EXPLICIT UPPER BOUNDS ON THE AVERAGE OF EULER-KRONECKER CONSTANTS OF NARROW RAY CLASS FIELDS

NEELAM KANDHIL, RASHI LUNIA AND JYOTHSNAA SIVARAMAN

ABSTRACT. For a number field $\mathbf{K}$, the Euler-Kronecker constant $\gamma_{\mathbf{K}}$ associated to $\mathbf{K}$ is an arithmetic invariant the size and nature of which is linked to some of the deepest questions in number theory. This theme was given impetus by Ihara who obtained bounds, both unconditional as well as under GRH for Dedekind zeta functions. In this note, we study the analogous constants associated to the narrow ray class fields of an imaginary quadratic field. Our goal for studying such families is twofold. First to show that for such families, the conditional bounds obtained by Ihara can be improved on the average, again under GRH for Dedekind zeta functions. Further, our family of number fields are non-abelian while such average bounds have earlier been studied for cyclotomic fields. The technical part of our work is to make the dependence of these upper bounds on the ambient number field explicit. Such explicit dependence is essential to further our objective.

1. INTRODUCTION

The Euler-Mascheroni constant γ, introduced by Euler [9] is given by following limit

$$\gamma = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right).$$

It is not known if γ is rational or irrational. It is perhaps prudent not to study γ in isolation, but as a member of some family whose behaviour can be studied en masse. One such family was envisaged by Ihara [19] in 2005, the members of which are referred to as Euler-Kronecker constants. To motivate these constants, let us note that γ is the constant term in the Laurent series expansion of the Riemann zeta function around $s = 1$. Following this line of thought, one can define the Euler-Kronecker constant of a number field $\mathbf{K}$ as follows.

The Dedekind zeta function of a number field $\mathbf{K}$ is given by

$$\zeta_{\mathbf{K}}(s) = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_{\mathbf{K}}\\ \mathfrak{a}\neq(0)}}\frac{1}{\mathfrak{N}\mathfrak{a}^s}, \qquad \Re(s) > 1.$$

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It is known to have a meromorphic continuation to the entire complex plane with only a simple pole at the point $s = 1$. Thus we have a Laurent series expansion for $\zeta_{\mathbf{K}}$ around $s = 1$. This is given by

$$\zeta_{\mathbf{K}}(s) = \frac{\rho_{\mathbf{K}}}{s-1} + c_{\mathbf{K}} + \mathrm{O}(s-1). \tag{1}$$

Following Ihara [19], we now define the Euler-Kronecker constant of $\mathbf{K}$ as

$$\gamma_{\mathbf{K}} := c_{\mathbf{K}}/\rho_{\mathbf{K}}.$$

Since $\gamma_{\mathbf{K}}$ is evidently real, it is natural to ask its sign. Ihara [20] conjectured that $\gamma_{\mathbf{K}}$ is positive when $\mathbf{K}$ is a cyclotomic field. However it has been shown by Ford, Luca and Moree [10] that $\gamma_{\mathbb{Q}(\zeta_p)}$ is negative for $p = 964477901$. Further, Ihara [20] showed that $\gamma_{\mathbf{K}}$ is negative infinitely often under the Generalised Riemann Hypothesis (GRH) for $\zeta_{\mathbf{K}}(s)$.

Another possible line of investigation is to obtain bounds for $\gamma_{\mathbf{K}}$ in terms of other invariants of the ambient number field $\mathbf{K}$. This was initiated by Ihara in his papers [19], [20].

For a number field $\mathbf{K}$ of degree at least 3, he showed under GRH that

$$-2(n_{\mathbf{K}}-1)\frac{D_{\mathbf{K}}-n_{\mathbf{K}}+1}{D_{\mathbf{K}}+n_{\mathbf{K}}-1}\left(\log\frac{D_{\mathbf{K}}}{n_{\mathbf{K}}-1}+1\right)-1 \le \gamma_{\mathbf{K}} \le \left(\frac{D_{\mathbf{K}}+1}{D_{\mathbf{K}}-1}\right)(2\log D_{\mathbf{K}}+1), \tag{2}$$

where $D_{\mathbf{K}} = \log\sqrt{|d_{\mathbf{K}}|}$ and $n_{\mathbf{K}}$ and $d_{\mathbf{K}}$ denote the degree and discriminant of $\mathbf{K}$ respectively.

Recently, Dixit [6] showed the following unconditional bounds for $|\gamma_{\mathbf{K}}|$ for certain families of number fields. We say that a number field $\mathbf{K}$ is almost normal if there is a tower of number fields

$$\mathbf{K} = \mathbf{K}_n \supset \mathbf{K}_{n-1} \supset \cdots \supset \mathbf{K}_1 = \mathbb{Q},$$

such that $\mathbf{K}_{i+1}/\mathbf{K}_i$ is Galois for all $1 \le i \le n$.

Theorem 1.1. (Dixit, [6]) *Let $\mathbf{K}$ be an almost normal number field, not containing any quadratic subfields. Then*

$$\gamma_{\mathbf{K}} \ll (\log|d_{\mathbf{K}}|)^4 n_{\mathbf{K}}^3.$$

On the other hand if $\mathbf{K}$ is a number field with solvable normal closure, not containing any quadratic subfields. Then

$$\gamma_{\mathbf{K}} \ll (\log|d_{\mathbf{K}}|)^{c\log\log|d_{\mathbf{K}}|}$$

for some absolute positive constant c.

For an arbitrary number field, Dixit and Ram Murty [7] related $\gamma_{\mathbf{K}}$ to the hypothetical Siegel zero of $\zeta_{\mathbf{K}}(s)$. They showed the following.

Theorem 1.2. (Dixit and Ram Murty, [7]) *For any number field $\mathbf{K}$, we have*

$$\gamma_{\mathbf{K}} \ll \log|d_{\mathbf{K}}|$$

if $\zeta_{\mathbf{K}}(s)$ has no Siegel zero. If it has a Siegel zero β_0, then the bound is

$$\gamma_{\mathbf{K}} = \frac{1}{2\beta_0(1-\beta_0)} + \mathrm{O}(\log |d_{\mathbf{K}}|).$$

In the case of cyclotomic fields, Ihara suggested the following:

Conjecture 1. (Ihara, [19]) *There are positive constants $0 < a_0, a_1 \leq 2$ such that for any m sufficiently large and any $\epsilon > 0$, we have*

$$(a_0 - \epsilon) \log m < \gamma_{\mathbb{Q}(\zeta_m)} < (a_1 + \epsilon) \log m,$$

where $\mathbb{Q}(\zeta_m)$ denotes the m-th cyclotomic field.

The results of Ford, Luca and Moree [10] indicate that $\gamma_{\mathbb{Q}(\zeta_m)}$ can be negative, thus the above lower bound suggested by Ihara does not hold.

Assuming GRH for Dedekind zeta functions of cyclotomic fields, Ihara, Kumar Murty and Shimura [21] proved that

$$\gamma_{\mathbb{Q}(\zeta_m)} \ll (\log m)^2.$$

In 2010, Badzyan [3] improved this result and showed that under GRH for Dedekind zeta functions of cyclotomic fields,

$$\gamma_{\mathbb{Q}(\zeta_m)} \ll (\log m)(\log \log m).$$

In 2011, Kumar Murty [23] proved that the upper bound of $|\gamma_{\mathbb{Q}(\zeta_p)}|$ coming from Ihara's conjecture is true on average when restricted to primes. More precisely,

Theorem 1.3. (Kumar Murty, [23]) *We have*

$$\frac{1}{\pi^*(Q)} \sum_{\frac{1}{2}Q < p \leq Q} |\gamma_{\mathbb{Q}(\zeta_p)}| \ll \log Q,$$

where the sum is over prime numbers p in the interval $(\frac{1}{2}Q, Q]$ and $\pi^*(Q)$ denotes the number of primes in this interval.

Fouvry in [11] studied the case of $\mathbb{Q}(\zeta_m)$ where m is not necessarily prime and showed the following.

Theorem 1.4. (Fouvry, [11]) *Uniformly for $Q \geq 3$, we have*

$$\frac{1}{Q} \sum_{\frac{1}{2}Q < m \leq Q} \gamma_{\mathbb{Q}(\zeta_m)} = \log Q + \mathrm{O}(\log \log Q).$$

The aforementioned result of Fouvry has recently been refined by Hong, Ono and Zhang [18] under the following conjecture.

Conjecture 2. [Elliott and Halberstam conjecture [8]] *For every real number $\theta < 1$ and for every positive integer $A > 0$, one has*

$$\sum_{q \le x^\theta} \max_{y \le x} \max_{(a,q)=1} \left| \sum_{\substack{n \le y \\ n \equiv a \bmod q}} \Lambda(n) - \frac{y}{\varphi(q)} \right| \ll_{A,\theta} \frac{x}{\log^A x}$$

for all real numbers $x > 2$. Here Λ is the Von Mangoldt function.

The precise result of Hong, Ono and Zhang is as follows.

Theorem 1.5. (Hong, Ono and Zhang, [18]) *Assume the Elliott-Halberstam conjecture, for $Q \to \infty$, we have*

$$\frac{1}{Q} \sum_{Q < m \le 2Q} |\gamma_{\mathbb{Q}(\zeta_m)} - \log m| = o(\log Q),$$

where the sum is over integers m.

We observe here that in a recently published work, Dixit and Ram Murty [7] have given an alternative proof of the Theorem 1.3 using certain new observations for arbitrary number fields. Further they use the same method to show the following result.

Theorem 1.6. (Dixit and Ram Murty, [7]) *Assuming the Elliott-Halberstam conjecture, we have*

$$\sum_{Q < p \le 2Q} |\gamma_{\mathbb{Q}(\zeta_p)} - \log p| = o(Q).$$

Since the above results focus on cyclotomic fields, the raison d'etre of our work is to carry out such investigations over families of Galois number fields which are non-abelian. One such family originates from narrow ray class fields of imaginary quadratic fields and it is the study of this family which we undertake. More precisely, we prove the following theorem.

Theorem 1.7. *Let $\mathbf{K}$ be an imaginary quadratic field with class number $h_{\mathbf{K}}$. For a non-zero principal prime ideal $\mathfrak{q}$ of $\mathcal{O}_{\mathbf{K}}$, let $\mathbf{K}(\mathfrak{q})$ be the narrow ray class field modulo $\mathfrak{q}$. Assume GRH for the Dedekind zeta functions of $\mathbf{K}(\mathfrak{q})$ for all non-zero principal prime ideals $\mathfrak{q}$. Then for $Q \ge 8\exp(8 \cdot 10^{45}|d_{\mathbf{K}}|)$, we have*

$$\frac{1}{\pi^*(Q)} \sideset{}{'}\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \le Q} |\gamma_{\mathbf{K}(\mathfrak{q})}| < |\gamma_{\mathbf{K}}| + (6000 h_{\mathbf{K}}^2 + 10^{17} h_{\mathbf{K}} + 11)\log Q, \tag{3}$$

where $'$ over the sum indicates that the sum is over principal prime ideals of $\mathcal{O}_{\mathbf{K}}$ and $\pi^*(Q)$ denotes the number of principal prime ideals $\mathfrak{q}$ of $\mathcal{O}_{\mathbf{K}}$ with norm in the interval $(\frac{1}{2}Q, Q]$.

At this juncture, it is perhaps worthwhile to highlight a few points about this result.

- The explicit dependence of the upper bound in (3) on $h_{\mathbf{K}}$ is what allows us to improve the lower bound in (2) for infinitely many ray class fields of $\mathbf{K}$. To the best of our knowledge, this is the first instance of a non-abelian family over $\mathbb{Q}$ for which the bound has been improved.
- Even though our final result is conditional under GRH, at various places, we have chosen unconditional bounds and not those under GRH unless essential.
- We have not striven to optimise the numerical constants in the above theorem.

This paper is organised as follows. In the next section we outline the strategy of our proof. In section 3, we list some results needed for our proofs while in section 4 we derive some lemmas and propositions needed to prove our theorem. The penultimate section gives a complete proof of Theorem 1.7. Finally in the last section we show that there are infinitely many non-abelian fields in the family of narrow ray class fields considered by us.

2. Strategy of proof

In this section, we quickly outline the strategy of the proof of Theorem 1.7. In [21], Ihara, Kumar Murty and Shimura define, for a number field $\mathbf{K}$ and Hecke character χ modulo $\mathfrak{q}$ of $\mathbf{K}$, (cf. section 3)

$$\Phi_{\mathbf{K},\chi}(x) = \frac{1}{x-1}\int_1^x \Big(\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\chi(\mathfrak{a})\Big) dt, \qquad (x>1)$$

and show that for $\chi \neq \chi_0$, (χ_0 is the principal Hecke character)

$$-\frac{L'}{L}(1,\chi) = \lim_{x\to\infty} \Phi_{\mathbf{K},\chi}(x),$$

where $L(s,\chi)$ is the Hecke L function corresponding to χ. We know that

$$\gamma_{\mathbf{K}(\mathfrak{q})} = \lim_{s\to 1}\left(\frac{\zeta'_{\mathbf{K}(\mathfrak{q})}}{\zeta_{\mathbf{K}(\mathfrak{q})}}(s) + \frac{1}{s-1}\right) \text{ and } \zeta_{\mathbf{K}(\mathfrak{q})}(s) = \zeta_{\mathbf{K}}(s)\prod_{\chi\neq\chi_0} L(s,\chi^*)$$

where χ^* is the primitive character inducing the Hecke character χ modulo $\mathfrak{q}$. It now follows that

$$|\gamma_{\mathbf{K}(\mathfrak{q})}| \le |\gamma_{\mathbf{K}}| + \left|\sum_{\chi\neq\chi_0}\Phi_{\mathbf{K},\chi}(x)\right| + \left|\sum_{\chi\neq\chi_0}\Big(\Phi_{\mathbf{K},\chi}(x) - \Phi_{\mathbf{K},\chi^*}(x)\Big)\right| + \sum_{\chi\neq\chi_0}\left|\frac{L'}{L}(1,\chi^*) + \Phi_{\mathbf{K},\chi^*}(x)\right|.$$

The crux of the proof lies in estimating each of these sums on average as we vary norm of $\mathfrak{q}$ between $Q/2$ and Q where Q is a large real parameter. In Theorem 5.2, we estimate the average of the second term on the right by breaking the average into two parts. For small primes, we use an analogue of the Brun-Titchmarsh theorem (see Theorem 4.7) and some elementary estimates. For large primes, we use the explicit Chebotarev density theorem under GRH due to Grenie and Molteni (see Theorem 3.12). The third term is relatively harmless and we estimate it directly (estimated after proof of Theorem 5.2). To estimate the last term we use an explicit formula for

$$\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x)$$

proved by Ihara, Kumar Murty and Shimura in [21] and deduce bounds under the assumption of GRH. Theorem 1.7 is obtained by combining all these estimates.

3. Preliminaries

3.1. **Notation.** Throughout this article, $\mathbf{K}$ will denote an algebraic number field of degree $n_{\mathbf{K}}$, contained in $\mathbb{C}$. Let $\mathcal{O}_{\mathbf{K}}$ be its ring of integers and $d_{\mathbf{K}}$ its discriminant with respect to $\mathbb{Q}$. Further, let $h_{\mathbf{K}}$ denote the class number of $\mathcal{O}_{\mathbf{K}}$ and $\mu_{\mathbf{K}}$ the set of roots of unity in $\mathcal{O}_{\mathbf{K}}$. For any finite set S, we use $|S|$ to denote its cardinality.

For an ideal $\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}$, let $\mathfrak{N}\mathfrak{a}$ denote the order of the finite group $\mathcal{O}_{\mathbf{K}}/\mathfrak{a}$. We use $\mathcal{P}_{\mathbf{K}}$ to denote the set of all prime elements of $\mathcal{O}_{\mathbf{K}}$. Further, $\mathfrak{p}$ and $\mathfrak{q}$ shall denote prime ideals in $\mathcal{O}_{\mathbf{K}}$. We define the generalized Euler-phi function as follows: For any non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}$,

$$\varphi_{\mathbf{K}}(\mathfrak{a}) = \mathfrak{N}\mathfrak{a} \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right).$$

Let $\rho_{\mathbf{K}}$ denote the residue of the Dedekind zeta function of $\mathbf{K}$ at $s = 1$. For an imaginary quadratic field $\mathbf{K}$, the class number formula gives us

$$\rho_{\mathbf{K}} = \frac{2\pi h_{\mathbf{K}}}{|\mu_{\mathbf{K}}|\sqrt{|d_{\mathbf{K}}|}}. \tag{4}$$

The following result gives bounds for $\rho_{\mathbf{K}}$ in terms of $d_{\mathbf{K}}$.

Lemma 3.1. (Deshouillers, Gun, Ramaré and Sivaraman [5]) *We have*

$$\frac{9}{25\sqrt{|d_{\mathbf{K}}|}} \leq \rho_{\mathbf{K}} \leq 6\left(\frac{2\pi^2}{5}\right)^{n_{\mathbf{K}}} |d_{\mathbf{K}}|^{1/4}.$$

We note that when $\mathbf{K}$ is imaginary quadratic, the above can be refined to obtain

$$\frac{\pi}{3\sqrt{|d_{\mathbf{K}}|}} \leq \rho_{\mathbf{K}} \leq 6\left(\frac{2\pi^2}{5}\right)^{2} |d_{\mathbf{K}}|^{1/4}.$$

For number fields $\mathbf{K} \subset \mathbf{L}$, we recall that the relative discriminant $\Delta_{\mathbf{L}/\mathbf{K}}$ is defined as the $\mathcal{O}_{\mathbf{K}}$-ideal generated by the discriminants of all bases of $\mathbf{L}$ over $\mathbf{K}$ contained in $\mathcal{O}_{\mathbf{L}}$. In this context, we record the following result.

Lemma 3.2. (Neukirch, page 202, [27]) *For a tower of fields $\mathbb{Q} \subset \mathbf{K} \subset \mathbf{L}$, let $[\mathbf{L} : \mathbf{K}]$ denote the degree of $\mathbf{L}/\mathbf{K}$ and $\Delta_{\mathbf{L}/\mathbf{K}}$ denote the relative discriminant of $\mathbf{L}$ over $\mathbf{K}$. Then we have the following relation*

$$|d_{\mathbf{L}}| = |d_{\mathbf{K}}|^{[\mathbf{L}\,:\,\mathbf{K}]}\, \mathfrak{N}\Delta_{\mathbf{L}/\mathbf{K}}.$$

Now we state the Conductor-Discriminant formula which will be used to bound the discriminant in Theorem 5.2.

Theorem 3.3. (Neukirch, page 534 (Conductor-Discriminant formula), [27]) *Let $\mathbf{L}/\mathbf{K}$ be a Galois extension of number fields and $\Delta_{\mathbf{L}/\mathbf{K}}$ denote its relative discriminant. Then we can express it by the following decomposition*

$$\Delta_{\mathbf{L}/\mathbf{K}} = \prod_{\chi \text{ irred.}} f(\chi)^{\chi(1)},$$

where χ varies over the irreducible characters of the Galois group of $\mathbf{L}/\mathbf{K}$ and $f(\chi)$ denotes its Artin conductor (see page 527 of [27]).

Recall that when $\mathbf{K} = \mathbb{Q}$, $\zeta_{\mathbf{K}}(s)$ is the Riemann zeta function $\zeta(s)$. Let $\frac{\Gamma'}{\Gamma}(s)$ and $\frac{\zeta'}{\zeta}(s)$ denote the logarithmic derivatives of the gamma function and Riemann zeta function evaluated at s respectively. Now we record the bounds on these functions.

Lemma 3.4. (Hall and Tenenbaum, page 146, [17]) *For real $\sigma > 1$, we have*

$$-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma - 1}.$$

Lemma 3.5. (Ahn and Kwon, [1]) *Assume that $\Re(s) > \frac{1}{2}$. We have*

$$\Re\left(\frac{\Gamma'}{\Gamma}(s)\right) \leq 1.08 \log(|s| + 2).$$

For an arithmetic function f and a positive arithmetic function g, $f(z) = \mathrm{O}^*(g(z))$ means that $|f(z)| \leq g(z)$.

3.2. **Narrow ray class groups and Hecke L-functions.** Let $\mathfrak{c}$ be a non-zero integral ideal of $\mathbf{K}$. For any element $\alpha \in \mathbf{K}$, we say that $\alpha \equiv 1 \bmod {}^*\mathfrak{c}$ if α satisfies the following two conditions:

1. The element α is of the form a/b for $a, b \in \mathcal{O}_{\mathbf{K}}$ where the ideals (a) and (b) are coprime to $\mathfrak{c}$ and $a \equiv b \bmod \mathfrak{c}$.
2. For any real embedding σ of $\mathbf{K}$, $\sigma(\alpha) > 0$.

Definition 1. *Let $I(\mathfrak{c})$ denote the set of non-zero fractional ideals relatively prime to $\mathfrak{c}$ and $P_{\mathfrak{c}}$ the group of non-zero principal fractional ideals $(\alpha) \subset \mathcal{O}_{\mathbf{K}}$ such that $\alpha \equiv 1 \bmod {}^*\mathfrak{c}$. Then the group $I(\mathfrak{c})/P_{\mathfrak{c}}$ is called the narrow ray class group of $\mathbf{K}$ modulo $\mathfrak{c}$, denoted by $H_{\mathfrak{c}}(\mathbf{K})$.*

Definition 2. *A character χ, of $H_{\mathfrak{c}}(\mathbf{K})$ will be called a generalized Dirichlet character modulo $\mathfrak{c}$. We shall use χ_0 to denote the principal generalized Dirichlet character.*

There is a natural restriction from $H_{\mathfrak{c}}(\mathbf{K})$ to $H_{\mathfrak{c}'}(\mathbf{K})$ for all $\mathfrak{c}' \mid \mathfrak{c}$.

Definition 3. *We say that a character of $H_{\mathfrak{c}}(\mathbf{K})$ is primitive if it does not factor through $H_{\mathfrak{c}'}(\mathbf{K})$ for every proper divisor $\mathfrak{c}'$ of $\mathfrak{c}$. Further, the conductor of a generalized Dirichlet character χ modulo $\mathfrak{c}$ is the smallest divisor $\mathfrak{c}'$ of $\mathfrak{c}$ such that χ factors through a character modulo $\mathfrak{c}'$.*

Further for a generalized Dirichlet character χ modulo $\mathfrak{c}$, we define the associated Hecke L-function as

$$L(s,\chi,\mathbf{K}) = \sum_{\substack{0\neq\mathfrak{a}\subseteq\mathcal{O}_{\mathbf{K}}\\ (\mathfrak{a},\mathfrak{c})=\mathcal{O}_{\mathbf{K}}}} \frac{\chi([\mathfrak{a}])}{\mathfrak{N}\mathfrak{a}^s}, \qquad \Re(s)>1,$$

where $[\mathfrak{a}]$ denotes the class of the ideal $\mathfrak{a}$ in $H_{\mathfrak{c}}(\mathbf{K})$. We will henceforth use $L(s,\chi)$ to denote $L(s,\chi,\mathbf{K})$. It is known that when χ is not the principal character, $L(s,\chi)$ has an analytic continuation to the entire complex plane. If χ is principal then $L(s,\chi)$ has a meromorphic continuation to the entire complex plane with only a simple pole at $s=1$. The Generalized Riemann Hypothesis (GRH) for Dedekind zeta functions states that the zeros of a Dedekind zeta function in the strip $0<\Re(s)<1$ lie on the line $\Re(s)=1/2$. Now we state the following lemma of Lagarias and Odlyzko.

Lemma 3.6. (Lagarias and Odlyzko, [24]) *Let χ be a generalized Dirichlet character modulo integral ideal $\mathfrak{c}$ and let $L(s,\chi)$ be the corresponding Hecke L-function. Then we have*

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(s,\overline{\chi}) = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right) - \log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}') - \mathbf{1}_{\chi}\left(\frac{2}{s}+\frac{2}{s-1}\right) - 2\,\frac{\Gamma'_{\chi}}{\Gamma_{\chi}}(s),$$

where the sum is over non-trivial zeros of $L(s,\chi)$, $\mathbf{1}_{\chi}$ is 1 if χ is a principal character otherwise it is 0, $\mathfrak{c}'$ is the conductor of χ and for some fixed integer $0\leq a_{\chi}\leq n_{\mathbf{K}}$,

$$\Gamma_{\chi}(s) := \left[\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right]^{a_{\chi}}\left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right]^{n_{\mathbf{K}}-a_{\chi}}. \tag{5}$$

Let $\chi\neq\chi_0$ be a generalized Dirichlet character on an imaginary quadratic field $\mathbf{K}$ and $L(s,\chi)$ be the corresponding Hecke L- function. Now we consider the following function

$$\Phi_{\chi}(x) := \frac{1}{x-1}\int_1^x \Big(\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}\mathfrak{a}\leq t}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\chi([\mathfrak{a}])\Big)dt, \tag{6}$$

as defined in [21]. Then we have the following result proved by Ihara, Kumar Murty and Shimura.

Theorem 3.7. (Ihara, Kumar Murty and Shimura, [21]) *For $x>1$ and any primitive generalized Dirichlet character χ, we have*

$$\frac{L'}{L}(1,\chi)+\Phi_{\chi}(x) = \frac{1}{x-1}\sum_{\rho}\frac{x^{\rho}-1}{\rho(1-\rho)} + \log\frac{x}{x-1} + \frac{1}{x-1}\log x,$$

where the sum is over all non-trivial zeros of $L(s,\chi)$ counted with multiplicities, and

$$\sum_{\rho}\frac{x^{\rho}-1}{\rho(1-\rho)} = \lim_{T\to\infty}\sum_{|\mathrm{Im}(\rho)|<\mathrm{T}}\frac{x^{\rho}-1}{\rho(1-\rho)}.$$

3.3. **Counting integral ideals.** We now state a result on counting the number of integral ideals of $\mathcal{O}_{\mathbf{K}}$ for number field $\mathbf{K}$ due to Gun, Ramaré and Sivaraman. For any embedding σ of $\mathbf{K}$, the Minkowski embedding θ of $\mathbf{K}$ to $\mathbb{R}^2$ maps x to $(\Re(\sigma(x)), \Im(\sigma(x)))$. In this setup, we have the following counting theorem.

Theorem 3.8. (Gun, Ramaré and Sivaraman, [15]) *Let $\mathfrak{a}, \mathfrak{q}$ be co-prime ideals of $\mathcal{O}_{\mathbf{K}}$, $\mathfrak{C}$ be the ideal class of $\mathfrak{a}\mathfrak{q}$ in the class group of $\mathcal{O}_{\mathbf{K}}$ and $\Lambda(\mathfrak{a}\mathfrak{q})$ be the lattice $\theta(\mathfrak{a}\mathfrak{q})$ in $\mathbb{R}^2$, where θ is as defined above. Also let*

$$S_\beta\left(\mathfrak{a}, \mathfrak{q}, t^2\right) = \{\alpha \in \mathfrak{a} \ : |\theta(\alpha)|^2 \leq t^2, \alpha \equiv \beta \bmod \mathfrak{q}\}$$

for some fix $\beta \in \mathcal{O}_{\mathbf{K}}$. Then for any real number $t \geq 1$, we have

$$\left|S_\beta\left(\mathfrak{a}, \mathfrak{q}, t^2\right)\right| = \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})} t^2 + \ \mathrm{O}^*\left(\frac{10^{13.66}\mathfrak{N}(\mathfrak{C}^{-1})}{|\mathfrak{N}(\mathfrak{a}\mathfrak{q})|^{\frac{1}{2}}} t + 1\right), \tag{7}$$

where

$$\mathfrak{N}(\mathfrak{C}^{-1}) = max_{\mathfrak{b} \in \mathfrak{C}^{-1}} \frac{1}{|\mathfrak{N}(\mathfrak{b})|^{\frac{1}{2}}}.$$

One can ignore 1 in the error term when $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$.

As a corollary the authors of [15, 16] deduce the following two theorems.

Theorem 3.9. (Gun, Ramaré and Sivaraman, [15]) *Let $\mathbf{K}$ be an imaginary quadratic field. Let $\mathfrak{q}$ be an integral ideal of $\mathbf{K}$ and $[\mathfrak{b}]$ be an element of $H_{\mathfrak{q}}(\mathbf{K})$. For any real number $x \geq 1$, we have*

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}} \\ [\mathfrak{a}] = [\mathfrak{b}] \\ \mathfrak{N}\mathfrak{a} \leq x}} 1 = \frac{\rho_{\mathbf{K}} \varphi(\mathfrak{q})}{|H_{\mathfrak{q}}(\mathbf{K})|} \frac{x}{\mathfrak{N}\mathfrak{q}} + \mathrm{O}^*\left(10^{21}\left(\frac{x}{\mathfrak{N}\mathfrak{q}}\right)^{1/2} + 4 \cdot 10^5\right).$$

Theorem 3.10. (Gun, Ramaré and Sivaraman, [16]) *Let $\mathbf{K}$ be an imaginary quadratic field. For any real number $x \geq 1$,*

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}\mathfrak{a} \leq x}} 1 = \rho_{\mathbf{K}} x + \mathrm{O}^*\left(10^{15}(h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2} x^{1/2}\right).$$

We now state a result of Garcia and Lee [12] which will be used in the proof of Theorem 1.7.

Theorem 3.11. *Let $\mathbf{K}$ be an imaginary quadratic field and $x \geq 2$. We have*

$$\sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = \log x + \mathrm{O}^*\left(3 + \frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log |d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}}\right).$$

Let $\mathbf{L}/\mathbf{K}$ be an abelian extension with Galois group G and let $n_{\mathbf{L}}$ and $n_{\mathbf{K}}$ be the respective degrees over $\mathbb{Q}$.

Definition 4. *Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_{\mathbf{K}}$, unramified in $\mathbf{L}$ and $\mathfrak{q}$ be a prime ideal above it in $\mathcal{O}_{\mathbf{L}}$. The unique Galois element σ in the Galois group of $\mathbf{L}/\mathbf{K}$ such that*

$$\sigma(a) \equiv a^{\mathfrak{N}(\mathfrak{p})} \bmod \mathfrak{q} \text{ for all } a \in \mathcal{O}_{\mathbf{L}}$$

is called the Artin symbol corresponding to $\mathfrak{p}$ and the extension $\mathbf{L}/\mathbf{K}$. The Artin symbol is extended multiplicatively to the set of all non-zero integral ideals of $\mathcal{O}_{\mathbf{K}}$ which are supported on the set of prime ideals which do not ramify in $\mathbf{L}/\mathbf{K}$. For such an integral ideal $\mathfrak{a}$, we use $(\mathbf{L}/\mathbf{K}, \mathfrak{a})$ to denote the Artin symbol.

Under the above notation, we have the following theorem from [14].

Theorem 3.12. (Grenié and Molteni, [14]) *Let σ be an element of the group G and let $\mathbf{1}_\sigma$ denote the characteristic function of σ. We define*

$$\psi(x, \mathbf{L}/\mathbf{K}, \sigma) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}} \\ \mathfrak{p} \text{ unramified } \forall\ \mathfrak{p} | \mathfrak{a} \\ 1 \le \mathfrak{N}\mathfrak{a} \le x}} \mathbf{1}_\sigma((\mathfrak{a}, \mathbf{L}/\mathbf{K}))\Lambda(\mathfrak{a}).$$

Under GRH, for all $x \ge 1$,

$$\Big| |G|\psi(x, \mathbf{L}/\mathbf{K}, \sigma) - x \Big| \le \sqrt{x}\left(\left(\frac{\log x}{2\pi} + 2\right)\log|d_{\mathbf{L}}| + \left(\frac{\log^2 x}{8\pi} + 2\right) n_{\mathbf{L}}\right).$$

The above result is an extension of their previous result for number fields [13] which we shall now state in a weaker form as required by our proof.

Theorem 3.13. (Grenié and Molteni, [13]) *Let $\mathbf{K}$ be a number field. We define $\psi(x) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}}, \\ 1 \le \mathfrak{N}\mathfrak{a} \le x}} \Lambda(\mathfrak{a})$.*

Under GRH, for all $x \ge 3$,

$$|\psi(x) - x| \le 7\log|d_{\mathbf{K}}|\sqrt{x}\log x + \sqrt{x}\log^2 x + 19\sqrt{x}.$$

Before we proceed further, we would like to introduce the notion of the narrow ray class field modulo a non-zero integral ideal $\mathfrak{c}$ of $\mathcal{O}_{\mathbf{K}}$. To do so, we first define the Artin map in the following manner : Given an abelian extension $\mathbf{L}/\mathbf{K}$ and a non-zero integral ideal $\mathfrak{c}$ such that every prime of $\mathbf{K}$ that ramifies in $\mathbf{L}$ divides $\mathfrak{c}$, we define $\Psi_{\mathbf{L}/\mathbf{K},\mathfrak{c}} : I(\mathfrak{c}) \to \mathrm{Gal}(\mathbf{L}/\mathbf{K})$ as the map

$$\prod_{\substack{\mathfrak{p} \nmid \mathfrak{c} \\ \mathfrak{p} \text{ prime ideal of } \mathcal{O}_{\mathbf{K}}}} \mathfrak{p}^{m_{\mathfrak{p}}} \mapsto \prod_{\substack{\mathfrak{p} \nmid \mathfrak{c} \\ \mathfrak{p} \text{ prime ideal of } \mathcal{O}_{\mathbf{K}}}} (\mathbf{L}/\mathbf{K}, \mathfrak{p})^{m_{\mathfrak{p}}}, \qquad m_{\mathfrak{p}} \in \mathbb{N}$$

where the integer $m_{\mathfrak{p}}$ is 0 for all but finitely many primes $\mathfrak{p}$. It is known from class field theory that this map is surjective.

Definition 5. *For a non-zero ideal $\mathfrak{c}$, the unique extension $\mathbf{L}$ of $\mathbf{K}$ such that*

(1) *every prime ideal of $\mathbf{K}$ that ramifies in $\mathbf{L}$ divides $\mathfrak{c}$ and*
(2) *the kernel of $\Psi_{\mathbf{L}/\mathbf{K},\mathfrak{c}}$ is $P(\mathfrak{c})$*

is called the narrow ray class field of $\mathbf{K}$ *with respect to* $\mathfrak{c}$*. We shall denote this field by* $\mathbf{K}(\mathfrak{c})$*.*

It is known that every finite abelian extension $\mathbf{L}$ of an imaginary quadratic field $\mathbf{K}$ is contained in a narrow ray class field $\mathbf{K}(\mathfrak{c})$ of $\mathbf{K}$ for some integral ideal $\mathfrak{c}$. The greatest common divisor of all such integral ideals is called the conductor of $\mathbf{L}$ with respect to $\mathbf{K}$.

Corollary 1.2 of [14] restates the above bounds in terms of the prime counting function $\pi(x, \mathbf{L}/\mathbf{K}, \sigma)$, where

$$\pi(x, \mathbf{L}/\mathbf{K}, \sigma) = \sum_{\substack{(0)\neq\mathfrak{p}\subset\mathcal{O}_{\mathbf{K}},\\ \mathfrak{p}\ \text{unramified}\\ 1\leq\mathfrak{N}\mathfrak{p}\leq x}} \mathbf{1}_{\sigma}((\mathfrak{p}, \mathbf{L}/\mathbf{K})).$$

In this setup, we now deduce a corollary about principal prime ideals. We first recall that the Hilbert class field of an imaginary quadratic field $\mathbf{K}$ is the narrow ray class field of $\mathbf{K}$ corresponding to the ideal $\mathcal{O}_{\mathbf{K}}$. By class field theory it is also the unique abelian extension of $\mathbf{K}$ where all the non-zero principal prime ideals of $\mathcal{O}_{\mathbf{K}}$ split completely. For a number field $\mathbf{K}$, we use $\mathbf{K}(\mathcal{O}_{\mathbf{K}})$ to denote its Hilbert class field.

Corollary 3.14. *Let* $\mathbf{K}$ *be an imaginary quadratic field and* $\mathbf{K}(\mathcal{O}_{\mathbf{K}})$ *the Hilbert class field of* $\mathbf{K}$*. Let* σ_0 *be the trivial element of the Galois group of* $\mathbf{K}(\mathcal{O}_{\mathbf{K}})/\mathbf{K}$*. Under GRH,* $\forall x \geq 2$*,*

$$\begin{aligned}
\pi(x, \mathbf{K}(\mathcal{O}_{\mathbf{K}})/\mathbf{K}, \sigma_0) &= \left|\{\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}} : 1 \leq \mathfrak{N}\mathfrak{p} \leq x, \mathfrak{p}\ \textit{splits completely in}\ \mathbf{K}(\mathcal{O}_{\mathbf{K}})\}\right| \\
&= \frac{1}{h_{\mathbf{K}}}\int_2^x \frac{1}{\log t}dt + \mathrm{O}^*\left(5\sqrt{x}\log|d_{\mathbf{K}}| + 2\sqrt{x}\left(\frac{\log x}{8\pi} + 9\right)\right).
\end{aligned}$$

4. Requisite Propositions and Lemmas

In this section, we state and prove some lemmas and propositions required in the proofs of our theorems. The first lemma gives us a bound on the cardinality of $H_{\mathfrak{q}}(\mathbf{K})$ for an imaginary quadratic field $\mathbf{K}$.

Lemma 4.1. *Let* $\mathbf{K}$ *be an imaginary quadratic field. For any non-zero prime ideal* $\mathfrak{q}$ *in* $\mathcal{O}_{\mathbf{K}}$*,*

$$\frac{1}{12}h_{\mathbf{K}}\mathfrak{N}\mathfrak{q} \leq |H_{\mathfrak{q}}(\mathbf{K})| < h_{\mathbf{K}}\mathfrak{N}\mathfrak{q}.$$

Proof. See Theorem 1 of Chapter VI [25] (page 127). □

We now state and prove some lemmas to compute the sum of the inverse of the function $\varphi_{\mathbf{K}}$ introduced earlier. For a number field $\mathbf{K}$ and a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}}$, we define

$$\sigma(\mathfrak{a}) := \sum_{\substack{\mathfrak{b}\subseteq\mathcal{O}_{\mathbf{K}}\\ \mathfrak{b}|\mathfrak{a}}} \mathfrak{N}\mathfrak{b}.$$

Lemma 4.2. *Let $\mathbf{K}$ be an imaginary quadratic field. Then for $x \geq 1$, we have*

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{\sigma(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \leq \zeta_{\mathbf{K}}(2)\rho_{\mathbf{K}} x + 10^{15}\zeta_{\mathbf{K}}(3/2)(h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2} x^{1/2}.$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{\sigma(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} &= \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{1}{\mathfrak{N}\mathfrak{a}} \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{b} | \mathfrak{a}}} \mathfrak{N}\mathfrak{b} = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{b} | \mathfrak{a}}} \frac{1}{\mathfrak{N}\mathfrak{b}} = \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{b} \leq x}} \frac{1}{\mathfrak{N}\mathfrak{b}} \sum_{\substack{\mathfrak{c} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{c} \leq \frac{x}{\mathfrak{N}\mathfrak{b}}}} 1 \\
&\leq \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{b} \leq x}} \frac{1}{\mathfrak{N}\mathfrak{b}} \left(\rho_{\mathbf{K}} \frac{x}{\mathfrak{N}\mathfrak{b}} + 10^{15} (h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2} \left(\frac{x}{\mathfrak{N}\mathfrak{b}}\right)^{1/2} \right) \\
&\leq \zeta_{\mathbf{K}}(2)\rho_{\mathbf{K}} x + 10^{15}\zeta_{\mathbf{K}}(3/2)(h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2} x^{1/2}.
\end{aligned}$$

Here we have used Theorem 3.10 in the penultimate step. Using equation (4), we get the desired result.

$\square$

Lemma 4.3. *For a number field $\mathbf{K}$ and $\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}}$, we have*

$$\frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})} \leq \zeta_{\mathbf{K}}(2) \frac{\sigma(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}.$$

Proof. For $\mathfrak{p} \mid \mathfrak{a}$, let $\alpha_{\mathfrak{p}}$ be such that $\mathfrak{p}^{\alpha_{\mathfrak{p}}} \| \mathfrak{a}$. We now consider

$$\frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})} = \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1} = \prod_{\mathfrak{p}|\mathfrak{a}} \frac{1 + \frac{1}{\mathfrak{N}\mathfrak{p}}}{1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2}} = \prod_{\mathfrak{p}|\mathfrak{a}} \frac{\mathfrak{N}\mathfrak{p}^{\alpha_{\mathfrak{p}}} + \mathfrak{N}\mathfrak{p}^{\alpha_{\mathfrak{p}}-1}}{\mathfrak{N}\mathfrak{p}^{\alpha_{\mathfrak{p}}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right)} \leq \zeta_{\mathbf{K}}(2) \frac{\sigma(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}.$$

$\square$

Lemma 4.4. *Let $\mathbf{K}$ be an imaginary quadratic field. We now have for $x \geq 3$,*

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{1}{\varphi_{\mathbf{K}}(\mathfrak{a})} \leq \zeta_{\mathbf{K}}(2)^2 \rho_{\mathbf{K}} \log ex + 2 \cdot 10^{15} \zeta_{\mathbf{K}}(3/2)\zeta_{\mathbf{K}}(2)(h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2}.$$

Proof. Consider

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{1}{\varphi_{\mathbf{K}}(\mathfrak{a})} = \sum_{1 \leq t \leq x} \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}\mathfrak{a} = t}} \frac{1}{\varphi_{\mathbf{K}}(\mathfrak{a})} = \sum_{1 \leq t \leq x} \frac{1}{t} \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}\mathfrak{a} = t}} \frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})}.$$

Let $a_t = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}\mathfrak{a} = t}} \frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})}$ and $A(x) = \sum_{1 \leq t \leq x} a_t$. We now apply partial summation formula to get

$$\sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{a} \leq x}} \frac{1}{\varphi_{\mathbf{K}}(\mathfrak{a})} = \sum_{1 \leq t \leq x} \frac{a_t}{t} = \frac{A(x)}{x} + \int_1^x \frac{A(u)}{u^2} du. \tag{8}$$

By Lemma 4.2 and Lemma 4.3,

$$A(x) = \sum_{1 \le t \le x} a_t = \sum_{1 \le t \le x} \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}\mathfrak{a} = t}} \frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})} = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}\mathfrak{a} \le x}} \frac{\mathfrak{N}\mathfrak{a}}{\varphi_{\mathbf{K}}(\mathfrak{a})}$$
$$\le \zeta_{\mathbf{K}}(2)^2 \rho_{\mathbf{K}} x + 10^{15} \zeta_{\mathbf{K}}(3/2) \zeta_{\mathbf{K}}(2) (h_{\mathbf{K}} \log(3h_{\mathbf{K}}))^{1/2} x^{1/2}.$$

Using the above bound in (8), we get the desired result.

□

Now we compute the upper bound of the absolute discriminant of the narrow ray class field $\mathbf{K}(\mathfrak{q})$ corresponding to modulus $\mathfrak{q}$ of number field $\mathbf{K}$.

Lemma 4.5. *Let $\mathbf{K}$ be an imaginary quadratic field. Let $\mathfrak{q}$ be a non-zero prime ideal in $\mathcal{O}_{\mathbf{K}}$ and $d_{\mathbf{K}(\mathfrak{q})}$ be the discriminant of $\mathbf{K}(\mathfrak{q})$. Then,*

$$\log |d_{\mathbf{K}(\mathfrak{q})}| \le \rho_{\mathbf{K}} \sqrt{|d_{\mathbf{K}}|} \mathfrak{N}\mathfrak{q} \log(|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q}).$$

Proof. Using Lemma 3.2, we have

$$|d_{\mathbf{K}(\mathfrak{q})}| = |d_{\mathbf{K}}|^{|H_{\mathfrak{q}}(\mathbf{K})|} \mathfrak{N}\Delta_{\mathbf{K}(\mathfrak{q})/\mathbf{K}}.$$

Since $\mathrm{Gal}(\mathbf{K}(\mathfrak{q})/\mathbf{K})$ is abelian, every irreducible character of $\mathrm{Gal}(\mathbf{K}(\mathfrak{q})/\mathbf{K})$ is of degree 1. If χ is a Hecke character of $\mathrm{Gal}(\mathbf{K}(\mathfrak{q})/\mathbf{K})$, $f(\chi)$ is also the conductor of a subfield of $\mathbf{K}(\mathfrak{q})$ which contains $\mathbf{K}$ (see page 535, Prop 11.10 of [27]). Therefore, $f(\chi)$ divides $\mathfrak{q}$. By Theorem 3.3, we have

$$\Delta_{\mathbf{K}(\mathfrak{q})/\mathbf{K}} = \prod_{\chi \text{ irred.}} f(\chi).$$

Taking norm on both sides, we get

$$\mathfrak{N}\Delta_{\mathbf{K}(\mathfrak{q})/\mathbf{K}} = \prod_{\chi \text{ irred.}} \mathfrak{N}f(\chi) \le \prod_{\chi \text{ irred.}} \mathfrak{N}\mathfrak{q} = \mathfrak{N}\mathfrak{q}^{|H_{\mathfrak{q}}(K)|}.$$

Using Lemma 4.1, we get

$$\log |d_{\mathbf{K}(\mathfrak{q})}| = |H_{\mathfrak{q}}(\mathbf{K})| \log |d_{\mathbf{K}}| + \log \mathfrak{N}\Delta_{\mathbf{K}(\mathfrak{q})/\mathbf{K}} \le \rho_{\mathbf{K}} \sqrt{|d_{\mathbf{K}}|} \mathfrak{N}\mathfrak{q} \log(|d_{\mathbf{K}}| \mathfrak{N}\mathfrak{q}).$$

□

4.1. **Selberg's sieve.** Let (t) be a non-zero principal integral ideal of $\mathcal{O}_{\mathbf{K}}$. For a non-zero integral ideal $\mathfrak{b}$, let $\mathfrak{b}_{(t)}$ denote the maximal divisor of $\mathfrak{b}$ co-prime to (t), that is

$$\mathfrak{b}_{(t)} = \prod_{(\mathfrak{p},(t))=1} \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{b})},$$

where $\nu_{\mathfrak{p}}(\mathfrak{b})$ denotes the valuation of $\mathfrak{b}$ at $\mathfrak{p}$. Also let $d(\mathfrak{b})$ denote the number of divisors of $\mathfrak{b}$ and $\Omega_{\mathbf{K}}(\mathfrak{b})$ denote the number of prime ideals dividing $\mathfrak{b}$ counted with multiplicity. For a non-zero integral ideal $\mathfrak{b}$, we define

$$\rho(\mathfrak{b}) = 2^{\Omega_{\mathbf{K}}(\mathfrak{b}_{(t)})}, \qquad f_{(t)}(\mathfrak{b}) = \frac{\mathfrak{N}\mathfrak{b}}{2^{\Omega_{\mathbf{K}}(\mathfrak{b}_{(t)})}} \qquad \text{and} \quad f_{1,(t)}(\mathfrak{b}) = \sum_{\mathfrak{a}|\mathfrak{b}} \mu(\mathfrak{a}) f_{(t)}\left(\frac{\mathfrak{b}}{\mathfrak{a}}\right).$$

Evidently all these functions are multiplicative. Let

$$\mathbb{P}(z) = \prod_{\substack{N\mathfrak{p} \le z \\ (\mathfrak{p},(2310))=1}} \mathfrak{p}.$$

Further, for an ideal $\mathfrak{e}$ of $\mathcal{O}_{\mathbf{K}}$ coprime to (2), we define

$$S_{\mathfrak{e}}(z) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\mathfrak{a}) \le z \\ (\mathfrak{a},(2310)\mathfrak{e}) = \mathcal{O}_K}} \frac{\mu^2(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})}, \quad \text{and} \quad \lambda_{\mathfrak{e}} = \mu(\mathfrak{e}) \frac{f_{(t)}(\mathfrak{e}) S_{\mathfrak{e}}(\frac{z}{\mathfrak{N}(\mathfrak{e})})}{f_{1,(t)}(\mathfrak{e}) S_{\mathcal{O}_{\mathbf{K}}}(z)}.$$

We observe here that 2310 is the product of all the primes upto 12. Working with primes greater than 12 allows us to get explicit lower bounds for certain sums which appear in the Selberg sieve (see equation (9) below). For non-zero integral ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$, we use $[\mathfrak{b}_1, \mathfrak{b}_2]$ to denote the least common multiple of the ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$. In this setup, we have the following result.

Lemma 4.6. *We have*

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2 | \mathbb{P}(z), \\ \mathfrak{N}\mathfrak{b}_i \le z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}| \frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} \le \zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{16} z.$$

Proof. The proof of this lemma follows along the same lines as Lemma 11 of [15]. We consider the sum

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2 | \mathbb{P}(z), \\ \mathfrak{N}\mathfrak{b}_i \le z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}| \frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} = \sum_{\substack{\mathfrak{d} | \mathbb{P}(z), \\ \mathfrak{N}\mathfrak{d} \le z}} \frac{\sqrt{\mathfrak{N}\mathfrak{d}}}{\rho(\mathfrak{d})} \sum_{\substack{\mathfrak{b}_i | \mathbb{P}(z), \\ \mathfrak{d} = (\mathfrak{b}_1,\mathfrak{b}_2) \\ \mathfrak{N}\mathfrak{b}_i \le z}} \frac{|\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}| \rho(\mathfrak{b}_1)\rho(\mathfrak{b}_2)}{\sqrt{\mathfrak{N}(\mathfrak{b}_1\mathfrak{b}_2)}}.$$

From the expression of $\lambda_{\mathfrak{b}}$ and with $y = z/\mathfrak{N}\mathfrak{d}$, we get

$$\sum_{\substack{\mathfrak{N}\mathfrak{c} \le y, \\ (\mathfrak{c},\mathfrak{d}(2310)) = \mathcal{O}_{\mathbf{K}}}} \frac{|\lambda_{\mathfrak{d}\mathfrak{c}}| \rho(\mathfrak{c})}{\sqrt{\mathfrak{N}\mathfrak{c}}} \le \frac{\sqrt{y}\ \mathfrak{N}\mathfrak{d}}{\rho(\mathfrak{d}) f_{1,(t)}(\mathfrak{d})} \prod_{\mathfrak{N}\mathfrak{p} > 11} \left(1 + \frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right).$$

We thus get

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2 | \mathbb{P}(z), \\ \mathfrak{N}\mathfrak{b}_i \le z}} |\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}| \frac{\rho([\mathfrak{b}_1,\mathfrak{b}_2])}{\sqrt{\mathfrak{N}([\mathfrak{b}_1,\mathfrak{b}_2])}} \le z \prod_{\mathfrak{N}\mathfrak{p} > 11} \left(1 + \frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))}\right)^2 \left(1 + \frac{\rho(\mathfrak{p})\sqrt{\mathfrak{N}\mathfrak{p}}}{(\mathfrak{N}\mathfrak{p} - \rho(\mathfrak{p}))^2}\right).$$

Note that

$$\prod_{\mathfrak{N}\mathfrak{p}>11}\left(1+\frac{\rho(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))}\right)\ \leq\ \zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{4}\ \text{and}\ \prod_{\mathfrak{N}\mathfrak{p}>11}\left(1+\frac{\rho(\mathfrak{p})\sqrt{\mathfrak{N}\mathfrak{p}}}{(\mathfrak{N}\mathfrak{p}-\rho(\mathfrak{p}))^{2}}\right)\ \leq\ \zeta_{\mathbf{K}}\left(\frac{3}{2}\right)^{8}.$$

This completes the proof of the lemma. □

Proposition 4.7. *Let* $\mathbf{K}$ *be an imaginary quadratic field and* t *be a fixed non-zero element in* $\mathcal{O}_{\mathbf{K}}$*, which is not a unit in* $\mathcal{O}_{\mathbf{K}}$*. We have for* $u\geq 2\exp\left(8\cdot 10^{45}\cdot|d_{\mathbf{K}}|\right)$

$$\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ \alpha,t\alpha+1\ \in\ \mathcal{P}_{\mathbf{K}}}}1\quad\leq\quad\left(\frac{2\cdot 10^{10}}{\sqrt{|d_{\mathbf{K}}|}\rho_{\mathbf{K}}^{2}}\right)\frac{\mathfrak{N}(t)u}{\varphi((t))\log^{2}\frac{u^{1/4}}{2}}+10^{30}u^{3/4}.$$

Proof. Let z be a real number such that $\sqrt{z}>12$. We would like to estimate the sum

$$\begin{aligned}\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ \alpha,t\alpha+1\in\mathcal{P}_{\mathbf{K}}}}1\quad&\leq\quad\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq z\\ \alpha,t\alpha+1\in\mathcal{P}_{\mathbf{K}}}}1+\sum_{\substack{z<\mathfrak{N}(\alpha)\leq u\\ 1\leq\mathfrak{N}(t\alpha+1)\leq z\\ \alpha,t\alpha+1\in\mathcal{P}_{\mathbf{K}}}}1+\sum_{\substack{z<\mathfrak{N}(\alpha)\leq u\\ ((\alpha(t\alpha+1)),\mathbb{P}(z))=1}}1\\ &\leq\quad\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ ((\alpha(t\alpha+1)),\mathbb{P}(z))=1}}1+4|\mu_{\mathbf{K}}|z.\end{aligned}$$

Let us consider the first sum

$$\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ ((\alpha(t\alpha+1)),\mathbb{P}(z))=1}}1=\sum_{1\leq\mathfrak{N}(\alpha)\leq u}\left(\sum_{\mathfrak{b}|((\alpha(t\alpha+1)),\mathbb{P}(z))}\mu(\mathfrak{b})\right)\leq\sum_{1\leq\mathfrak{N}(\alpha)\leq u}\left(\sum_{\mathfrak{b}|((\alpha(t\alpha+1)),\mathbb{P}(z))}\lambda_{\mathfrak{b}}\right)^{2}.$$

Rearranging the sums we get

$$\sum_{1\leq\mathfrak{N}(\alpha)\leq u}\left(\sum_{\mathfrak{b}|((\alpha(t\alpha+1)),\mathbb{P}(z))}\lambda_{\mathfrak{b}}\right)^{2}=\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)\\ \mathfrak{N}\mathfrak{b}_i\leq z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ \mathfrak{b}_i|(\alpha(t\alpha+1))}}1.$$

Note that $\mathfrak{b}=[\mathfrak{b}_1,\mathfrak{b}_2]$ is squarefree. If $\mathfrak{b}_{(t)}$ is the maximal divisor of $\mathfrak{b}$ co-prime to (t), we need to count α in $2^{\Omega_{\mathbf{K}}(\mathfrak{b}_{(t)})}$ classes in $\mathcal{O}_{\mathbf{K}}/\mathfrak{b}$. Using Theorem 3.8 for $\mathfrak{a}=\mathcal{O}_{\mathbf{K}}$, $\mathfrak{q}=\mathfrak{b}$, we get for $z\leq u^{\frac{1}{2}}$

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)\\ \mathfrak{N}\mathfrak{b}_i\leq z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\sum_{\substack{1\leq\mathfrak{N}(\alpha)\leq u\\ [\mathfrak{b}_1,\mathfrak{b}_2]|(\alpha(t\alpha+1))}}1=\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)\\ \mathfrak{N}\mathfrak{b}_i\leq z}}\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}\left(\frac{c_{\mathbf{K}}2^{\Omega_{\mathbf{K}}([\mathfrak{b}_1,\mathfrak{b}_2]_{(t)})}u}{\mathfrak{N}[\mathfrak{b}_1,\mathfrak{b}_2]}+\mathrm{O}^*\left(10^{14}2^{\Omega_{\mathbf{K}}([\mathfrak{b}_1,\mathfrak{b}_2]_{(t)})}\sqrt{\frac{u}{\mathfrak{N}[\mathfrak{b}_1,\mathfrak{b}_2]}}\right)\right),$$

where $c_{\mathbf{K}}$ is the constant given by $\frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}}$. We first deal with the error term. By Lemma 4.6, we get

$$\sum_{\substack{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)\\ \mathfrak{N}\mathfrak{b}_i\leq z}}|\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}|\frac{2^{\Omega_{\mathbf{K}}([\mathfrak{b}_1,\mathfrak{b}_2]_{(t)})}}{\sqrt{\mathfrak{N}[\mathfrak{b}_1,\mathfrak{b}_2]}}\leq 3^{32}\cdot z.$$

The main term now becomes $\displaystyle\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f_{(t)}([\mathfrak{b}_1,\mathfrak{b}_2])}$ which in turn is

$$\sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2} f_{(t)}((\mathfrak{b}_1,\mathfrak{b}_2))}{f_{(t)}(\mathfrak{b}_1)f_{(t)}(\mathfrak{b}_2)} = \sum_{\mathfrak{b}_1,\mathfrak{b}_2|\mathbb{P}(z)} \frac{\lambda_{\mathfrak{b}_1}\lambda_{\mathfrak{b}_2}}{f_{(t)}(\mathfrak{b}_1)f_{(t)}(\mathfrak{b}_2)} \sum_{\mathfrak{a}|(\mathfrak{b}_1,\mathfrak{b}_2)} f_{1,(t)}(\mathfrak{a}) = \sum_{\mathfrak{a}|\mathbb{P}(z)} f_{1,(t)}(\mathfrak{a}) \left(\sum_{\substack{\mathfrak{c}|\mathbb{P}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f_{(t)}(\mathfrak{c})} \right)^2 .$$

Further, we observe that

$$\sum_{\substack{\mathfrak{c}|\mathbb{P}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f_{(t)}(\mathfrak{c})} = S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \sum_{\substack{\mathfrak{c}|\mathbb{P}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\mu(\mathfrak{c})}{f_{1,(t)}(\mathfrak{c})} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{g}) \le \frac{z}{\mathfrak{N}(\mathfrak{c})}\\ (\mathfrak{g},(2310)\mathfrak{c})=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_{1,(t)}(\mathfrak{g})}.$$

Writing $\mathfrak{c} = \mathfrak{h}\mathfrak{a}$ with $(\mathfrak{h},\mathfrak{a}) = \mathcal{O}_{\mathbf{K}}$, we get

$$\begin{aligned} &\frac{\mu(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})} S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{h}) \le \frac{z}{\mathfrak{N}(\mathfrak{a})}\\ \mathfrak{h}|\mathbb{P}(z)\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \frac{\mu(\mathfrak{h})}{f_{1,(t)}(\mathfrak{h})} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{g}) \le \frac{z}{\mathfrak{N}(\mathfrak{h}\mathfrak{a})}\\ (\mathfrak{g},(2310)\mathfrak{h}\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{g})}{f_{1,(t)}(\mathfrak{g})} \\ = &\frac{\mu(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})} S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{h}) \le \frac{z}{\mathfrak{N}(\mathfrak{a})}\\ \mathfrak{h}|\mathbb{P}(z)\\ (\mathfrak{h},\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{g}) \le \frac{z}{\mathfrak{N}(\mathfrak{h}\mathfrak{a})}\\ (\mathfrak{g},(2310)\mathfrak{h}\mathfrak{a})=\mathcal{O}_{\mathbf{K}}}} \mu(\mathfrak{h}) \frac{\mu^2(\mathfrak{g}\mathfrak{h})}{f_{1,(t)}(\mathfrak{g}\mathfrak{h})}. \end{aligned}$$

Setting $\mathfrak{a}_1 = \mathfrak{g}\mathfrak{h}$ gives

$$\sum_{\substack{\mathfrak{c}|\mathbb{P}(z)\\ \mathfrak{a}|\mathfrak{c}}} \frac{\lambda_{\mathfrak{c}}}{f_{(t)}(\mathfrak{c})} = \frac{\mu(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})} S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}_1) \le \frac{z}{\mathfrak{N}(\mathfrak{a})}\\ (\mathfrak{a}_1,(2310)\mathfrak{a})=\mathcal{O}_{\mathbf{K}}\\ \mathfrak{a}_1|\mathbb{P}(z)}} \frac{\mu^2(\mathfrak{a}_1)}{f_{1,(t)}(\mathfrak{a}_1)} \sum_{\mathfrak{h}|\mathfrak{a}_1} \mu(\mathfrak{h}) = S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \frac{\mu(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})}.$$

Combining the above, the main term is $c_{\mathbf{K}} u S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1}$. However,

$$\begin{aligned} S_{\mathcal{O}_{\mathbf{K}}}(z) &= \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})} = \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_{(t)}(\mathfrak{a})} \cdot \frac{f_{(t)}(\mathfrak{a})}{f_{1,(t)}(\mathfrak{a})} \\ &= \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{\mu^2(\mathfrak{a})}{f_{(t)}(\mathfrak{a})} \cdot \prod_{\mathfrak{p}|\mathfrak{a}} \frac{f_{(t)}(\mathfrak{p})}{\mu(\mathfrak{p}) + f_{(t)}(\mathfrak{p})} \\ &\ge \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{a})} - \sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ \mathfrak{a} \text{ non-sq-free}\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{a})}. \end{aligned}$$

Consider

$$\sum_{\substack{1\le \mathfrak{N}(\mathfrak{a}) \le z\\ \mathfrak{a} \text{ non-sq-free}\\ (\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{a})} \le \sum_{\substack{1\le \mathfrak{N}\mathfrak{b} \le z\\ (\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}} \sum_{\substack{\mathfrak{N}\mathfrak{p} \le \sqrt{z}\\ (\mathfrak{p},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{b}) f_{(t)}(\mathfrak{p})^2} = \sum_{\substack{1\le \mathfrak{N}\mathfrak{b} \le z\\ (\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{b})} \left(\sum_{\substack{\mathfrak{N}\mathfrak{p} \le \sqrt{z}\\ (\mathfrak{p},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{f_{(t)}(\mathfrak{p})^2} \right).$$

Looking at the sum

$$\sum_{\substack{\mathfrak{N}\mathfrak{p}\leq\sqrt{z}\\(\mathfrak{p},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{1}{f_{(t)}(\mathfrak{p})^2}\leq\sum_{\substack{\mathfrak{N}\mathfrak{p}\leq\sqrt{z}\\(\mathfrak{p},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{4}{\mathfrak{N}(\mathfrak{p})^2}\leq\frac{8}{11}.$$

Therefore,

$$S_{\mathcal{O}_{\mathbf{K}}}(z)\geq\frac{3}{11}\sum_{\substack{1\leq\mathfrak{N}(\mathfrak{a})\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{1}{f_{(t)}(\mathfrak{a})}. \tag{9}$$

Let us consider

$$\sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{1}{f_{(t)}(\mathfrak{a})}=\sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{2^{\Omega_{\mathbf{K}}(\mathfrak{a}_{(t)})}}{\mathfrak{N}\mathfrak{a}}\geq\sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{d_{(t)}(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}.$$

Here $d_{(t)}(\mathfrak{a})$ is the number of ideals dividing $\mathfrak{a}$ and co-prime to (t). We now look at

$$\begin{aligned}\prod_{\mathfrak{p}|(t)}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1}\sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{d_{(t)}(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} &= \sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{d_{(t)}(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\sum_{\mathfrak{p}|\mathfrak{b}\Rightarrow\mathfrak{p}|(t)}\frac{1}{\mathfrak{N}\mathfrak{b}}\\ &= \sum_{(0)\neq\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}}\frac{1}{\mathfrak{N}\mathfrak{b}}\sum_{\substack{1\leq\mathfrak{N}\mathfrak{a}\leq z,\mathfrak{a}|\mathfrak{b}\\(\mathfrak{a},(2310))=\mathcal{O}_{\mathbf{K}}\\ \mathfrak{p}|\frac{\mathfrak{b}}{\mathfrak{a}}\Rightarrow\mathfrak{p}|(t)}}d_{(t)}(\mathfrak{a})\\ &\geq \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq z\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}\frac{d(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}}.\end{aligned}$$

In order to compute the above sum, we consider the following

$$\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq z\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}d(\mathfrak{b}) = \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq z\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}\sum_{\mathfrak{e}|\mathfrak{b}}1=\sum_{\substack{\mathfrak{N}\mathfrak{e}\leq z\\(\mathfrak{e},(2310))=1}}\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq z\\\mathfrak{e}|\mathfrak{b},(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}1=\sum_{\substack{\mathfrak{N}\mathfrak{e}\leq z\\(\mathfrak{e},(2310))=1}}\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq\frac{z}{\mathfrak{N}\mathfrak{e}}\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}1.$$

We have by Theorem 3.9

$$\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq\frac{z}{\mathfrak{N}\mathfrak{e}}\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}1=\frac{\rho_{\mathbf{K}}\varphi((2310))z}{2310^2\mathfrak{N}\mathfrak{e}}+O^*\left(3\cdot10^{24}h_{\mathbf{K}}\left(\frac{z}{\mathfrak{N}\mathfrak{e}}\right)^{\frac{1}{2}}+3\cdot10^{12}h_{\mathbf{K}}\right).$$

Therefore, we have

$$\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\1\leq\mathfrak{N}\mathfrak{b}\leq\frac{z}{\mathfrak{N}\mathfrak{e}}\\(\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}}1\geq\frac{\rho_{\mathbf{K}}\varphi((2310))z}{2310^2\mathfrak{N}\mathfrak{e}}-3\cdot10^{24}h_{\mathbf{K}}\left(\frac{z}{\mathfrak{N}\mathfrak{e}}\right)^{\frac{1}{2}}-3\cdot10^{12}h_{\mathbf{K}}.$$

By partial summation formula,

$$\sum_{\substack{\mathfrak{e}\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}\mathfrak{e}\le z\\ (\mathfrak{e},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{\mathfrak{N}\mathfrak{e}} \ge \left(\frac{\rho_{\mathbf{K}}\varphi((2310))}{2310^2}\right)\log ez - 7\cdot 10^{24}h_{\mathbf{K}}.$$

Similarly, again by partial summation, we get

$$\sum_{\substack{\mathfrak{e}\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}\mathfrak{e}\le z\\ (\mathfrak{e},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{1}{\sqrt{\mathfrak{N}\mathfrak{e}}} \le \frac{2\rho_{\mathbf{K}}\varphi((2310))}{2310^2}\sqrt{z} + 6\cdot 10^{24}h_{\mathbf{K}}\log z.$$

Combining the above inequalities,

$$\begin{aligned}\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{b}\le z\\ (\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}} d(\mathfrak{b}) &\ge \sum_{\substack{\mathfrak{N}\mathfrak{e}\le z\\ (\mathfrak{e},(2310))=\mathcal{O}_{\mathbf{K}}}} \left(\frac{\rho_{\mathbf{K}}\varphi((2310))z}{2310^2\mathfrak{N}\mathfrak{e}} - 4\cdot 10^{24}h_{\mathbf{K}}\left(\frac{z}{\mathfrak{N}\mathfrak{e}}\right)^{\frac{1}{2}}\right)\\ &\ge \left(\frac{\rho_{\mathbf{K}}\varphi((2310))}{2310^2}\right)^2 z\log ez - 4.81\cdot 10^{49}\rho_{\mathbf{K}}h_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}z\\ &\ge \frac{1}{2}\left(\frac{\rho_{\mathbf{K}}\varphi((2310))}{2310^2}\right)^2 z\log ez,\end{aligned}$$

for $\log ez \ge \frac{2310^2\cdot 10^{50}h_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}}{\rho_{\mathbf{K}}\varphi((2310))^2}$. Now, using the fact that $h_{\mathbf{K}} \ge \rho_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}/(2\pi)$ and that $\varphi((2310)) \ge 480^2$, we get $\log ez \ge 2\cdot 10^{45}|d_{\mathbf{K}}|$. We now apply partial summation once again to get

$$\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{b}\le z\\ (\mathfrak{b},(2310))=\mathcal{O}_{\mathbf{K}}}} \frac{d(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} \ge \frac{(6\rho_{\mathbf{K}})^2}{2\cdot 10^{10}}\log^2 z \quad\text{for}\quad \log ez \ge 2\cdot 10^{45}|d_{\mathbf{K}}|.$$

Combining the above for $\log ez \ge 2\cdot 10^{45}|d_{\mathbf{K}}|$,

$$S_{\mathcal{O}_{\mathbf{K}}}(z)^{-1} \le \prod_{\mathfrak{p}|(t)}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1}\frac{22\cdot 10^{10}}{3\,(6\rho_{\mathbf{K}})^2\log^2 z}.$$

If we suppose further that $z \le \sqrt{u}$, we get

$$\begin{aligned}\sum_{\substack{1\le\mathfrak{N}(\alpha)\le u\\ \alpha,t\alpha+1\,\in\,\mathcal{P}_{\mathbf{K}}}} 1 &\le 4|\mu_{\mathbf{K}}|z + \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}}\prod_{\mathfrak{p}|(t)}\left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1}\frac{22\cdot 10^{10}u}{3\,(6\rho_{\mathbf{K}})^2\log^2 z} + 10^{14}\cdot 3^{32}\cdot z\sqrt{u}\\ &\le \frac{2\pi}{\sqrt{|d_{\mathbf{K}}|}}\cdot\frac{8\cdot 10^{10}\mathfrak{N}(t)u}{\varphi((t))\,(6\rho_{\mathbf{K}})^2\log^2 z} + 10^{30}z\sqrt{u}.\end{aligned}$$

Further, since we have $\log z \leq 2\sqrt{z}$, we get for $z = \dfrac{u^{1/4}}{2}$,

$$\sum_{\substack{1 \leq \mathfrak{N}(\alpha) \leq u \\ \alpha, t\alpha+1 \, \in \, \mathcal{P}_{\mathbf{K}}}} 1 \quad \leq \quad \left(\frac{2 \cdot 10^{10}}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|}} \right) \frac{\mathfrak{N}(t) u}{\varphi((t)) \log^2 \frac{u^{1/4}}{2}} + 10^{30} u^{3/4}.$$

□

Lemma 4.8. *For $Q \geq \exp(10^{45}|d_{\mathbf{K}}|)$, we have*

$$h_{\mathbf{K}} \pi^*(Q) \geq \frac{2Q}{25 \log Q}.$$

Proof. We first observe that for $y \geq 8$, we have

$$\int_2^y \frac{1}{\log t} dt \geq \frac{y}{\log y}.$$

For $y \geq 200^5$, we have $4/\log 2 \leq y^{1/5}/\log y$ and this implies

$$\int_2^y \frac{1}{\log t} dt \leq \int_2^{y^{4/5}} \frac{1}{\log 2} dt + \int_{y^{4/5}}^y \frac{1}{\log y^{4/5}} dt \leq \frac{y^{4/5}}{\log 2} + \frac{5y}{4 \log y} \leq \frac{3y}{2 \log y}.$$

We have

$$\pi^*(Q) = \pi(Q, \mathbf{K}(\mathcal{O}_{\mathbf{K}})/\mathbf{K}, \sigma_0) - \pi\left(Q/2, \mathbf{K}(\mathcal{O}_{\mathbf{K}})/\mathbf{K}, \sigma_0\right),$$

where σ_0 is the trivial element of the Galois group of $\mathbf{K}(\mathcal{O}_{\mathbf{K}})/\mathbf{K}$. We now get by Corollary 3.14,

$$\begin{aligned} h_{\mathbf{K}} \pi^*(Q) \quad &\geq \quad \frac{Q}{\log Q} - 5h_{\mathbf{K}} \sqrt{Q} \log |d_{\mathbf{K}}| - 2h_{\mathbf{K}} \sqrt{Q} \left(\frac{\log Q}{8\pi} + 9 \right) \\ &- \quad \frac{3Q}{4 \log Q/2} - 5h_{\mathbf{K}} \sqrt{\frac{Q}{2}} \log |d_{\mathbf{K}}| - 2h_{\mathbf{K}} \sqrt{\frac{Q}{2}} \left(\frac{\log \frac{Q}{2}}{8\pi} + 9 \right). \end{aligned}$$

Further since $Q \geq 2^6$, we get

$$\begin{aligned} h_{\mathbf{K}} \pi^*(Q) \quad &\geq \quad \frac{Q}{\log Q} - 5h_{\mathbf{K}} \sqrt{Q} \log |d_{\mathbf{K}}| - 2h_{\mathbf{K}} \sqrt{Q} \left(\frac{\log Q}{8\pi} + 9 \right) \\ &- \quad \frac{18Q}{20 \log Q} - 5h_{\mathbf{K}} \sqrt{\frac{Q}{2}} \log |d_{\mathbf{K}}| - 2h_{\mathbf{K}} \sqrt{\frac{Q}{2}} \left(\frac{\log \frac{Q}{2}}{8\pi} + 9 \right). \end{aligned}$$

Simplifying the above and using Lemma 3.1 and Lemma 4.1, we get

$$h_{\mathbf{K}} \pi^*(Q) \quad \geq \quad \frac{Q}{\log Q} \left(\frac{1}{10} - \frac{10^3 |d_{\mathbf{K}}|^{3/4} \log |d_{\mathbf{K}}| \log Q}{\sqrt{Q}} - \frac{400 |d_{\mathbf{K}}|^{3/4} \log Q}{\sqrt{Q}} \left(\frac{\log Q}{8\pi} + 9 \right) \right).$$

We now bound each of the negative terms separately using the lower bound for Q as follows. Using the fact that $\log x/\sqrt{x}$ is a decreasing function for $x \geq 8$, for the first term, we get

$$\frac{10^3 |d_{\mathbf{K}}|^{3/4} \log |d_{\mathbf{K}}| \log Q}{\sqrt{Q}} \leq \frac{10^{48} |d_{\mathbf{K}}|^{7/4} \log |d_{\mathbf{K}}|}{\exp(\frac{10^{45}|d_{\mathbf{K}}|}{2})} \leq \frac{1}{100}.$$

Similarly, for the second term,

$$\frac{400|d_{\mathbf{K}}|^{3/4}\log Q}{\sqrt{Q}}\left(\frac{\log Q}{8\pi}+9\right) \leq \frac{400\cdot 10^{90}|d_{\mathbf{K}}|^{11/4}}{8\pi\exp(\frac{10^{45}|d_{\mathbf{K}}|}{2})}+\frac{3600\cdot 10^{45}|d_{\mathbf{K}}|^{7/4}}{\exp(\frac{10^{45}|d_{\mathbf{K}}|}{2})} \leq \frac{1}{100}.$$

This gives us the lemma. □

Next two lemmas are devoted to the proof of Proposition 4.11. We first compute an upper bound of the real part of the function $\frac{\Gamma'_\chi}{\Gamma_\chi}(s)$ for $s\in\mathbb{C}$ (see equation (5) for its definition).

Lemma 4.9. *For $\Re(s)>1$ and number field $\mathbf{K}$, we have*

$$\Re\left(\frac{\Gamma'_\chi}{\Gamma_\chi}(s)\right) < n_{\mathbf{K}}\left(\log\left(\frac{|s+1|}{2}+2\right)-\frac{\log\pi}{2}\right).$$

Proof. Taking the logarithmic derivative on both sides of (5), we obtain

$$\frac{\Gamma'_\chi}{\Gamma_\chi}(s) = -\frac{n_{\mathbf{K}}}{2}\log\pi+\frac{a_\chi}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)+\frac{(n_{\mathbf{K}}-a_\chi)}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).$$

Using Lemma 3.5, for $\Re(s)>1$, we get

$$\begin{aligned}\Re\left(\frac{\Gamma'_\chi}{\Gamma_\chi}(s)\right) &< -\frac{n_{\mathbf{K}}}{2}\log\pi + a_\chi\log\left(\frac{|s+1|}{2}+2\right)+(n_{\mathbf{K}}-a_\chi)\log\left(\frac{|s|}{2}+2\right)\\ &< n_{\mathbf{K}}\left(\log\left(\frac{|s+1|}{2}+2\right)-\frac{\log\pi}{2}\right).\end{aligned}$$

□

The following lemma gives us an upper bound of the absolute value of the logarithmic derivative of Hecke L- functions.

Lemma 4.10. *Let $L(s,\chi)$ be the Hecke L-function corresponding to a generalized Dirichlet character χ modulo integral ideal $\mathfrak{c}$. Then for $\Re(s)=\sigma>1$ and number field $\mathbf{K}$, we have*

$$\left|\frac{L'}{L}(s,\chi)\right| < \frac{n_{\mathbf{K}}}{\sigma-1}.$$

Proof. Taking logarithmic derivative of Euler product of Hecke L function and Dedekind zeta function, we obtain

$$\left|\frac{L'}{L}(s,\chi)\right| = \Big|\sum_{\substack{\mathfrak{p}\nmid\mathfrak{c}\\ \mathfrak{p}\neq(0)}}\frac{\log\mathfrak{N}\mathfrak{p}}{\frac{(\mathfrak{N}\mathfrak{p})^s}{\chi([\mathfrak{p}])}-1}\Big| \leq \sum_{\mathfrak{p}\neq(0)}\frac{\log\mathfrak{N}\mathfrak{p}}{(\mathfrak{N}\mathfrak{p})^\sigma-1} = -\frac{\zeta'_{\mathbf{K}}}{\zeta_{\mathbf{K}}}(\sigma).$$

Similarly, we note that

$$-\frac{\zeta'_{\mathbf{K}}}{\zeta_{\mathbf{K}}}(\sigma) \leq -n_{\mathbf{K}}\frac{\zeta'}{\zeta}(\sigma) < \frac{n_{\mathbf{K}}}{\sigma-1}, \quad \sigma>1.$$

The last inequality follows from Lemma 3.4. Now combining the above identities, we get

$$\left|\frac{L'}{L}(s,\chi)\right| < \frac{n_{\mathbf{K}}}{\sigma-1}.$$

□

Now let $L(s,\chi)$ be the Hecke L-function corresponding to generalized Dirichlet character χ modulo $\mathfrak{c}$ and $\mathfrak{c}'$ be the conductor of χ. Then following Lagarias and Odlyzko [24, Lemma 5.4], we have the following result which gives us an explicit bound on the number of zeros of $L(s,\chi)$.

Proposition 4.11. *For a real number t, let $N_{t,\chi}$ denote the number of zeros (denoted $\rho=\alpha+i\beta$) of $L(s,\chi)$ with $0<\alpha<1$ and $|\beta-t|\le 1$. Then*

$$N_{t,\chi} < 5(3(n_{\mathbf{K}}+1)+\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}')+2n_{\mathbf{K}}\log(|t|+2)).$$

Moreover, when $\mathbf{K}\neq\mathbb{Q}$ then

$$N_{t,\chi} < 50 n_{\mathbf{K}}\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}'\ (|t|+2)).$$

Proof. From Lemma 3.6, we have

$$\Re\left(\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right)\right) = \Re\left(\frac{L'}{L}(s,\chi)\right)+\Re\left(\frac{L'}{L}(s,\overline{\chi})\right)+\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}')$$
$$+\mathbf{1}_{\chi}\Re\left(\frac{2}{s}+\frac{2}{s-1}\right)+2\Re\left(\frac{\Gamma_{\chi}'}{\Gamma_{\chi}}(s)\right).$$

Evaluating it at $s=2+it$ and using Lemma 4.9 and Lemma 4.10, we obtain

$$\begin{aligned}\Re\left(\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right)\right) &< n_{\mathbf{K}}(2-\log\pi)+\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}')+2\Re\left(\frac{1}{2+it}+\frac{1}{1+it}\right)\\ &\quad+2n_{\mathbf{K}}\log\left(\frac{|3+it|}{2}+2\right)\\ &< 3n_{\mathbf{K}}+3+\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}')+2n_{\mathbf{K}}\log(|t|+2). \qquad (10)\end{aligned}$$

Clearly, when $\mathbf{K}\neq\mathbb{Q}$, the above quantity is bounded above by $10n_{\mathbf{K}}\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{c}'\ (|t|+2))$. As noticed by Lagarias and Odlyzko [24, Lemma 5.4], we have

$$\Re\left(\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right)\right) \ge \sum_{\substack{\rho\\ |\beta-t|\le 1}}\frac{2-\alpha}{(2-\alpha)^2+(t-\beta)^2} > \frac{1}{5}N_{t,\chi}, \qquad (11)$$

since α is between 0 and 1. Substituting (11) into (10), we get our desired results. □

5. Bounds on the average value of the Euler-Kronecker constant

Throughout this section, we will use the following notations. Let $\mathbf{K}$ be an imaginary quadratic number field and $\mathfrak{q}$ be a non-zero principal prime ideal of $\mathbf{K}$. Let χ be a generalized

Dirichlet character $\bmod \mathfrak{q}$. Now after summing over all non-trivial characters in the following function (see equation (6))

$$\Phi_\chi(x) = \frac{1}{x-1}\int_1^x \Big(\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\chi([\mathfrak{a}])\Big)dt,$$

we get by orthogonality of characters of a finite group,

$$\sum_{\chi\neq\chi_0}\Phi_\chi(x) = \frac{1}{x-1}\int_1^x \Big(|H_\mathfrak{q}(\mathbf{K})| \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t\\ \mathfrak{a}\in[1]_\mathfrak{q}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} - \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t\\ (\mathfrak{a},\mathfrak{q})=1}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\Big)dt, \tag{12}$$

where $[1]_\mathfrak{q}$ is the trivial element of the group $H_\mathfrak{q}(\mathbf{K})$. Let

$$\psi(x,\mathfrak{q},[1]_\mathfrak{q}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x\\ \mathfrak{a}\in[1]_\mathfrak{q}}} \Lambda(\mathfrak{a}) \quad \text{and} \quad \psi(x) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x}} \Lambda(\mathfrak{a}).$$

Since the Galois group of $\mathbf{K}(\mathfrak{q})/\mathbf{K}$ is isomorphic to the group $H_\mathfrak{q}(\mathbf{K})$ under the Artin map, we can identify the trivial element of the Galois group of $\mathbf{K}(\mathfrak{q})/\mathbf{K}$ with the trivial element $[1]_\mathfrak{q}\in H_\mathfrak{q}(\mathbf{K})$. Therefore, $\psi(x,\mathfrak{q},[1]_\mathfrak{q}) = \psi(x,\mathbf{K}(\mathfrak{q})/\mathbf{K},\sigma_o)$ where σ_o is the identity element of the Galois group of $\mathbf{K}(\mathfrak{q})/\mathbf{K}$. For a non-zero prime ideal $\mathfrak{q}$,

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x\\ (\mathfrak{a},\mathfrak{q})=1}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \ge \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} - \frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}-1}. \tag{13}$$

By Abel's summation formula, we get

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x\\ \mathfrak{a}\in[1]_\mathfrak{q}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} = \frac{\psi(x,\mathfrak{q},[1]_\mathfrak{q})}{x} + \int_1^x \frac{\psi(u,\mathfrak{q},[1]_\mathfrak{q})}{u^2}du \quad \text{and} \quad \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le x}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(u)}{u^2}du.$$

Integrating over t on both sides of (13), we get

$$\begin{aligned}\int_1^x \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t\\ (\mathfrak{a},\mathfrak{q})=1}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}dt \quad &\ge \quad \int_1^x \frac{\psi(t)}{t}dt + \int_1^x\int_1^t \frac{\psi(u)}{u^2}du\, dt - \frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}-1}(x-1)\\ &\ge \quad \int_1^x \frac{\psi(t)}{t}dt + \int_1^x\int_u^x \frac{\psi(u)}{u^2}dt\, du - \frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}-1}(x-1)\\ &\ge \quad x\int_1^x \frac{\psi(u)}{u^2}du - \frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}-1}(x-1).\end{aligned}$$

Similarly, we get

$$\int_1^x \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\le\mathfrak{N}\mathfrak{a}\le t\\ \mathfrak{a}\in[1]_\mathfrak{q}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}dt = x\int_1^x \frac{\psi(u,\mathfrak{q},[1]_\mathfrak{q})}{u^2}du.$$

Hence, (12) becomes

$$\sum_{\chi \neq \chi_0} \Phi_\chi(x) \leq \frac{x}{x-1} \int_1^x \left(|H_{\mathfrak{q}}(\mathbf{K})| \frac{\psi(u, \mathfrak{q}, [1]_{\mathfrak{q}})}{u^2} - \frac{\psi(u)}{u^2} \right) du + \frac{\log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q} - 1}. \tag{14}$$

Lemma 5.1. *We have*

$$\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q} \frac{\log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q} - 1} \leq 14 + 4 \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}},$$

where the sum runs over prime ideals $\mathfrak{q}$ with $\mathfrak{N}\mathfrak{q}$ in the interval $(\frac{1}{2}Q, Q]$.

Proof. Using Theorem 3.11, we get

$$\left| \sum_{1 \leq \mathfrak{N}\mathfrak{q} \leq x} \frac{\log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}} - \log x \right| \leq 3 + \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}}.$$

It implies that

$$\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q} \frac{\log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q} - 1} \leq \sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q} \frac{2 \log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}} \leq 2 \log 2 + 12 + 4 \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}}.$$

$\square$

Remark 5.1. *We note, from the above, that we have*

$$\sum_{\frac{Q}{2} \leq \mathfrak{N}\mathfrak{q} \leq Q} \frac{\log \mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q} - 1} < 8 \cdot 10^{45} |d_{\mathbf{K}}|.$$

Theorem 5.2. *Assuming GRH, for $x > 2$ and $Q \geq 8 \exp(8 \cdot 10^{45} |d_{\mathbf{K}}|)$, we have*

$$\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q}{}^{'} \left| \sum_{\chi \neq \chi_0} \Phi_\chi(x) \right| \leq (2^{13} \cdot 10^{13} h_{\mathbf{K}} + 6000 h_{\mathbf{K}}^2 + 10) \pi^*(Q) \log Q.$$

where the outer sum is over principal prime ideals and $\pi^*(Q)$ denotes the number of principal prime ideals with norm in the interval $(\frac{1}{2}Q, Q]$.

Proof. From now onwards, we will drop the prime over summation and always assume the sum to be over principal prime ideals in the given interval.
From equation (14), for $x \geq 2$, we get

$$\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q} \left| \sum_{\chi \neq \chi_0} \Phi_\chi(x) \right| \leq 2 \int_1^x \frac{\sum_{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q} \left| |H_{\mathfrak{q}}(\mathbf{K})| \psi(u, \mathfrak{q}, [1]_{\mathfrak{q}}) - \psi(u) \right|}{u^2} du + C_1(\mathbf{K}), \tag{15}$$

where $C_1(\mathbf{K}) = 14 + 4 \frac{e^{75} |d_{\mathbf{K}}|^{1/3} (\log |d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}}$.
For any principal ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}}$, let $\mathcal{T}_{\mathfrak{a}}$ be the set of all principal prime ideals $\mathfrak{q}$ such that there

exists a generator $\pi_{\mathfrak{a}}$ of $\mathfrak{a}$ such that $\pi_{\mathfrak{a}} \equiv 1 \bmod \mathfrak{q}$. Then,

$$\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\le Q} |H_{\mathfrak{q}}(\mathbf{K})|\psi(u,\mathfrak{q},[1]_{\mathfrak{q}}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}} \\ 1\le\mathfrak{N}(\mathfrak{a})\le u \\ \mathfrak{a}\text{ principal}}} \Lambda(\mathfrak{a}) \sum_{\substack{\mathfrak{q}\in\mathcal{T}_{\mathfrak{a}} \\ \frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\le Q}} |H_{\mathfrak{q}}(\mathbf{K})|. \tag{16}$$

First we shall consider the case $u \le Q^{5/2}$.

Contribution from prime powers $\mathfrak{p}^k$ for $k \ge 2$: Consider a principal ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbf{K}}$, $1 \le \mathfrak{N}\mathfrak{a} \le u$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$ be the distinct prime ideals with norm in $(Q/2, Q]$ such that $\mathfrak{a} \in [1]_{\mathfrak{p}_i}$ for all $1 \le i \le j$. Therefore for each $1 \le i \le j$, we have $\pi_{\mathfrak{a},\mathfrak{p}_i}$, generators of $\mathfrak{a}$, such that $\pi_{\mathfrak{a},\mathfrak{p}_i} \equiv 1 \bmod \mathfrak{p}_i$. We observe that since $\mathfrak{a}$ is a principal ideal, all the $\pi_{\mathfrak{a},\mathfrak{p}_i}$ are associates. Then we have

$$\prod_{i=1}^{j} \mathfrak{p}_i \mid \prod_{w\in\mu_{\mathbf{K}}} (w\pi_{\mathfrak{a},\mathfrak{p}_1} - 1).$$

Let $c = |\mu_{\mathbf{K}}|$. Then

$$\frac{Q^j}{2^j} \le \mathfrak{N}(\mathfrak{p}_1 \ldots \mathfrak{p}_j) \le \prod_{w\in\mu_{\mathbf{K}}} \mathfrak{N}(w\pi_{\mathfrak{a},\mathfrak{p}_1} - 1) \le (4\mathfrak{N}(\pi_{\mathfrak{a},\mathfrak{p}_1}))^c \le (4u)^c \le (4Q^{5/2})^c, \tag{17}$$

where we have used the fact that $\mathfrak{N}(\alpha - 1) \le 4\mathfrak{N}(\alpha)$ for any non-zero integral element α in an imaginary quadratic field. For $Q > 8$, comparing both sides of (17), we note that

$$2^{6j-15c} < Q^{2j-5c} \le 2^{2j+4c}.$$

This implies that $j \le 5|\mu_{\mathbf{K}}|$. So, the number of terms in the inner sum of (16) is bounded above by $5|\mu_{\mathbf{K}}|$ for $Q > 8$. Using Lemma 4.1, we see that the contribution of powers of prime ideals to the sum (16) is

$$\sum_{\substack{\mathfrak{a}=\mathfrak{p}^k\subset\mathcal{O}_{\mathbf{K}} \\ 1\le\mathfrak{N}\mathfrak{p}^k\le u \\ \mathfrak{p}^k\text{ principal} \\ k\ge2}} \Lambda(\mathfrak{a}) \sum_{\substack{\mathfrak{q}\in\mathcal{T}_{\mathfrak{a}} \\ \frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\le Q}} |H_{\mathfrak{q}}(\mathbf{K})| \;\le\; (5|\mu_{\mathbf{K}}|)h_{\mathbf{K}}Q \sum_{\substack{1\le\mathfrak{N}\mathfrak{p}^k\le u \\ k\ge2}} \log\mathfrak{N}\mathfrak{p} \;\le\; 10h_{\mathbf{K}}|\mu_{\mathbf{K}}|Q\sqrt{u}\log u.$$

Putting this in (15), we get

$$2\int_1^{Q^{5/2}} \left(\sum_{\substack{\mathfrak{a}=\mathfrak{p}^k\subset\mathcal{O}_{\mathbf{K}} \\ 1\le\mathfrak{N}\mathfrak{p}^k\le u \\ \mathfrak{p}^k\text{ principal} \\ k\ge2}} \Lambda(\mathfrak{a}) \sum_{\substack{\mathfrak{q}\in\mathcal{T}_{\mathfrak{a}} \\ \frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\le Q}} |H_{\mathfrak{q}}(\mathbf{K})| \right) \frac{du}{u^2} \;\le\; 20h_{\mathbf{K}}|\mu_{\mathbf{K}}|Q \int_1^{Q^{5/2}} \frac{\log u}{u^{3/2}} du \le 480h_{\mathbf{K}}Q,$$

where we use (4) and the fact that for $\Re(s) > 0$,

$$\int_1^{\infty} \frac{\log^n t}{t^{s+1}} dt = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{18}$$

Since $Q > \exp(10^{45}|d_{\mathbf{K}}|)$, using Lemma 4.8 and the fact that $\log|d_{\mathbf{K}}| \geq \log 2$, we get

$$2\int_1^{Q^{5/2}} \left(\sum_{\substack{\mathfrak{p}^k \subset \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{p}^k \leq u \\ \mathfrak{p}^k \text{ principal} \\ k \geq 2}} \Lambda(\mathfrak{a}) \sum_{\substack{\mathfrak{q} \in \mathcal{T}_{\mathfrak{a}} \\ \frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q}} |H_{\mathfrak{q}}(\mathbf{K})| \right) \frac{du}{u^2} \leq 6000 h_{\mathbf{K}}^2 \pi^*(Q) \log Q. \tag{19}$$

Contribution from primes $\mathfrak{p}$: We now want to compute the contribution of prime ideals to (16). To do so, we consider the sum

$$\sum_{\substack{\mathfrak{p} \subset \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{p} \leq u \\ \mathfrak{p} \text{ principal}}} \log \mathfrak{N}\mathfrak{p} \sum_{\substack{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q \\ \mathfrak{p} \in [1]_{\mathfrak{q}}}} |H_{\mathfrak{q}}(\mathbf{K})|.$$

To bound this term for each $\mathfrak{p}$ in the outer sum, we choose a generator $\pi_{\mathfrak{p}}$ and count the number of ideals $\mathfrak{q}$ such that $\pi_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{q}$. Therefore

$$\sum_{\substack{\mathfrak{p} \subset \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{p} \leq u \\ \mathfrak{p} \text{ principal}}} \log \mathfrak{N}\mathfrak{p} \sum_{\substack{\mathfrak{p} \in [1]_{\mathfrak{q}} \\ \frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q}} |H_{\mathfrak{q}}(\mathbf{K})| \leq \sum_{\substack{\pi_{\mathfrak{p}} \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\pi_{\mathfrak{p}}) \leq u \\ \pi_{\mathfrak{p}} \in \mathcal{P}_{\mathbf{K}}}} \log \mathfrak{N}(\pi_{\mathfrak{p}}) \sum_{\substack{\pi_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{q} \\ \frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q}} |H_{\mathfrak{q}}(\mathbf{K})|.$$

For each principal prime ideal $\mathfrak{q}$, we now fix a generator $\pi_{\mathfrak{q}}$. The condition $\pi_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{q}$ is equivalent to the existence of a $\delta \in \mathcal{O}_{\mathbf{K}}$ with $\mathfrak{N}(\delta) \leq 8u/Q$ such that $\pi_{\mathfrak{p}} - 1 = \delta\pi_{\mathfrak{q}}$. Therefore, we have

$$\begin{aligned}
\sum_{\substack{\mathfrak{p} \subset \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}\mathfrak{p} \leq u \\ \mathfrak{p} \text{ principal}}} \log \mathfrak{N}\mathfrak{p} \sum_{\substack{\mathfrak{p} \in [1]_{\mathfrak{q}} \\ \frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q}} |H_{\mathfrak{q}}(\mathbf{K})| &\leq \sum_{\substack{\pi_{\mathfrak{p}} \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\pi_{\mathfrak{p}}) \leq u \\ \pi_{\mathfrak{p}} \in \mathcal{P}_{\mathbf{K}}}} \sum_{\substack{\frac{1}{2}Q < \mathfrak{N}\mathfrak{q} \leq Q \\ \pi_{\mathfrak{p}} - 1 = \pi_{\mathfrak{q}}\delta \\ \text{for some } \delta \in \mathcal{O}_{\mathbf{K}}}} |H_{\mathfrak{q}}(\mathbf{K})| \log \mathfrak{N}\pi_{\mathfrak{p}} \\
&\leq \sum_{\substack{\pi_{\mathfrak{p}} \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\pi_{\mathfrak{p}}) \leq u \\ \pi_{\mathfrak{p}} \in \mathcal{P}_{\mathbf{K}}}} \sum_{\substack{\frac{1}{2}Q < \mathfrak{N}(\pi_{\mathfrak{q}}) \leq Q \\ \pi_{\mathfrak{p}} - 1 = \pi_{\mathfrak{q}}\delta \\ \text{for some } \delta \in \mathcal{O}_{\mathbf{K}}}} |H_{(\pi_{\mathfrak{q}})}(\mathbf{K})| \log \mathfrak{N}(\pi_{\mathfrak{p}}) \\
&\leq \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) \leq \frac{8u}{Q}}} \sum_{\substack{\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}} \in \mathcal{P}_{\mathbf{K}} \\ \mathfrak{N}(\pi_{\mathfrak{p}}) \leq u \\ \frac{1}{2}Q < \mathfrak{N}(\pi_{\mathfrak{q}}) \leq Q \\ \pi_{\mathfrak{p}} - 1 = \delta\pi_{\mathfrak{q}}}} |H_{(\pi_{\mathfrak{q}})}(\mathbf{K})| \log \mathfrak{N}(\pi_{\mathfrak{p}}). \qquad (20)
\end{aligned}$$

We split the outer summation as

$$\sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) \leq \frac{8u}{Q}}} = \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) < \frac{u}{Q}}} + \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ \frac{u}{Q} \leq \mathfrak{N}(\delta) \leq \frac{8u}{Q}}}$$

Now we shall compute the following integrals

$$I_1 = \int_1^{Q^{5/2}} \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < \frac{u}{Q}}} \sum_{\substack{\pi_\mathfrak{p}, \pi_\mathfrak{q} \in \mathcal{P}_{\mathbf{K}} \\ \mathfrak{N}(\pi_\mathfrak{p}) \le u \\ \frac{1}{2}Q < \mathfrak{N}(\pi_\mathfrak{q}) \le Q \\ \pi_\mathfrak{p} - 1 = \delta \pi_\mathfrak{q}}} |H_{(\pi_\mathfrak{q})}(\mathbf{K})| \log \mathfrak{N}(\pi_\mathfrak{p}) \frac{du}{u^2} \tag{21}$$

$$I_2 = \int_1^{Q^{5/2}} \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ \frac{u}{Q} \le \mathfrak{N}(\delta) \le \frac{8u}{Q}}} \sum_{\substack{\pi_\mathfrak{p}, \pi_\mathfrak{q} \in \mathcal{P}_{\mathbf{K}} \\ \mathfrak{N}(\pi_\mathfrak{p}) \le u \\ \frac{1}{2}Q < \mathfrak{N}(\pi_\mathfrak{q}) \le Q \\ \pi_\mathfrak{p} - 1 = \delta \pi_\mathfrak{q}}} |H_{(\pi_\mathfrak{q})}(\mathbf{K})| \log \mathfrak{N}(\pi_\mathfrak{p}) \frac{du}{u^2}. \tag{22}$$

Estimating I_1 : For $\delta \in \mathcal{O}_{\mathbf{K}} \backslash \{0\}$, we note that counting the number of pairs of prime elements $(\pi_\mathfrak{p}, \pi_\mathfrak{q})$ such that $\pi_\mathfrak{p} - 1 = \delta \pi_\mathfrak{q}$ in the inner sum in (21) is same as counting number of $\alpha \in \mathcal{P}_{\mathbf{K}}$ such that $\delta\alpha + 1$ also belongs to $\mathcal{P}_{\mathbf{K}}$. Using the conditions $\pi_\mathfrak{p} - 1 = \delta \pi_\mathfrak{q}$ and $Q/2 < \mathfrak{N}(\pi_\mathfrak{q}) \le Q$, we get $\mathfrak{N}(\pi_\mathfrak{p}) \le 4\mathfrak{N}(\pi_\mathfrak{p} - 1) \le 4\mathfrak{N}(\delta)Q$. Also, $\mathfrak{N}(\alpha) = \mathfrak{N}(\pi_\mathfrak{q}) \le Q$. Using Proposition 4.7 we bound this term by

$$\sum_{\substack{\pi_\mathfrak{p}, \pi_\mathfrak{q} \in \mathcal{P}_{\mathbf{K}} \\ \mathfrak{N}(\pi_\mathfrak{p}) \le u \\ \frac{1}{2}Q < \mathfrak{N}(\pi_\mathfrak{q}) \le Q \\ \pi_\mathfrak{p} - 1 = \delta \pi_\mathfrak{q}}} 1 \le \sum_{\substack{1 \le \mathfrak{N}(\alpha) \le Q \\ \alpha, \delta\alpha + 1 \in \mathcal{P}_{\mathbf{K}}}} 1 \le \frac{2^5 \cdot 10^{10} \cdot \mathfrak{N}(\delta) Q}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{Q}{16}} + 10^{30} Q^{3/4}.$$

Using equation (5) and Lemma 4.1, we get

$$I_1 \le h_{\mathbf{K}} Q \int_1^{Q^{5/2}} \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < \frac{u}{Q}}} \left(\frac{2^5 \cdot 10^{10} \cdot \mathfrak{N}(\delta) Q}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{Q}{16}} + 10^{30} Q^{3/4} \right) \log 4Q\mathfrak{N}(\delta) \frac{du}{u^2}.$$

Interchanging sum and integral, we get

$$\begin{aligned} I_1 &\le h_{\mathbf{K}} Q \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < Q^{3/2}}} \left(\frac{2^5 \cdot 10^{10} \cdot \mathfrak{N}(\delta) Q}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{Q}{16}} + 10^{30} Q^{3/4} \right) \log 4Q\mathfrak{N}(\delta) \int_{Q\mathfrak{N}(\delta)}^{\infty} \frac{du}{u^2} \\ &\le h_{\mathbf{K}} Q \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < Q^{3/2}}} \left(\frac{2^5 \cdot 10^{10} \cdot \mathfrak{N}(\delta) Q}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{Q}{16}} + 10^{30} Q^{3/4} \right) \frac{\log 4Q\mathfrak{N}(\delta)}{Q\mathfrak{N}(\delta)}. \end{aligned}$$

Let

$$I_{11} = h_{\mathbf{K}} Q \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < Q^{3/2}}} \left(\frac{2^5 \cdot 10^{10} \cdot \mathfrak{N}(\delta) Q}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{Q}{16}} \right) \frac{\log 4Q\mathfrak{N}(\delta)}{Q\mathfrak{N}(\delta)}$$

and

$$I_{12} = h_{\mathbf{K}} Q \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \le \mathfrak{N}(\delta) < Q^{3/2}}} 10^{30} Q^{3/4} \frac{\log 4Q\mathfrak{N}(\delta)}{Q\mathfrak{N}(\delta)}.$$

Computing the summation in the first term and using Lemma 4.4, we get

$$\begin{aligned} I_{11} &\leq \frac{2^4 \cdot 5 \cdot 10^{10} h_{\mathbf{K}} Q \log(2Q)}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \log^2 \frac{Q}{16}} \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) < Q^{3/2}}} \frac{1}{\varphi((\delta))} \\ &\leq \frac{2^4 \cdot 5 \cdot 10^{10} h_{\mathbf{K}} Q \log(2Q)}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \log^2 \frac{Q}{16}} \left(\zeta_{\mathbf{K}}(2)^2 \rho_{\mathbf{K}} \log 4Q^{3/2} + \sqrt{3} \cdot 2 \cdot 10^{15} \zeta_{\mathbf{K}}(3/2) \zeta_{\mathbf{K}}(2) h_{\mathbf{K}} \right). \end{aligned}$$

For $Q > 2^9$ we have $\log 2Q \leq 2 \log \frac{Q}{16}$. Further, since $h_{\mathbf{K}} \leq \rho_{\mathbf{K}} \sqrt{|d_{\mathbf{K}}|}$, we get

$$I_{11} \leq 2^8 \cdot 3 \cdot 10^{11} Q + \frac{10^{40} h_{\mathbf{K}} Q}{\rho_{\mathbf{K}} \log \frac{Q}{16}}.$$

Since $Q \geq 8 \exp(8 \cdot 10^{45} \cdot |d_{\mathbf{K}}|)$, we have $\log Q/16 \geq 10^{45} \cdot |d_{\mathbf{K}}|$. This gives us

$$I_{11} \leq 2^8 \cdot 3 \cdot 10^{11} Q + \frac{Q}{\sqrt{|d_{\mathbf{K}}|}}.$$

Using Lemma 4.8, we get

$$I_{11} \leq \left(2^5 \cdot 3 \cdot 10^{13} h_{\mathbf{K}} + \frac{1}{2} \right) \pi^*(Q) \log Q. \tag{23}$$

We now consider

$$I_{12} \leq \frac{5}{2} \cdot 10^{30} h_{\mathbf{K}} Q^{3/4} \log 2Q \sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) < Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)}.$$

By partial summation formula, we have

$$\sum_{\substack{\delta \in \mathcal{O}_{\mathbf{K}} \\ 1 \leq \mathfrak{N}(\delta) < Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)} \leq 2 \rho_{\mathbf{K}} \log 2Q + 2 \cdot \sqrt{3} \cdot 10^{15} h_{\mathbf{K}}. \tag{24}$$

We now substitute the bound (24) in bound for I_{12}. Using Lemma 4.8, the fact that $\log 2Q \leq 16(2Q)^{1/16}$, and the bound $Q \geq 8 \exp(8 \cdot 10^{45} \cdot |d_{\mathbf{K}}|)$ we get

$$I_{12} \leq 10^{35} \rho_{\mathbf{K}} h_{\mathbf{K}}^2 \frac{\pi^*(Q) \log Q}{Q^{1/4}} + 10^{49} h_{\mathbf{K}}^3 \frac{\pi^*(Q) \log Q}{Q^{1/4}} \leq \frac{1}{2} \pi^*(Q) \log Q. \tag{25}$$

Therefore,

$$I_1 \leq (2^5 \cdot 3 \cdot 10^{13} h_{\mathbf{K}} + 1) \pi^*(Q) \log Q. \tag{26}$$

Estimating I_2 : In the same way as before, for $\delta \in \mathcal{O}_{\mathbf{K}} \backslash \{0\}$, counting the number of pairs of prime elements $(\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}})$ such that $\pi_{\mathfrak{p}} - 1 = \delta \pi_{\mathfrak{q}}$ in the inner sum in (22) is same as counting number of $\alpha \in \mathcal{P}_{\mathbf{K}}$ such that $\delta\alpha + 1$ also belongs to $\mathcal{P}_{\mathbf{K}}$. Using the condition $\pi_{\mathfrak{p}} - 1 = \delta \pi_{\mathfrak{q}}$we get $\mathfrak{N}(\delta\alpha) = \mathfrak{N}(\delta\pi_{\mathfrak{q}}) = \mathfrak{N}(\pi_{\mathfrak{p}} - 1) \leq 4\mathfrak{N}(\pi_{\mathfrak{p}}) \leq 4u$. Using Proposition 4.7 we bound this term by

$$\sum_{\substack{\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}} \in \mathcal{P}_{\mathbf{K}} \\ \mathfrak{N}(\pi_{\mathfrak{p}}) \leq u \\ \frac{1}{2} Q < \mathfrak{N}(\pi_{\mathfrak{q}}) \leq Q \\ \pi_{\mathfrak{p}} - 1 = \delta \pi_{\mathfrak{q}}}} 1 \leq \sum_{\substack{1 \leq \mathfrak{N}(\alpha) \leq 4u/\mathfrak{N}(\delta) \\ \alpha, \delta\alpha + 1 \in \mathcal{P}_{\mathbf{K}}}} 1 \leq \frac{2^7 \cdot 10^{10} u}{\rho_{\mathbf{K}}^2 \sqrt{|d_{\mathbf{K}}|} \varphi((\delta)) \log^2 \frac{u}{4\mathfrak{N}(\delta)}} + 10^{30} \left(\frac{4u}{\mathfrak{N}(\delta)} \right)^{3/4}. \tag{27}$$

Using Lemma 4.1 and (27), we get

$$I_2 \;\leq\; \frac{2^7\cdot 10^{10} h_{\mathbf{K}} Q}{\rho_{\mathbf{K}}^2\sqrt{|d_{\mathbf{K}}|}\log^2(\frac{Q}{32})}\int_1^{Q^{5/2}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ \frac{u}{Q}\leq\mathfrak{N}(\delta)\leq\frac{8u}{Q}}} \frac{u^{-1}\log u}{\varphi_{\mathbf{K}}((\delta))}du \;+\; 3\cdot 10^{30} h_{\mathbf{K}} Q\int_1^{Q^{5/2}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ \frac{u}{Q}\leq\mathfrak{N}(\delta)\leq\frac{8u}{Q}}} \frac{\log u}{\mathfrak{N}(\delta)^{3/4}u^{5/4}}du.$$

We shall denote henceforth by

$$I_{21} = \frac{2^7\cdot 10^{10} h_{\mathbf{K}} Q}{\rho_{\mathbf{K}}^2\sqrt{|d_{\mathbf{K}}|}\log^2(\frac{Q}{32})}\int_1^{Q^{5/2}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ \frac{u}{Q}\leq\mathfrak{N}(\delta)\leq\frac{8u}{Q}}} \frac{u^{-1}\log u}{\varphi_{\mathbf{K}}((\delta))}du$$

and

$$I_{22} = 3\cdot 10^{30} h_{\mathbf{K}} Q\int_1^{Q^{5/2}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ \frac{u}{Q}\leq\mathfrak{N}(\delta)\leq\frac{8u}{Q}}} \frac{\log u}{u^{5/4}\mathfrak{N}(\delta)^{3/4}}du.$$

Let us first consider I_{21}, interchanging sum and integral and using the bound $h_{\mathbf{K}} \leq \rho_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}$, we get

$$I_{21} \leq \frac{2^7\cdot 10^{10} Q}{\rho_{\mathbf{K}}\log^2(\frac{Q}{32})} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\varphi_{\mathbf{K}}((\delta))}\int_{\frac{Q\mathfrak{N}(\delta)}{8}}^{Q\mathfrak{N}(\delta)} \frac{\log u}{u}du.$$

Integrating and using Lemma 4.4 and the fact that $\log(3h_{\mathbf{K}}) \leq 8(3h_{\mathbf{K}})^{1/8}$, we get

$$\begin{aligned}
I_{21} &\leq \frac{2^6\cdot 10^{10}\cdot\log 8\cdot Q}{\rho_{\mathbf{K}}\log^2(\frac{Q}{32})} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\varphi_{\mathbf{K}}((\delta))}\log\left(\frac{Q^2\mathfrak{N}(\delta)^2}{8}\right)\\
&\leq \frac{2^9\cdot 10^{11}\cdot\log 8\cdot Q\log^2(2Q)}{\log^2(\frac{Q}{32})} + \frac{2^8\cdot 3^4\cdot 10^{26}\cdot Q\log(2Q)}{\rho_{\mathbf{K}}\log^2(\frac{Q}{32})}(h_{\mathbf{K}}\log(3h_{\mathbf{K}}))^{1/2}\\
&\leq \frac{2^9\cdot 10^{11}\cdot\log 8\cdot Q\log^2(2Q)}{\log^2(\frac{Q}{32})} + \frac{2^{11}\cdot 3^4\cdot 10^{26}\cdot Q\log(2Q)}{\rho_{\mathbf{K}}\log^2(\frac{Q}{32})}h_{\mathbf{K}}^{9/16}.
\end{aligned}$$

Using the fact that $2\log(Q/32) \geq \log(2Q)$ for $Q \geq 2^{11}$, we get

$$I_{21} \leq 2^{13}\cdot 10^{11}\cdot\log 8\cdot Q + \frac{2^{12}\cdot 3^4\cdot 10^{26}Q}{\rho_{\mathbf{K}}\log(\frac{Q}{32})}h_{\mathbf{K}}^{9/16}.$$

Since $Q \geq 8\exp(8\cdot 10^{45}|d_{\mathbf{K}}|)$ and bounding $h_{\mathbf{K}}$, we get

$$I_{21} \leq 2^{13}\cdot 10^{11}\cdot\log 8\cdot Q + \frac{Q}{2\cdot 10^{10}\rho_{\mathbf{K}}^{7/16}|d_{\mathbf{K}}|^{23/32}}.$$

We now apply Lemma 4.8 and we bound $\rho_{\mathbf{K}}$ using Lemma 3.1 to get

$$I_{21} \leq 2^{10}\cdot 10^{13}\cdot\log 8\cdot h_{\mathbf{K}}\pi^*(Q)\log Q + \frac{1}{2}\pi^*(Q)\log Q. \tag{28}$$

Let us now consider I_{22}. We have

$$\begin{aligned}
I_{22} &\leq 6\cdot 10^{30} h_{\mathbf{K}} Q \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq \mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)Q^{1/4}} \int_{\frac{Q\mathfrak{N}(\delta)}{8}}^{Q\mathfrak{N}(\delta)} \frac{\log u}{u} du \\
&\leq 3\cdot \log 8\cdot 10^{30} h_{\mathbf{K}} Q \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq \mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)Q^{1/4}} \log \frac{Q^2\mathfrak{N}(\delta)^2}{8} \\
&\leq 15\cdot \log 8\cdot 10^{30} h_{\mathbf{K}} \frac{Q\log(2Q)}{Q^{1/4}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq \mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)}.
\end{aligned}$$

Since $\log(2Q)\leq 20Q^{1/8}$, we have

$$I_{22}\leq 3\cdot\log 8\cdot 10^{32} h_{\mathbf{K}} \frac{Q}{Q^{1/8}} \sum_{\substack{\delta\in\mathcal{O}_{\mathbf{K}}\\ 1\leq \mathfrak{N}(\delta)\leq 8Q^{3/2}}} \frac{1}{\mathfrak{N}(\delta)}.$$

Now, using (24), we get

$$\begin{aligned}
I_{22} &\leq 10^{33} h_{\mathbf{K}} \frac{Q}{Q^{1/8}} (2\rho_{\mathbf{K}} \log 8Q + 2\cdot\sqrt{3}\cdot 10^{15} h_{\mathbf{K}}) \\
&\leq 10^{35} \rho_{\mathbf{K}} h_{\mathbf{K}} \frac{Q}{Q^{1/16}} + \cdot 10^{49} h_{\mathbf{K}}^2 \frac{Q}{Q^{1/8}}.
\end{aligned}$$

By Lemma 3.1, Lemma 4.8, the bound on $h_{\mathbf{K}}$ and the fact that $Q\geq 8\exp(8\cdot 10^{45}|d_{\mathbf{K}}|)$, we get

$$I_{22}\leq \frac{1}{2}\pi^*(Q)\log Q. \tag{29}$$

Combining (28) and (29), we get

$$I_2\leq (2^{12}\cdot 10^{13}\cdot h_{\mathbf{K}}+1)\pi^*(Q)\log Q. \tag{30}$$

Finally from (26) and (30), we get

$$2\int_1^{Q^{5/2}} \left(\sum_{\substack{\mathfrak{p}\subset\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}\mathfrak{p}\leq u\\ \mathfrak{p}\text{ principal}}} \Lambda(\mathfrak{a}) \sum_{\substack{\mathfrak{q}\in\mathcal{T}_{\mathfrak{a}}\\ \frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}} |H_{\mathfrak{q}}(\mathbf{K})| \right) \frac{du}{u^2} \leq (2^{13}\cdot 10^{13}\cdot h_{\mathbf{K}}+2)\pi^*(Q)\log Q. \tag{31}$$

Contribution from $\psi(u)$: Using Theorem 3.13, under the Riemann hypothesis we have for $u\geq 3$,

$$\psi(u)\leq u+7\log|d_{\mathbf{K}}|\sqrt{u}\log u+\sqrt{u}\log^2 u+19\sqrt{u}.$$

Putting this in (15), we get

$$2\int_1^{Q^{5/2}} \frac{\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}\psi(u)}{u^2} du \leq (5\log Q+56\log|d_{\mathbf{K}}|+109)\,\pi^*(Q)\leq 6\cdot\pi^*(Q)\log Q. \tag{32}$$

We will now consider the case $u > Q^{5/2}$. Since

$$\psi(u,\mathfrak{q},[1]_{\mathfrak{q}}) \quad = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}\mathfrak{a}\leq u\\ \mathfrak{a}\in[1]_{\mathfrak{q}}}}\Lambda(\mathfrak{a}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}\mathfrak{a}\leq u\\ (\mathfrak{a},\mathfrak{q})=\mathcal{O}_{\mathbf{K}}\\ (\mathfrak{a},\mathbf{K}(\mathfrak{q})/\mathbf{K})=\sigma_0}}\Lambda(\mathfrak{a}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}}\\ 1\leq\mathfrak{N}\mathfrak{a}\leq u\\ (\mathfrak{a},\mathfrak{q})=\mathcal{O}_{\mathbf{K}}}}\mathbf{1}_{\sigma_o}((\mathfrak{a},\mathbf{K}(\mathfrak{q})/\mathbf{K}))\Lambda(\mathfrak{a}),$$

where $(\mathfrak{a},\mathbf{K}(\mathfrak{q})/\mathbf{K})$ denotes the Artin symbol of $\mathfrak{a}$ and σ_o the trivial element of $\mathrm{Gal}(\mathbf{K}(\mathfrak{q})/\mathbf{K})$ with $\mathbf{1}_{\sigma_o}$ as its characteristic function. Using Theorem 3.12, Lemma 4.1, Lemma 4.5 and equation (4), we get

$$\begin{aligned}\Big||H_{\mathfrak{q}}(\mathbf{K})|\psi(u,\mathfrak{q},[1]_{\mathfrak{q}})-u\Big| &\leq \sqrt{u}\left(\left(\frac{\log u}{2\pi}+2\right)\log|d_{\mathbf{K}(\mathfrak{q})}|+\left(\frac{\log^2 u}{4\pi}+4\right)|H_{\mathfrak{q}}(\mathbf{K})|\right)\\ &\leq 2\rho_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}\mathfrak{N}\mathfrak{q}\Big(\log(\mathfrak{N}\mathfrak{q}|d_{\mathbf{K}}|)\log u+\log^2 u\Big)\sqrt{u}.\end{aligned}$$

Therefore, for $Q \geq 8\exp(8\cdot 10^{45}|d_{\mathbf{K}}|)$,

$$\begin{aligned}&2\int_{Q^{5/2}}^{x}\frac{\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}\Big||H_{\mathfrak{q}}(\mathbf{K})|\psi(u,\mathfrak{q},[1]_{\mathfrak{q}})-u\Big|}{u^2}du\\ \leq\quad & 4\rho_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}Q\pi^*(Q)\int_{Q^{5/2}}^{x}\frac{\log(Q|d_{\mathbf{K}}|)\log u+\log^2 u}{u^{3/2}}du\\ \leq\quad & \frac{1}{2}\pi^*(Q)\log Q,\end{aligned}$$

where in the last step we have used Lemma 3.1. By Theorem 3.13, for $Q \geq 3$, we get

$$|\psi(u)-u| \leq 7(\log|d_{\mathbf{K}}|)\sqrt{u}\log u+\sqrt{u}\log^2 u+19\sqrt{u}.$$

This implies that for $Q \geq 8\exp(8\cdot 10^{45}|d_{\mathbf{K}}|)$,

$$2\int_{Q^{5/2}}^{x}\frac{\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}|\psi(u)-u|}{u^2}du \leq (56\log|d_{\mathbf{K}}|+108)\pi^*(Q) \leq \frac{1}{2}\pi^*(Q)\log Q,$$

where we have used equation (18). Thus, the contribution of $Q^{5/2} < u < x$ to (15) is

$$2\int_{Q^{5/2}}^{x}\frac{\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}\Big||H_{\mathfrak{q}}(\mathbf{K})|\psi(u,\mathfrak{q},[1]_{\mathfrak{q}})-\psi(u)\Big|}{u^2}du \leq \pi^*(Q)\log Q. \tag{33}$$

Combining Lemma 5.1, (19), (31), (32) and (33), we get

$$\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}\Big|\sum_{\chi\neq\chi_0}\Phi_\chi(x)\Big| \leq (2^{13}\cdot 10^{13}h_{\mathbf{K}}+6000h_{\mathbf{K}}^2+10)\pi^*(Q)\log Q.$$

□

Let $\mathfrak{q}$ be a non-zero principal prime ideal of $\mathcal{O}_{\mathbf{K}}$. We note that given a generalized Dirichlet character χ modulo $\mathfrak{q}$ it is either primitive or induced by a character of the class group (also called a generalized Dirichlet character modulo $\mathcal{O}_{\mathbf{K}}$). We now consider the sum

$$\begin{aligned}\sum_{\chi\neq\chi_0}(\Phi_{\chi^*}(x)-\Phi_\chi(x)) &= \sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi^* \bmod \mathfrak{q}}}(\Phi_{\chi^*}(x)-\Phi_\chi(x)) + \sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi^* \bmod \mathcal{O}_{\mathbf{K}}\\ \chi\neq\chi_0}}(\Phi_{\chi^*}(x)-\Phi_\chi(x))\\ &= \sum_{\substack{\chi \bmod \mathfrak{q}\\ \chi^* \bmod \mathcal{O}_{\mathbf{K}}\\ \chi\neq\chi_0}}(\Phi_{\chi^*}(x)-\Phi_\chi(x))\\ &= \sum_{\substack{\chi^* \bmod \mathcal{O}_{\mathbf{K}}\\ \chi\neq\chi_0}}\frac{1}{x-1}\int_1^x\left(\sum_{\substack{\mathfrak{N}\mathfrak{a}\leq t,\\ \mathfrak{q}\mid\mathfrak{a}}}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\chi^*(\mathfrak{a})\right)dt.\end{aligned}$$

Since $\mathfrak{q}$ is a prime ideal, it follows that the inner sum only consists of terms for which $\mathfrak{a}=\mathfrak{q}^k$. Further since $\mathfrak{q}$ is principal so is $\mathfrak{a}$. Therefore, we get

$$\begin{aligned}\sum_{\chi\neq\chi_0}(\Phi_{\chi^*}(x)-\Phi_\chi(x)) &= \frac{1}{x-1}\int_1^x\left(\sum_{\substack{\mathfrak{N}\mathfrak{a}\leq t,\\ \mathfrak{q}\mid\mathfrak{a}}}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\sum_{\substack{\chi^* \bmod \mathcal{O}_{\mathbf{K}}\\ \chi\neq\chi_0}}\chi^*(\mathfrak{a})\right)dt\\ &\leq \frac{h_{\mathbf{K}}}{x-1}\int_1^x\left(\sum_{\substack{k\geq 1\\ \mathfrak{N}\mathfrak{q}^k\leq t}}\frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}^k}\right)dt\\ &\leq \frac{h_{\mathbf{K}}x}{x-1}\cdot\frac{\log\mathfrak{N}\mathfrak{q}}{\mathfrak{N}\mathfrak{q}-1}.\end{aligned}$$

Therefore for $x\geq 2$, using Lemma 5.1, we obtain

$$\sum_{\frac{Q}{2}<\mathfrak{N}\mathfrak{q}\leq Q}\left|\sum_{\chi\neq\chi_0}(\Phi_\chi(x)-\Phi_{\chi^*}(x))\right|\leq 2h_{\mathbf{K}}\left(14+4\frac{e^{75}|d_{\mathbf{K}}|^{1/3}(\log|d_{\mathbf{K}}|)^2}{\rho_{\mathbf{K}}}\right). \tag{34}$$

Proposition 5.3. *Let $Q\geq 2$ and $x\geq Q^4$. Let χ be a generalized Dirichlet character modulo $\mathfrak{q}$, where $\mathfrak{q}$ is a non-zero principal prime ideal in $\mathcal{O}_{\mathbf{K}}$. Also let $\frac{L'}{L}(1,\chi)$ denote the logarithmic derivative of the corresponding Hecke L-function evaluated at 1. Then under GRH, we have*

$$\sideset{}{'}\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q}\sum_{\chi\neq\chi_0}\left|\frac{L'}{L}(1,\chi^*)+\Phi_{\chi^*}(x)\right|<2010\rho_{\mathbf{K}}\sqrt{|d_{\mathbf{K}}|}\frac{\pi^*(Q)\log(5|d_{\mathbf{K}}|Q)}{Q},$$

where the outer sum is over principal prime ideals in the interval $(\frac{1}{2}Q,Q]$ and $\pi^*(Q)$ denotes the number of principal prime ideals with norm in the interval $(\frac{1}{2}Q,Q]$.

Proof. Using Theorem 3.7, we have

$$\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x) = \frac{1}{x-1}\sum_{\rho_{\chi^*}} \frac{x^{\rho_{\chi^*}}-1}{\rho_{\chi^*}(1-\rho_{\chi^*})} + \log\frac{x}{x-1} + \frac{1}{x-1}\log x, \tag{35}$$

where the sum is over all non trivial zeros of $L(s,\chi^*)$ counted with multiplicities. Using Proposition 4.11, we get

$$N_{t,\chi^*} = \#\{\rho = \frac{1}{2} + i\beta : |\beta - t| \le 1, L(\rho,\chi^*) = 0\} < 100\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}\ (|t|+2)).$$

For a non-trivial zero $\rho_{\chi^*} = \frac{1}{2} + i\beta$ of $L(s,\chi^*)$, we have $|\rho_{\chi^*}(1-\rho_{\chi^*})| = \left(\frac{1}{2}\right)^2 + \beta^2$. For $|\beta| \le 1$, we get

$$\sum_{\substack{\rho_{\chi^*}=1/2+i\beta \\ |\beta|\le 1}} \frac{1}{|\rho_{\chi^*}(1-\rho_{\chi^*})|} \le 4N_{0,\chi^*} < 400\log(2|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}). \tag{36}$$

Also, for $|\beta| > 1$, we get

$$\begin{aligned}\sum_{\substack{\rho_{\chi^*}=1/2+i\beta \\ |\beta|> 1}} \frac{1}{|\rho_{\chi^*}(1-\rho_{\chi^*})|} &\le \sum_{k=1}^{\infty} \sum_{\substack{\rho_{\chi^*}=1/2+i\beta \\ k<|\beta|\le k+1}} \frac{1}{\beta^2} \le \sum_{k=1}^{\infty}\frac{1}{k^2}(N_{k,\chi^*} + N_{-k,\chi^*}) \\ &< 200\sum_{k=1}^{\infty}\frac{\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}\ (k+2))}{k^2}.\end{aligned}$$

We have

$$\sum_{k=1}^{\infty}\frac{\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}\ (k+2))}{k^2} \quad\le\quad \zeta(2)\log(|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}) + \sum_{k=1}^{\infty}\frac{\log(k+2)}{k^2} \le 3\log(5|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}).$$

It implies that

$$\sum_{\substack{\rho_{\chi^*}=1/2+i\beta \\ |\beta|> 1}} \frac{1}{|\rho_{\chi^*}(1-\rho_{\chi^*})|} \le 600\log(5|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}). \tag{37}$$

Plugging (36) and (37) in (35), we get

$$\begin{aligned}\left|\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x)\right| &< \frac{\sqrt{x}+1}{x-1}\sum_{\rho_{\chi^*}}\frac{1}{|\rho_{\chi^*}(1-\rho_{\chi^*})|} + \log\frac{x}{x-1} + \frac{2\log x}{x} \\ &< \frac{2000}{\sqrt{x}}\log(5|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}) + \frac{1}{x-1} + \frac{4}{\sqrt{x}} \\ &< \frac{2010}{\sqrt{x}}\log(5|d_{\mathbf{K}}|\mathfrak{N}\mathfrak{q}).\end{aligned}$$

Summing over $\chi \neq \chi_0$ modulo $\mathfrak{q}$ and then over the principal prime ideals $\mathfrak{q}$ with $\mathfrak{N}\mathfrak{q} \in (\frac{1}{2}Q, Q]$, we get

$$\begin{aligned}\sideset{}{'}\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q} \sum_{\chi\neq\chi_0} \left|\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x)\right| &< \frac{2010}{\sqrt{x}} h_{\mathbf{K}} Q \pi^*(Q) \log(5|d_{\mathbf{K}}|Q) \\ &\leq \frac{2010}{\sqrt{x}} \rho_{\mathbf{K}} \sqrt{|d_{\mathbf{K}}|} Q \pi^*(Q) \log(5|d_{\mathbf{K}}|Q),\end{aligned}$$

where we use equation (4) in the last step. Choosing $x \geq Q^4$, we get our desired result. $\square$

5.1. **Proof of Theorem 1.7.** Let χ be a generalized Dirichlet character modulo $\mathfrak{q}$, where $\mathfrak{q}$ is a non-zero principal prime ideal in $\mathcal{O}_{\mathbf{K}}$. We have (see Chapter VIII, [4] for more details)

$$\zeta_{\mathbf{K}(\mathfrak{q})}(s) = \prod_{\chi \bmod \mathfrak{q}} L(s, \chi^*), \tag{38}$$

where the product is over all generalized Dirichlet characters modulo $\mathfrak{q}$. For any number field $\mathbf{K}$, an equivalent definition of $\gamma_{\mathbf{K}}$ is given by

$$\gamma_{\mathbf{K}} = \lim_{s\to 1}\left(\frac{\zeta'_{\mathbf{K}}}{\zeta_{\mathbf{K}}}(s) + \frac{1}{s-1}\right).$$

Therefore, taking the logarithmic derivative on both sides of (38), we get

$$\gamma_{\mathbf{K}(\mathfrak{q})} = \gamma_{\mathbf{K}} + \sum_{\chi\neq\chi_0} \frac{L'}{L}(1,\chi^*).$$

It now follows that

$$\gamma_{\mathbf{K}(\mathfrak{q})} = \gamma_{\mathbf{K}} - \sum_{\chi\neq\chi_0} \Phi_\chi(x) + \sum_{\chi\neq\chi_0} \Big(\Phi_\chi(x) - \Phi_{\chi^*}(x)\Big) + \sum_{\chi\neq\chi_0} \Big(\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x)\Big).$$

Therefore,

$$|\gamma_{\mathbf{K}(\mathfrak{q})}| \leq |\gamma_{\mathbf{K}}| + \left|\sum_{\chi\neq\chi_0} \Phi_\chi(x)\right| + \left|\sum_{\chi\neq\chi_0} \Big(\Phi_\chi(x) - \Phi_{\chi^*}(x)\Big)\right| + \left|\sum_{\chi\neq\chi_0} \Big(\frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x)\Big)\right|.$$

Using Theorem 5.2, (34) Proposition 5.3 and the fact that $Q \geq 8\exp(8\cdot 10^{45}|d_{\mathbf{K}}|)$, we get

$$\begin{aligned}\sideset{}{'}\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q} |\gamma_{\mathbf{K}(\mathfrak{q})}| &< |\gamma_{\mathbf{K}}|\pi^*(Q) + (6000h_{\mathbf{K}}^2 + 2^{13}\cdot 10^{13} h_{\mathbf{K}} + 10)\pi^*(Q)\log Q \\ &+ 2h_{\mathbf{K}}\log Q + \pi^*(Q)\log Q.\end{aligned}$$

This implies that

$$\frac{1}{\pi^*(Q)} \sideset{}{'}\sum_{\frac{1}{2}Q<\mathfrak{N}\mathfrak{q}\leq Q} |\gamma_{\mathbf{K}(\mathfrak{q})}| < |\gamma_{\mathbf{K}}| + (6000h_{\mathbf{K}}^2 + 10^{17}h_{\mathbf{K}} + 11)\log Q.$$

6. Concluding Remarks

We conclude by showing that infinitely many narrow ray class fields corresponding to principal prime moduli of an imaginary quadratic field are non-abelian. Given any narrow ray class field $\mathbf{K}(\mathfrak{q})$ of $\mathbf{K}$ modulo an integral ideal $\mathfrak{q}$, the Hilbert class field of $\mathbf{K}$ is a subfield of $\mathbf{K}(\mathfrak{q})$ containing $\mathbf{K}$. By a result of Ankeny and Chowla [2], we know that there are infinitely many imaginary quadratic fields whose class number is divisible by 3. By genus theory (see chapter 1 of [22]) if the Hilbert class field of $\mathbf{K}$ is abelian over $\mathbb{Q}$, the class number must be a power of 2. Therefore for the family of fields suggested by the result of Ankeny and Chowla, the Hilbert class field must be non-abelian and therefore all the narrow ray class fields of such an imaginary quadratic are also non-abelian.

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NEELAM KANDHIL,
MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, D-53111 BONN, GERMANY.

RASHI LUNIA,
THE INSTITUTE OF MATHEMATICAL SCIENCES, A CI OF HOMI BHABHA NATIONAL INSTITUTE, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.

JYOTHSNAA SIVARAMAN,
STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE, MYSORE RD, RVCE POST, BENGALURU, KARNATAKA 560059, INDIA.

Email address: `kandhil@mpim-bonn.mpg.de`

Email address: `rashisl@imsc.res.in`

Email address: `jyothsnaa.s@gmail.com`