap \Pi_1^0.$

From eqn ③

$\Delta_1 = CE \cap co\Sigma_1 = CE \cap coCE$ ~~[illegible]~~.

(From eqn ①)

By observation ①,

we know that $DEC \subseteq CE$ i.e $\Delta_n^0 \subseteq \Sigma_n^0$ ... Eqn ⑨

By observation ⑤,

If C and D are class of languages.

then $C \subseteq D \Rightarrow co\,C \subseteq co D$

$\therefore$ Eqn 9 $\Rightarrow$ $DEC \subseteq CE \Rightarrow co\,DEC \subseteq co\,CE.$

From lemma 2, $coDEC = DEC.$

which implies $DEC \subseteq coCE.$ ⑩ Therefore from ⑨ and ⑩ we can conclude eqn ①.

2) From equation ③

$$\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$$

From the diagram, we know

$$\Pi_n^0 \subseteq \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$$

$$\Pi_n^0 \subseteq \exists\Pi_n^0 \cap \Pi_{n+1} \quad \cdots \text{(i)} \qquad (\text{Since } \Sigma_{n+1}^0 = \exists\Pi_n^0).$$

We also know that,

$\Pi_n^0 \subseteq \exists\Pi_n^0 \cap \forall\Pi_n^0$, from this equation we can arrive on

$$\Pi_n^0 \subseteq \exists\Pi_n \quad \cdots \text{(ii)}$$

Now we need to prove, $\Pi_n^0 \subseteq \Pi_{n+1}^0$.

To prove the above statement we should prove the base case, where $\Pi_1 \subseteq \Pi_2$, so that we can prove $\Pi_n^0 \subseteq \Pi_{n+1}^0$

Now, $\Pi_1^0 \subseteq \Pi_2^0 \Rightarrow \forall DEC \subseteq \forall\exists DEC$ from eqn ⑤ and ⑧

We know $\exists DEC = CE$, from Theorem 29,

$\Pi_1^0 \subseteq \Pi_2^0 \rightarrow \forall DEC \subseteq \forall CE$, which we can prove ~~prove~~ and generalizes that $\Pi_n^0 \subseteq \Pi_{n+1}^0$.

3) from ~~eqn~~, the diagram we know.

$$\Sigma_n^0 \subseteq \Sigma_{n+1}^0 \cap \Pi_{n+1}$$

From eqn (ii), $\Pi_{n+1}^0 = \forall\Sigma_n^0$,

$\Rightarrow \Sigma_n^0 \subseteq \Sigma_{n+1}^0 \cap \forall\Sigma_n^0$ ...(iii)

$\Rightarrow$ We know that $\Sigma_n^0 \subseteq \forall\Sigma_n^0 \cap \exists\Sigma_n^0$.

$\Rightarrow \Sigma_n^0 \subseteq \forall\Sigma_n^0$ (iv)

Therefore, from eqn (iii) and (iv) we can prove:

$\Sigma_n^0 \subseteq \Sigma_{n+1}^0 \longrightarrow \Sigma_n^0 \subseteq \exists\Pi_n^0$ from eqn (1)

From eqn (2),

$\Sigma_n^0 \subseteq \exists co\Sigma_n^0 \longrightarrow \Sigma_n^0 \subseteq co\forall\Sigma_n^0$ (Since $\exists co = co\forall$)

Completing the equation.

$co\Sigma_n^0 \subseteq \forall\Sigma_n^0 \Rightarrow \Pi_n^0 \subseteq \Pi_{n+1}^0$

Thus we have proved the induction. $\square$.

Hence we have proved all the ~~inclusions~~ inclusions in the diagram.

Problem 10

(a) Given: 1. C and D are sets of languages $g: C \xrightarrow{onto} D$ .. ①

& $g: C \xrightarrow{onto} D$ ... ②

To prove: If C is countable then D is countable.

Proof: According to the definition for the countability of a set,

If C is a countable set, then there exists a function, $f: N \xrightarrow{onto}$

and thus we can define the cardinality of the sets as

$|D| \leq |C|$.

(i) If $D = \phi$, then we can say D is countable by the definition

(ii) If $D \neq \phi$, then taking an element $b \in \overset{C}{\cancel{D}}$. If D were to be countable, then ~~f(b)=D~~, $f(b) = D$, from eqn ①. Since C is countable, we can define a function $h: N \xrightarrow{onto} D$;

$$h(n) = \begin{cases} f(n) & \text{if } f(n) \in D \\ a, & \text{if } f(n) \notin A. \end{cases}$$

which proves that D is countable.

b) We know that C is countable,

To prove: $f: C \xrightarrow{\text{onto}} \exists C$. & $g: C \xrightarrow{\text{onto}} \forall C$.

~~as~~ we need to show $\exists C$ and $\forall C$ is countable.

To prove $C \xrightarrow{\text{onto}} \exists C$, we need to show $x \in \exists C$. i.e.

$\langle x, w \rangle \in C$ where $f(\langle x, w \rangle) = x$.

$$\exists C = \{ x \in \Sigma^* \mid \exists w \in \Sigma^*, \langle x, w \rangle \in C \}. \quad \text{Eqn } ①$$

The existential projection of the class, $\exists C$ contains the x and $\langle x, w \rangle$ is in the class C, where w witness for x.

This proves that $f: C \xrightarrow{\text{onto}} \exists C$. Eqn ① also solidifies our proof where every element in the co-domain has a pre-image in the domain.

The same proof can be used for $\forall C$.

where $\forall C = \{ x \in \Sigma^* \mid \exists w \in \Sigma^*, \langle x, w \rangle \in C \}$,

where $\forall C$ contains x where $\langle x, w \rangle$ is in C.

$\therefore g: C \xrightarrow{\text{onto}} \forall C$. $\square$

Problem 16.

Given: * $A = L(U)$ be the universel ce language.

We know > From Theorem 32, CE paramaterizes CE. ... Eqn ①

* Let $B \subseteq \Sigma^*$

To Prove: If $A \leq_m B$ and $\Sigma^* \setminus A \leq_m B$, then B is neither c.e nor co.ce.

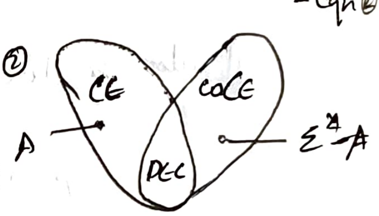
Proof:

We know from theorem 36,

> The universal c.e language $A = L(U)$ of theeorem 32, is c.e but not decidable.

> We can from Corallary 37 state that, if $A = L(U)$ is the universal c.e language of theorem 32, then $\Sigma^* - A \in coCE \setminus DEC$. - Eqn ②

→ $A \in c.e \setminus DEC$. - Eqn ②



Let us break the proof into two contradicting parts by assuming B is ce and B is co.CE.

(i) B is c.e ($B \in CE$)

* According to the definition, if a class of languages is closed under $\leq_m$ reductions for all $A, B \leq \Sigma^*$,

then $A \leq_m B \in C \Rightarrow A \in C$. ... Eqn ③.

* From the above statement we can say that if $A \leq_m B$ and $\Sigma^* - A \leq_m B$ we can conclude from eqn ①&③ saying that $A \in CE$ and $\Sigma^* - A \in CE$.

* Since A is in ce and containg DEC (Eqn②) ad $\Sigma^* \setminus A$ is in co.ce not containg DEC (co-c.e $\setminus$ DEC) which is a clear contradiction.

* Therefore, B cannot be in CE, ie $B \notin CE$.

(ii) $B \in coCE$.

> With $A \leq_m B$, $A \in co\text{-}CE$ which is not DEC ($co.CE - DEC$). However, Eqn 2 states ~~that~~ otherwise, where $A \in CE$, which is a clear condriction. Hence $B \notin coCE$.

From (i) and (ii) we can conclude that B is neither CE nor $coCE$.

Problem 15:

Given: A is c.e. and also $A \subseteq \Sigma^*$

B is an infinite decidable subset of A... Eqn (a)

To prove:

A is undecidable then A-B is also undecidable... Eqn(b)

Solution:

We are to prove the proof using contradiction.

The condraction will be A-B is decidable.

We know from Eqn (a),

that B is decidable. The union of any two decidable sets will also be decidable. From the given statement, then.

$$(A-B) \cup B \longrightarrow A \quad \text{... Eqn(c)}$$

(decidable) (decidable) (Decidable)

However, the statement given is A is undecidable, which contradicts.

Therefore, we can say from Eqn (b) and Eqn(c) that A-B can not be decidable.