Cheng Wui Sum 22074221D Comp1433 assignment Q3

Q3a

Derive negative binomial distribution from binomial and geometric distribution

General form of Binomial distribution	General form of Geometric distribution
$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$	$$P(X = y) = (1-p)^{y-1} p$$
Where	Where
n is the total number of trials	p is the probability of success
p is the probability of success	y is the trial on which the first success occurs
x is the number of successes in n trials	

For negative binomial, we are finding the number of trials X that must occur until r success, assuming having the same probability of p

as the parameters r and k represent the number of successes and failures, they needed to achieve a certain number of successes before stopping the sequence of Bernoulli trials

We need to have r-1 success in k-1 trials, followed by r^{th} success on the k^{th} trial

Sub $r - 1 = x$ in the general binomial distribution formula

Sub $k - 1 = n$ in the general binomial distribution formula

as r-1 must occur in the first k-1 trial and the k^{th} trial must occur on the k^{th} trial

Sub $y = 1$ in the general geometric distribution formula

as we want the first trial to be a success in the equation

The equation of negative binomial is the product of the binomial distribution and geometric distribution

Hence the Negative binomial equation will be as follow

$$P(X = k; r, p) = \binom{k-1}{r-1} \times p^{r-1} \times (1-p)^{(k-1)-(r-1)} \times p \times (1-p)^{1-1}$$

$$= \binom{k-1}{r-1} \times p^{r-1} \times (1-p)^{(k-r)} \times p \times (1-p)^0$$

$$= \binom{k-1}{r-1} \times p^{r-1+1} \times (1-p)^{(k-r)}$$

$$= \binom{k-1}{r-1} \times p^{r} \times (1-p)^{(k-r)}$$

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Derive the expected value of negative binomial distribution

Convert the negative binomial distribution into another form for easier derive

$$X = k = r + z$$

$$P(Z = z; r, p) = \binom{r+z-1}{z} \times p^r \times (1-p)^z$$

$$P(X = k; r, p) = \binom{k-1}{r-1} \times p^r \times (1-p)^{(k-r)}$$

Let

$y = z - 1$

$q = r + 1$

Finding the expectation value of the negative binomial in the other form

$$\begin{aligned}
\mu &= E(Z) \\
&= \sum_{z=1}^{\infty} z \times \binom{r+z-1}{r} \times p^r \times (1-p)^z \\
&= \sum_{z=1}^{\infty} z \times \frac{(r+z-1)!}{(r-1)!\,z!} \times p^r \times (1-p)^z \\
&= \sum_{z=1}^{\infty} z \times \frac{(r+z-1)!}{\frac{r!}{r} z(z-1)!} \times p^r \times (1-p)^z \\
&= \sum_{z=1}^{\infty} r \times \frac{(r+z-1)!}{r!\,(z-1)!} \times p^r \times (1-p)^z \\
&= \sum_{z=1}^{\infty} r \times \binom{r+z-1}{z-1} \times p^r \times (1-p)^z
\end{aligned}$$

sub the y= z-1 in and q=r+1 in

$$\begin{aligned}
&= r \times \sum_{y=0}^{\infty} \binom{q-1+y-1-1}{y} \times p^{q-1} \times (1-p)^{y+1} \\
&= r \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q-1} \times (1-p)^{y+1} \\
&= r \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q} \times p^{-1} \times (1-p)^{y} \times (1-p) \\
&= \frac{r(1-p)}{p} \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q} \times (1-p)^{y}
\end{aligned}$$

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In here we are making $\sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^q \times (1-p)^y = 1$ to simplified the equation

$$= \frac{r(1-p)}{p} \times 1$$

$$= \frac{r(1-p)}{p}$$

Convert the expectation value of the negative binomial to its original form

$$E(X) = E(r+Z)$$

$$= E(r) + E(Z)$$

$$= \frac{r(1-p)}{p} + r\left(\frac{p}{p}\right)$$

$$= \frac{r}{p}$$

Derive the variance of negative binomial distribution

Let

$y = z - 2$

$q = r + 2$

$$\sigma^2 = Var(X) = Var(Z)$$

$$= E(Z^2) - E((Z))^2$$

$$= E[Z(Z-1)] + E(Z) - (E(Z))^2$$

$$= (\sum_{z=2}^{\infty} z \times (z-1) \times \binom{r+z-1}{r} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

$$= (\sum_{z=2}^{\infty} z \times (z-1) \times \frac{(r+z-1)!}{(r-1)!\, z!} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

$$= (\sum_{z=2}^{\infty} z \times (z-1) \times \frac{(r+z-1)!}{\frac{(r+1)!}{r(r+1)} \times z \times (z-1) \times (z-2)!} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

$$= (\sum_{z=2}^{\infty} z \times (r+1) \times \frac{(r+z-1)!}{(r+1)! \times (z-2)!} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

$$= (\sum_{z=2}^{\infty} z \times (r+1) \times \binom{r+z-1}{z-2} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

$$= (r \times (r+1) \times \sum_{z=2}^{\infty} \binom{r+z-1}{z-2} \times p^r \times (1-p)^z) + E(Z) - (E(Z))^2$$

sub the y= z-2 in and q=r+2 in

$$= (r \times (r+1) \times \sum_{y=0}^{\infty} \binom{q-2+y+2-1}{y} \times p^{q-2} \times (1-p)^{y+2}) + E(Z) - (E(Z))^2$$

$$= (r \times (r+1) \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q-2} \times (1-p)^{y+2}) + E(Z) - (E(Z))^2$$

$$= (r \times (r+1) \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q} \times p^{-2} \times (1-p)^{y} \times (1-p)^{2}) + E(Z) - (E(Z))^2$$

$$= (\frac{r \times (r+1) \times (1-p)^2}{p^2} \times \sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q} \times (1-p)^{y}) + E(Z) - (E(Z))^2$$

In here we are making $\sum_{y=0}^{\infty} \binom{q+y-1}{y} \times p^{q} \times (1-p)^{y}) = 1$ to simplified the equation

$$= (\frac{r \times (r+1) \times (1-p)^2}{p^2} \times 1) + E(Z) - (E(Z))^2$$

$$= \frac{r \times (r+1) \times (1-p)^2}{p^2} + \frac{r(1-p)}{p} - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \frac{r(1-p)}{p} \times \left[\frac{(r+1)(1-p) + p + r(1-p)}{p}\right]$$

$$= \frac{r(1-p)}{p^2} \times (r - rp + 1 - p + p - r + rp)$$

$$= \frac{r(1-p)}{p^2}$$

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Q3c

Equation list for negative binomial distribution		
Formula	$P(X = k; r, p) = \binom{k-1}{r-1} \times p^r \times (1-p)^{(k-r)}$	$P(Z = z; r, p) = \binom{r+z-1}{z} \times p^r \times (1-p)^z$
Mean (expected value)	$\mu = \frac{r}{p}$	$\mu = \frac{r(1-p)}{p}$
Standard Deviation	$\sigma = \sqrt{\frac{r(1-p)}{p^2}}$	

Let the cat die is a successful trial

Number of successful trials $r = 9$

Probability of success $p = \frac{1}{20}$

number of weeks Montgomery will survive is denoted by X

Expected value are calculated as follows

$\mu = E(Z)$ $= \frac{9\left(1-\frac{1}{20}\right)}{\frac{1}{20}}$ $= \frac{9 \times 0.95}{0.05}$ $= 171$	$\mu = E(X)$ $= \frac{9}{\frac{1}{20}}$ $= 180$

The expected life expectancy of Montgomery will be 171 or 180 weeks.

Standard Deviation are calculated as follows

$$\sigma = \sqrt{\frac{9\left(1-\frac{1}{20}\right)}{\frac{1}{20}^2}}$$

$$= \sqrt{\frac{9 \times 0.95}{0.05^2}}$$

$$= \sqrt{3420}$$

$$= 58.48076607$$

$$\approx 58.5(correct\ to\ 3\ significant\ figure)$$

The standard deviation of Montgomery life expectancy will be 58.5

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Q3d

Let the cat die is a successful trial

Let

$p = \frac{1}{20}$

$k = 104$

The probability of Montgomery will survive for another 2 years if he has 1 life left

Let

$r = 1$

$$
\begin{aligned}
P(X \geq k; r, p) &= 1 - P(X < k - 1; r, p) \\
&= 1 - \sum_{k=0}^{k-1} \binom{r + k - 1}{k} \times p^r \times (1 - p)^k \\
&= 1 - \sum_{k=0}^{103} \binom{k}{k} \times 0.05^1 \times (1 - 0.05)^k \\
&= 1 - (0.05 + 0.0475 + 0.0045125 + \cdots) \\
&\approx 0.00482
\end{aligned}
$$

The probability of Montgomery will survive for another 2 years if he has 1 life left is 0.00482

The probability of Montgomery will survive for another 2 years if he has 9 lives left

$r = 9$

$$
\begin{aligned}
P(X \geq k; r, p) &= 1 - P(X < k - 1; r, p) \\
&= 1 - \sum_{k=0}^{k-1} \binom{r + k - 1}{k} \times p^r \times (1 - p)^k \\
&= 1 - \sum_{k=0}^{103} \binom{8 + k}{k} \times 0.05^9 \times (1 - 0.05)^k \\
&= 1 - \left(\left(\binom{8 + 0}{0} \times 0.05^9 \times (1 - 0.05)^0 \right) + \left(\binom{8 + 1}{1} \times 0.05^9 \times (1 - 0.05)^1 \right) + \cdots \right) \\
&= 1 - ((1.953125 \times 10^{-12}) + (1.669921875 \times 10^{-11}) + \cdots) \\
&\approx 0.89117
\end{aligned}
$$

The probability of Montgomery will survive for another 2 years if he has 9 lives left is 0.89117