

REPRESENTING IDEAL CLASSES OF RAY CLASS GROUPS BY PRODUCT OF PRIME IDEALS OF SMALL SIZE

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ABSTRACT. We prove that, for every modulus \mathfrak{q} , every class of the narrow ray class group $H_{\mathfrak{q}}(\mathbf{K})$ of an arbitrary number field \mathbf{K} contains a product of three unramified prime ideals \mathfrak{p} of degree one with $\mathfrak{N}\mathfrak{p} \leq (t(\mathbf{K})\mathfrak{N}\mathfrak{q})^3$, where $t(\mathbf{K})$ is an explicit function of \mathbf{K} described in (1). To achieve this result, we first obtain a sharp explicit Brun-Titchmarsh Theorem for ray classes and then an equally explicit improved Brun-Titchmarsh Theorem for large subgroups of narrow ray class groups. En route, we deduce an explicit upper bound for the least prime ideal in a quadratic subgroup of a narrow ray class group and also for the size of the least ideal that is a product of degree one primes in any given class of $H_{\mathfrak{q}}(\mathbf{K})$.

1. INTRODUCTION AND STATEMENTS OF THE THEOREMS

A classical theorem of Linnik [15, 16] states that there exists an absolute constant $b > 0$ such that, given a positive integer q and an invertible residue class $a \bmod q$, the smallest prime in the aforementioned residue class is at most q^b . Several efforts have been made to bound b from above when q is large. The best result to date is that of Xylouris [36, 35] who proved that b can be taken to be 5 provided q is large enough.

Let \mathbf{K} be a number field, $\mathcal{O}_{\mathbf{K}}$ be its ring of integers and let \mathfrak{q} be an integral ideal of \mathbf{K} . Let us recall the definition of the (narrow) ray class group. Let $I(\mathfrak{q})$ be the group of fractional ideals of \mathbf{K} which are co-prime to \mathfrak{q} and $P_{\mathfrak{q}}$ be the subgroup of $I(\mathfrak{q})$ consisting of principal ideals (α) satisfying $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{q})$ for all prime ideals \mathfrak{p} dividing \mathfrak{q} and $\sigma(\alpha) > 0$ for all embeddings σ of \mathbf{K} in \mathbb{R} . This is the ray class group associated to the modulus $\mathfrak{q}\mathfrak{q}_{\infty}$, where \mathfrak{q}_{∞} is the set of all real places of \mathbf{K} . However for notational convenience, we shall simply refer to this as the narrow ray class group of modulus \mathfrak{q} . We set $H_{\mathfrak{q}}(\mathbf{K}) = I(\mathfrak{q})/P_{\mathfrak{q}}$. When $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$, the group $H_{\mathfrak{q}}(\mathbf{K})$ is the usual class group in the narrow sense.

One way to generalize Linnik's problem to number fields is the following: Given an integral ideal \mathfrak{q} in $\mathcal{O}_{\mathbf{K}}$ and a class C of $H_{\mathfrak{q}}(\mathbf{K})$, find a prime ideal $\mathfrak{p} \in C$ such that $\mathfrak{N}(\mathfrak{p}) \leq A \mathfrak{N}(\mathfrak{q})^b$. This problem was first considered by Fogels [3] which was then refined by Weiss [33, 32] who proved the following theorem.

Theorem 1. (Weiss [33, 32] 1983) *There exists an effectively computable constant a such that the following holds. In any number field \mathbf{K} of discriminant $d_{\mathbf{K}}$, every element of $H_{\mathfrak{q}}(\mathbf{K})$ contains a prime ideal \mathfrak{p} of \mathbf{K} such that $\mathfrak{N}\mathfrak{p} \leq (|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q})^a$.*

Here and thereafter, \mathfrak{N} denotes the absolute norm of \mathbf{K} over \mathbb{Q} . Note that the exponent a is not explicit.

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There is another generalization of Linnik's problem which was initiated by Lagarias-Montgomery-Odlyzko [11]. By class field theory, finding a prime ideal in a class of the ray class group $H_{\mathfrak{q}}(\mathbf{K})$ is the same as finding a prime ideal with a given Artin symbol in the Galois group of the corresponding ray class field $\mathbf{K}_{\mathfrak{q}}$ over the number field \mathbf{K} . In this set-up, Lagarias, Montgomery and Odlyzko derived an upper bound for the least norm of such a prime ideal in terms of $|d_{\mathbf{K}_{\mathfrak{q}}/\mathbb{Q}}|^a$ with a inexplicit. For explicit a , see the works of Ahn-Kwon [1], Kadiri-Wong [9], Thorner-Zaman [29] and Zaman [37]. All these bounds are however large in terms of the exponent of the norm of the modulus \mathfrak{q} . For example, if one takes a rational prime p , then the ray class field is the p -th cyclotomic field whose discriminant has absolute value p^{p-2} .

In this article, we would like to show that if one considers a product of three prime ideals in place of a single prime ideal, one can find prime ideals with much smaller norm when compared to the norm of the modulus. Furthermore, we shall make the dependence on the ground field \mathbf{K} explicit.

Our notation recalled below is classical. In brief, $n_{\mathbf{K}}$, $h_{\mathbf{K}}$, $R_{\mathbf{K}}$ and $d_{\mathbf{K}}$ are respectively the degree, the class number, the regulator and the discriminant of a number field \mathbf{K} while $\alpha_{\mathbf{K}}$ is the residue of the Dedekind zeta-function at 1. In this set up, we have the following theorem.

Theorem 2. *Let \mathbf{K} be a number field, \mathfrak{q} be an integral ideal of \mathbf{K} . Set*

$$(1) \quad t(\mathbf{K}) = \max\left(n_{\mathbf{K}}^{48n_{\mathbf{K}}^3} |d_{\mathbf{K}}|^6 (R_{\mathbf{K}} h_{\mathbf{K}})^{n_{\mathbf{K}}}, \exp(|d_{\mathbf{K}}|^{30})\right).$$

Each element of $H_{\mathfrak{q}}(\mathbf{K})$ contains a product of three degree one primes, each of norm at most $t(\mathbf{K})^3 \mathfrak{N}\mathfrak{q}^3$.

As the case $\mathbf{K} = \mathbb{Q}$ has already been treated in [24, 23] by Ramaré together with Serra, Srivastav and Walker, we may assume that $\mathbf{K} \neq \mathbb{Q}$ and we do so hereafter.

Remark 1.1. *Three features of this result should be underlined: the dependence on the field \mathbf{K} is completely explicit in terms of classical invariants of the field, and even numerically so. However, we did not strive to get small constants. The second point is that the exponent in \mathfrak{q} is relatively small. The third point is more technical; the dependence in \mathfrak{q} has the form $\mathfrak{N}\mathfrak{q}^3$ and not $\mathfrak{N}\mathfrak{q}^3$ times a power of $\log \mathfrak{N}\mathfrak{q}$ and this precision comes from a much more refined treatment. The dependence in \mathbf{K} is most probably dominated by the term $\exp(|d_{\mathbf{K}}|^{30})$ that comes from a possible real zero abnormally close to 1. We control such a zero by Lemma 37 due to Kadiri and Wong [9] (see also [1]).*

Theorem 2 relies on four ingredients of independent interest. We first need a Brun-Titchmarsh Theorem for elements of $H_{\mathfrak{q}}(\mathbf{K})$.

Theorem 3. *Let $\mathfrak{b}, \mathfrak{q}$ be integral ideals with $(\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}$ and $[\mathfrak{b}] \in H_{\mathfrak{q}}(\mathbf{K})$. Then*

$$\sum_{\substack{\mathfrak{p} \in [\mathfrak{b}], \\ \mathfrak{N}\mathfrak{p} \leq X}} 1 \leq \frac{2X}{|H_{\mathfrak{q}}(\mathbf{K})| \log\left(\frac{X}{u(\mathbf{K}) \mathfrak{N}\mathfrak{q}}\right)}, \quad u(\mathbf{K}) = n_{\mathbf{K}}^{48n_{\mathbf{K}}^3} |d_{\mathbf{K}}|^6 (R_{\mathbf{K}} h_{\mathbf{K}})^{n_{\mathbf{K}}}$$

provided the denominator is > 0 .

This is the number field analogue of the classical Brun-Titchmarsh Theorem for the initial interval, see for instance [17, Theorem 2] by Montgomery and Vaughan. A precursor of this result can be found in [7, Theorem 4] by Hinz and Lodemann, though without the dependence in \mathbf{K} and with a slightly worse upper bound. As these two authors, we rely on the Selberg sieve, though with an improved treatment of the

error term, see Theorem 32, and on an estimate for the number of integral ideals recalled as Theorem 8 below.

Our second ingredient is a Brun-Titchmarsh Theorem valid for cosets, the analogue of [23, Theorem 1.2] when $\mathbf{K} = \mathbb{Q}$, and which is the topic of Section 6. We notice here that in Theorem 3, we *parametrized* the class while for Theorem 35, we *capture* elements of a subgroup by using multiplicative characters. Our third ingredient is less novel.

Theorem 4. *Every element in $H_{\mathfrak{q}}(\mathbf{K})$ contains an integral ideal \mathfrak{a} such that $\mathfrak{N}\mathfrak{a} \leq 10^{25n_{\mathbf{K}}} n_{\mathbf{K}}^{7n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^3$ and \mathfrak{a} is product of degree one primes.*

Theorem 5. *Every element in $H_{\mathfrak{q}}(\mathbf{K})$ contains an integral ideal \mathfrak{a} such that*

$$\mathfrak{N}\mathfrak{a} \leq F_1(\mathfrak{q}) \mathfrak{N}\mathfrak{q} \log(3F(\mathfrak{q}))^{n_{\mathbf{K}}} \log \log(B(\mathbf{K})F(\mathfrak{q}) \mathfrak{N}\mathfrak{q})^2, \quad B(\mathbf{K}) = n_{\mathbf{K}}^{50n_{\mathbf{K}}} (E(\mathbf{K}) \sqrt{|d_{\mathbf{K}}|})^{n_{\mathbf{K}}},$$

and \mathfrak{a} is product of degree one primes. Here $F(\mathfrak{q}) = 2^{r_1} h_{\mathbf{K}} \phi(\mathfrak{q}) / h_{\mathbf{K}, \mathfrak{q}}$, $F_1(\mathfrak{q}) = 2^{r_1} h_{\mathbf{K}} \mathfrak{N}\mathfrak{q}$ and the constant $E(\mathbf{K})$ is equal to $1000 n_{\mathbf{K}}^{12n_{\mathbf{K}}} (R_{\mathbf{K}} / |\mu_{\mathbf{K}}|)^{1/n_{\mathbf{K}}} [\log((2n_{\mathbf{K}})^{4n_{\mathbf{K}}} R_{\mathbf{K}} / |\mu_{\mathbf{K}}|)]^{n_{\mathbf{K}}}$.

Theorem 4 is enough for our purpose, but our proof gives only the bound $\mathfrak{N}\mathfrak{a} \ll_{\mathbf{K}, \varepsilon} \mathfrak{N}\mathfrak{q} |H_{\mathfrak{q}}(\mathbf{K})|^{2(1+\varepsilon)}$, while $|H_{\mathfrak{q}}(\mathbf{K})|$ should be enough. By using some techniques from sieve method, Theorem 5 corrects this defect as far as the dependence in $\mathfrak{N}\mathfrak{q}$ is concerned, but the dependence in $d_{\mathbf{K}}$ becomes much worse.

Our fourth and last ingredient shows that quadratic subgroups of $H_{\mathfrak{q}}(\mathbf{K})$ contain small degree one prime ideal.

Theorem 6. *Let χ be a quadratic character on $H_{\mathfrak{q}}(\mathbf{K})$. There exists a prime ideal \mathfrak{p} of degree one in \mathbf{K} such that $(\mathfrak{p}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}$, $\chi(\mathfrak{p}) = 1$ and $\mathfrak{N}\mathfrak{p} \leq 8 \cdot (10^{31} n_{\mathbf{K}}^7)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^2$.*

This bound is modest as far as the exponent of $\mathfrak{N}\mathfrak{q}$ is concerned, but it is completely explicit. When $\mathbf{K} = \mathbb{Q}$, the question has been treated by Linnik and Vinogradov in [31]. Their better exponent comes from the usage of the Burgess bounds and Siegel's Theorem while we only rely on convexity (or equivalently, on a Polya-Vinogradov inequality). The exponent we get is $3/2 + \varepsilon$ for any positive ε .

2. NOTATION AND PRELIMINARIES

Notation. Let $\mathbf{K} \neq \mathbb{Q}$ be a number field with discriminant $|d_{\mathbf{K}}| \geq 3$ (by Minkowski's bound). Also let us set $n_{\mathbf{K}} = [\mathbf{K} : \mathbb{Q}] \geq 2$ and \mathfrak{q} be an (integral) ideal of \mathbf{K} . The number of real embeddings of \mathbf{K} is denoted by r_1 whereas the number of complex ones are denoted by $2r_2$. The ring of integers of \mathbf{K} is denoted by $\mathcal{O}_{\mathbf{K}}$, the narrow ray class group modulo \mathfrak{q} is denoted by $H_{\mathfrak{q}}(\mathbf{K})$ and the (absolute) norm is denoted by \mathfrak{N} . Throughout the article \mathfrak{p} will denote a prime ideal in $\mathcal{O}_{\mathbf{K}}$, p will denote a rational prime number and for any integral ideals $\mathfrak{a}, \mathfrak{b}$, their lcm and gcd in $\mathcal{O}_{\mathbf{K}}$ will be denoted by $[\mathfrak{a}, \mathfrak{b}]$ and $(\mathfrak{a}, \mathfrak{b})$ respectively. Further an element of $H_{\mathfrak{q}}(\mathbf{K})$ containing an integral ideal \mathfrak{a} will be denoted by $[\mathfrak{a}]$.

A sum over degree one prime ideals will be denoted by $\sum_{\mathfrak{p}}^b$, and $\prod_{\mathfrak{p}}^b$ will denote a product over degree one prime ideals. Similarly $\sum_{\mathfrak{p}}^{\sharp}$ and $\prod_{\mathfrak{p}}^{\sharp}$ denotes respectively a sum and a product over primes \mathfrak{p} that are *not* of degree one. As a generalisation, the sign $\sum_{\mathfrak{a}}^b$ denotes a summation over integral ideals \mathfrak{a} whose prime factors are all of degree one.

Smoothings. We shall work with a generic smoothing function $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with the following properties:

- $w(t) = 0$ when $t \geq 1$ and $t \leq 1/10$,
- w does not vanish uniformly and $|w(t)| \leq 1$ throughout,
- w is at least $n_{\mathbf{K}} + 3$ times continuously differentiable,
- For every $m \leq n_{\mathbf{K}} + 2$, we have $w^{(m)}(\frac{1}{10}) = w^{(m)}(1) = 0$, where $w^{(m)}$ denotes the m -th derivative of w .

We will henceforth refer to this function as ‘the smoothing function’. Its Mellin transform \check{w} is defined by

$$(2) \quad \check{w}(s) = \int_0^\infty w(t)t^{s-1}dt.$$

We show in Lemma 11 that this analytic function decreases at least like $1/(1+|s|)^{n_{\mathbf{K}}+3}$ uniformly in any vertical strip.

For applications, we select the special function w_0 described below. Let

$$f_k(t) = \begin{cases} (4t(1-t))^k & \text{when } t \in [0, 1], \\ 0 & \text{otherwise} \end{cases}$$

be as defined in page 348 of [22] with $k = n_{\mathbf{K}} + 4$. We set

$$(3) \quad w_0(t) = f_{n_{\mathbf{K}}+4} \left(\frac{10}{9} \left(t - \frac{1}{10} \right) \right).$$

Lemma 7. *We have $\|w_0\|_\infty = 1$, $10\sqrt{n_{\mathbf{K}}}\check{w}_0(1) \in [2, 15]$, $\|w'_0\|_1 = 2$, $\|w_0^{(n_{\mathbf{K}}+3)}\|_\infty \leq 4(40n_{\mathbf{K}})^{n_{\mathbf{K}}+3}$.*

Proof. Indeed, by [22, Lemma 2.2], we have $\check{w}_0(1) = \|w_0\|_1 = \frac{9}{10} \frac{2^{2k} \cdot k!^2}{(2k+1)!}$ with $k = n_{\mathbf{K}} + 4$. Applying the classical explicit Stirling’s formula

$$(4) \quad n! = (n/e)^n \sqrt{2\pi n} e^{\frac{\theta_+(n)}{12n}}, \quad (\theta_+(n) \in [0, 1]),$$

we find that

$$\|w_0\|_1 = \frac{9}{10} \frac{\sqrt{\pi k}}{(2k+1)} e^{\frac{\theta_+(k)}{6k}} - \frac{\theta_+(2k)}{24k} = \frac{\xi(n_{\mathbf{K}})}{10\sqrt{n_{\mathbf{K}}}}, \quad 2 \leq \xi(n_{\mathbf{K}}) \leq 15.$$

Next we check that

$$\|f'_k\|_1 = 2 \int_0^{1/2} f'_k(t)dt = 2.$$

Also, by Leibniz Formula, we find that

$$f_k^{(k-1)}(t) = 4^k \sum_{0 \leq \ell \leq k-1} \binom{k-1}{\ell} \frac{k!}{(k-\ell)!} t^{k-\ell} (-1)^{k-1-\ell} \frac{(k-1)!}{(k-(k-1-\ell))!} (1-t)^{k-(k-1-\ell)}$$

so that

$$|w_0^{(k-1)}(t)| \leq \left(\frac{10}{9} \right)^{k-1} 4^k (k-1)! \sum_{0 \leq \ell \leq k-1} \binom{k-1}{\ell} \binom{k}{k-\ell} = \left(\frac{10}{9} \right)^{k-1} 4^k (k-1)! \binom{2k-1}{k}$$

by Vandermonde’s identity. We bound this almost trivially:

$$\left(\frac{10}{9} \right)^{k-1} 4^k (k-1)! \binom{2k-1}{k} = \left(\frac{10}{9} \right)^{k-1} 4^k \frac{(2k-1)!}{k!} \leq 4^k (2k-1)^{k-1} \leq 4(40n_{\mathbf{K}})^{n_{\mathbf{K}}+3}.$$

□

Explicit number of ideals in a ray class below some bound. We now state the main theorem in [5] which is required for the proofs of Theorem 3 and Theorem 5 .

Theorem 8. *Let \mathfrak{q} be a modulus of \mathbf{K} and $[\mathfrak{a}]$ be an element of $H_{\mathfrak{q}}(\mathbf{K})$. For any real number $X \geq 1$, we get*

$$\sum_{\substack{\mathfrak{b} \in [\mathfrak{a}] \\ \mathfrak{b} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N} \mathfrak{b} \leq X}} 1 = \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q})}{h_{\mathbf{K}, \mathfrak{q}}} \frac{X}{\mathfrak{N}(\mathfrak{q})} + O^* \left(E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n_{\mathbf{K}}}} \log(3F(\mathfrak{q}))^{n_{\mathbf{K}}} \left(\frac{X}{\mathfrak{N}(\mathfrak{q})} \right)^{1 - \frac{1}{n_{\mathbf{K}}}} + n_{\mathbf{K}}^{8n_{\mathbf{K}}} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathfrak{q}) \right).$$

where $F(\mathfrak{q}) = 2^{r_1} \frac{h_{\mathbf{K}} \phi(\mathfrak{q})}{h_{\mathbf{K}, \mathfrak{q}}} \geq 1$, $E(\mathbf{K}) = 1000 n_{\mathbf{K}}^{12n_{\mathbf{K}}} (R_{\mathbf{K}}/|\mu_{\mathbf{K}}|)^{1/n_{\mathbf{K}}} [\log((2n_{\mathbf{K}})^{4n_{\mathbf{K}}} R_{\mathbf{K}}/|\mu_{\mathbf{K}}|)]^{n_{\mathbf{K}}}$ and the notation O^* denotes that the implied constant is less than or equal to 1.

Lower bound for the root discriminant. The root discriminant is defined by $|d_{\mathbf{K}}|^{1/n_{\mathbf{K}}}$.

Lemma 9. *We have $|d_{\mathbf{K}}|^{1/n_{\mathbf{K}}} \geq \pi/2$.*

Proof. In this proof, we denote $n_{\mathbf{K}}$ by n . By Minkowski's bound, we find that

$$\rho = |d_{\mathbf{K}}|^{1/n_{\mathbf{K}}} \geq \frac{\pi}{4} n^2 / n!^{2/n}.$$

The explicit Stirling Formula recalled in (4) yields $n!^{2/n} \leq \frac{n^2}{e^2} (\sqrt{2\pi n} e^{\theta_+(n)/(12n)})^{2/n} \leq n^2/2$ when $n \geq 3$ while $2!^{2/2} = 2 \leq 2^2/2$. We note that, when $n \geq 5$, the Minkowski bound is superseded by the bound given in [14, Eq. (2)] by Liang and Zassenhaus (the quantity V_{r_1, r_2} is given on line 14, page 18 of their paper, and $\mu_n \geq 1$). \square

The Dedekind zeta-function. For $\Re s = \sigma > 1$, the Dedekind zeta-function is defined by

$$\zeta_{\mathbf{K}}(s) = \sum_{\substack{\mathfrak{a} \in \mathcal{O}_{\mathbf{K}}, \\ \mathfrak{a} \neq (0)}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where \mathfrak{a} ranges over the integral ideals of $\mathcal{O}_{\mathbf{K}}$. It has only a simple pole at $s = 1$ of residue $\alpha_{\mathbf{K}}$, say. We know from the analytic class number formula that

$$(5) \quad \alpha_{\mathbf{K}} = \frac{2^{r_1} (2\pi)^{r_2} h_{\mathbf{K}} R_{\mathbf{K}}}{|\mu_{\mathbf{K}}| \sqrt{|d_{\mathbf{K}}|}},$$

where $h_{\mathbf{K}}, R_{\mathbf{K}}, d_{\mathbf{K}}$ are as defined in the introduction and $\mu_{\mathbf{K}}$ is the group of roots of unity in \mathbf{K} . As we will see later (Lemma 14) its order in vertical strips is sufficiently moderate so that multiplication by $\check{w}(s)$ makes it an L^1 -function on any line $\Re s = \sigma \geq -1/2$.

Euler-Kronecker constant. For a number field \mathbf{K} , the Euler-Kronecker constant is defined as

$$\gamma_{\mathbf{K}} = \lim_{s \rightarrow 1} \left(\frac{\zeta'_{\mathbf{K}}(s)}{\zeta_{\mathbf{K}}(s)} + \frac{1}{s-1} \right).$$

We also know that in a neighbourhood of $s = 1$

$$\zeta_{\mathbf{K}}(s) = \frac{\alpha_{\mathbf{K}}}{s-1} + \alpha_{\mathbf{K}} \gamma_{\mathbf{K}} + O(s-1),$$

where the constant in O depends on \mathbf{K} . The constant $\gamma_{\mathbf{K}}$ is called the ‘Euler-Kronecker constant’ in [8] by Ihara. We use Proposition 3, page 431 of this paper, namely the inequality

$$(6) \quad \gamma_{\mathbf{K}} \geq -\frac{1}{2} \log |d_{\mathbf{K}}|$$

(where $\alpha_{\mathbf{K}}$ in Ihara's paper is given by (1.2.2), $\beta_{\mathbf{K}}$ by (1.2.3) and $c_{\mathbf{K}} = 1$ by (1.3.12)). The conclusion we need is also restated as (0.7) therein.

The narrow ray-class group. By narrow ray class group $H_{\mathbf{q}}(\mathbf{K})$, we consider those ray class groups where the integral ideal \mathfrak{q} is completed with all real archimedean places. We have

$$(7) \quad h_{\mathbf{K},1} = |H_1(\mathbf{K})| \left| \frac{|H_{\mathbf{q}}(\mathbf{K})|}{2^{r_1} \phi(\mathfrak{q}) |H_1(\mathbf{K})|} \right|,$$

where

$$(8) \quad \phi(\mathfrak{q}) = \mathfrak{N}(\mathfrak{q}) \prod_{\mathfrak{p}|\mathfrak{q}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})} \right)$$

and $H_1(\mathbf{K})$ denotes the narrow ray class group corresponding to $\mathcal{O}_{\mathbf{K}}$. A good reference for this are the notes of Sutherland [28]. We also have the following theorem in this context.

Lemma 10 (Lang, [13], page 127). *Let \mathfrak{q} be a modulus of \mathbf{K} , $h_{\mathbf{K},\mathfrak{q}} = |H_{\mathbf{q}}(\mathbf{K})|$ and $r_1, h_{\mathbf{K}}$ be as defined earlier. Then $h_{\mathbf{K},\mathfrak{q}} \leq 2^{r_1} \phi(\mathfrak{q}) h_{\mathbf{K}}$.*

Characters on the narrow ray-class group. A manner to work with the narrow ray class group $H_{\mathbf{q}}(\mathbf{K})$ is to consider its character group. When lifted to the set of all ideals, these are characters that vanish on ideals which are not co-prime to \mathfrak{q} .

An excellent reference is the report [12] where Landau explains in detail and refines Hecke's theory. We are only interested in the extensions of the true characters of $H_{\mathbf{q}}(\mathbf{K})$ and these are the ones that have finite order. These are the ones that Hecke considers, while Landau [18, Lemma 6.34 and (6.7)] considers an extended class of characters that may have infinite orders.

In our case, the notion of conductor of a character goes through, and the functional equation of the Hecke L -function of a primitive ("eigentlicher" in Landau's paper) is given in [12, Theorem LVI].

3. SOME GENERAL LEMMAS

On Mellin transform.

Lemma 11. *When $\Re s = 0$ and $|\Im s| \geq 1$ or when $\Re s \geq 1/2$, we have*

$$|\check{w}(s)| \leq \frac{2^{\frac{n_{\mathbf{K}}}{2}+3} \|w^{(n_{\mathbf{K}}+3)}\|_{\infty}}{(1+|s|)^{n_{\mathbf{K}}+3}}.$$

Proof. We set $A = n_{\mathbf{K}} + 3$ and $t = \Im s$ for typographical simplification. Integrating by parts A times and noting that $w^{(m)}(1) = w^{(m)}(0) = 0$ for $0 \leq m \leq A$, we get

$$\left| \check{w}(s) = \frac{(-1)^A}{s(s+1) \cdots (s+A-1)} \int_{1/10}^1 w^{(A)}(u) u^{s+A-1} du \right| \leq \frac{\|w^{(n_{\mathbf{K}}+3)}\|_{\infty}}{|s||s+1| \cdots |s+A-1|}.$$

In case $\Re s = 0$, we note that $|t| \geq 1$ implies $1 + |t| \leq 2|t|$. Furthermore we have $\sqrt{2} |m + it| \geq |t| + 1$ for all $m \geq 1$. This yields the constant $2^{\frac{n_{\mathbf{K}}}{2}+2}$.

In case $\Re s \geq 1/2$, we first notice that $3|s| \geq |s| + 1$, and then we prove that $\sqrt{2} |s+m| \geq |s| + 1$ for $m \geq 1$ as before. To prove this inequality, set $\sigma = \Re s$. It is enough to prove that $t^2 + 2(\sigma + m)^2 \geq \sigma^2 + 2|s| + 1$. As $|s| \leq |t| + |\sigma|$, it is enough to prove that $t^2 - 2|t| + 1 \geq -\sigma^2 + 2(1 - 2m)\sigma + 2(1 - m^2)$ which is obviously

true. This yields the constant $3 \cdot 2^{(n_{\mathbf{K}}+2)/2}$. We majorize the constant in both cases by $2^{\frac{n_{\mathbf{K}}}{2}+3}$ and this concludes this lemma. \square

Lemma 12. *For $\varepsilon \in (0, 1/2]$, we define*

$$(9) \quad M(w, \varepsilon) = \int_{-\infty}^{\infty} |\ddot{w}(it)|(1 + |t|)^{\frac{1+\varepsilon}{2}n_{\mathbf{K}}} dt.$$

We have $M(w, \varepsilon) \leq 2^{2+\frac{\varepsilon n_{\mathbf{K}}}{2}} (\|w^{(n_{\mathbf{K}}+3)}\|_{\infty} + 10 \cdot 2^{\frac{n_{\mathbf{K}}}{2}} \|w\|_1)$.

Proof. Set $n = n_{\mathbf{K}}$. We split this integral according to whether $|t| \geq 1$ or not. When $|t| \geq 1$, Lemma 11 applies. When $|t| \leq 1$, we simply use $|\ddot{w}(it)| \leq 10\|w\|_1$. This gives us

$$M(w, \varepsilon) \leq 2 \frac{2^{\frac{n}{2}+3} \|w^{(n+3)}\|_{\infty}}{2^{\frac{1-\varepsilon}{2}n+2} (\frac{1-\varepsilon}{2}n+2)} + 10\|w\|_1 2^{\frac{1+\varepsilon}{2}n+2} \leq 2^{2+\frac{\varepsilon n}{2}} (\|w^{(n+3)}\|_{\infty} + 10 \cdot 2^{\frac{n}{2}} \|w\|_1).$$

This completes the proof of the lemma. \square

Lemma 13. *For $\varepsilon \in (0, 1/2]$ and $r \in \{1, 2\}$, we define*

$$(10) \quad M^*(\varepsilon, r) = \int_{\frac{1+\varepsilon}{2}-i\infty}^{\frac{1+\varepsilon}{2}+i\infty} |\ddot{w}_0(s)|(1 + |s|)^{\frac{1+\varepsilon}{2r}n_{\mathbf{K}}} ds.$$

We have $M^*(\varepsilon, r) \leq 12(57n_{\mathbf{K}})^{n_{\mathbf{K}}+3}$.

Proof. Set $n = n_{\mathbf{K}}$ and $\sigma = (1 + \varepsilon)/2$. Lemma 11 applies and gives us

$$M^*(\varepsilon, r) \leq 2 \cdot 2^{\frac{n}{2}+3} \|w_0^{(n+3)}\|_{\infty} \int_0^{\infty} \frac{dt}{(1 + |\sigma + it|)^{\frac{(2r-1-\varepsilon)n}{2r}+3}}.$$

Furthermore

$$\int_0^{\infty} \frac{dt}{(1 + |\sigma + it|)^{\frac{(2r-1-\varepsilon)n}{2r}+3}} \leq \int_0^{\infty} \frac{dt}{(1 + t)^{\frac{(2r-1-\varepsilon)n}{2r}+3}} \leq 1/2.$$

Lemma 7 leads to the bound $12(40\sqrt{2}n)^{n+3}$ which is indeed not more than $12(57n)^{n+3}$. \square

On the Dedekind zeta-function.

Lemma 14. *Let $0 < \varepsilon \leq 1/2$. In the strip $-\varepsilon \leq \sigma \leq 1 + \varepsilon$, the Dedekind zeta-function $\zeta_{\mathbf{K}}(s)$ satisfies the inequality*

$$|\zeta_{\mathbf{K}}(s)| \leq 3\zeta(1 + \varepsilon)^{n_{\mathbf{K}}} \left| \frac{s+1}{s-1} \right| (|d_{\mathbf{K}}|(1 + |s|)^{n_{\mathbf{K}}})^{\frac{1+\varepsilon-\sigma}{2}}.$$

Proof. This is Theorem 4 of [21] by Rademacher. \square

As a corollary, we deduce the following lemma.

Lemma 15. *We have $\frac{9 \cdot 2^{n_{\mathbf{K}}} h_{\mathbf{K}}}{100\sqrt{|d_{\mathbf{K}}|}} \leq \alpha_{\mathbf{K}} \leq 6 \left(\frac{2\pi^2}{5} \right)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^{1/4}$ and $h_{\mathbf{K}} \leq 67(\pi^2/5)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^{3/4}$.*

See the book [20] by Pohst and Zassenhaus for more on lower bounds for $\alpha_{\mathbf{K}}$.

Proof. Lemma 14 with the choices $\varepsilon = 1/2$ and $s = 1$ gives the upper bound. As $2^{r_1}(2\pi)^{r_2} \geq 2^{r_1}2^{2r_2} \geq 2^{n_{\mathbf{K}}}$ and the ratio of the regulator to $\mu_{\mathbf{K}}$ is bounded below absolutely by 0.09 (see [4] by E. Friedman), Eq. (5) provides us with the lower bound for $\alpha_{\mathbf{K}}$. Concerning the upper bound for $h_{\mathbf{K}}$, we use (5) again and this time, derive from it that $\alpha_{\mathbf{K}} \geq (9/100) 2^{n_{\mathbf{K}}} h_{\mathbf{K}} / \sqrt{|d_{\mathbf{K}}|}$, from which the last bound follows immediately. \square

Next we deduce similar bounds for Hecke L -functions corresponding to primitive characters χ of finite order. For a Hecke character χ defined modulo \mathfrak{q} , the Hecke L -function associated to χ is defined as follows;

$$L_{\mathfrak{q}}(s, \chi) = \sum_{\substack{\mathfrak{a} \in \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \neq 0}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\Re s > 1$. We now state a result which bounds the growth of the Hecke L -series using the Phragmén-Lindelöf principle.

Lemma 16 ([21], Theorem 5). *Let $0 < \varepsilon \leq 1/2$. In the strip $-\varepsilon \leq \sigma \leq 1 + \varepsilon$, the Hecke L -series associated with the primitive character χ of finite order and conductor \mathfrak{q} satisfies the inequality*

$$|L_{\mathfrak{q}}(s, \chi)| \leq \zeta(1 + \varepsilon)^{n_{\mathbf{K}}} (|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q}) (1 + |s|)^{n_{\mathbf{K}}})^{\frac{1+\varepsilon-\sigma}{2}}.$$

Proof. This is a direct consequence of [21, Theorem 5] with $\eta = \varepsilon$, but one should be mindful of the notation, since Rademacher considers general Hecke Grossencharakteren, not necessarily of finite order. Things are rather clear when we inspect the gamma-factor given in [21, Bottom of page 202]. We have $a_p = 0$ for every complex place, $v_p = 0$ for every place and $a_p \in \{0, 1\}$ for real places, q of them taking the value 1. \square

On rational primes.

Lemma 17. *For $x \geq 1$, we have $\sum_{\substack{p^k \leq x, \\ k \geq 2}} 1 \leq \frac{5\sqrt{x}}{4}$.*

Proof. We first check this property by Pari/GP for $t \leq 10^7$. To extend this result, let us denote by S the sum to be bounded above. Inequality [25, (3.32)] by Rosser and Schoenfeld tells us that

$$(11) \quad \sum_{p \leq x} \log p \leq 1.02x \quad \text{for } x > 0.$$

We then readily check that

$$\begin{aligned} S &\leq \sum_{p \leq \sqrt{x}} \frac{\log x}{\log p} \leq 2(\log x) \int_2^{\sqrt{x}} \sum_{p \leq u} \log p \frac{du}{u(\log u)^3} + (\log x) \sum_{p \leq \sqrt{x}} \frac{\log p}{(\frac{1}{2} \log x)^2} \\ &\leq 2.04(\log x) \int_2^{\sqrt{x}} \frac{du}{(\log u)^3} + 4.08 \frac{\sqrt{x}}{\log x} \leq \sqrt{x} \end{aligned}$$

when $x \geq 10^7$. The lemma follows readily. \square

Lemma 18. *For $x \geq 100$, we have $\sum_{p \leq x} \frac{1}{p} \leq 2 \log \log x$.*

Proof. This is readily checked by Pari/GP for $x \leq 10^8$. We conclude the proof by appealing to [25, Theorem 5] by Rosser and Schoenfeld. \square

On the Möbius function. For an ideal \mathfrak{b} of $\mathcal{O}_{\mathbf{K}}$, we define the Möbius function as

$$(12) \quad \mu(\mathfrak{b}) = \begin{cases} 1 & \text{if } \mathfrak{b} = \mathcal{O}_{\mathbf{K}}, \\ (-1)^r & \text{if } \mathfrak{b} = \mathfrak{p}_1 \cdots \mathfrak{p}_r, \text{ where } \mathfrak{p}_i \text{ are distinct prime ideals,} \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer R and for an ideal \mathfrak{b} of $\mathcal{O}_{\mathbf{K}}$, we define the truncated Möbius function μ_R in the following manner;

$$\mu_R(\mathfrak{b}) = \begin{cases} \mu(\mathfrak{b}) & \text{if } \omega(\mathfrak{b}) \leq R \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi_R(\mathfrak{b}) = \sum_{\partial|\mathfrak{b}} \mu_R(\partial)$. Applying Möbius inversion formula and the fact that $\sum_{\partial|\mathfrak{b}} \mu(\partial) = 1$ if and only if $\mathfrak{b} = \mathcal{O}_{\mathbf{K}}$, we get $\mu_R(\mathfrak{b}) = \sum_{\partial|\mathfrak{b}} \mu(\frac{\mathfrak{b}}{\partial}) \psi_R(\partial)$. In this context, we have the following Lemma.

Lemma 19. *For any integral ideal $\partial \neq \mathcal{O}_{\mathbf{K}}$, we have*

$$|\psi_R(\partial)| \leq \binom{\omega(\partial) - 1}{R}.$$

Proof. Applying induction on R , we get

$$\psi_R(\partial) = \sum_{0 \leq k \leq R} (-1)^k \binom{\omega(\partial)}{k} = (-1)^R \binom{\omega(\partial) - 1}{R}$$

(see H. Halberstam and H. Richert [6, p. 46/47]). □

4. DEGREE ONE PRIMES IN RAY CLASS GROUPS

Let \mathfrak{q} be a modulus and let T be a subgroup of index 2 of $H_{\mathfrak{q}}(\mathbf{K})$. Let χ be the quadratic character whose kernel is T . We want to show that there exists a degree one prime \mathfrak{p} with small norm such that $\chi(\mathfrak{p}) = -1$ and another prime \mathfrak{p}' with small norm such that $\chi(\mathfrak{p}') = 1$.

L -series for degree one primes. For a Hecke character of finite order modulo \mathfrak{q} , we define

$$(13) \quad F(s, \chi) = \prod_{\mathfrak{p} \nmid \mathfrak{q}} \frac{1}{1 - \chi(\mathfrak{p}) \mathfrak{N}(\mathfrak{p})^{-s}}.$$

Lemma 20. *When $\Re s = (1 + \varepsilon)/2$ for some $0 < \varepsilon \leq 1/2$ and χ is a Hecke character of finite order modulo \mathfrak{q} , we have*

$$|F(s, \chi)| \leq \zeta(1 + \varepsilon)^{\frac{3n_{\mathbf{K}}}{2}} (|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q}))^{\frac{1+\varepsilon}{4}} \theta(\mathfrak{q}) (1 + |s|)^{\frac{(1+\varepsilon)n_{\mathbf{K}}}{4}},$$

where $\theta(\mathfrak{q}) = \prod_{\mathfrak{p}|\mathfrak{q}} \frac{\sqrt{\mathfrak{N}\mathfrak{p}}}{\sqrt{\mathfrak{N}\mathfrak{p}} - 1}$. When $\chi = \chi_{0, \mathfrak{q}}$ is the trivial character modulo \mathfrak{q} , we have

$$|F(s, \chi_{0, \mathfrak{q}})| \leq 27\zeta(1 + \varepsilon)^{\frac{3n_{\mathbf{K}}}{2}} |d_{\mathbf{K}}|^{\frac{1+\varepsilon}{4}} \theta(\mathfrak{q}) (1 + |s|)^{\frac{(1+\varepsilon)n_{\mathbf{K}}}{4}}.$$

Proof. To find an upper bound for $F(s, \chi)$, we first reduce it to Hecke L -series using the following product $F(s, \chi) = L_{\mathfrak{q}}(s, \chi) J(s, \chi)$, where

$$(14) \quad J(s, \chi) = \prod_{\mathfrak{p} \nmid \mathfrak{q}}^{\#} (1 - \chi(\mathfrak{p}) \mathfrak{N}(\mathfrak{p})^{-s}).$$

We readily find that, when $\Re s = (1 + \varepsilon)/2$, we have

$$(15) \quad |J(s, \chi)| \leq \prod_p (1 + p^{-2\sigma})^{\frac{n_{\mathbf{K}}}{2}} \leq \zeta(1 + \varepsilon)^{\frac{n_{\mathbf{K}}}{2}}.$$

The next step is to reduce $L_{\mathbf{q}}(s, \chi)$ to $L_{\mathfrak{f}}(s, \chi^*)$, where χ^* is the primitive character, say modulo \mathfrak{f} , inducing χ . We directly see that

$$L_{\mathbf{q}}(s, \chi) = L_{\mathfrak{f}}(s, \chi^*) \prod_{\substack{\mathfrak{p}|\mathbf{q} \\ \mathfrak{p} \nmid \mathfrak{f}}} (1 - \chi(\mathfrak{p})\mathfrak{N}(\mathfrak{p})^{-s}).$$

Therefore, applying Lemma 16 when χ^* is not equal to the constant character 1, and Lemma 14 otherwise, together with the bound in (15), we get the desired result. \square

Products of degree one primes in ray classes modulo \mathbf{q} , sieve approach.

Lemma 21. *Let \mathfrak{b} be an integral ideal co-prime to \mathbf{q} . When $F_1(\mathbf{q}) = 2^{r_1} h_{\mathbf{K}} \mathfrak{N} \mathbf{q}$ and*

$$X \geq \log(3F(\mathbf{q}))^{n_{\mathbf{K}}} B(\mathbf{K}) F_1(\mathbf{q}) \mathfrak{N} \mathbf{q} \log \log(B(\mathbf{K}) F(\mathbf{q}) \mathfrak{N} \mathbf{q})^2, \quad B(\mathbf{K}) = (n_{\mathbf{K}}^{50n_{\mathbf{K}}} E(\mathbf{K}) \sqrt{|d_{\mathbf{K}}|})^{n_{\mathbf{K}}},$$

we have

$$\sum_{\substack{\mathfrak{p} \\ \mathfrak{N} \mathfrak{p} \geq 40^{n_{\mathbf{K}}} \\ \deg \mathfrak{p} \geq 2}} \sum_{\substack{\mathfrak{p} | \mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{a}] = [\mathfrak{b}] \\ \mathfrak{N} \mathfrak{a} \leq X}} 1 \leq \frac{\alpha_{\mathbf{K}} \phi(\mathbf{q}) X}{2^{n_{\mathbf{K}}} \mathfrak{N} \mathbf{q} |H_{\mathbf{q}}(\mathbf{K})|}.$$

We can remove the term $\log \log(B(\mathbf{K}) F(\mathbf{q}) \mathfrak{N} \mathbf{q})^2$ when $n_{\mathbf{K}} \geq 3$.

Proof. We shorten $n_{\mathbf{K}}$ by n and set $D = 40^n$. On calling S the sum on the left hand side, Theorem 8 gives us the upper bound, with $Y = \alpha_{\mathbf{K}} \phi(\mathbf{q}) X / (\mathfrak{N} \mathbf{q} h_{\mathbf{K}, \mathbf{q}})$,

$$S \leq \sum_{\substack{\mathfrak{p} \\ \mathfrak{N} \mathfrak{p} \geq D, \\ \deg \mathfrak{p} \geq 2}} \frac{Y}{\mathfrak{N} \mathfrak{p}} + E(\mathbf{K}) F(\mathbf{q})^{\frac{1}{n}} \log(3F(\mathbf{q}))^n \sum_{\substack{\mathfrak{p} \\ X \geq \mathfrak{N} \mathfrak{p} \geq D, \\ \deg \mathfrak{p} \geq 2}} \left(\frac{X}{\mathfrak{N} \mathbf{q} \mathfrak{N} \mathfrak{p}} \right)^{1 - \frac{1}{n}} + n^{8n} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathbf{q}) \sum_{\substack{\mathfrak{p} \\ \mathfrak{N} \mathfrak{p} \leq X, \\ \deg \mathfrak{p} \geq 2}} 1.$$

We now have to study each of these three sums. We readily bound above the first one by

$$Y \sum_{\substack{p^k \geq D, \\ k \geq 2}} \frac{n/2}{p^k} \leq \frac{nY}{2} \int_D^\infty \left(\sum_{\substack{D \leq p^k \leq t, \\ k \geq 2}} 1 \right) \frac{dt}{t^2} \leq \frac{5nY}{8} \int_D^\infty \frac{dt}{t^{3/2}} \leq \frac{5nY}{4\sqrt{D}}$$

by Lemma 17. We notice that $(5/4)40^{-n/2} \leq 1/2^{2n+1}$ when $n \geq 2$. The same Lemma 17 yields the bound $\frac{5}{8} n^{8n+1} R_{\mathbf{K}} F(\mathbf{q}) \sqrt{X} / |\mu_{\mathbf{K}}|$ for the third term. We further find that

$$\begin{aligned} \frac{5}{8} n^{8n+1} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathbf{q}) \sqrt{X} &= \frac{5n^{8n+1}}{8} \frac{R_{\mathbf{K}} \mathfrak{N} \mathbf{q} h_{\mathbf{K}, \mathbf{q}} F(\mathbf{q}) Y \sqrt{X}}{\alpha_{\mathbf{K}} |\mu_{\mathbf{K}}| \phi(\mathbf{q}) X} = \frac{5n^{8n+1}}{8} \frac{2^{r_1} h_{\mathbf{K}} R_{\mathbf{K}} \mathfrak{N} \mathbf{q} Y}{\alpha_{\mathbf{K}} |\mu_{\mathbf{K}}| \sqrt{X}} \\ &\leq \frac{5n^{8n+1}}{8} (2\pi)^{-r_2} \sqrt{|d_{\mathbf{K}}|} \frac{\mathfrak{N} \mathbf{q} Y}{\sqrt{X}} \leq n^{9n} \sqrt{|d_{\mathbf{K}}|} \frac{\mathfrak{N} \mathbf{q} Y}{\sqrt{X}} \end{aligned}$$

by applying the definition of $F(\mathbf{q})$ from Theorem 8 and the expression of $\alpha_{\mathbf{K}}$ in terms of the invariants of the field mentioned in (5). Let us now examine the second term above, a task for which we distinguish

between the case when $n = 2$ and $n \geq 3$. In the latter situation, we find that

$$(16) \quad \sum_{\substack{\mathfrak{p} \\ X \geq \mathfrak{N}\mathfrak{p} \geq D, \\ \deg \mathfrak{p} \geq 2}} \frac{1}{(\mathfrak{N}\mathfrak{p})^{1-\frac{1}{n}}} \leq \frac{n}{2} \sum_{p \geq 2} \frac{1}{p^{2(1-1/3)}} \leq n$$

by appealing to Pari/GP. In this case, we thus find that

$$\begin{aligned} S/Y &\leq \frac{n}{2^{2n+1}} + E(\mathbf{K})F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \frac{nh_{\mathbf{K},\mathfrak{q}}}{\alpha_{\mathbf{K}}\phi(\mathfrak{q})} \left(\frac{\mathfrak{N}\mathfrak{q}}{X}\right)^{1/n} + n^{9n} \sqrt{|d_{\mathbf{K}}|} \frac{\mathfrak{N}\mathfrak{q}}{\sqrt{X}} \\ &\leq \frac{1}{2^{n+1}} + E(\mathbf{K}) \log(3F(\mathfrak{q}))^n \frac{n2^{r_1}h_{\mathbf{K}}}{\alpha_{\mathbf{K}}} \left(\frac{F(\mathfrak{q})\mathfrak{N}\mathfrak{q}}{X}\right)^{1/n} + n^{9n} \sqrt{|d_{\mathbf{K}}|} \frac{\mathfrak{N}\mathfrak{q}}{\sqrt{X}} \\ &\leq \frac{1}{2^{n+1}} + \frac{100n}{9} E(\mathbf{K}) \log(3F(\mathfrak{q}))^n \sqrt{|d_{\mathbf{K}}|} \left(\frac{F(\mathfrak{q})\mathfrak{N}\mathfrak{q}}{X}\right)^{1/n} + n^{9n} \sqrt{|d_{\mathbf{K}}|} \frac{\mathfrak{N}\mathfrak{q}}{\sqrt{X}} \end{aligned}$$

by Lemma 10 and 15. Since we have assumed that

$$(17) \quad X \geq B(\mathbf{K})F_1(\mathfrak{q})\mathfrak{N}\mathfrak{q} \log(3F(\mathfrak{q}))^{n^2}, \quad B(\mathbf{K}) = (n^{50n^2} E(\mathbf{K}) \sqrt{|d_{\mathbf{K}}|})^n$$

we find that $S \leq Y/2^n$ when $n \geq 3$. When $n = 2$, we only have to replace the upper bound n in (16) by $n \log \log X = 2 \log \log X$ by Lemma 18. Proceeding as above we reach the inequality

$$S/Y \leq \frac{1}{4} + \frac{200}{9} E(\mathbf{K}) \log(3F(\mathfrak{q}))^2 \log \log X \sqrt{|d_{\mathbf{K}}|} \left(\frac{F(\mathfrak{q})\mathfrak{N}\mathfrak{q}}{X}\right)^{1/2} + 2^{18} \frac{\mathfrak{N}\mathfrak{q}}{\sqrt{X}} \sqrt{|d_{\mathbf{K}}|}.$$

Our hypothesis on X reads

$$X \geq \log(3F(\mathfrak{q}))^4 B(\mathbf{K})F_1(\mathfrak{q})\mathfrak{N}\mathfrak{q} \log \log(B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q})^2.$$

We just need to notice that $\log(3F(\mathfrak{q}))^4 \log \log(B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q})^2 \leq (B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q})^6$, so that

$$\begin{aligned} \log \log X &\leq \log \log((B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q})^7) \\ &\leq \left(1 + \frac{\log 7}{\log \log(B(\mathbf{K}))}\right) \log \log(B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q}) \leq 2 \log \log(B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q}). \end{aligned}$$

This is enough to conclude. \square

Lemma 22. *Let \mathfrak{b} be an integral ideal co-prime to \mathfrak{q} . When $F_1(\mathfrak{q}) = 2^{r_1} h_{\mathbf{K}} \mathfrak{N}\mathfrak{q}$ and*

$$X \geq \log(3F(\mathfrak{q}))^{n_{\mathbf{K}}^2} B(\mathbf{K})F_1(\mathfrak{q})\mathfrak{N}\mathfrak{q} \log \log(B(\mathbf{K})F(\mathfrak{q})\mathfrak{N}\mathfrak{q})^2, \quad B(\mathbf{K}) = (n_{\mathbf{K}}^{50n_{\mathbf{K}}^2} E(\mathbf{K}) \sqrt{|d_{\mathbf{K}}|})^{n_{\mathbf{K}}},$$

we have

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{a}] = [\mathfrak{b}] \\ \mathfrak{N}\mathfrak{a} \leq X}} 1 \geq (1/4) \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) X}{\mathfrak{N}\mathfrak{q} |H_{\mathfrak{q}}(\mathbf{K})|}.$$

Proof. We denote $n_{\mathbf{K}}$ by n and set $D = 40^n$. Let M be the product of the prime ideals in $\mathcal{O}_{\mathbf{K}}$ of degree greater than or equal to 2, co-prime to \mathfrak{q} and of norm at most D . Denoting the sum to be evaluated by S , a simple combinatorial argument together with Lemma 21 gives us

$$S \geq \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{a}] = [\mathfrak{b}] \\ \mathfrak{N}\mathfrak{a} \leq X \\ (\mathfrak{a}, M) = 1}} 1 - \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) X}{2^n \mathfrak{N}\mathfrak{q} |H_{\mathfrak{q}}(\mathbf{K})|} = S(M) - \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) X}{2^n \mathfrak{N}\mathfrak{q} |H_{\mathfrak{q}}(\mathbf{K})|}$$

say. We detect the coprimality condition with an elementary instance of the Brun sieve. We select an odd positive integer R and write

$$1_{(a,M)=1} = \sum_{\mathfrak{d}|(M,a)} \mu(\mathfrak{d}) \geq \sum_{\substack{\mathfrak{d}|(M,a), \\ \omega(\mathfrak{d}) \leq R}} \mu(\mathfrak{d}).$$

We deduce

$$S(M) \geq \sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} \mu(\mathfrak{d}) \sum_{\substack{\mathfrak{a}' \subseteq \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{d}\mathfrak{a}'] = [\mathfrak{b}] \\ \mathfrak{N}\mathfrak{a}' \leq \frac{X}{\mathfrak{N}\mathfrak{d}}}} 1.$$

Theorem 8 gives us the lower bound, with $Y = \alpha_{\mathbf{K}} X \phi(\mathfrak{q}) / (\mathfrak{N}\mathfrak{q} h_{\mathbf{K},\mathfrak{q}})$,

$$S(M) \geq Y \sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} \frac{\mu(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} - E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} \left(\frac{X}{\mathfrak{N}\mathfrak{q}\mathfrak{N}\mathfrak{d}} \right)^{1-\frac{1}{n}} - n^{8n} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathfrak{q}) \sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} 1.$$

Concerning the main term, we can write

$$\begin{aligned} \sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} \frac{\mu(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} &= \sum_{\mathfrak{d}|M} \frac{\mu_R(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} = \sum_{\mathfrak{d}|M} \frac{1}{\mathfrak{N}\mathfrak{d}} \sum_{\mathfrak{b}|\mathfrak{d}} \mu\left(\frac{\mathfrak{d}}{\mathfrak{b}}\right) \psi_R(\mathfrak{b}) = \sum_{\mathfrak{b}|M} \frac{\psi_R(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} \sum_{\mathfrak{c}|\frac{M}{\mathfrak{b}}} \frac{\mu(\mathfrak{c})}{\mathfrak{N}\mathfrak{c}} \\ (18) \qquad \qquad \qquad &= \prod_{\mathfrak{p}|M} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) \sum_{\mathfrak{b}|M} \frac{\psi_R(\mathfrak{b})}{\phi(\mathfrak{b})}. \end{aligned}$$

Applying Lemma 19, we have

$$\sum_{\substack{\mathfrak{b}|M \\ \mathfrak{N}\mathfrak{b} > 1}} \frac{\psi_R(\mathfrak{b})}{\phi_{\mathbf{K}}(\mathfrak{b})} \leq \sum_{\substack{\mathfrak{b}|M \\ \mathfrak{N}\mathfrak{b} > 1}} \binom{\omega(\mathfrak{b})-1}{R} \frac{1}{\phi(\mathfrak{b})} \leq \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \binom{m-1}{R} \sum_{\substack{\mathfrak{b}|M \\ \mathfrak{N}\mathfrak{b} > 1 \\ \omega(\mathfrak{b})=m}} \frac{1}{\phi(\mathfrak{b})},$$

where $\pi_{\mathbf{K}}(x)$ be the number of prime ideals in $\mathcal{O}_{\mathbf{K}}$ with norm less than or equal to x . The last quantity is equal to

$$\begin{aligned} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \binom{m-1}{R} \left(\sum_{\mathfrak{p}|M} \frac{1}{\phi(\mathfrak{p})} \right)^m \frac{1}{m!} &\leq \frac{1}{R!} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \frac{1}{(m-R)!} \left(\sum_{\mathfrak{p}|M} \frac{1}{\mathfrak{N}\mathfrak{p}-1} \right)^m \\ &\leq \frac{1}{R!} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \frac{1}{(m-R)!} \left(\sum_{p < 40^n} \frac{n}{2(p^2-1)} \right)^m \\ &\leq \frac{1}{R!} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \frac{1}{(m-R)!} \left(\sum_{p < 40^n} \frac{n}{p^2} \right)^m. \end{aligned}$$

We have, however, that

$$\sum_{p < 40^n} \frac{1}{p^2} \leq \int_1^{40^n} \frac{dt}{t^2} \leq 1.$$

Therefore, we get

$$\sum_{\substack{\mathfrak{b}|M \\ \mathfrak{N}\mathfrak{b} > 1}} \frac{\psi_R(\mathfrak{b})}{\phi(\mathfrak{b})} \leq \frac{1}{R!} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \frac{n^m}{(m-R)!} \leq \frac{n^R}{R!} \sum_{m=R+1}^{\pi_{\mathbf{K}}(40^n)} \frac{n^{m-R}}{(m-R)!} \leq \frac{n^R e^n}{R!} \leq \frac{e^{n+R \log n}}{R!}.$$

We know that $R! \geq (\frac{R}{e})^R$ and this implies that

$$\sum_{\substack{\mathfrak{b}|M \\ \mathfrak{N}\mathfrak{b} > 1}} \frac{\psi_R(\mathfrak{b})}{\phi(\mathfrak{b})} \leq \frac{e^{n+R \log n}}{R^R} e^R \leq \exp(n + R - R \log R + R \log n).$$

We select

$$(19) \quad R = 2[5n \log n] + 1$$

so that

$$(20) \quad \exp\left(n - R \log \frac{R}{en}\right) \leq \exp\left(n - (9n \log n) \log \frac{9 \log n}{e}\right) \leq \exp(-4n) \leq 50^{-n}.$$

Combining (18) and (20), we get

$$\sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} \frac{\mu(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} \geq \prod_{\mathfrak{p}|M} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) (1 - 50^{-n}) \geq \left(\frac{6}{\pi^2}\right)^{\frac{n}{2}} (1 - 10^{-n}),$$

since all the prime ideals dividing M have degree at least 2. This takes care of the main term. Concerning the error term for $S(M)$, we first notice that the number of rational primes less than D is at most $3D/(2 \log D)$ (see [25, (3.6)]). This implies that the number of prime ideals in $\mathcal{O}_{\mathbf{K}}$ of norms at most D and of degree ≥ 2 is at most $[n/2]3D/(2 \log D)$ which is not more than $2D/3$. Whence

$$\sum_{\substack{\mathfrak{d}|M, \\ \omega(\mathfrak{d}) \leq R}} 1 \leq (2D/3)^R.$$

We have thus reached the inequality

$$S(M) \geq Y(1 - 10^{-n})(6/\pi^2)^{n/2} - (2D/3)^R \left(E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \left(\frac{X}{\mathfrak{N}\mathfrak{q}}\right)^{1-\frac{1}{n}} + n^{8n} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathfrak{q}) \right).$$

We get

$$\begin{aligned} \frac{1}{Y} (2D/3)^R E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \left(\frac{X}{\mathfrak{N}\mathfrak{q}}\right)^{1-\frac{1}{n}} &\leq 40^{12n^2 \log n} \frac{h_{\mathbf{K},\mathfrak{q}}}{\alpha_{\mathbf{K}} \phi(\mathfrak{q})} \left(\frac{F(\mathfrak{q}) \mathfrak{N}\mathfrak{q}}{X}\right)^{\frac{1}{n}} E(\mathbf{K}) \log(3F(\mathfrak{q}))^n \\ &\leq 40^{12n^2 \log n} \frac{2^{r_1} h_{\mathbf{K}}}{\alpha_{\mathbf{K}}} \left(\frac{F(\mathfrak{q}) \mathfrak{N}\mathfrak{q}}{X}\right)^{\frac{1}{n}} E(\mathbf{K}) \log(3F(\mathfrak{q}))^n \\ &\leq 40^{12n^2 \log n} \frac{100}{9} \sqrt{|d_{\mathbf{K}}|} \left(\frac{F(\mathfrak{q}) \mathfrak{N}\mathfrak{q}}{X}\right)^{\frac{1}{n}} E(\mathbf{K}) \end{aligned}$$

by Lemma 10 and 15. Our lower bound on X ensures that this upper bound is $\leq 1/10^n$. The second error term is treated similarly. We write

$$\begin{aligned} \frac{D^R n^{8n}}{Y} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathfrak{q}) &\leq \frac{40^{12n^2 \log n} n^{8n} \mathfrak{N}\mathfrak{q} h_{\mathbf{K},\mathfrak{q}}}{\alpha_{\mathbf{K}} X \phi(\mathfrak{q})} \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} F(\mathfrak{q}) \\ &= \frac{40^{12n^2 \log n} n^{8n} \mathfrak{N}\mathfrak{q} h_{\mathbf{K},\mathfrak{q}}}{X h_{\mathbf{K}} \phi(\mathfrak{q}) 2^{r_1} (2\pi)^{r_2}} \sqrt{|d_{\mathbf{K}}|} \frac{2^{r_1} \phi(\mathfrak{q}) h_{\mathbf{K}}}{h_{\mathbf{K},\mathfrak{q}}} \leq 40^{12n^2 \log n} n^{8n} \frac{\mathfrak{N}\mathfrak{q}}{X} \sqrt{|d_{\mathbf{K}}|} \end{aligned}$$

by using (5) and the definition of $F(\mathfrak{q})$. The lower bound for X again implies that this quantity is at most $1/10^n$. Combining all our estimates, we find that

$$S/Y \geq (1 - 10^{-n})(6/\pi^2)^{n/2} - \frac{1}{10^n} - \frac{1}{10^n} - \frac{1}{2^n} \geq \frac{1}{4}(6/\pi^2)^{n/2}$$

as required. This concludes the proof. \square

Products of degree one primes in ray classes modulo \mathfrak{q} , analytic approach.

Lemma 23. *Let \mathfrak{b} be an integral ideal co-prime to \mathfrak{q} . When $X \geq 10^{25n_{\mathbf{K}}} n_{\mathbf{K}}^{7n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^3$, we have*

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{a}] = [\mathfrak{b}]}}^{\mathfrak{b}} w_0 \left(\frac{\mathfrak{N}(\mathfrak{a})}{X} \right) \geq \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) \check{w}_0(1) X}{2(1.3)^{n_{\mathbf{K}}} \mathfrak{N}\mathfrak{q} |H_{\mathfrak{q}}(\mathbf{K})|}.$$

Proof. On calling S the sum on the left hand side, the orthogonality of characters readily gives us that (recall (13))

$$\begin{aligned} S &= \frac{1}{|H_{\mathfrak{q}}(\mathbf{K})|} \sum_{\chi \in \hat{H}_{\mathfrak{q}}(\mathbf{K})} \bar{\chi}(\mathfrak{b}) \sum_{(\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}^{\mathfrak{b}} w_0 \left(\frac{\mathfrak{N}(\mathfrak{a})}{X} \right) \chi(\mathfrak{a}), \\ &= \frac{1}{|H_{\mathfrak{q}}(\mathbf{K})|} \sum_{\chi \in \hat{H}_{\mathfrak{q}}(\mathbf{K})} \frac{\bar{\chi}(\mathfrak{b})}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s, \chi) \check{w}_0(s) X^s ds. \end{aligned}$$

On using Lemma 20, we find that

$$\begin{aligned} S &= \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) J(1, \chi_{0, \mathfrak{q}})}{\mathfrak{N}\mathfrak{q}} \frac{\check{w}_0(1) X}{|H_{\mathfrak{q}}(\mathbf{K})|} + \frac{1}{|H_{\mathfrak{q}}(\mathbf{K})|} \sum_{\chi \in \hat{H}_{\mathfrak{q}}(\mathbf{K})} \frac{\bar{\chi}(\mathfrak{b})}{2\pi i} \int_{\frac{1+\varepsilon}{2}-i\infty}^{\frac{1+\varepsilon}{2}+i\infty} F(s, \chi) \check{w}_0(s) X^s ds \\ &= \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) J(1, \chi_{0, \mathfrak{q}})}{\mathfrak{N}\mathfrak{q}} \frac{\check{w}_0(1) X}{|H_{\mathfrak{q}}(\mathbf{K})|} + O^* \left(5\zeta(1+\varepsilon)^{\frac{3n_{\mathbf{K}}}{2}} (|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q})^{\frac{1+\varepsilon}{4}} \theta(\mathfrak{q}) X^{\frac{1+\varepsilon}{2}} M^*(\varepsilon, 2) \right) \end{aligned}$$

where M^* is as defined in Lemma 13. Proceeding as in (15), we find that

$$(21) \quad J(1, \chi_{0, \mathfrak{q}}) \geq \zeta(2)^{-n_{\mathbf{K}}/2}$$

while Lemma 15 provides us with a lower bound for $\alpha_{\mathbf{K}}$ and, combined with (7), an upper bound for $|H_{\mathfrak{q}}(\mathbf{K})|$. This together with Lemma 7 tells us that the main term above is at least, with the notation $n = n_{\mathbf{K}}$,

$$\frac{9X}{33500 \sqrt{\zeta(2)^n (\pi^2/5)^n \sqrt{n} |d_{\mathbf{K}}|^{5/4} \mathfrak{N}\mathfrak{q}}} \geq \frac{X}{3723 (\pi^2/5)^{3n/2} \sqrt{n} |d_{\mathbf{K}}|^{5/4} \mathfrak{N}\mathfrak{q}}.$$

This is larger than twice the above error term provided we have

$$(22) \quad X^{\frac{1-\varepsilon}{2}} \geq 83 \cdot 10^9 n^{\frac{7}{2}} (159n)^n \zeta(1+\varepsilon)^{\frac{3n}{2}} |d_{\mathbf{K}}|^{\frac{6+\varepsilon}{4}} \theta(\mathfrak{q}) \mathfrak{N}\mathfrak{q}^{\frac{5+\varepsilon}{4}}.$$

We select $\varepsilon = 1/10$. The previous inequality is implied by

$$X \geq 19 \cdot 10^{23} n^{\frac{70}{9}} (5476n)^{\frac{20n}{9}} |d_{\mathbf{K}}|^{\frac{61}{18}} \theta(\mathfrak{q})^{\frac{20}{9}} \mathfrak{N}\mathfrak{q}^{\frac{17}{6}}.$$

Now $(\frac{\sqrt{p}}{\sqrt{p}-1})^{20/9} \leq p^{1/6}$ when $p > 19$, while $\prod_{p \leq 19} \theta(p)^{20/9} / p^{1/6} \leq 1200$. As a conclusion, we derive that $\theta(\mathfrak{q})^{20/9} \leq 1200^n \mathfrak{N}\mathfrak{q}^{1/6}$. Some numerical computations end the proof of our lemma. \square

Proof of Theorems 4 and 5. Theorem 4 follows as an easy consequence of Lemma 23, and similarly, Theorem 5 follows from Lemma 22. \square

Degree one primes in quadratic ray subgroups modulo \mathfrak{q} .

Lemma 24. *Let χ be quadratic character on $H_{\mathfrak{q}}(\mathbf{K})$. We have*

$$L_{\mathfrak{q}}(1, \chi) \geq \frac{9 \cdot 2^{2n_{\mathbf{K}}}}{100\alpha_{\mathbf{K}}|d_{\mathbf{K}}|\sqrt{\mathfrak{N}(\mathfrak{q})}} \quad \text{and} \quad F(1, \chi) \geq \frac{9 \cdot 2^{2n_{\mathbf{K}}}}{100\alpha_{\mathbf{K}}|d_{\mathbf{K}}|\sqrt{\zeta(2)^{n_{\mathbf{K}}}\mathfrak{N}(\mathfrak{q})}}$$

where $d_{\mathbf{K}}$ is the discriminant of \mathbf{K} , $\alpha_{\mathbf{K}}$ is the residue of the Dedekind zeta function at $s = 1$.

This lemma improves on [7, Lemma 2] of Hinz and Lodemann in two ways: the dependence in the base field is explicit and the lower bound is in $1/\sqrt{\mathfrak{N}\mathfrak{q}}$ instead of $1/[\sqrt{\mathfrak{N}\mathfrak{q}}(\log \mathfrak{N}\mathfrak{q})^2]$.

Proof. We note that the product $L_{\mathfrak{f}}(s, \chi)\zeta_{\mathbf{K}}(s)$, where $\mathfrak{f}|\mathfrak{q}$ is the Dedekind zeta function of a quadratic extension \mathbf{M} of \mathbf{K} . By (5), we have

$$L_{\mathfrak{f}}(1, \chi)\alpha_{\mathbf{K}} = \alpha_{\mathbf{M}} = \frac{2^{r_1}(2\pi)^{r_2}h_{\mathbf{M}}R_{\mathbf{M}}}{|\mu_{\mathbf{M}}|\sqrt{|d_{\mathbf{M}}|}},$$

where r_1 and $2r_2$ are the number of real and complex embeddings of \mathbf{M} . Since this field is of degree $2n_{\mathbf{K}}$ over \mathbb{Q} , Lemma 15 gives us

$$(23) \quad \alpha_{\mathbf{M}} \geq \frac{9 \cdot 2^{n_{\mathbf{M}}}}{100\sqrt{|d_{\mathbf{M}}|}}.$$

Finally, since the extension \mathbf{M}/\mathbf{K} has conductor with finite part \mathfrak{f} and is a quadratic extension, by the conductor discriminant formula ([19, Chapter VII, Point (11.9)] by Neukirch) the relative discriminant of \mathbf{M}/\mathbf{K} is also \mathfrak{f} . However we also have that ([19, Chapter III, Corollary (2.10)]) $\sqrt{|d_{\mathbf{M}}|} = |d_{\mathbf{K}}|\sqrt{\mathfrak{N}(\mathfrak{f})}$. This gives that

$$|L_{\mathfrak{q}}(1, \chi)| \geq |L_{\mathfrak{f}}(1, \chi)| \prod_{\substack{\mathfrak{p}|\mathfrak{q} \\ \mathfrak{p} \nmid \mathfrak{f}}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right).$$

Since

$$\frac{\sqrt{\mathfrak{N}\mathfrak{q}}}{\sqrt{\mathfrak{N}\mathfrak{f}}} \prod_{\substack{\mathfrak{p}|\mathfrak{q} \\ \mathfrak{p} \nmid \mathfrak{f}}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) \geq \frac{\sqrt{\mathfrak{N}\mathfrak{q}} \prod_{\mathfrak{p}|\mathfrak{q}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)}{\sqrt{\mathfrak{N}\mathfrak{f}} \prod_{\mathfrak{p}|\mathfrak{f}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)} \geq 1,$$

we get the desired bound. To extend it to $F(1, \chi)$, we notice that

$$(24) \quad \left| \frac{L_{\mathfrak{q}}(1, \chi)}{F(1, \chi)} \right| \leq \prod_{\mathfrak{p}|\mathfrak{q}}^{\#} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1} \leq \zeta(2)^{\frac{n_{\mathbf{K}}}{2}}.$$

□

Lemma 25. *Let χ be a quadratic character on $H_{\mathfrak{q}}(\mathbf{K})$. We have*

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ (\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}}^{\mathfrak{b}} (\mathbf{1} \star \chi)(\mathfrak{a}) w_0 \left(\frac{\mathfrak{N}(\mathfrak{a})}{X} \right) > \frac{X}{27\sqrt{\mathfrak{N}\mathfrak{q}}|d_{\mathbf{K}}|^2} \frac{\phi(\mathfrak{q})}{\mathfrak{N}\mathfrak{q}}$$

provided that $X \geq 8 \cdot (10^{31}n_{\mathbf{K}}^7)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^2$.

Proof. Let us denote the sum on the left hand side by $S_1(w_0)$ and the principal Hecke character modulo \mathfrak{q} by $\chi_{0,\mathfrak{q}}$. By mimicking the proof of Lemma 23, we find that

$$\begin{aligned} S_1(w_0) &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s, \chi_{0,\mathfrak{q}}) F(s, \chi) \check{w}_0(s) X^s ds \\ &= \alpha_{\mathbf{K},\mathfrak{q}} F(1, \chi) \check{w}_0(1) X + \frac{1}{2\pi i} \int_{\frac{1+\varepsilon}{2}-i\infty}^{\frac{1+\varepsilon}{2}+i\infty} F(s, \chi_{0,\mathfrak{q}}) F(s, \chi) \check{w}_0(s) X^s ds \\ &= \alpha_{\mathbf{K},\mathfrak{q}} F(1, \chi) \check{w}_0(1) X + O^*\left(27\zeta(1+\varepsilon)^{3n\kappa} |d_{\mathbf{K}}|^{\frac{1+\varepsilon}{2}} \mathfrak{N}\mathfrak{q}^{\frac{1+\varepsilon}{4}} \theta(\mathfrak{q})^2 X^{\frac{1+\varepsilon}{2}} M^*(\varepsilon, 1)\right), \end{aligned}$$

where $\alpha_{\mathbf{K},\mathfrak{q}} = \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) J(1, \chi_{0,\mathfrak{q}})}{\mathfrak{N}\mathfrak{q}}$ is the residue of $F(s, \chi_{0,\mathfrak{q}})$ at $s = 1$ and M^* is as defined in Lemma 13. We also have

$$\alpha_{\mathbf{K},\mathfrak{q}} = \alpha_{\mathbf{K}} J(1, \chi_{0,\mathfrak{q}}) \frac{\phi(\mathfrak{q})}{\mathfrak{N}\mathfrak{q}} \geq \frac{\phi(\mathfrak{q})}{\mathfrak{N}\mathfrak{q}} \frac{\alpha_{\mathbf{K}}}{\sqrt{\zeta(2)^{n\kappa}}}$$

where we have used the inequality (21). The above is valid for a general smoothing function w but we restrict ourselves to $w = w_0$.

Applying Lemma 7 and using yet again the notation n for $n_{\mathbf{K}}$, we get

$$S_1(w_0) \geq \frac{9(4/\zeta(2))^n X}{500 \cdot \sqrt{n} \mathfrak{N}\mathfrak{q} |d_{\mathbf{K}}|} \frac{\phi(\mathfrak{q})}{\mathfrak{N}\mathfrak{q}} - 324 \zeta(1+\varepsilon)^{3n} (57n)^{n+3} |d_{\mathbf{K}}|^{\frac{1+\varepsilon}{2}} \mathfrak{N}\mathfrak{q}^{\frac{1+\varepsilon}{4}} \theta(\mathfrak{q})^2 X^{\frac{1+\varepsilon}{2}}$$

and the first summand is larger than twice the second one provided that

$$X^{\frac{1-\varepsilon}{2}} \geq 7 \cdot 10^9 n^{\frac{7}{2}} (24n)^n \zeta(1+\varepsilon)^{3n} |d_{\mathbf{K}}|^{\frac{3+\varepsilon}{2}} \theta(\mathfrak{q})^2 \frac{\mathfrak{N}\mathfrak{q}}{\phi(\mathfrak{q})} \mathfrak{N}\mathfrak{q}^{\frac{3+\varepsilon}{4}}.$$

We select $\varepsilon = 1/10$. The previous inequality is implied by

$$X \geq 8 \cdot 10^{21} \cdot n^{\frac{70}{9}} (31944n)^{\frac{20n}{9}} |d_{\mathbf{K}}|^{\frac{31}{9}} \left(\theta(\mathfrak{q})^2 \frac{\mathfrak{N}\mathfrak{q}}{\phi(\mathfrak{q})} \right)^{\frac{20}{9}} \mathfrak{N}\mathfrak{q}^{\frac{31}{18}}.$$

Since $\theta(\mathfrak{q})^2 \frac{\mathfrak{N}\mathfrak{q}}{\phi(\mathfrak{q})} = \prod_{\mathfrak{p}|\mathfrak{q}} \frac{\mathfrak{N}\mathfrak{p}^2}{(\mathfrak{N}\mathfrak{p}-1)(\sqrt{\mathfrak{N}\mathfrak{p}}-1)^2}$ and

$$\left(\frac{p^2}{(p-1)(\sqrt{p}-1)^2} \right)^{20/9} \leq p^{1/6} \text{ when } p > 56 \quad \text{and} \quad \prod_{p \leq 56} \frac{1}{p^{1/6}} \left(\frac{p^2}{(p-1)(\sqrt{p}-1)^2} \right)^{20/9} \leq 9 \cdot 10^9,$$

we have

$$X \geq 8 \cdot (10^{31} n^7)^n |d_{\mathbf{K}}|^{\frac{31}{9}} \mathfrak{N}\mathfrak{q}^{\frac{34}{18}}.$$

We simplify the final statement by noticing that $\frac{9(4/\zeta(2))^n}{1000\sqrt{n}} \geq \frac{1}{27}$. □

We may now complete the proof of Theorem 6.

Proof of Theorem 6. Suppose that the theorem is not true. Then every degree one prime ideal \mathfrak{p} co-prime to \mathfrak{q} with norm at most $X = \mathfrak{N}\mathfrak{p} \leq 8 \cdot (10^{31} n^7)^{n\kappa} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^2$ satisfies the property that $\chi(\mathfrak{p}) = -1$. Then for every non square-full ideal $\mathfrak{a} (\neq \mathcal{O}_{\mathbf{K}})$ which decomposes only as a product of prime ideals of degree one and of norm at most X , we have $(1 \star \chi)(\mathfrak{a}) = 0$. By Lemma 25, we get a contradiction, and this completes the proof of the theorem. □

5. SELBERG SIEVE FOR NUMBER FIELDS IN SIEVE DIMENSION ONE

In this section, we derive some lemmas which are required to prove the number field analogues of two versions of the Brun-Titchmarsh Theorem. Throughout this section, z will denote a real number greater than one and all ideals considered are integral ideals. For a fixed integral ideal \mathfrak{q} , we define

$$(25) \quad V(z) = \prod_{\substack{\mathfrak{p} | \mathcal{P}(z) \\ (\mathfrak{p}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \mathfrak{p} \quad \text{where} \quad \mathcal{P}(z) = \prod_{\substack{\mathfrak{p} \\ \Re(\mathfrak{p}) \leq z}} \mathfrak{p}.$$

Recall the definition of the Möbius function in (12). Further, for any ideal \mathfrak{c} of $\mathcal{O}_{\mathbf{K}}$, we define

$$(26) \quad G_{\mathfrak{c}}(z) = \sum_{\substack{0 < \Re(\mathfrak{a}) \leq z, \\ (\mathfrak{a}, \mathfrak{c}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{\phi(\mathfrak{a})}, \quad G(z) = G_{\mathcal{O}_{\mathbf{K}}}(z).$$

For some fixed integral ideal \mathfrak{q} , we further set

$$(27) \quad \lambda_{\mathfrak{c}}(\mathfrak{q}) = \mu(\mathfrak{c}) \frac{\Re(\mathfrak{c}) G_{\mathfrak{c}\mathfrak{q}}\left(\frac{z}{\Re(\mathfrak{c})}\right)}{\phi(\mathfrak{c}) G_{\mathfrak{q}}(z)}, \quad \lambda_{\mathfrak{c}} = \lambda_{\mathfrak{c}}(\mathcal{O}_{\mathbf{K}}).$$

We set $\lambda_{\mathfrak{c}}(\mathfrak{q}) = 0$ whenever $\Re(\mathfrak{c}) > z$ or $(\mathfrak{c}, \mathfrak{q}) \neq \mathcal{O}_{\mathbf{K}}$. In this set-up, we recall the following two lemmas from [26] by Schaal. The reader may also refer to the beginning of Section 4 of the paper by Debaene [2].

Lemma 26. *For any ideal $\mathfrak{c} \mid V(z)$, one has $|\lambda_{\mathfrak{c}}(\mathfrak{q})| \leq 1$.*

Lemma 27. *We have $G_{\mathfrak{q}}(z)^{-1} = \sum_{\substack{\mathfrak{c}_1, \mathfrak{c}_2 \\ \mathfrak{c}_i \mid V(z)}} \frac{\lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q})}{\Re[\mathfrak{c}_1, \mathfrak{c}_2]}$.*

Proof. In [26], the required definitions are in (1.25) and in (3.1) except that we do not consider a more severe sieving restriction, so that $\rho = x$ and this latter quantity is called z in our setting. The bound for $|\lambda_{\mathfrak{c}}(\mathfrak{q})|$ is given in (3.2). The expression for $G_{\mathfrak{q}}(z)$ is contained in the last displayed equations at the bottom of page 293 with the remark of eq. (3.3). \square

 5.1. A lower bound for $G_{\mathfrak{q}}(z)$.

Lemma 28. *When $y > 0$ and k is a positive integer, we have*

$$\max(0, 1 - y)^k = \frac{1}{2\pi i} \int_{\Re s = 2} y^{-s} \frac{k! ds}{s(s+1) \cdots (s+k)}.$$

The following theorem gives a lower bound for $G_{\mathfrak{q}}(z)$.

Theorem 29. *When $z \geq (10^6 n_{\mathbf{K}})^{4n_{\mathbf{K}}} |d_{\mathbf{K}}|^3$, we have $G_{\mathfrak{q}}(z) \geq \alpha_{\mathbf{K}} \frac{\phi(\mathfrak{q})}{\Re(\mathfrak{q})} \log \frac{z}{e^2 n_{\mathbf{K}} \sqrt{|d_{\mathbf{K}}|}}$.*

This is a version of Lemma 5 of [27] where the dependence in the field is explicit. In Lemma 14 of [2], a similar result is proved, but it relies on $R_{\mathbf{K}} h_{\mathbf{K}}$ when we prefer to rely on $d_{\mathbf{K}}$.

Proof. We first remove the dependence in \mathfrak{q} by using the following inequality from [27], page 266, obtained by combining the points (a) and (b) therein:

$$(28) \quad G_{\mathfrak{q}}(z) \geq \frac{\phi(\mathfrak{q})}{\Re(\mathfrak{q})} \sum_{0 < \Re(\mathfrak{a}) \leq z} \frac{1}{\Re(\mathfrak{a})}.$$

We note in passing that it is straightforward to adapt the inequality [30, (1.3)] by Van Lint and Richert to prove that $G_q(z) \geq \frac{\phi(q)}{\mathfrak{N}(q)} G(z)$, which is more refined than (28). To handle right hand side of (28), we use Lemma 28 with $y = \mathfrak{N}(\mathfrak{a})/z$ to get

$$\begin{aligned} \sum_{0 < \mathfrak{N}(\mathfrak{a}) \leq z} \frac{1}{\mathfrak{N}(\mathfrak{a})} &\geq \sum_{\substack{\mathfrak{a} \\ \mathfrak{a} \neq 0}} \frac{1}{\mathfrak{N}(\mathfrak{a})} \max \left(0, \left(1 - \frac{\mathfrak{N}(\mathfrak{a})}{z} \right) \right)^k \\ &= \frac{1}{2\pi i} \sum_{\substack{\mathfrak{a} \\ \mathfrak{a} \neq 0}} \frac{1}{\mathfrak{N}(\mathfrak{a})} \int_{\Re s=2} \left(\frac{\mathfrak{N}(\mathfrak{a})}{z} \right)^{-s} \frac{k!}{s(s+1) \cdots (s+k)} ds. \end{aligned}$$

Since we are in the region of absolute convergence of $\zeta_{\mathbf{K}}(s)$, this leads to

$$\sum_{0 < \mathfrak{N}(\mathfrak{a}) \leq z} \frac{1}{\mathfrak{N}(\mathfrak{a})} \geq \frac{1}{2\pi i} \int_{\Re s=2} \zeta_{\mathbf{K}}(1+s) \frac{k!z^s}{s(s+1) \cdots (s+k)} ds.$$

We note that the integrand has a double pole at $s = 0$ and move the line of integration to $\Re s = -1/4$. In the neighborhood of $s = 0$, we find that

$$s^2 \zeta_{\mathbf{K}}(1+s) \frac{k!z^s}{s(s+1) \cdots (s+k)} = \frac{\alpha_{\mathbf{K}} k! z^s}{(s+1) \cdots (s+k)} + \frac{\alpha_{\mathbf{K}} \gamma_{\mathbf{K}} k! s z^s}{(s+1) \cdots (s+k)} + O(s^2).$$

The residue at $s = 0$ is thus given by

$$r = \lim_{s \rightarrow 0} \frac{d}{ds} \left(s^2 \zeta_{\mathbf{K}}(1+s) \frac{k!z^s}{s(s+1) \cdots (s+k)} \right) = \alpha_{\mathbf{K}} \left(\gamma_{\mathbf{K}} + \log z - \sum_{\ell=1}^k \frac{1}{\ell} \right).$$

The remaining integral is now

$$I = \frac{1}{2\pi i} \int_{\Re s=-\frac{1}{4}} \frac{k! \zeta_{\mathbf{K}}(1+s) z^s}{s(s+1) \cdots (s+k)} ds.$$

Further Lemma 14 with $\varepsilon = 1/4$ gives us

$$|\zeta_{\mathbf{K}}(3/4 + it)| \leq 27 \zeta(5/4)^{n_{\mathbf{K}}} (|d_{\mathbf{K}}| (7/4 + |t|)^{n_{\mathbf{K}}})^{1/4}.$$

This gives

$$|I| \leq 5 \zeta(5/4)^{n_{\mathbf{K}}} k! \left(\frac{|d_{\mathbf{K}}|}{z} \right)^{1/4} \int_{\Re s=-\frac{1}{4}} \frac{(7/4 + |t|)^{\frac{n_{\mathbf{K}}}{4}} |ds|}{|s(s+1) \cdots (s+k)|}.$$

To estimate the integral, say I_0 , choose $T > 0$, $k > n_{\mathbf{K}}/4$ and write

$$\begin{aligned} \frac{1}{2} I_0 &= \int_{t=0, \Re s=-\frac{1}{4}}^T \frac{(7/4 + t)^{\frac{n_{\mathbf{K}}}{4}} |ds|}{|s(s+1) \cdots (s+k)|} + \int_{t=T, \Re s=-\frac{1}{4}}^{\infty} \frac{(7/4 + t)^{\frac{n_{\mathbf{K}}}{4}} |ds|}{|s(s+1) \cdots (s+k)|} \\ &\leq \frac{8}{(k-1)!} \int_0^T \left(\frac{7}{4} + t \right)^{\frac{n_{\mathbf{K}}}{4}} dt + \int_T^{\infty} \left(\frac{7}{4t} + 1 \right)^{\frac{n_{\mathbf{K}}}{4}} \frac{dt}{t^{k+1-\frac{n_{\mathbf{K}}}{4}}} \\ &\leq \frac{32}{(k-1)!(4+n_{\mathbf{K}})} (7/4 + T)^{1+\frac{n_{\mathbf{K}}}{4}} + \left(\frac{7}{4T} + 1 \right)^{\frac{n_{\mathbf{K}}}{4}} \frac{1}{(k - \frac{n_{\mathbf{K}}}{4}) T^{k-\frac{n_{\mathbf{K}}}{4}}}. \end{aligned}$$

Hence, with $T = k/e$,

$$\begin{aligned} k! I_0 &\leq (7/4 + k/e)^{1+\frac{n_{\mathbf{K}}}{4}} \left(\frac{64k}{4+n_{\mathbf{K}}} + \frac{8 \cdot k!}{(4k - n_{\mathbf{K}})(k/e)^k} \right) \\ &\leq (7/4 + k/e)^{1+\frac{n_{\mathbf{K}}}{4}} \left(\frac{64k}{4+n_{\mathbf{K}}} + \frac{8 \cdot \sqrt{2\pi k} e^{\frac{1}{12k}}}{(4k - n_{\mathbf{K}})} \right) \end{aligned}$$

by using the explicit Stirling Formula recalled in (4). Therefore, with the choice $k = n_{\mathbf{K}}$, we find that

$$\begin{aligned} |I| &\leq \left(5^{1/n_{\mathbf{K}}} \zeta(5/4) \left(\frac{7}{4n_{\mathbf{K}}} + \frac{1}{e}\right)^{\frac{1}{n_{\mathbf{K}}} + \frac{1}{4}} \left(\frac{64n_{\mathbf{K}}}{4+n_{\mathbf{K}}} + \frac{8 \cdot \sqrt{2\pi} e^{\frac{1}{12n_{\mathbf{K}}}}}{3\sqrt{n_{\mathbf{K}}}}\right)^{1/n_{\mathbf{K}}} n_{\mathbf{K}}^{\frac{1}{n_{\mathbf{K}}} + \frac{1}{4}}\right)^{n_{\mathbf{K}}} \left(\frac{|d_{\mathbf{K}}|}{z}\right)^{\frac{1}{4}} \\ &\leq (10^3 n_{\mathbf{K}})^{n_{\mathbf{K}}} \left(\frac{|d_{\mathbf{K}}|}{z}\right)^{\frac{1}{4}}. \end{aligned}$$

Whence

$$\begin{aligned} \frac{\mathfrak{N}_{\mathbf{q}}}{\phi(\mathbf{q})} G_{\mathbf{q}}(z) &\geq \alpha_{\mathbf{K}} (\gamma_{\mathbf{K}} + \log z - \log n_{\mathbf{K}} - 1) - (10^3 n_{\mathbf{K}})^{n_{\mathbf{K}}} \left(\frac{|d_{\mathbf{K}}|}{z}\right)^{\frac{1}{4}} \\ &\geq \alpha_{\mathbf{K}} \left(\log z - \log \sqrt{|d_{\mathbf{K}}|} - \log n_{\mathbf{K}} - 1 - (10^3 n_{\mathbf{K}})^{n_{\mathbf{K}}} \frac{100 \sqrt{|d_{\mathbf{K}}|}}{9 \cdot 2^{n_{\mathbf{K}}}} \left(\frac{|d_{\mathbf{K}}|}{z}\right)^{\frac{1}{4}} \right) \end{aligned}$$

by Lemma 15 and the inequality $\gamma_{\mathbf{K}} \geq -\log \sqrt{|d_{\mathbf{K}}|}$ from (6). This completes the proof of Theorem 29. \square

5.2. Controlling the error term in Selberg's sieve. Our first lemma borrows from [6] by Halberstam and Richert.

Lemma 30. *When $x \geq 1$ and $\alpha \in [0, 1)$, we have $\sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^{\alpha}} \leq \frac{1.02 n_{\mathbf{K}} x^{1-\alpha}}{1-\alpha}$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ be prime ideals of $\mathcal{O}_{\mathbf{K}}$ lying over a rational prime p . We then have

$$\sum_{1 \leq i \leq g} e_i \log \mathfrak{N}\mathfrak{p}_i = n_{\mathbf{K}} \log p,$$

where $e_i \geq 1$ is the ramification index of \mathfrak{p}_i above p . This implies that $\sum_{1 \leq i \leq g} \log \mathfrak{N}\mathfrak{p}_i \leq n_{\mathbf{K}} \log p$. Inequality (11) concludes the proof of the lemma when $\alpha = 0$. For the general case, we use summation by parts to write

$$\begin{aligned} \sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^{\alpha}} &= \alpha \int_1^x \sum_{\mathfrak{N}\mathfrak{p} \leq t} \log \mathfrak{N}\mathfrak{p} \frac{dt}{t^{1+\alpha}} + \sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\log \mathfrak{N}\mathfrak{p}}{x^{\alpha}} \\ &\leq 1.02 n_{\mathbf{K}} \left(\alpha \int_1^x \frac{dt}{t^{\alpha}} + x^{1-\alpha} \right) \leq 1.02 n_{\mathbf{K}} \frac{x^{1-\alpha}}{1-\alpha} \end{aligned}$$

as required. \square

Lemma 31. *When $x \geq 1$ and $\alpha \in [0, 1)$, we have*

$$\sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \mu^2(\mathfrak{a}) \left(\frac{x}{\mathfrak{N}\mathfrak{a}}\right)^{\alpha} \leq \frac{(1 + 1.02 n_{\mathbf{K}})x}{(1-\alpha)(1+\log x)} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}.$$

Proof. We set $S(x) = \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \mu^2(\mathfrak{a}) (x/\mathfrak{N}\mathfrak{a})^{\alpha}$. We use $\log y \leq (y^{1-\alpha} - 1)/(1-\alpha)$ when $y > 0$ and readily find that

$$\begin{aligned} S(x) \log x &= \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \mu^2(\mathfrak{a}) \left(\frac{x}{\mathfrak{N}\mathfrak{a}}\right)^{\alpha} \log \frac{x}{\mathfrak{N}\mathfrak{a}} + \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \mu^2(\mathfrak{a}) \left(\frac{x}{\mathfrak{N}\mathfrak{a}}\right)^{\alpha} \log \mathfrak{N}\mathfrak{a} \\ &\leq \frac{x}{1-\alpha} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} - S(x) + \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \mu^2(\mathfrak{a}) \left(\frac{x}{\mathfrak{N}\mathfrak{a}}\right)^{\alpha} \sum_{\mathfrak{p}|\mathfrak{a}} \log \mathfrak{N}\mathfrak{p} \\ &\leq \frac{x}{1-\alpha} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} - S(x) + \sum_{\mathfrak{N}\mathfrak{p} \leq x} \log \mathfrak{N}\mathfrak{p} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x/\mathfrak{N}\mathfrak{p}} \mu^2(\mathfrak{a}) \left(\frac{x/\mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{a}}\right)^{\alpha}. \end{aligned}$$

We invert the summation in the last term and then appeal to Lemma 30 to get

$$S(x)(1 + \log x) \leq \frac{x}{1 - \alpha} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} + \frac{1.02n_{\mathbf{K}} x}{1 - \alpha} \sum_{0 < \mathfrak{N}\mathfrak{a} \leq x} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}$$

and our lemma follows readily. \square

Theorem 32. *When $z \geq 1$ and $\alpha \in [0, 1]$, we have*

$$\sum_{\mathfrak{e}|V(z)} \frac{|\lambda_{\mathfrak{e}}(\mathfrak{q})|}{\mathfrak{N}\mathfrak{e}^{\alpha}} \leq \frac{3.1 n_{\mathbf{K}} z^{1-\alpha}}{(1 - \alpha)(2 + \log z)} c_1(\alpha)^{n_{\mathbf{K}}} + z^{3(1-\alpha)/4} c_2(\alpha)^{n_{\mathbf{K}}},$$

where

$$c_1(\alpha) = \prod_p \left(1 + \frac{1 + p^{\alpha}}{(p - 1)p} \right), \quad c_2(\alpha) = \prod_p \left(1 + \frac{1 + p^{\alpha}}{(p - 1)p^{\frac{1+3\alpha}{4}}} \right).$$

This theorem replaces Lemma 15 in [2], but notice that we avoid the high power of $\log z$ and even save an additional $\log z$. We also mention for future use that

$$(29) \quad \frac{c_2(\alpha)}{\zeta(\frac{5+3\alpha}{4})\zeta(\frac{5-\alpha}{4})} = \prod_p \left(1 + \frac{(p^{1/4} - p^{\alpha/4})p^{\alpha/4}}{p^{5/2} - p^{3/2}} + \frac{-(p^{5/4} + p^{1/4})p^{\alpha} + p^{\frac{6+3\alpha}{4}} + p^{\frac{\alpha}{4}} + p^{\frac{5\alpha}{4}} - p^{5/4}}{p^{\frac{3\alpha}{2}}(p^{15/4} - p^{11/4})} \right).$$

where the right hand side is uniformly bounded above for $\alpha \in [0, 1]$.

Proof. Set $T = \sum_{\mathfrak{e}|V(z)} |\lambda_{\mathfrak{e}}(\mathfrak{q})|/\mathfrak{N}\mathfrak{e}^{\alpha}$. Then from the definition of $\lambda_{\mathfrak{e}}(\mathfrak{q})$, we get

$$\begin{aligned} G_{\mathfrak{q}}(z)T &= \sum_{\mathfrak{e}|V(z)} \frac{\mathfrak{N}\mathfrak{e}^{1-\alpha}}{\phi(\mathfrak{e})} \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq \frac{z}{\mathfrak{N}\mathfrak{e}}, \\ (\mathfrak{a}, \mathfrak{e}\mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{\phi(\mathfrak{a})} = \sum_{\mathfrak{e}|V(z)} \mathfrak{N}\mathfrak{e}^{1-\alpha} \sum_{\substack{\mathfrak{N}\mathfrak{b} \leq z, \\ (\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}, \\ \mathfrak{e}|\mathfrak{b}}} \frac{\mu^2(\mathfrak{b})}{\phi(\mathfrak{b})} \\ &= \sum_{\substack{\mathfrak{N}\mathfrak{b} \leq z, \\ (\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{b})}{\phi(\mathfrak{b})} \sum_{\mathfrak{e}|\mathfrak{b}} \mathfrak{N}\mathfrak{e}^{1-\alpha}. \end{aligned}$$

We note that, when \mathfrak{b} is squarefree, we have

$$\frac{\mathfrak{N}\mathfrak{b}^{\alpha}}{\phi(\mathfrak{b})} \sum_{\mathfrak{e}|\mathfrak{b}} \mathfrak{N}\mathfrak{e}^{1-\alpha} = \prod_{\mathfrak{p}|\mathfrak{b}} \left(\frac{\mathfrak{N}\mathfrak{p} + \mathfrak{N}\mathfrak{p}^{\alpha}}{\mathfrak{N}\mathfrak{p} - 1} \right) = \prod_{\mathfrak{p}|\mathfrak{b}} \left(1 + \frac{1 + \mathfrak{N}\mathfrak{p}^{\alpha}}{\mathfrak{N}\mathfrak{p} - 1} \right) = \sum_{\mathfrak{f}|\mathfrak{b}} g_{\alpha}(\mathfrak{f})$$

where $g_{\alpha}(\mathfrak{f})$ is the completely multiplicative function defined on primes by $g_{\alpha}(\mathfrak{p}) = \frac{1 + \mathfrak{N}\mathfrak{p}^{\alpha}}{\mathfrak{N}\mathfrak{p} - 1}$. Only the values on squarefree ideals are required but to define a unique function, we added that it is a ‘completely’ multiplicative function. Therefore

$$\begin{aligned} G_{\mathfrak{q}}(z)T &= \sum_{\substack{\mathfrak{N}\mathfrak{b} \leq z, \\ (\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}^{\alpha}} \sum_{\mathfrak{f}|\mathfrak{b}} g_{\alpha}(\mathfrak{f}) = \sum_{\mathfrak{N}\mathfrak{f} \leq z} \mu^2(\mathfrak{f}) g_{\alpha}(\mathfrak{f}) \sum_{\substack{\mathfrak{N}\mathfrak{b} \leq z, \\ (\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}, \\ \mathfrak{f}|\mathfrak{b}}} \frac{\mu^2(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}^{\alpha}} \\ &\leq \sum_{\substack{\mathfrak{N}\mathfrak{f} \leq z, \\ (\mathfrak{f}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{f}) g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}^{\alpha}} \sum_{\mathfrak{N}\mathfrak{b}_1 \leq \frac{z}{\mathfrak{N}\mathfrak{f}}} \frac{\mu^2(\mathfrak{b}_1)}{\mathfrak{N}\mathfrak{b}_1^{\alpha}}. \end{aligned}$$

At this level, we use the trivial bound

$$G_{\mathfrak{q}}(z) = \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z \\ (\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \prod_{\mathfrak{p}|\mathfrak{a}} \sum_{k \geq 0} \mathfrak{N}\mathfrak{p}^{-k} \geq \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z \\ (\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{1}{\mathfrak{N}(\mathfrak{a})}$$

and Lemma 31 with $x = z/\mathfrak{N}\mathfrak{f}$ for the inner sum of the right-hand side when $\mathfrak{N}\mathfrak{f} \leq \sqrt{z}$, and with $x = \sqrt{z}$ for the remaining sum. This first gives us

$$\begin{aligned} \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq z, \\ (\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{1}{\mathfrak{N}(\mathfrak{a})} T &\leq \frac{(1 + 1.02n_{\mathbf{K}})z^{1-\alpha}}{1-\alpha} \sum_{\substack{\mathfrak{N}\mathfrak{f} \leq \sqrt{z}, \\ (\mathfrak{f}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}(1 + \log(z/\mathfrak{N}\mathfrak{f}))} \sum_{\substack{\mathfrak{N}\mathfrak{b}_1 \leq \frac{z}{\mathfrak{N}\mathfrak{f}}, \\ (\mathfrak{b}_1, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{b}_1)}{\mathfrak{N}\mathfrak{b}_1} \\ &\quad + z^{(1-\alpha)/2} \sum_{\substack{z \geq \mathfrak{N}\mathfrak{f} > \sqrt{z}, \\ (\mathfrak{f}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}^{\alpha}} \sum_{\substack{\mathfrak{N}(\mathfrak{a}) \leq \sqrt{z}, \\ (\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{1}{\mathfrak{N}(\mathfrak{a})}. \end{aligned}$$

We readily deduce from this the inequality

$$T \leq \frac{3.1 n_{\mathbf{K}} z^{1-\alpha}}{(1-\alpha)(2 + \log z)} \sum_{\substack{\mathfrak{N}\mathfrak{f} \leq z, \\ (\mathfrak{f}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}} + z^{(1-\alpha)/2} \sum_{\substack{z \geq \mathfrak{N}\mathfrak{f} > \sqrt{z}, \\ (\mathfrak{f}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}^{\alpha}}.$$

To proceed, we forget about the coprimality in \mathfrak{f} and use Rankin's trick for the second part, noticing that $(z/\mathfrak{N}\mathfrak{f})^{(1-\alpha)/4} \geq 1$ therein. This leads to

$$T \leq \frac{3.1 n_{\mathbf{K}} z^{1-\alpha}}{(1-\alpha)(2 + \log z)} C_1(\alpha) + z^{3(1-\alpha)/4} C_2(\alpha).$$

Concerning the constants $C_1(\alpha)$ and $C_2(\alpha)$, we readily find that

$$\begin{cases} C_1(\alpha) = \sum_{\substack{\mathfrak{f} \\ \mathfrak{f} \neq 0}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}} = \prod_{\mathfrak{p}} \left(1 + \frac{1 + \mathfrak{N}\mathfrak{p}^{\alpha}}{(\mathfrak{N}\mathfrak{p} - 1)\mathfrak{N}\mathfrak{p}} \right) \leq \prod_{\mathfrak{p}} \left(1 + \frac{1 + p^{\alpha}}{(p-1)p} \right)^{n_{\mathbf{K}}}, \\ C_2(\alpha) = \sum_{\substack{\mathfrak{f} \\ \mathfrak{f} \neq 0}} \frac{\mu^2(\mathfrak{f})g_{\alpha}(\mathfrak{f})}{\mathfrak{N}\mathfrak{f}^{\frac{3\alpha+1}{4}}} = \prod_{\mathfrak{p}} \left(1 + \frac{1 + \mathfrak{N}\mathfrak{p}^{\alpha}}{(\mathfrak{N}\mathfrak{p} - 1)\mathfrak{N}\mathfrak{p}^{\frac{3\alpha+1}{4}}} \right) \leq \prod_{\mathfrak{p}} \left(1 + \frac{1 + p^{\alpha}}{(p-1)p^{\frac{3\alpha+1}{4}}} \right)^{n_{\mathbf{K}}}. \end{cases}$$

The lemma follows readily. \square

Corollary 33. *When $z \geq 1$, we have $\sum_{\substack{\mathfrak{e} | V(z), \\ (\mathfrak{e}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} |\lambda_{\mathfrak{e}}(\mathfrak{q})| \leq 6 \cdot 89^{n_{\mathbf{K}}} \frac{z}{2 + \log z}.$*

Proof. We use Theorem 32 with $\alpha = 0$. We numerically find that

$$1.6 n_{\mathbf{K}} \prod_{\mathfrak{p}} \left(1 + \frac{2}{(p-1)p} \right)^{n_{\mathbf{K}}} \leq \prod_{\mathfrak{p}} \left(1 + \frac{2}{(p-1)p^{1/4}} \right)^{n_{\mathbf{K}}} \leq (88.2)^{n_{\mathbf{K}}}.$$

Furthermore $z^{3/4} \leq 4z/(2 + \log z)$ and $6(88.2)^{n_{\mathbf{K}}} \leq 6 \cdot 89^{n_{\mathbf{K}}}$. \square

Corollary 34. *When $z \geq 1$, we have $\sum_{\substack{\mathfrak{e} | V(z), \\ (\mathfrak{e}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}}} \frac{|\lambda_{\mathfrak{e}}(\mathfrak{q})|}{\mathfrak{N}\mathfrak{e}^{1-\frac{1}{n_{\mathbf{K}}}}} \leq n_{\mathbf{K}}^{9n_{\mathbf{K}}} \frac{z^{1/n_{\mathbf{K}}}}{2 + \log z}.$*

Proof. We use Theorem 32 with $\alpha = 1 - 1/n_{\mathbf{K}}$. Let us call T the quantity to be bounded above. Theorem 32 gives the upper bound, with $n = n_{\mathbf{K}}$,

$$\begin{aligned} \frac{3.1 n^2 z^{1/n}}{2 + \log z} c_1 (1 - 1/n)^n + z^{3/(4n)} c_2 (1 - 1/n)^n \\ = \frac{n z^{1/n}}{2 + \log z} \left(3.1 n c_1 (1 - 1/n)^n + 4 c_2 (1 - 1/n)^n \frac{\frac{1}{2n} + \log(z^{1/(4n)})}{z^{1/(4n)}} \right). \end{aligned}$$

The maximum of $y \mapsto (\frac{1}{4} + \log y)/y$ is attained at $y = e^{3/4}$ and $c_1(\alpha) \leq c_2(\alpha)$, so we can simplify this upper bound to

$$\frac{n z^{1/n}}{2 + \log z} (3.1 n + 1.9) c_2 (1 - 1/n)^n.$$

Now we want to find an upper bound for $c_2(1 - 1/n)$. Applying inequality (29) and the fact that $\alpha \geq 1/2$ in our case, we get

$$\frac{c_2(1 - 1/n)}{\zeta(1 + \frac{1}{4n})\zeta(2 - \frac{3}{4n})} \leq \prod_p \left(1 + \frac{p^{1/2}}{p^{5/2} - p^{3/2}} + \frac{p^2 - p^{3/2} - p^{1/2} + 1}{p^{1/2}(p^{15/4} - p^{11/4})} \right).$$

A direct computation shows that right hand side of the product for $p \leq 10^4$ is bounded by 2.3, and then using calculus we derive that $c_2(1 - 1/n) \leq 11.5n\zeta(13/8)$. This leads to

$$c_2(1 - 1/n) \leq 26.45 n.$$

Furthermore the quantity $n(3.1n + 1.9)(26.45n)^n$ may further be bounded above by n^{9n} . This establishes this lemma. \square

6. BRUN-TITCHMARSH THEOREM FOR COSETS OF RAY CLASS GROUPS

In this section, we prove a number field analogue of Brun-Titchmarsh theorem for cosets. This result, while being of independent interest, is also crucial in proving our main theorem.

Theorem 35. *Let $\mathfrak{a}, \mathfrak{q}$ be integral ideals with $(\mathfrak{a}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}$. Also let H be a subgroup of $H_{\mathfrak{q}}(\mathbf{K})$ with index Y . When $X/Y \geq (10^6 n_{\mathbf{K}})^{8n_{\mathbf{K}}+11} |d_{\mathbf{K}}|^6 \sqrt{|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q})} \log(|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q})^{n_{\mathbf{K}}})$, we have*

$$\sum_{\mathfrak{p} \in [\mathfrak{a}]H} w \left(\frac{\mathfrak{N}(\mathfrak{p})}{X} \right) \leq \frac{2\|w\|_1 X}{Y \log \frac{u^*(w, \mathbf{K}) X}{Y \sqrt{\mathfrak{N}(\mathfrak{q})} \log(|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q})^{n_{\mathbf{K}}})}}, \quad u^*(w, \mathbf{K}) = \frac{\|w\|_1 / (\|w^{(n_{\mathbf{K}}+3)}\|_{\infty} + 5\|w\|_1)}{20000 \cdot 2^{22n_{\mathbf{K}}} |d_{\mathbf{K}}|^{3/2}}.$$

for any non-negative smoothing function w as defined in Section 2.

When compared to the usual Brun-Titchmarsh Theorem, this result has the surprising feature to be sometimes valid for instance when $X < \mathfrak{N}(\mathfrak{q})$. For instance, when $Y = 2$, the bound X can be as small as $\mathcal{O}_{\mathbf{K}}(\mathfrak{N}(\mathfrak{q})^{\frac{1}{2}+\epsilon})$.

Proof. We use the notation $G = H_{\mathfrak{q}}(\mathbf{K})$ and $n = n_{\mathbf{K}}$. For any ideal \mathfrak{b} of $\mathcal{O}_{\mathbf{K}}$, we define

$$\delta(\mathfrak{b}) = \frac{1}{Y} \sum_{\chi \in \widehat{G/H}} \chi([\mathfrak{b}\mathfrak{a}^{-1}]H) \quad \text{and} \quad \delta^*(\mathfrak{b}) = \frac{1}{Y} \sum_{\chi \in \widehat{G/H}} \chi^*([\mathfrak{b}\mathfrak{a}^{-1}]H)$$

where $\widehat{G/H}$ denotes the group of characters of G/H , χ^* is the primitive character inducing χ . For any integral ideal \mathfrak{b} of $\mathcal{O}_{\mathbf{K}}$ with $(\mathfrak{b}, \mathfrak{q}) = \mathcal{O}_{\mathbf{K}}$, $\bar{\mathfrak{b}}$ denotes an element of G/H . In this case, $\delta(\mathfrak{b}) = \delta^*(\bar{\mathfrak{b}})$. For the case $(\mathfrak{b}, \mathfrak{q}) \neq \mathcal{O}_{\mathbf{K}}$, let $\mathfrak{q}(\mathfrak{b})$ be the largest divisor of \mathfrak{q} co-prime to \mathfrak{b} and let $L_{\mathfrak{q}(\mathfrak{b})}$ be the image of H in $H_{\mathfrak{q}(\mathfrak{b})}(\mathbf{K}) = G_{\mathfrak{q}(\mathfrak{b})}$, say. Note that

$$\delta^*(\mathfrak{b}) = \frac{1}{Y} \sum_{\chi \in \widehat{G_{\mathfrak{q}(\mathfrak{b})}/L_{\mathfrak{q}(\mathfrak{b})}}} \chi^*([\mathfrak{b}\mathfrak{a}^{-1}]H) = \frac{|G_{\mathfrak{q}(\mathfrak{b})}/L_{\mathfrak{q}(\mathfrak{b})}|}{Y} \mathbf{1}_{[\mathfrak{a}]L_{\mathfrak{q}(\mathfrak{b})}},$$

where $\mathbf{1}_{[\mathfrak{a}]L_{\mathfrak{q}(\mathfrak{b})}}$ is the characteristic function of $[\mathfrak{a}]L_{\mathfrak{q}(\mathfrak{b})}$. This shows that $\delta^*(\mathfrak{b})$ is non-negative whenever $(\mathfrak{b}, \mathfrak{q}) \neq \mathcal{O}_{\mathbf{K}}$. Therefore $\delta^*(\mathfrak{b}) \geq \delta(\mathfrak{b})$.

Consider the sum $T_1 = \sum_{\mathfrak{p}} \delta(\mathfrak{p})w(\mathfrak{N}(\mathfrak{p})/X)$. As $|w| \leq 1$, we readily find that

$$\begin{aligned} T_1 &= \sum_{\mathfrak{N}(\mathfrak{p}) \leq z} \delta(\mathfrak{p})w\left(\frac{\mathfrak{N}(\mathfrak{p})}{X}\right) + \sum_{\mathfrak{N}(\mathfrak{p}) > z} \delta(\mathfrak{p})w\left(\frac{\mathfrak{N}(\mathfrak{p})}{X}\right) \\ &\leq nz + \sum_{(\mathfrak{b}, V(z)) = \mathcal{O}_{\mathbf{K}}} \delta^*(\mathfrak{b})w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right) \leq nz + \sum_{\mathfrak{b}} \delta^*(\mathfrak{b})w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right) \sum_{\mathfrak{e} | (\mathfrak{b}, V(z))} \mu(\mathfrak{e}), \end{aligned}$$

where $V(z)$ is as in Section 5. Since $\sum_{\mathfrak{e} | \mathfrak{b}} \mu(\mathfrak{e}) \leq (\sum_{\mathfrak{e} | \mathfrak{b}} \lambda_{\mathfrak{e}})^2$, we have

$$T_1 - nz \leq \sum_{\mathfrak{b}} \delta^*(\mathfrak{b})w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right) \left(\sum_{\mathfrak{e} | (\mathfrak{b}, V(z))} \lambda_{\mathfrak{e}} \right)^2 \leq \sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2} \sum_{[\mathfrak{e}_1, \mathfrak{e}_2] | \mathfrak{b}} \delta^*(\mathfrak{b})w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right).$$

Replacing the definition of $\delta^*(\mathfrak{b})$, we get

$$\begin{aligned} T_1 - nz &\leq \sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \frac{\lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2}}{Y} \sum_{[\mathfrak{e}_1, \mathfrak{e}_2] | \mathfrak{b}} w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right) \sum_{\chi \in \widehat{G/H}} \chi^*([\mathfrak{b}\mathfrak{a}^{-1}]H) \\ &\leq \sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \frac{\lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2}}{Y} \sum_{\chi \in \widehat{G/H}} \chi^*([\mathfrak{a}^{-1}]H) \sum_{[\mathfrak{e}_1, \mathfrak{e}_2] | \mathfrak{b}} w\left(\frac{\mathfrak{N}(\mathfrak{b})}{X}\right) \chi^*([\mathfrak{b}]H) \\ &\leq \sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \frac{\lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2}}{Y} \sum_{\chi \in \widehat{G/H}} \chi^*([\mathfrak{a}^{-1}]H) \sum_{\mathfrak{m}} w\left(\frac{\mathfrak{N}(\mathfrak{m}[\mathfrak{e}_1, \mathfrak{e}_2])}{X}\right) \chi^*([\mathfrak{m}[\mathfrak{e}_1, \mathfrak{e}_2]]H). \end{aligned}$$

Applying Mellin transforms, we get

$$\sum_{\mathfrak{m}} w\left(\frac{\mathfrak{N}(\mathfrak{m}[\mathfrak{e}_1, \mathfrak{e}_2])}{X}\right) \chi^*([\mathfrak{m}[\mathfrak{e}_1, \mathfrak{e}_2]]H) = \chi^*([\mathfrak{e}_1, \mathfrak{e}_2]]H) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \tilde{w}(s) L_{\mathfrak{q}^*}(s, \chi^*) \frac{X^s ds}{\mathfrak{N}([\mathfrak{e}_1, \mathfrak{e}_2])^s}.$$

where \mathfrak{q}^* is the conductor of χ^* . We shift the line of integration to the line $\Re s = 0$. This leads to the upper bound of $T_1 - nz$ given by

$$\sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \frac{\lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2}}{Y} \left\{ \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q}) \tilde{w}(1) X}{\mathfrak{N}(\mathfrak{q}) \mathfrak{N}([\mathfrak{e}_1, \mathfrak{e}_2])} + \sum_{\chi \in \widehat{G/H}} \frac{\chi^*([\mathfrak{a}^{-1}[\mathfrak{e}_1, \mathfrak{e}_2]]H)}{2\pi i} \int_{-i\infty}^{i\infty} \tilde{w}(s) L_{\mathfrak{q}^*}(s, \chi^*) \frac{X^s ds}{\mathfrak{N}([\mathfrak{e}_1, \mathfrak{e}_2])^s} \right\}.$$

On using Lemma 16 when $\chi^* \neq 1$ and Lemma 14 when $\chi^* = 1$ on the line $\Re s = 0$, we see that $T_1 - nz$ is less than

$$\frac{\alpha_{\mathbf{K}} \tilde{w}(1) \phi(\mathfrak{q}) X}{\mathfrak{N}(\mathfrak{q}) Y} \sum_{\substack{\mathfrak{e}_1, \mathfrak{e}_2 \\ \mathfrak{e}_i | V(z)}} \frac{\lambda_{\mathfrak{e}_1} \lambda_{\mathfrak{e}_2}}{\mathfrak{N}([\mathfrak{e}_1, \mathfrak{e}_2])} + O^*\left(3M(w, \varepsilon) \zeta(1 + \varepsilon)^{n_{\mathbf{K}}} (|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q}))^{\frac{1+\varepsilon}{2}} \left(\sum_{\mathfrak{e} | V(z)} |\lambda_{\mathfrak{e}}| \right)^2\right),$$

where $0 < \varepsilon < 1/24$ and $M(w, \varepsilon)$ is defined and bounded in Lemma 12. The sum over \mathfrak{e} may be treated by Corollary 33. As a partial conclusion, we reach

$$T_1 \leq nz + \frac{\alpha_{\mathbf{K}} \tilde{w}(1) X}{Y G(z)} + 12 \cdot 2^{\frac{\varepsilon}{2}} n (\|w^{(n+3)}\|_{\infty} + 10 \cdot 2^{\frac{n}{2}} \|w\|_1) \zeta(1 + \varepsilon)^n (|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q}))^{\frac{1+\varepsilon}{2}} 36 \cdot 89^{2n} \frac{z^2}{\log^2 z}.$$

Let us assume that $z \geq (10^6 n)^{4n} |d_{\mathbf{K}}|^3$ to use Theorem 29. This leads to

$$T_1 \leq nz + \frac{\tilde{w}(1) X}{Y \log \frac{z}{n_{\mathbf{K}}^4 \sqrt{|d_{\mathbf{K}}|}}} + 432 \cdot 2^{\frac{28+\varepsilon}{2}} n (\|w^{(n+3)}\|_{\infty} + 10 \cdot 2^{\frac{n}{2}} \|w\|_1) \zeta(1 + \varepsilon)^n (|d_{\mathbf{K}}| \mathfrak{N}(\mathfrak{q}))^{\frac{1+\varepsilon}{2}} \frac{z^2}{\log^2 z}.$$

We select $\varepsilon = \min(1/2, 2/\log(|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q})))$ and use

$$\zeta(1 + \varepsilon)^n (|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q}))^{\frac{1+\varepsilon}{2}} \leq e(2\log(|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q})))^n \sqrt{|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q})}$$

to derive the bound

$$\frac{Y \log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}}{X \|w\|_1} T_1 \leq 1 + 2600 \cdot 2^{16n} \cdot \frac{(\|w^{(n+3)}\|_{\infty} + 5\|w\|_1)}{\|w\|_1} \log(|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q}))^n \sqrt{|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q})} \frac{Y z^2}{X \log z}.$$

We select

$$z^2 = \frac{X \|w\|_1 / Y}{2600 \cdot 2^{16n} (\|w^{(n+3)}\|_{\infty} + 5\|w\|_1) \log(|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q}))^n \sqrt{|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q})}}.$$

This gives us the inequality

$$\frac{Y \log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}}{X \|w\|_1} T_1 \leq 1 + \frac{1}{\log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}} \leq \frac{1}{1 - \log^{-1} \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}},$$

where we have used the inequality $1 + x \leq 1/(1 - x)$ for $x \in [0, 1)$. Thus

$$\sum_{\mathfrak{p} \in [\mathfrak{a}]^H} w \left(\frac{\mathfrak{N}(\mathfrak{p})}{X} \right) \leq \frac{2\|w\|_1 X}{Y \log \frac{X/(Y \sqrt{\mathfrak{N}(\mathfrak{q})} \log(|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{q}))^n)}{|d_{\mathbf{K}}|^{3/2} 20000 \cdot 2^{22n} \cdot (\|w^{(n+3)}\|_{\infty} + 5\|w\|_1)/\|w\|_1}}},$$

where we have used the inequality $2^{16n} n^8 \leq 2^{22n}$. This completes the proof of our theorem. \square

7. BRUN-TITCHMARSH THEOREM FOR SINGLE CLASS OF RAY CLASS GROUPS

Applying Theorem 35 for the trivial subgroup and forgetting the field-dependence leads to the upper bound $\frac{2X}{h_{\mathbf{K}, \mathfrak{q}} \log(X/\mathfrak{N}(\mathfrak{q})^{3/2+\epsilon})}$ while the usual Brun-Titchmarsh inequality has $\log(X/\mathfrak{N}(\mathfrak{q})^{1+\epsilon})$.

Proof of Theorem 3. As before, we denote $n_{\mathbf{K}}$ by n . We use a Selberg sieve of dimension one as presented in Section 5 with coefficients $(\lambda_{\mathfrak{c}}(\mathfrak{q}))_{\mathfrak{N}(\mathfrak{c}) \leq z}$. On denoting by T_1 the sum to evaluate, this gives us

$$T_1 \leq \sum_{\substack{\mathfrak{a} \in [\mathfrak{b}], \\ \mathfrak{N}(\mathfrak{a}) \leq X}} \left(\sum_{\mathfrak{c} | (\mathfrak{a}, V(z))} \lambda_{\mathfrak{c}}(\mathfrak{q}) \right)^2 + nz \leq \sum_{\mathfrak{c}_1, \mathfrak{c}_2} \lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q}) \sum_{\substack{[\mathfrak{c}_1, \mathfrak{c}_2] | \mathfrak{a} \in [\mathfrak{b}], \\ \mathfrak{N}(\mathfrak{a}) \leq X}} 1 + nz.$$

We write $\mathfrak{a} = [\mathfrak{c}_1, \mathfrak{c}_2] \mathfrak{c}$ where \mathfrak{c} is an integral ideal in the class of $\mathfrak{b}[\mathfrak{c}_1, \mathfrak{c}_2]^{-1}$ (this is legal since the lcm $[\mathfrak{c}_1, \mathfrak{c}_2]$ is indeed prime to \mathfrak{q}), and of norm $\leq X/\mathfrak{N}[\mathfrak{c}_1, \mathfrak{c}_2]$. We now apply Theorem 8. This gives us

$$\begin{aligned} T_1 - nz &\leq \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q})}{h_{\mathbf{K}, \mathfrak{q}}} \frac{X}{\mathfrak{N}(\mathfrak{q})} \sum_{\mathfrak{c}_1, \mathfrak{c}_2} \frac{\lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q})}{\mathfrak{N}[\mathfrak{c}_1, \mathfrak{c}_2]} + \left(\sum_{\mathfrak{c}} |\lambda_{\mathfrak{c}}(\mathfrak{q})| \right)^2 n^{8n} R_{\mathbf{K}} F(\mathfrak{q}) \\ &\quad + E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \left(\frac{X}{\mathfrak{N}(\mathfrak{q})} \right)^{1-\frac{1}{n}} \sum_{\mathfrak{c}_1, \mathfrak{c}_2} \frac{|\lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q})|}{\mathfrak{N}[\mathfrak{c}_1, \mathfrak{c}_2]^{1-\frac{1}{n}}}. \end{aligned}$$

The first term on the right hand side is handled by Lemma 27, the second term is controlled by Corollary 33 and we now show that the third one can be controlled by Corollary 34. Indeed, Selberg's diagonalisation process gives us, with $\alpha = 1 - \frac{1}{n}$,

$$\sum_{\mathfrak{c}_1, \mathfrak{c}_2} \frac{|\lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q})|}{\mathfrak{N}[\mathfrak{c}_1, \mathfrak{c}_2]^{\alpha}} = \sum_{\mathfrak{c}_1, \mathfrak{c}_2} \mathfrak{N}(\mathfrak{c}_1, \mathfrak{c}_2)^{\alpha} \frac{|\lambda_{\mathfrak{c}_1}(\mathfrak{q}) \lambda_{\mathfrak{c}_2}(\mathfrak{q})|}{\mathfrak{N}(\mathfrak{c}_1)^{\alpha} \mathfrak{N}(\mathfrak{c}_2)^{\alpha}} \leq \sum_{\substack{\partial, \\ \mathfrak{N}(\partial) \leq z}} \phi_{\alpha}(\partial) \left(\sum_{\partial | \mathfrak{c}} \frac{|\lambda_{\mathfrak{c}}(\mathfrak{q})|}{\mathfrak{N}(\mathfrak{c})^{\alpha}} \right)^2,$$

where $\phi_\alpha(\partial) = \mathfrak{N}(\partial)^\alpha$. We now note that

$$\begin{aligned} \sum_{\partial|\mathfrak{e}} \frac{|\lambda_{\mathfrak{e}}(\mathfrak{q})|}{\mathfrak{N}\mathfrak{e}^\alpha} &\leq \sum_{(\mathfrak{e}, \partial\mathfrak{q})=1} \frac{\mathfrak{N}\mathfrak{e}\mathfrak{N}\partial}{\phi(\mathfrak{e})\phi(\partial)\mathfrak{N}\partial^\alpha\mathfrak{N}\mathfrak{e}^\alpha} \frac{G_{\mathfrak{e}\partial\mathfrak{q}}(z/\mathfrak{N}\mathfrak{e})}{G_{\mathfrak{q}}(z)} \\ &\leq \frac{G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)}{G_{\mathfrak{q}}(z)} \frac{\mathfrak{N}\partial^{1-\alpha}}{\phi(\partial)} \sum_{(\mathfrak{e}, \partial\mathfrak{q})=1} \frac{\mathfrak{N}\mathfrak{e}}{\phi(\mathfrak{e})\mathfrak{N}\mathfrak{e}^\alpha} \frac{G_{\mathfrak{e}\partial\mathfrak{q}}(z/\mathfrak{N}\mathfrak{e})}{G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)} \\ &\leq \frac{G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)}{G_{\mathfrak{q}}(z)} \frac{\mathfrak{N}\partial^{1-\alpha}}{\phi(\partial)} \sum_{(\mathfrak{e}, \partial\mathfrak{q})=1} \frac{|\lambda_{\mathfrak{e}}(\mathfrak{q}, z/\mathfrak{N}\partial)|}{\mathfrak{N}\mathfrak{e}^\alpha}, \end{aligned}$$

where $\lambda_{\mathfrak{e}}(\mathfrak{q}, z/\mathfrak{N}\partial) = \frac{\mathfrak{N}\mathfrak{e}}{\phi(\mathfrak{e})} \frac{G_{\mathfrak{e}\partial\mathfrak{q}}(z/\mathfrak{N}\mathfrak{e})}{G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)}$. By Corollary 34, this is bounded above by

$$n^{9n} \frac{\mathfrak{N}\partial}{\phi(\partial)} \frac{G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)}{G_{\mathfrak{q}}(z)} \frac{z^{1-\alpha}}{(2 + \log(z/\mathfrak{N}\partial))\mathfrak{N}\partial}.$$

We use the upper bound $G_{\mathfrak{q}}(z)$ for $(\mathfrak{N}\partial/\phi(\partial))G_{\partial\mathfrak{q}}(z/\mathfrak{N}\partial)$ which follows along the lines of the inequality [30, (1.3)] by Van Lint and Richert. Hence we have

$$\begin{aligned} \sum_{\mathfrak{e}_1, \mathfrak{e}_2} \frac{|\lambda_{\mathfrak{e}_1}(\mathfrak{q})\lambda_{\mathfrak{e}_2}(\mathfrak{q})|}{\mathfrak{N}[\mathfrak{e}_1, \mathfrak{e}_2]^\alpha} &\leq n^{18n} z^{2(1-\alpha)} \sum_{\substack{\partial, \\ 0 < \mathfrak{N}\partial \leq z}} \mu^2(\partial) \frac{\phi_\alpha(\partial)}{(2 + \log(z/\mathfrak{N}\partial))^2 \mathfrak{N}\partial^2} \\ &\leq \frac{n^{20n} z^{2(1-\alpha)}}{(\log z)^2} \sum_{\substack{\partial, \\ 0 < \mathfrak{N}\partial \leq z^\alpha}} \mu^2(\partial) \frac{\phi_\alpha(\partial)}{\mathfrak{N}\partial^2} + \frac{n^{18n} z^{2(1-\alpha)}}{4} \sum_{\substack{\partial, \\ z^\alpha < \mathfrak{N}\partial \leq z}} \mu^2(\partial) \frac{\phi_\alpha(\partial)}{\mathfrak{N}\partial^2} \left(\frac{\mathfrak{N}\partial}{z^\alpha}\right)^{\frac{1-\alpha}{2}} \\ &\leq n^{20n} \frac{z^{2(1-\alpha)}}{(\log z)^2} \zeta_{\mathbf{K}}(2-\alpha) + n^{18n} \frac{z^{\frac{(4-\alpha)}{2}(1-\alpha)}}{4} \zeta_{\mathbf{K}}\left(1 + \frac{1-\alpha}{2}\right). \end{aligned}$$

We simplify the upper bound as follows;

$$n^{20n} \frac{z^{2/n}}{(\log z)^2} \left((n+1)^n + \frac{(\log z)^2}{4z^{\frac{n-1}{2n^2}}} (2n+1)^n \right)$$

which is bounded above by

$$n^{20n} \frac{z^{2/n}}{(\log z)^2} \left((n+1)^n + \frac{(\frac{4n^2}{n-1} \log z^{\frac{n-1}{4n^2}})^2}{4z^{\frac{n-1}{2n^2}}} (2n+1)^n \right) \leq n^{27n} \frac{z^{2/n}}{(\log z)^2}.$$

Let us resume our study of T_1 . The estimates above lead to

$$\begin{aligned} \frac{h_{\mathbf{K}, \mathfrak{q}}}{X} T_1 &\leq \frac{\alpha_{\mathbf{K}} \phi(\mathfrak{q})}{\mathfrak{N}\mathfrak{q} G_{\mathfrak{q}}(z)} + 36 \cdot 2^{14n} n^{8n} R_{\mathbf{K}} F(\mathfrak{q}) \frac{z^2 h_{\mathbf{K}, \mathfrak{q}}}{\mathfrak{N}\mathfrak{q}} \frac{\mathfrak{N}\mathfrak{q}/X}{(2 + \log z)^2} \\ &\quad + n \frac{h_{\mathbf{K}, \mathfrak{q}}}{\mathfrak{N}\mathfrak{q}} \frac{z \mathfrak{N}\mathfrak{q}}{X} + n^{27n} E(\mathbf{K}) F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n \frac{h_{\mathbf{K}, \mathfrak{q}}}{\mathfrak{N}\mathfrak{q}(\log z)^2} \left(\frac{z^2 \mathfrak{N}\mathfrak{q}}{X} \right)^{\frac{1}{n}}, \end{aligned}$$

where $E(\mathbf{K}) = 1000n^{12n^2} R_{\mathbf{K}}^{1/n} [\log((2n)^{4n} R_{\mathbf{K}})]^n$ and $F(\mathfrak{q}) = 2^{r_1} \phi(\mathfrak{q}) h_{\mathbf{K}}/h_{\mathbf{K}, \mathfrak{q}}$ are defined in Theorem 8. We employ Lemma 29 to bound $G_{\mathfrak{q}}(z)$ from below, on assuming that $z \geq (10^6 n)^{4n} |d_{\mathbf{K}}|^3$, getting

$$\begin{aligned} \frac{h_{\mathbf{K}, \mathfrak{q}}}{X} T_1 &\leq \frac{1}{\log \frac{z}{e^{4n^4} \sqrt{|d_{\mathbf{K}}|}}} \left(1 + 36 \cdot 2^{14n} n^{8n} R_{\mathbf{K}} 2^{r_1} h_{\mathbf{K}} \frac{z^2 \mathfrak{N}\mathfrak{q}/X}{2 + \log z} \right. \\ &\quad \left. + n^{2r_1} h_{\mathbf{K}} \frac{z \mathfrak{N}\mathfrak{q} \log z}{X} + n^{27n} E(\mathbf{K}) \frac{F(\mathfrak{q})^{\frac{1}{n}} \log(3F(\mathfrak{q}))^n}{F(\mathfrak{q})} \frac{2^{r_1} h_{\mathbf{K}}}{\log z} \left(\frac{z^2 \mathfrak{N}\mathfrak{q}}{X} \right)^{\frac{1}{n}} \right). \end{aligned}$$

We notice that $z \log z \leq \frac{2}{3} z^2 / \log z$ when $z > 1$ and that

$$\begin{aligned}
 \frac{y^{\frac{1}{n}} \log(3y)^n}{y} &= \left(\frac{\log(3y)}{y^{\frac{1}{n} - \frac{1}{n^2}}} \right)^n = 3^{1 - \frac{1}{n}} \left(\frac{\log(3y)}{(3y)^{\frac{1}{n} - \frac{1}{n^2}}} \right)^n \\
 (30) \quad &= 3^{1 - \frac{1}{n}} \frac{n^{2n}}{(n-1)^n} \left(\frac{\log((3y)^{\frac{1}{n} - \frac{1}{n^2}})}{(3y)^{\frac{1}{n} - \frac{1}{n^2}}} \right)^n \leq 3 \frac{n^{2n}}{e^n (n-1)^n} \leq n^{2n}.
 \end{aligned}$$

On assuming that $z^2 \leq X/\mathfrak{N}\mathfrak{q}$, we reach the upper bound

$$\begin{aligned}
 \frac{h_{\mathbf{K},\mathfrak{q}}}{X} T_1 &\leq \frac{1}{\log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}} \left(1 + \left(36n^{23n} R_{\mathbf{K}} + n^{2n} + E(\mathbf{K}) n^{30n} \right) \frac{h_{\mathbf{K}}}{\log z} \left(\frac{z^2 \mathfrak{N}\mathfrak{q}}{X} \right)^{\frac{1}{n}} \right) \\
 &\leq \frac{1}{\log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}} \left(1 + \left(y + 1 + 1000y^{1/n} (\log y)^n \right) \frac{n^{27n^2} h_{\mathbf{K}}}{\log z} \left(\frac{z^2 \mathfrak{N}\mathfrak{q}}{X} \right)^{\frac{1}{n}} \right)
 \end{aligned}$$

with $y = (2n)^{4n} R_{\mathbf{K}}$. Recall that $R_{\mathbf{K}} \geq 1/5$ by [4], so that $y \geq 13000$. A reasoning very similar to the one that led to inequality (30) applies, getting

$$\frac{h_{\mathbf{K},\mathfrak{q}}}{X} T_1 \leq \frac{1}{\log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}}} \left(1 + \frac{n^{35n^2} R_{\mathbf{K}} h_{\mathbf{K}}}{\log z} \left(\frac{z^2 \mathfrak{N}\mathfrak{q}}{X} \right)^{\frac{1}{n}} \right).$$

Choosing $z^2 = \frac{X}{\mathfrak{N}\mathfrak{q}} (n^{35n^2} R_{\mathbf{K}} h_{\mathbf{K}})^{-n}$, we get

$$\frac{h_{\mathbf{K},\mathfrak{q}}}{X} T_1 \leq \frac{1}{\log \frac{z}{n^4 \sqrt{|d_{\mathbf{K}}|}} - 1} \leq \frac{2}{\log \frac{X}{\mathfrak{N}\mathfrak{q}} - \log(n^8 |d_{\mathbf{K}}| n^{35n^3} (R_{\mathbf{K}} h_{\mathbf{K}})^n)}$$

and this completes the proof of our theorem. \square

8. FINDING ENOUGH PRIMES

Winckler in his PhD thesis (see Theorem 1.7 of [34]) proved an explicit version of Tchebotarev density Theorem. The case $L = \mathbf{K}$ provides us with the following explicit version of Landau's prime ideal theorem.

Theorem 36 (Winckler). *For every $x \geq \exp(110000n_{\mathbf{K}}(\log(9|d_{\mathbf{K}}|^8))^2)$, we have*

$$\sum_{\mathfrak{N}\mathfrak{p} \leq x} 1 = \text{Li}(x) + O^*(\text{Li}(x^\beta)) + O^*\left(10^{14} x \exp \frac{-\sqrt{\log x}}{12}\right),$$

where β is the possible largest real zero of $\zeta_{\mathbf{K}}$ and $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$ is the usual logarithmic integral function.

This possible exceptional zero β can be controlled by the next lemma.

Lemma 37 (Kadiri and Wong [9], 2022). *Any positive real zero ρ of $\zeta_{\mathbf{K}}(s)$ satisfies $1 - \rho \geq 1/|d_{\mathbf{K}}|^{12}$.*

We need a lower bound for the number of prime ideals of degree one, meaning we need to remove the primes of degree > 1 from the estimate of Theorem 36. This, and more, is achieved by using the next Lemma.

Lemma 38. *We have*

$$\sum_{\substack{\mathfrak{p}, k, \\ \mathfrak{N}\mathfrak{p}^k > p, \\ \mathfrak{N}\mathfrak{p} \leq X}} 1 \leq 3.64 n_{\mathbf{K}} \sqrt{X}.$$

The summation is over powers $k \geq 2$ for primes of any degree, or of powers $k \geq 1$ for primes of degree > 1 . We denote by p the rational prime below \mathfrak{p} .

Proof. Each prime p has at most $n_{\mathbf{K}}$ prime ideals above itself. For each such prime, say of norm p^f , the contribution to the sum is at most $\frac{\log X}{f \log p} \leq (\log X)/\log 2$ and a prime p that contributes verifies $p \leq \sqrt{X}$. We use the bound for $\pi(x) \leq 1.26x/\log x$ from [25, Corollary 1] by Rosser and Schoenfeld valid for $x = \sqrt{X} > 1$. The lemma follows. \square

Lemma 39. *When $x \geq \exp(|d_{\mathbf{K}}|^{30})$, we have*
$$\sum_{\mathfrak{p}}^b 1 \geq \frac{x}{\log x} + \frac{(1 - 10^{-10})x}{(\log x)^2}.$$

Proof. Let $\pi_{\mathbf{K}}^b(x)$ be the quantity to estimate. We start by combining Theorem 36 together with Lemma 38 to obtain

$$\pi_{\mathbf{K}}^b(x) \geq \text{Li}(x) - \text{Li}(x^\beta) - 10^{14}x \exp\left(\frac{-\sqrt{\log x}}{12}\right) - 3.64n_{\mathbf{K}}\sqrt{x}.$$

By Theorem 37, we have $\beta \leq 1 - |d_{\mathbf{K}}|^{-115}$, while, by Lemma 9, we have $n_{\mathbf{K}} \leq 3 \log |d_{\mathbf{K}}|$. This gives us a lower bound solely in terms of the discriminant. We also need to handle the logarithmic integral. Using integration by parts, we find that

$$\text{Li}(y) = \frac{y}{\log y} - \frac{2}{\log 2} + \int_2^y \frac{dt}{\log^2 t} \geq \frac{y}{\log y} - \frac{2}{\log 2} + \frac{\text{Li}(y)}{\log y}.$$

We first deduce from this that $\text{Li}(y) \geq y/\log y$ when $y \geq 7.5$, by neglecting the additional term $\text{Li}(y)/\log y$. On incorporating it, we get $\text{Li}(y) \geq \frac{y}{\log y} + \frac{y}{(\log y)^2}$. As for an upper bound, we work rather trivially:

$$\text{Li}(y) \leq \int_2^{\sqrt{y}} \frac{dt}{\log 2} + \int_{\sqrt{y}}^y \frac{dt}{\log \sqrt{y}} \leq \frac{2y}{\log y} + \frac{\sqrt{y}}{\log 2} \leq \frac{3y}{\log y}$$

when $y \geq 20$. This leads to, with $d = |d_{\mathbf{K}}|$,

$$\begin{aligned} \left(\frac{\log x}{x} \pi_{\mathbf{K}}^b(x) - 1\right) \log x &\geq 1 - \frac{3e^{-\frac{\log x}{d^{12}}}}{\beta} \log x - 10^{14}(\log x)^2 e^{-\frac{\sqrt{\log x}}{12}} - 12(\log d) \frac{(\log x)^2}{\sqrt{x}}, \\ &\geq 1 - 6d^{30}e^{-d^{18}} - 10^{14}d^{60}e^{-\frac{d^{15}}{12}} - 12(\log d)d^{60}e^{-\frac{d^{30}}{2}} \\ &\geq 1 - 10^{-10} \end{aligned}$$

since $d \geq 2$. This completes the proof of the lemma. \square

Lemma 40. *When $x \geq \exp(|d_{\mathbf{K}}|^{30})$, we have*
$$\sum_{\mathfrak{p}}^b w_0(\mathfrak{N}\mathfrak{p}/x) \geq \frac{x\|w_0\|_1}{\log x} + \frac{x\|w_0\|_1}{5(\log x)^2}.$$

Proof. Since w_0 is bounded above by 1 and has support within $[0, 1]$, we may again use Lemma 38 to handle the condition ‘ \mathfrak{p} of degree 1’. Whence, on denoting $T(w_0)$ the sum to be studied, we find that

$$T(w_0) \geq \sum_{\mathfrak{p}} w_0(\mathfrak{N}\mathfrak{p}/x) - 3.64n_{\mathbf{K}}\sqrt{x}.$$

As $w_0(y) = -\int_y^1 w'_0(t)dt$, we get

$$T(w_0) \geq -\int_0^1 \left(\sum_{\mathfrak{np} \leq tx} 1 \right) w'_0(t)dt - 3.64n_{\mathbf{K}}\sqrt{x}.$$

Of course $w'_0(t) = 0$ when $t \leq 1/10$, so we may assume that $tx \geq x/10$ which is thus larger than $\exp(110000n_{\mathbf{K}}(\log(9|d_{\mathbf{K}}|^8))^2)$. Theorem 36 applies and yields

$$\begin{aligned} T(w_0) &\geq -\int_0^1 \text{Li}(xt)w'_0(t)dt - \|w'_0\|_1 \left(\text{Li}(x^\beta) + 10^{14}x \exp \frac{-\sqrt{\log x}}{12} \right) - 3.64n_{\mathbf{K}}\sqrt{x} \\ &\geq \int_0^1 \frac{xw_0(t)dt}{\log(xt)} - \|w'_0\|_1 \left(\text{Li}(x^\beta) + 10^{14}x \exp \frac{-\sqrt{\log x}}{12} \right) - 3.64n_{\mathbf{K}}\sqrt{x} \\ &\geq \frac{x\|w_0\|_1}{\log x} + \int_0^1 \frac{xw_0(t)\log(1/t)dt}{(\log x)\log(xt)} - \|w'_0\|_1 \left(\text{Li}(x^\beta) + 10^{14}x \exp \frac{-\sqrt{\log x}}{12} \right) - 3.64n_{\mathbf{K}}\sqrt{x}. \end{aligned}$$

All of that is valid for a rather general non-negative function w . When it comes to $w = w_0$, we have $10\sqrt{n_{\mathbf{K}}}\|w_0\|_1 \in [2, 15]$ by Lemma 7 and $\|w'_0\|_1 = 2$. Further, by applying Lemma 9, we get $\sqrt{n_{\mathbf{K}}} \leq \sqrt{(\log |d_{\mathbf{K}}|)/\log(\pi/2)}$ which is not more than $|d_{\mathbf{K}}|^2$. We finally notice that

$$\|w_0 \log(1/t)\|_1 \geq \int_{1/10}^{11/20} w_0(t) \log(1/t)dt \geq \log(20/11) \frac{\|w_0\|_1}{2} \geq \frac{\|w_0\|_1}{4}.$$

Hence, and following estimates very similar to the ones done during the proof of Lemma 39, we find that

$$\begin{aligned} T(w_0) &\geq \frac{x\|w_0\|_1}{\log x} + \frac{x\|w_0\|_1}{4(\log x)^2} \left(1 - 832\sqrt{\log |d_{\mathbf{K}}|}x^{\beta-1} \log x - 757(\log |d_{\mathbf{K}}|)^{3/2}x^{-1/2}(\log x)^2 - 10^{-6} \frac{\sqrt{\log |d_{\mathbf{K}}|}}{\log x} \right) \\ &\geq \frac{x\|w_0\|_1}{\log x} + \frac{x\|w_0\|_1}{5(\log x)^2} \end{aligned}$$

as required. \square

9. PROOF OF THEOREM 2

In order to prove Theorem 2, we need the following lemmas.

Lemma 41. *We have $n_{\mathbf{K}}^{48n_{\mathbf{K}}^3}(R_{\mathbf{K}}h_{\mathbf{K}})^{n_{\mathbf{K}}} \geq 10^{25n_{\mathbf{K}}}n_{\mathbf{K}}^{7n_{\mathbf{K}}}$.*

Proof. By [4], we have $R_{\mathbf{K}}/|\mu_{\mathbf{K}}| \geq 9/100$ which implies $R_{\mathbf{K}} \geq 9/100$. It is thus enough to check the inequality

$$n_{\mathbf{K}}^{48n_{\mathbf{K}}^3} \left(\frac{9}{100} \right)^{n_{\mathbf{K}}} \geq 10^{25n_{\mathbf{K}}}n_{\mathbf{K}}^{7n_{\mathbf{K}}}$$

which is readily seen to hold true as $n_{\mathbf{K}} \geq 2$. \square

Lemma 42 (Kneser's Theorem [10], 1953). *Let G be a finite abelian group and \mathcal{B} be a non-empty subset of G . Also let $H = \{g \in G \mid g + \mathcal{B} + \mathcal{B} = \mathcal{B} + \mathcal{B}\}$ be the stabiliser of the set $\mathcal{B} + \mathcal{B}$. If \mathcal{B} intersects λ many cosets of H , then*

$$|\mathcal{B} + \mathcal{B}| \geq (2\lambda - 1)|H|.$$

Proof of Theorem 2. For any un-ramified prime ideal \mathfrak{p} of degree one, we denote $\mathfrak{N}(\mathfrak{p})$ by p . Also let \mathcal{A} be the subset of $H_{\mathfrak{q}}(\mathbf{K})$ defined by

$$(31) \quad \mathcal{A} = \{[\mathfrak{a}] \in H_{\mathfrak{q}}(\mathbf{K}) \mid \exists \mathfrak{p} \text{ with } \mathfrak{N}(\mathfrak{p}) = p < X, [\mathfrak{p}] = [\mathfrak{a}]\}.$$

Let us set $u(\mathbf{K}) = n_{\mathbf{K}}^{48n_{\mathbf{K}}^3} |d_{\mathbf{K}}|^6 (R_{\mathbf{K}} h_{\mathbf{K}})^{n_{\mathbf{K}}}$. When $X > u(\mathbf{K}) \mathfrak{N}_{\mathfrak{q}}$, Theorem 3 gives us that

$$\sum_{\mathfrak{N}\mathfrak{p} \leq X}^b 1 \leq \frac{2|\mathcal{A}|X}{h_{\mathbf{K},\mathfrak{q}} \log \frac{X}{u(\mathbf{K}) \mathfrak{N}_{\mathfrak{q}}}}.$$

On the other side and when $X \geq \exp(|d_{\mathbf{K}}|^{30})$, Lemma 39 ensures us that

$$\frac{X}{2(\log X)^2} + \frac{X}{\log X} \leq \sum_{\mathfrak{N}\mathfrak{p} \leq X}^b 1.$$

On assuming that the required conditions hold true, a comparison of both inequalities gives us

$$\frac{|\mathcal{A}|}{h_{\mathbf{K},\mathfrak{q}}} \geq \frac{\log \frac{X}{u(\mathbf{K}) \mathfrak{N}_{\mathfrak{q}}}}{2 \log X} + \frac{\log \frac{X}{u(\mathbf{K}) \mathfrak{N}_{\mathfrak{q}}}}{4(\log X)^2}.$$

Take $X = (t(\mathbf{K}) \mathfrak{N}_{\mathfrak{q}})^3$, where $t(\mathbf{K})$ is defined in (1). The above inequality gives us

$$(32) \quad \frac{|\mathcal{A}|}{h_{\mathbf{K},\mathfrak{q}}} \geq \frac{1}{3} + \frac{1}{18 \log t(\mathbf{K}) + 18 \log \mathfrak{N}_{\mathfrak{q}}}.$$

From now onwards, let us set $G = H_{\mathfrak{q}}(\mathbf{K})$. Also let H be the stabilizer of $\mathcal{A} \cdot \mathcal{A}$ in G , y be the index of H in G and λ be the number of cosets of H that intersect \mathcal{A} .

By Kneser's Theorem, we have

$$|\mathcal{A} \cdot \mathcal{A}| \geq (2\lambda - 1)|H| = \frac{2\lambda - 1}{y} |G|.$$

We further know that

$$\lambda \geq \left\lceil \frac{|\mathcal{A}|}{|H|} \right\rceil$$

and that λ is an integer. Furthermore, to be sure that $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G$, we only need

$$(33) \quad |\mathcal{A}| + \frac{2\lambda - 1}{y} |G| > |G| \quad \text{i.e.} \quad \frac{|\mathcal{A}|}{|G|} + \frac{2\lambda - 1}{y} > 1.$$

This follows from the following observation. Given any $[\mathfrak{b}] \in G$, we consider the set

$$[\mathfrak{b}]\mathcal{A}^{-1} = \{[\mathfrak{b}][\mathfrak{a}] : [\mathfrak{a}^{-1}] \in \mathcal{A}\}.$$

For any $[\mathfrak{b}] \in G$ if $[\mathfrak{b}]\mathcal{A}^{-1} \cap \mathcal{A} \cdot \mathcal{A} \neq \emptyset$, then $[\mathfrak{b}] \in \mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A}$. However $|[\mathfrak{b}]\mathcal{A}^{-1}| = |\mathcal{A}|$. Therefore by Pigeon-hole principle if $|\mathcal{A}| + |\mathcal{A} \cdot \mathcal{A}| > |G|$, then $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G$.

From the lower bound $\lambda \geq |\mathcal{A}|/|H|$, we observe that it suffices to show that

$$(34) \quad 3 \frac{|\mathcal{A}|}{|G|} - \frac{1}{y} > 1.$$

Let us discuss the possible values of y .

■ *Large values of y .* By (32), the above inequality (34) will be satisfied if we have

$$\frac{1}{9 \log t(\mathbf{K}) + 9 \log \mathfrak{N}_{\mathfrak{q}}} - \frac{1}{y} > 0$$

This implies that the inequality (34) holds when $y > 9 \log t(\mathbf{K}) + 9 \log \mathfrak{N}_{\mathfrak{q}}$.

■ *The case $y = 1$.* This is the case when $H = G$ and thus $\mathcal{A} \cdot \mathcal{A} = G$. Hence $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G$.

■ *The case $y = 2$.* So H is a quadratic subgroup of G . Using Lemma 41, we have

$$X \geq (10^{25n_{\mathbf{K}}} n_{\mathbf{K}}^{7n_{\mathbf{K}}} |d_{\mathbf{K}}|^{4/3} \mathfrak{N}\mathfrak{q})^3 \geq 10^{25n_{\mathbf{K}}} n_{\mathbf{K}}^{7n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^3.$$

Theorem 4 tells us that the subgroup generated by \mathcal{A} is G . It follows that \mathcal{A} contains an element of $G \setminus H$. By Theorem 6, we have that the subset \mathcal{A} also contains an element of H . Indeed, we have $X \geq 8(10^{31} n_{\mathbf{K}}^7)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^4 \mathfrak{N}\mathfrak{q}^2$. As $\mathcal{A} \cdot \mathcal{A}$ is a union of cosets mod H , we conclude that $\mathcal{A} \cdot \mathcal{A} = G$, whence $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G$.

■ *Medium values of $y \not\equiv 2 \pmod{3}$.* Since $\mathcal{A} \cdot \mathcal{A}$ is a union of cosets modulo H , it is enough to check that $\mathcal{A}/H \cdot (\mathcal{A} \cdot \mathcal{A}/H)$ covers G/H . This means that it is enough to assume that \mathcal{A} is a union of $(\lambda$ many) cosets modulo H , from which we infer that we only need to prove that $\frac{\lambda}{y} + \frac{2\lambda-1}{y} > 1$, i.e. $\lambda > (y+1)/3$. Note that (32) implies that $\lambda > y/3$. When $y \equiv 0 \pmod{3}$, then $\lambda \geq y/3 + 1$. When $y \equiv 1 \pmod{3}$, the integer part of $y/3$ is at least $(y+2)/3$, so that the inequality $\lambda > (y+1)/3$ is satisfied. There remains the values of $y \geq 3$ that are $2 \pmod{3}$ and below $9 \log t(\mathbf{K}) + 9 \log \mathfrak{N}\mathfrak{q}$.

■ *Medium values of $y \equiv 2 \pmod{3}$ and $y \geq 5$.* We use Theorem 35 for $y \leq 9 \log t(\mathbf{K}) + 9 \log \mathfrak{N}\mathfrak{q}$ together with Lemma 40. This gives us, with $n_{\mathbf{K}} = n$,

$$\begin{aligned} \frac{X \|w_0\|_1}{\log X} + \frac{X \|w_0\|_1}{5(\log X)^2} &\leq \sum_{\mathfrak{p}}^b w_0(\mathfrak{N}\mathfrak{p}/X) \leq \sum_{[\mathfrak{a}]H \in \mathcal{A}/H} \sum_{\mathfrak{p} \in [\mathfrak{a}]H} w_0(\mathfrak{N}\mathfrak{p}/X) \\ &\leq \lambda \frac{2 \|w_0\|_1 X}{y \log \frac{X \|w_0\|_1 / [20000(\|w_0^{(n+3)}\|_{\infty} + 5 \|w_0\|_1) \log(|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q})^n]}{2^{22n} y \sqrt{|d_{\mathbf{K}}|^3 \mathfrak{N}(\mathfrak{q})}}}. \end{aligned}$$

Lemma 7 gives us

$$\frac{\|w_0\|_1}{\|w_0^{(n+3)}\|_{\infty} + 5 \|w_0\|_1} \geq \frac{2\sqrt{n}}{(40n)^{n+4} + 75\sqrt{n}} \geq \frac{\sqrt{n}}{(40n)^{n+4}}.$$

We have thus obtained the inequality

$$\log \frac{\sqrt{n}}{20000(40n)^{n+4} 2^{22n} |d_{\mathbf{K}}|^{3/2}} \frac{X}{9 \log(t(\mathbf{K}) \mathfrak{N}\mathfrak{q}) \sqrt{\mathfrak{N}(\mathfrak{q})} \log(|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q})^n} \leq \frac{2\lambda}{y} \log X$$

i.e. also

$$1 - \frac{1}{\log X} \log \left(180000 \cdot 2^{22n} (40n)^{n+4} |d_{\mathbf{K}}|^{3/2} \log(t(\mathbf{K}) \mathfrak{N}\mathfrak{q}) \sqrt{\mathfrak{N}(\mathfrak{q})} \log(|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q})^n \right) \leq \frac{2\lambda}{y}.$$

We readily check that $180000 \cdot 2^{22n} (40n)^{n+4} |d_{\mathbf{K}}|^{3/2} \leq \sqrt{t(\mathbf{K})}$, so that the above inequality implies that

$$1 - \frac{1}{3 \log V} \log(\sqrt{V} (\log V)^{1+n}) \leq \frac{2\lambda}{y},$$

where $V = t(\mathbf{K}) \mathfrak{N}\mathfrak{q}$, while we are to prove that $\lambda > (y+1)/3$ (see the discussion of the case “Medium values of $y \not\equiv 2 \pmod{3}$ ”). We should thus prove that $\log(\sqrt{V} (\log V)^{1+n}) < \frac{3}{5} \log V$, i.e.

$$(\log V)^{10+10n} < V.$$

Again by using Lemma 9, we see that $n \leq \frac{1}{30 \log(\pi/2)} \log \log t(\mathbf{K}) \leq \frac{1}{13} \log \log V$. With $W = \log V$, we should thus prove that

$$10 \log W + \frac{10}{13} (\log W)^2 < W.$$

This inequality happens to be always satisfied, which concludes our proof. \square

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