COMP 1433: Introduction to Data Analytics &
COMP 1003: Statistical Tools and Applications

Lecture 2 – Probability Basics for Data Analytics

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Looking Back (Last Lecture)

- Course arrangement plan and structure
- Introduction to Data Analytics (An Introduction)
  - **Key concepts**, *data*, *data analytics*, etc.
  - Applications, e.g., *search engine, product recommendation, spam detection*, etc.
  - Advance Technology, e.g., *classification, regression, clustering, matching, time-series analysis*, etc.
- Some basic concepts of *data mining* and *big data*.
  - Good to have an impression about what they are.
  - You'll have chances to learn more about them later!

Probability VS. Data Analytics

- In analytics process, we usually use random variables to describe the data. ***Why***?
  - Mathematical process to solve real-life problem
  - To make data computable.
  - To discover the patterns and trends behind data.
  - Allows the analysis of distribution and statistics.
- Probability helps predict the how likely that an event will happen.
  - E.g., *weather prediction*, *product recommendation*.

What is probability?

- *Probability* is a numerical description of *how likely an event is to occur* and or *how likely that a proposition is true*. --- From Wikipedia.
- We run a random experiment n times, during which an event A occurs m times, then we say the *frequency* of A's occurrence is $f_A = \frac{m}{n}$.
- When n is large enough, f_A will be very close to a value p, which is defined as the probability of A to occur, i.e., $\lim_{n \to +\infty} f_A \equiv P(A) = p$
  - When we toss a coin, the probability of "heads up" is 0.5

Roadmap

- Basic Concepts of Probability
- Conditional Probability and Bayes' Formula
- Data Analytics Applications with Probability
  - Naïve Bayes
  - Probabilistic Language Model

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What is *sample space*?

- The set of all possible outcomes of an experiment
- Example: *coin flipping*
- What are the possible outcomes?
  - The coin lands *Heads up* - H
  - The coin lands *Tails up* - T
- The sample space: $S = \{H, T\}$
- For fair coins
  - What are the chances of H and T happening?
  - Empirically, H and T have 50-50 chance to happen.
  - The *probability* for H to happen is 50% (0.5), so does T

What is an *event*?

- Subsets of the sample space
- Example: *rolling a die once*
  - The sample space $S = \{1,2,3,4,5,6\}$
  - An example event $E = \{1,3,4\}$
  - If we *roll the die once*, and the it lands with 1 or 3 or 4 up, then we say the event E *occurs.*

- The probability that E *occurs* is:
  - $P(E) = \frac{1+1+1}{6} = \frac{3}{6} = \frac{1}{2}$

What is *probability*?

- A *probability function* P that assigns a real number (the probability of E) to every event $E \subset S$
- P *must* satisfy the following basic properties:
  - $0 \leq P(E) \leq 1$
  - $P(S) = 1$
- ***An important property to speed up computing***:
  - For any *disjoint events*, E_i $(i = 1,2, \dots, n)$, we have $P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$

Probability Distribution

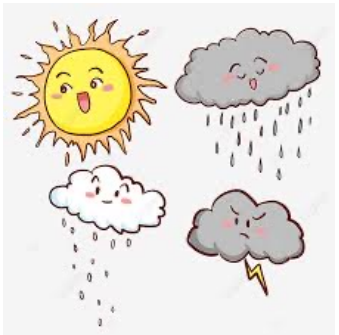
- A probability distribution is a listing of probabilities for every possible value the random variable might take.

Rolling a dice once

X	$P(X)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Weather Prediction

W	$P(W)$
Sunny	0.48
Cloudy	0.12
Rainy	0.25
Snowy	0
Foggy	0.15



Some Notions

- $P(E \cup F)$: the probability that E or F occurs
- $P(E, F)$ or $P(EF)$: the probability that both E and F occurs.
- $P(E^c)$: the probability that E does not occur.

Revisit the Coin Flipping Example

- **Scenario 1:** if the coin is flipped twice, what is the probability of two *heads*?
  - $P(H, H) = \frac{1}{4}$
- **Scenario 2:** if the coin is flipped twice, what is the probability of two *heads*, given that *we know the first toss gave a head*.
  - $P(H, H | H \ at \ the \ first \ toss) = \frac{1}{2}$

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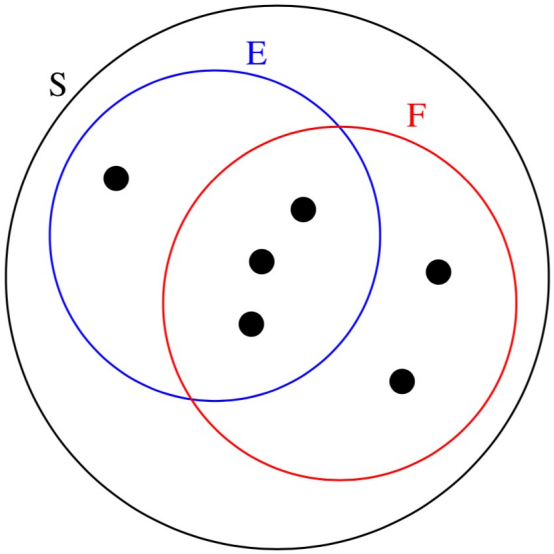
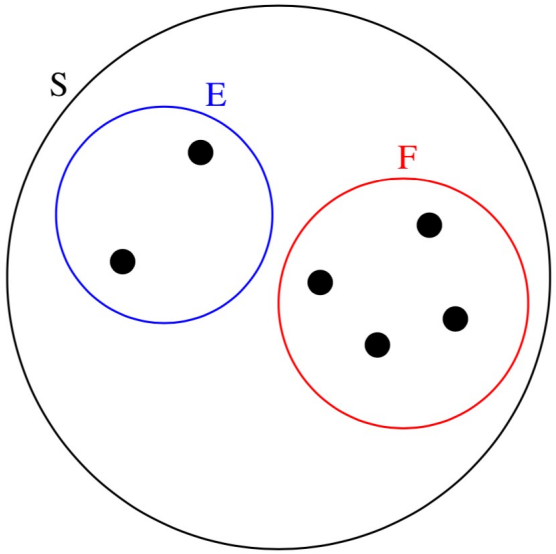
Conditional Probability

- If E and F are events, then $P(E|F)$ is the *conditional probability* of E, given F.
- $P(E|F) \equiv \frac{P(E,F)}{P(F)}$, assuming that $P(F) \neq 0$





What is $P(E|F)$ for the two cases?

Example: Conditional Probability

- Suppose we draw a card from a shuffled set of 52 playing cards.
- What is the probability of drawing a *Queen*, given that the card drawn is of suits *Hearts*.
  - $P(Q|H) = \frac{P(Q,H)}{P(H)} = \frac{1/52}{1/4} = \frac{1}{13}$
- What is the probability of drawing a *Queen*, given that the card drawn is a *Face* card?
  - $P(Q|F) = \frac{P(Q,F)}{P(F)} = \frac{P(Q)}{P(F)} = \frac{4/52}{12/52} = \frac{1}{3}$

Discussion: Teddy and Charlie

- It is well known that *Uncle Bob* has two beautiful children, *Teddy* and *Charlie*. However, no one knows whether they are *sons* or *daughters*.
- One day, you met *Bob* in the park. He told you that he has at least a *daughter*. Can you estimate the probability both *Teddy* and *Charlie* are daughters?
- On the other day, you met *Uncle Bob* again. He was with his beautiful daughter and introduced that she was *Teddy*. Now, can you estimate again the probability both *Teddy* and *Charlie* are daughters?

Law of Total Probability

- Sometimes, the computation of $P(E)$ will be easier if we condition E on another event F, namely, from
- $P(E) = P\big(E(F \cup F^c)\big) = P(E, F) + P(E, F^c)$
- Also, $P(E, F) = P(E|F)P(F)$

  and $P(E, F^c) = P(E|F^c)P(F^c)$
- $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$

Example: Law of Total Probability

- An insurance company holds the following data concerning the probability of an *insurance claim*:
  - For people under age 30, the probability is 4%
  - For people over age 30, the probability is 2%
- And it is known that 30% of the targeted population is under age 30.
- What is the probability of an insurance claim for a randomly chosen person?

Example: Law of Total Probability

- $S = \{all\ persons\ under\ consideration\}$
- $C = \{persons\ filing\ a\ claim\}$
- $U = \{persons\ under\ age\ 30\}$
- Thus, $P(C) = P(C|U)\ P(U) + P(C|U^c)\ P(U^c)$

$$= \frac{4}{100}\frac{3}{10} + \frac{2}{100}\frac{7}{10}$$

$$= \frac{26}{1000} = 2.6\% .$$

Bayes' formula

- Sometimes, we need a formula that *inverts conditioning*, e.g., predicts an event conditioned on some observations.
- Since $P(EF) = P(E|F)P(F)$ *and* $P(EF) = P(F|E)P(E)$
- Then we have

$$P(F|E) = \frac{P(E,F)}{P(E)} = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

Law of total probability

Example: Bayes' formula

- Suppose 1 in 1,000 persons has a certain disease.
- For 99% of the diseased persons, a test will yield positive results.
- For 5% of the healthy persons, a test will also yield positive results (false alarm).
- What is the probability of a positive test diagnosing the disease?
  - $D = \{Diseased\ persons\}$
  - $H = \{Healthy\ persons\}$
  - $+ = \{Persons\ with\ positive\ test\ results\}$

Example: Bayes' formula

- What is the probability of a positive test diagnosing the disease?
  - $D = \{Diseased\ persons\}$
  - $H = \{Healthy\ persons\}$
  - $+ = \{Persons\ with\ positive\ test\ results\}$
- By the given statistics, we have
  - $P(D) = 0.001, P(+|D) = 0.99, P(+|H) = 0.05$
- With Bayes' formula:

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|H) \cdot P(H)}$$

$$= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.05 \cdot 0.999} \cong 0.0194$$

Independent Events

- Two events E and F are *independent* if
  - $P(E, F) = P(E)P(F)$
- In this case:
  - $P(E|F) = \frac{P(E,F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$, assuming $P(F) \neq 0$
- In other words,
  - *knowing F occurred doesn't change the probability of E.*

Example: Independent Events

- Two different numbers are drawn at random from $\{1,2,3,4\}$
- If we don't consider the order of two numbers, e.g., (1,2) is identical to (2,1), then what is the sample space S?
- Define the following functions on S:
  - $X(\{i,j\}) = i + j$
  - $Y(\{i,j\}) = |i - j|$
- Which of the following pairs are independent?
  - $X = 5$ and $Y = 2$
  - $X = 5$ and $Y = 1$

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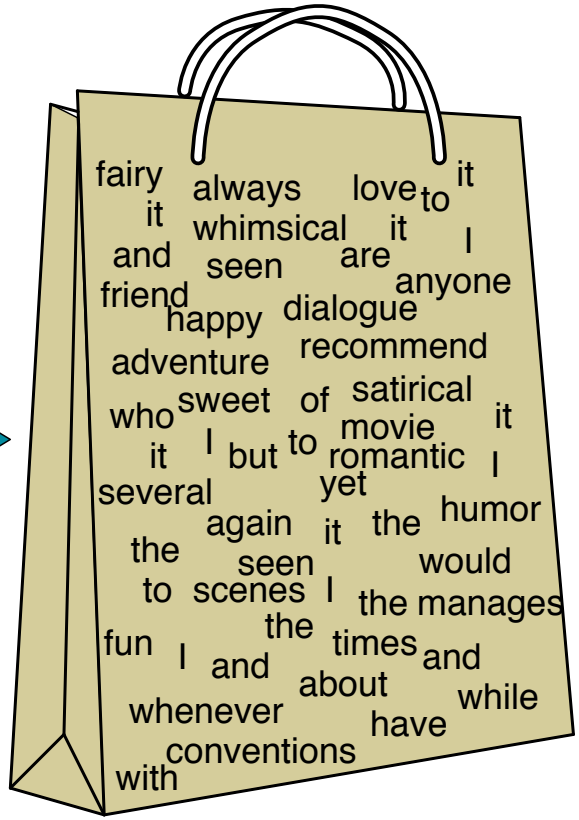
Classification Methods: Supervised Machine Learning

- *Input:*
  - a document *d*
  - a fixed set of classes $C = \{c_1, c_2, \ldots, c_J\}$
  - A training set of *m* hand-labeled documents $(d_1, c_1), \ldots., (d_m, c_m)$
- *Output:*
  - a learned classifier *γ:d* → *c*

The Bag of Words Representation

I love this movie! It's sweet, but with satirical humor. The dialogue is great and the adventure scenes are fun... It manages to be whimsical and romantic while laughing at the conventions of the fairy tale genre. I would recommend it to just about anyone. I've seen it several times, and I'm always happy to see it again whenever I have a friend who hasn't seen it yet!



it	6
I	5
the	4
to	3
and	3
seen	2
yet	1
would	1
whimsical	1
times	1
sweet	1
satirical	1
adventure	1
genre	1
fairy	1
humor	1
have	1
great	1
…	…

The bag of words representation

$\gamma($

seen	2
sweet	1
whimsical	1
recommend	1
happy	1
...	...

$)=c$

Bayes' Rule Applied to Documents and Classes

- For a document *d* and a class *c*

$$P(c \mid d) = \frac{P(d \mid c)P(c)}{P(d)}$$

Our goal is to maximize $P(c \mid d)$ with a $c \in C$

Naive Bayes Classifier (I)

argmax: the arguments of the maximum (e.g., find the optimal value of c which maximizes $P(c|d)$)

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(c \mid d)$$

MAP is "maximum a posteriori" = most likely class

$$= \underset{c \in C}{\operatorname{argmax}} \frac{P(d \mid c)P(c)}{P(d)}$$

Bayes' Formula

$$= \underset{c \in C}{\operatorname{argmax}} P(d \mid c)P(c)$$

Dropping the denominator

Naive Bayes Classifier (II)

"Likelihood" "Prior"

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(d \mid c) P(c)$$

$$= \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \cdots, x_n \mid c) P(c)$$

Document d represented as features $x_1, x_2, \ldots, x_n$

Naive Bayes Classifier (III)

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \cdots, x_n \mid c) P(c)$$

$|X|^n \bullet |C|$ parameters

How often does this class occur?

Could only be estimated if a very, very large number of training examples was available.

We can just count the relative frequencies in a corpus

Multinomial Naive Bayes Independence Assumptions

$$P(x_1, x_2, \ldots, x_n \mid c)$$

- **Bag of Words assumption**: Assume position doesn't matter
- **Conditional Independence**: Assume the feature probabilities $P(x_i \mid c_j)$ are independent given the class *c.*

$$P(x_1, \ldots, x_n \mid c) = P(x_1 \mid c) \bullet P(x_2 \mid c) \bullet P(x_3 \mid c) \bullet \ldots \bullet P(x_n \mid c)$$

Multinomial Naive Bayes Classifier

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \ldots, x_n \mid c) P(c)$$

$$c_{NB} = \underset{c \in C}{\operatorname{argmax}} P(c_j) \prod_{x \in X} P(x \mid c)$$

$$X = \{x_1, x_2, \ldots, x_n\}$$

Applying Multinomial Naive Bayes Classifiers to Text Classification

positions ← all word positions in the test document

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j) \prod_{i \in positions} P(x_i \mid c_j)$$

Problems with multiplying lots of probabilities

- There's a problem with this:

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j) \prod_{i \in positions} P(x_i \mid c_j)$$

- Multiplying lots of probabilities can result in floating-point underflow!
- Luckily, $\log(ab) = \log(a) + \log(b)$
- Let's sum logs of probabilities instead of multiplying probabilities!

Put them in the log space

Instead of this:

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j) \prod_{i \in positions} P(x_i \mid c_j)$$

This:

$$c_{\mathrm{NB}} = \underset{c_j \in C}{\operatorname{argmax}} \left[\log P(c_j) + \sum_{i \in \mathrm{positions}} \log P(x_i | c_j) \right]$$

This is ok since log doesn't change the ranking of the classes (class with highest prob still has highest log prob)

Model is now just max of sum of weights: a *linear* function of the inputs

So naive bayes is a *linear classifier*

Put them in the log space

Instead of this:

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j) \prod_{i \in positions} P(x_i \mid c_j)$$

Parameters to be estimated from the data

This:

$$c_{\mathrm{NB}} = \underset{c_j \in C}{\operatorname{argmax}} \left[\log P(c_j) + \sum_{i \in \mathrm{positions}} \log P(x_i | c_j) \right]$$

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Probabilistic Language Models

- **Machine Translation**:
  - $P(high\ winds\ tonight) > P(large\ winds\ tonight)$
- **Spell Correction**
  - The office is about fifteen minuets from my house
  - $P(about\ fifteen\ minutes\ from) > P(about\ fifteen\ minuets\ from)$
- **Speech Recognition**
  - $P(I\ saw\ a\ van) >> P(eyes\ awe\ of\ an)$
- **Summarization**, **question-answering**, etc.

Probabilistic Language Models

- **Goal**: compute the *probability* of a sentence or sequence of words:
- $P(W) = P(w_1, w_2, w_3, w_4, w_5 \ldots w_n)$
- **Related task**: probability of an *upcoming word*:
- $P(w_5|w_1, w_2, w_3, w_4)$
- A model that computes either of these:
- $P(W)$ or $P(w_n|w1, w2 \ldots w_{n-1})$ is called a **language model**.
- **Better**: the grammar (fit more people's behavior)

How to compute $P(W)$

- How to compute this *joint probability*:
  - $P(its, water, is, so, transparent, that)$
- **Intuition**: let's rely on the **Chain Rule** of Probability

Two variables: $P(A,B) = P(A)P(B|A)$

More variables: $P(A,B,C,D) = P(A)P(B|A)P(C|A,B)P(D|A,B,C)$

General Form: $P(x_1, x_2, x_3, \dots, x_n) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \dots P(x_n|x_1, \dots, x_{n-1})$

Example

$$P(w_1 w_2 \ldots w_n) = \prod_i P(w_i \mid w_1 w_2 \ldots w_{i-1})$$

$$P(\text{“}its\ water\ is\ so\ transparent\text{”}) =$$
$$P(its) \cdot P(water|its)$$
$$\cdot\ P(is|its\ water) \cdot P(so|its\ water\ is)$$
$$\cdot P(transparent|its\ water\ is\ so)$$

How to estimate each of these probability terms?

A straightforward solution

- Could we just count and divide?

$$P(\text{the} \mid \text{its water is so transparent that}) = \frac{Count(\text{its water is so transparent that the})}{Count(\text{its water is so transparent that})}$$

- No! Too many words in the sentence!
- We'll never see enough data for estimating these…

Andrei Markov

Markov Assumption

Simplifying assumption:

$$P(w_i \mid w_1 w_2 \ldots w_{i-1}) \approx P(w_i \mid w_{i-k} \ldots w_{i-1})$$

$$P(\text{the} \mid \text{its water is so transparent that}) \approx P(\text{the} \mid \text{that})$$

Or maybe

$$P(\text{the} \mid \text{its water is so transparent that}) \approx P(\text{the} \mid \text{transparent that})$$

Simplest Case: Unigram Model

$$P(w_1 w_2 \ldots w_n) \approx \prod_i P(w_i)$$

Some automatically generated sentences from a unigram model

```
fifth, an, of, futures, the, an, incorporated, a,
a, the, inflation, most, dollars, quarter, in, is,
mass

thrift, did, eighty, said, hard, 'm, july, bullish

that, or, limited, the
```

Bigram Model

- Condition on the previous word:

$$P(w_i \mid w_1 w_2 \ldots w_{i-1}) \approx P(w_i \mid w_{i-1})$$

texaco, rose, one, in, this, issue, is, pursuing, growth, in, a, boiler, house, said, mr., gurria, mexico, 's, motion, control, proposal, without, permission, from, five, hundred, fifty, five, yen

outside, new, car, parking, lot, of, the, agreement, reached

this, would, be, a, record, november

N-gram Models

- We can extend to trigrams, 4-grams, 5-grams
- In general, this is an insufficient model of language

because language has **long-distance dependencies**:

- "*The computer which I had just put into the machine room on the fifth floor crashed*."

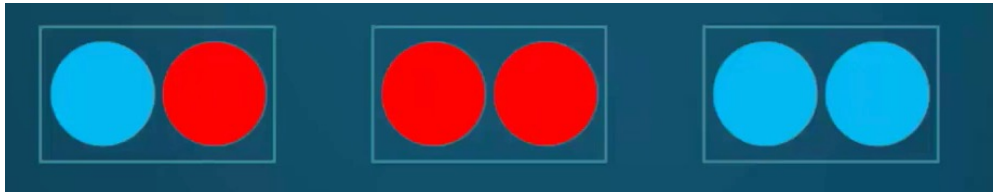
- But it is still very helpful even in advanced models.

One Slide to Takeaway

- What is a **sample space**, **event**, **probability**?
- What is **conditional probability**?
- What is **Bayes' formula**?
- How to formulate a **Naïve Bayes classifier**?
- How to formulate a **Probabilistic Language Model**?

Exercise: Blue and Red Balls

- There are three boxes below with red and/or blue balls:



- We randomly draw a box and then draw a red ball from it.
  - Assume each box and each ball has equal chances to be sampled.
- What is the probability that the other ball in the box is also red?