Topic 1

Probability

Section 1 – Experiments and Sample Space

Definitions:

An **experiment** is defined to be any process which generates well defined outcomes. By this we mean that on any single repetition of the experiment one and only one of the possible experimental outcomes will occur.

The **sample space** is defined as the set of all possible experimental outcomes. Any one particular experimental outcome is referred to as a **sample point** and is an element of the sample space.

Section 2 – Simple and Joint Events

Definitions:

An **event** is a collection of one or more of the outcomes of an experiment. Usually, the event is denoted by capital letters such as *A*, *B*, *C*.

A **simple event** can be described by a single characteristic.

A **joint event** is an event that has two or more characteristic.

Section 2 – Simple and Joint Events

Example:

Suppose that a card is randomly selected from a deck.

The **event** that the selected card is black is a **simple event**.

The event that the selected card is a black ace is a **joint event**.

Section 3 – Probability

Definitions:

Probability is a numerical measure of the likelihood that a specific event will occur.

Let A be an event.
The probability that event A will occur is denoted by $P(A)$.

Section 3 – Probability

Two Axioms of Probability

1. The probability of an event always lies in the range 0 to 1.

$$0 \le P(A) \le 1$$

Impossible event: An event that cannot occur. $P(A) = 0$

Sure event: An event that is certain to occur. $P(A) = 1$

2. Let S be the sample space of an experiment.

$$P(S) = 1$$

Section 4 – Three Conceptual Approaches to Probability

(1) Classical Probability

Definition: Outcomes that have the same probability of occurrence are called **equally likely outcome**.

The classical probability rule is applied to compute the probabilities of events for an experiment all of whose outcomes are equally likely.

Classical Probability Rule

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}$$

Section 4 – Three Conceptual Approaches to Probability

Example:

Find the probability of obtaining an even number in one roll of a die.

Solution:

6 outcomes: 1, 2, 3, 4, 5, 6 (equally likely)

Let A be an event that an even number is observed on the die. → $A = \{2, 4, 6\}$

$$P(A) = \frac{3}{6} = 0.50$$

Section 4 – Three Conceptual Approaches to Probability

(2) Relative Frequency Concept of Probability

Suppose we want to calculate the following probabilities:

•The probability that an 80-year-old person will live for at least one more year.

•The probability that the tossing of an unbalanced coin will result in a head.

•The probability that we will observe a 1-spot if we roll a loaded die.

Section 4 – Three Conceptual Approaches to Probability

These probabilities cannot be computed using the classical probability rule because the various outcomes for the corresponding experiments are not equally likely.

Relative Frequency as an Approximation of Probability

If an experiment is repeated n times and an event A is observed f times, then, according to the relative frequency concept of probability:

$$\boxed{P(A) = \frac{f}{n}}$$

Section 4 – Three Conceptual Approaches to Probability

Example:

Ten of the 500 randomly selected cars manufactured at a certain auto factory are found to be defective.

Assuming that the defective cars are manufactured randomly, what is the probability that the next car manufactured at this auto factory is defective?

Section 4 – Three Conceptual Approaches to Probability

Solution:

Let n denote the total number of cars in the sample and f the number of defective cars in n.

$$P(\text{next car is a defective}) = \frac{f}{n} = \frac{10}{500} = 0.02$$

This probability is actually the relative frequency of defective cars in 500 cars.

Section 4 – Three Conceptual Approaches to Probability

Note:
The relative frequencies are not probabilities but approximate probabilities. However, if the experiment is repeated again and again, this approximate probability of an outcome obtained from the relative frequency will approach the actual probability of that outcome. This is called the **Law of Large Numbers**.

Law of Large Numbers

If an experiment is repeated again and again, the probability of an event obtained from the relative frequency approaches the actual or theoretical probability.

Section 4 – Three Conceptual Approaches to Probability

(3) Subjective Probability

Many times we face experiments that neither have equally likely outcomes nor can be repeated to generate data.

In such cases, we cannot compute the probabilities of events using the classical probability rule or the relative frequency concept.

Section 4 – Three Conceptual Approaches to Probability

Consider the following probabilities of events:

- The probability that Carol, who is taking statistics this semester, will earn an A in this course.
- The probability that the Hang Seng Index will be higher at the end of the next trading day.
- The probability that the Manchester United will win the English Premier League next season.

Neither the classical probability rule nor the relative frequency concept of probability can be applied to calculate probabilities for these examples.

All these examples belong to experiments that have neither equally likely outcomes nor the potential of being repeated.

Definition: **Subjective probability** is the probability assigned to an event based on subjective judgment, experience, information, and belief.

Section 5 – Counting Rule
Section 5.1 Multiplicative Rule

You have k sets of different elements, n_1 in the first set, n_2 in the second set, …, n_k in the k^{th} set. Suppose you want to form a sample of k elements by taking one element from each of the k sets. The number of different samples that can be formed is the product

$$\boxed{n_1 \times n_2 \times \cdots \times n_k}$$

Section 5 – Counting Rule
Section 5.1 Multiplicative Rule

Example:

Suppose that you have twenty candidates for three different executive positions, E_1, E_2 and E_3. How many different ways could you fill the positions?

Solution:

Set 1: The candidates available to fill position $E_1 \Rightarrow n_1 = 20$

Set 2: The candidates remaining (after filling E_1) that are available to fill $E_2 \Rightarrow n_2 = 19$

Set 3: The candidates remaining (after filling E_1 and E_2) that are available to fill $E_3 \Rightarrow n_3 = 18$

The number of different ways to fill the three positions is $20 \times 19 \times 18 = 6840$

Section 5 – Counting Rule
Section 5.2 Permutations Rule

Definition: A **permutation** of a set of objects in any arrangement of these objects in a definite order.

Example:

$$S = \{\, a, b, c \,\}$$

Permutations of the elements of S:

$$abc, acb, bac, bca, cab, cba$$

Section 5 – Counting Rule
Section 5.2 Permutations Rule

Factorial – The symbol $n!$ represents the product of all integers from n to 1. That is,

$$\boxed{n! = n(n-1)(n-2)\cdots(3)(2)(1)}$$

and by definition, $0! = 1$

Section 5 – Counting Rule

Section 5.2 Permutations Rule

- The number of permutations (arrangements) of n different elements is $n!$
- The number of permutations of n distinct objects taken r at a time where $1 \le r \le n$ is

$$\boxed{{}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}}$$

Section 5 – Counting Rule

Section 5.2 Permutations Rule

Example:

Suppose that two letters are to be selected from A, B, C, D and arranged in order.

(a) How many permutations are possible?

(b) What is the probability that the selection will contain letter "A"?

Section 5 – Counting Rule
Section 5.2 Permutations Rule

Solution:

(a) How many permutations are possible?

$$_{4}P_{2} = \frac{4!}{(4-2)!} = 12$$

(b) What is the probability that the selection will contain letter "*A*"?

A	

$$_{3}P_{1} = \frac{3!}{(3-1)!} = 3$$

	A

$$_{3}P_{1} = \frac{3!}{(3-1)!} = 3$$

Required probability:

$$\frac{6}{12} = 0.5$$

Section 5 – Counting Rule
Section 5.3 Combinations Rule

Definition: A **combination** of a set of objects is a group or subset of the objects disregarding their order.

That is, sampling is equivalent to partitioning a set of n elements into 2 groups: elements that appear in the sample and those that do not.

Section 5 – Counting Rule
Section 5.3 Combinations Rule

Let r be the number of elements in the sample, and $n-r$ be the number of elements remaining.

Then the number of different samples of r elements that can be selected from n is

$$\boxed{{}_nC_r \text{ or } \binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

Section 5 – Counting Rule
Section 5.3 Combinations Rule

Example:

How many ways can an executive committee of 5 be chosen from a board of directors consisting of 15 members?

Solution:

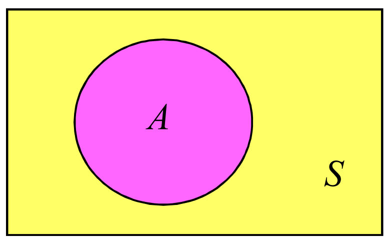
$${}_{15}C_5 = \frac{15!}{5!\,(15-5)!} = 3003$$

Section 5 – Counting Rule

Section 5.4 Venn Diagram

A **Venn diagram** shows a sample space and events within the space.

The sample space is represented by a rectangle.
A simple event is shown as a shaded circle.



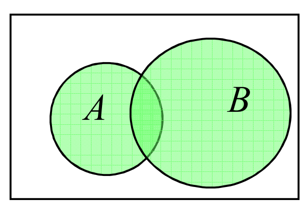
A: Simple event *A*

S: Sample space

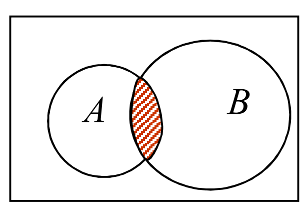
Section 5 – Counting Rule

Section 5.5 Basic Probability Laws

Definition: The **union** of two events *A* and *B* is the set of all outcomes that are included in either *A* or *B* or both.
The union is denoted by $A \cup B$.



Definition: The **intersection** of two events *A* and *B* is the set of all outcomes that are included in both *A* and *B*.
The intersection is denoted by $A \cap B$.



Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Example:

If event B is getting an even number on a die toss, and the event A is getting a number 1 or 2.

Then,

$$A \cup B = \{1, 2, 4, 6\}$$

$$A \cap B = \{2\}$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Definition: A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be **collectively exhaustive** if one of the events must occur.
That is, sample space $S = A_1 \cup A_2 \cup \ldots \cup A_n$

Example:

Suppose that a student is randomly selected from a class.

Let M be the event that the selected student is male.
Let F be the event that the selected student is female.

Then M and F are **collectively exhaustive**.

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Definition: The probability of the intersection of two events is called their **joint probability**.

It is written as $P(A \text{ and } B)$ or $P(A \cap B)$.

Addition Rule

Consider two events A and B, the probability of the union of A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Example:

A hamburger chain found that 65% of all customers order french fries, 78% order soft drink, and 55% order both.

What is the probability that a customer will order at least one of these?

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Solution:

Let A be the event "Customer orders french fries" and B be the event "Customer orders soft drink".

From the given information,

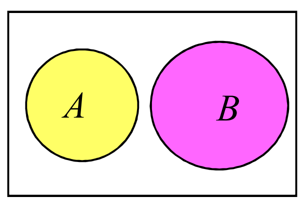
$$P(A) = 0.65,\ P(B) = 0.78,\ P(A \cap B) = 0.55$$

Required probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.65 + 0.78 - 0.55 = 0.88$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Definition: Two events are said to be **mutually exclusive** if, when one of the two events occurs in an experiment, the other cannot occur. That is, $P(A \cap B) = 0$.



Addition Rule – Mutually Exclusive

If the events, A and B, are mutually exclusive, the probability that either event occurs is

$$P(A \cup B) = P(A) + P(B)$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Example:

A box contains two white balls, four yellow balls and five red balls. A ball is selected from the box, what is the probability that the selected ball is red or yellow?

Solution:

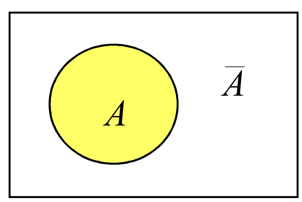
Let W be the event "the selected ball is white", R be the event "the selected ball is red" and Y be the event "the selected ball is yellow".

Required probability:

$$P(R \cup Y) = P(R) + P(Y) = \frac{5}{11} + \frac{4}{11} = \frac{9}{11}$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Definition: The **complement** of A, denoted by $\bar{A}$ or A', is defined to be the event consisting of all sample points that are not in A.



For any event A,

$$\boxed{P(A) = 1 - P(\bar{A})}$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Example:

If event A is getting a number 3 or 4 on a die toss, then the complement of event A is $\overline{A} = \{1, 2, 5, 6\}$.

Example:

A university is hiring candidates for four lecturer positions. The candidates are five men and three women. Assuming that every combination of men and women is equally likely to be chosen, what is the probability that at least one woman will be selected?

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Solution:

P(at least one woman will be selected)

$$\frac{\binom{5}{3}\times\binom{3}{1}}{\binom{8}{4}}+\frac{\binom{5}{2}\times\binom{3}{2}}{\binom{8}{4}}+\frac{\binom{5}{1}\times\binom{3}{3}}{\binom{8}{4}}=\frac{13}{14}$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Alternatively,

Let A be the event that at least a woman is selected.

$$P(\bar{A}) = \frac{\binom{5}{4} \times \binom{3}{0}}{\binom{8}{4}} = \frac{1}{14}$$

$$\Rightarrow \; P(A) = 1 - P(\bar{A}) = \frac{13}{14}$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Example:

A director has two assistants helping her with her business. The probability that the older of the two assistants will be absent on any given day is 0.08, the probability that the younger of the two will be absent on any given day is 0.05, and the probability that they will both absent on any given day is 0.02. Find the probabilities that

(a) either or both of the assistants will be absent on any given day;

(b) at least one of the two assistants will not be absent on any given day;

(c) only one of the two assistants will be absent on any given day.

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

Solution:

Let A be the event the older of the two assistants will be absent on any given day, and
B be the event the younger of the two assistants will be absent on any given day.

From the given information,

$$P(A) = 0.08,\ P(B) = 0.05,\ P(A \cap B) = 0.02$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

(a) the probability that either or both of the assistants will be absent on any given day

Use the addition rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.08 + 0.05 - 0.02 = 0.11$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

(b) the probability that at least one of the two assistants will not be absent on any given day

Use the complement rule, we have

$$P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - 0.02 = 0.98$$

Section 5 – Counting Rule
Section 5.5 Basic Probability Laws

(c) the probability that only one of the two assistants will be absent on any given day

The required probability is

$$P(A \cup B) - P(A \cap B)$$

$$= 0.11 - 0.02 = 0.09$$

Section 5 – Counting Rule
Section 5.6 Marginal Probability

Definition: **Marginal probability** is the probability of a single event without consideration of any other event. Marginal probability is also called **simple probability**.

Section 5 – Counting Rule
Section 5.6 Marginal Probability

Example:

All 420 employees of a company were asked if they are smokers or non-smokers and whether or not they are university graduates. Based on this information, the following two-way classification table was prepared.

	University graduates	Not a University graduates	Total
Smoker	35	80	115
Non-smoker	130	175	305
Total	165	255	420

Section 5 – Counting Rule

Section 5.6 Marginal Probability

If one employee is selected at random from this company, find the probability that this employee is

(a) a university graduate;

(b) a non-smoker;

(c) a university graduate and a smoker; and

(d) a university graduate or a non-smoker.

Note:

The probabilities in (a) and (b) are simple probabilities while the probability in (c) is a joint probability.

Section 5 – Counting Rule

Section 5.6 Marginal Probability

Solution:

(a) a university graduate

$$P(\text{University graduate}) = \frac{35+130}{420} = \frac{165}{420} = 0.3929$$

(b) a non-smoker

$$P(\text{Non-smoker}) = \frac{130+175}{420} = \frac{305}{420} = 0.7262$$

Section 5 – Counting Rule

Section 5.6 Marginal Probability

(c) a university graduate and a smoker

$$P(\text{University graduate and smoker}) = \frac{35}{420} = 0.8333$$

(d) a university graduate or a non-smoker

$$P(\text{University graduate or Non-smoker})$$

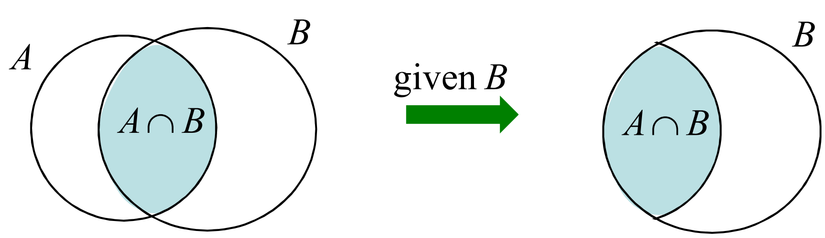
$$= P(\text{University graduate}) + P(\text{Non-smoker}) - P(\text{University graduate and Non-smoker})$$

$$= \frac{165}{420} + \frac{305}{420} - \frac{130}{420} = 0.8095$$

Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

Definition: The probability of an event A given that an event B has occurred, is called the **conditional probability** of A given B and is denoted by the symbol $P(A \mid B)$ and read as 'the probability of A given that B has already occurred'.



Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

If A and B are two events with $P(A) \neq 0$ and $P(B) \neq 0$, then

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

The probability that both of the two events A and B occur is

$$\boxed{P(A \cap B) = P(A) \cdot P(B \mid A) \quad \text{and} \quad P(A \cap B) = P(B) \cdot P(A \mid B)}$$

Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

Example:

Suppose a couple has two children.
The sample space is

$$S = \{bb, bg, gb, gg\}$$

where we assume an equiprobable space, that is, we assume probability 1/4 for each point.

Find the probability that both children are boys if it is known that:

(a) At least one of the children is a boy.

(b) The older child is a boy.

Section 5 – Counting Rule
Section 5.7 Conditional Probability and Independent

Solution:

(a) The probability that both children are boys given that at least one of the children is a boy

Here the reduced sample space consists of three elements $\{bb, bg, gb\}$.

Let E denotes the event that both children are boys and F the event that at least one of them is a boy, we have

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Section 5 – Counting Rule
Section 5.7 Conditional Probability and Independent

Solution:

(b) The probability that both children are boys given that the older child is a boy

Here the reduced sample space consists of two elements $\{bb, bg\}$.

Let E denotes the event that both children are boys and G the event that the older child is a boy, we have

$$P(E \mid G) = \frac{P(E \cap G)}{P(G)} = \frac{1/4}{2/4} = \frac{1}{2}$$

Section 5 – Counting Rule
Section 5.7 Conditional Probability and Independent

Example:

Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement.

If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both drawn balls are red?

Section 5 – Counting Rule
Section 5.7 Conditional Probability and Independent

Solution:

Let R_1 and R_2 denote, respectively, the events that the first and second ball drawn is red.

We have $P(R_1) = \dfrac{8}{12}$ and $P(R_2 \mid R_1) = \dfrac{7}{11}$

The desired probability is

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1)$$
$$= \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33} = 0.4242$$

Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

Definition: Two events A and B are said to be **independent** if the occurrence of one does not affect the probability of the occurrence of the other.

In other words, A and B are independent events if

$$\boxed{P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)}$$

If two events A and B are independent, then

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

Example:

A box contains a total of 100 DVDs that were manufactured on two machines. Of them, 60 were manufactured on Machine I. Of the total DVDs, 15 are defective. Of the 60 DVDs that were manufactured on Machine I, 9 are defective.

Let D be the event that a randomly selected DVD is defective and A be the event that a randomly selected DVD was manufactured on Machine I.

Are events D and A independent?

Section 5 – Counting Rule

Section 5.7 Conditional Probability and Independent

Solution:

From the given information,

$$P(D) = \frac{15}{100} = 0.15 \quad \text{and} \quad P(D \mid A) = \frac{9}{60} = 0.15$$

$\Rightarrow \; P(D) = P(D \mid A)$

$\Rightarrow$ Events D and A are independent.

Section 5 – Counting Rule

Section 5.8 Law of Total Probability

Law of Total Probability

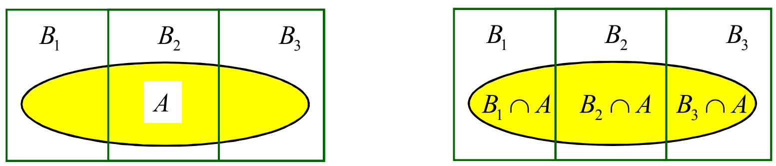
Assume that $B_1, B_2, \ldots, B_n$ are collectively exhaustive events where $P(B_i) > 0$, for $i = 1, 2, \ldots, n$ and B_i and B_j are mutually exclusive events for $i \neq j$.

Then for any event A

$$P(A) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + \cdots + P(B_n)P(A \mid B_n)$$

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

For instance, when $n = 3$, that is, B_1, B_2, B_3 , are collectively exhaustive and mutually exclusive events.



$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A)$$

$$= P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)$$

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

Example:

In a college, the number of junior students is five times of the number of senior students. The probability that a senior student will be taking computer science is 0.20 and the probability that a junior student will be taking computer science is 0.30.

If a student is selected randomly, find the probability that
(a) he/she is a senior computer science student?
(b) he/she is a computer science major student?
(c) he/she is a junior given that he/she is a computer science student?

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

Solution:

Let
A denotes the event that the selected student is a senior;
B denotes the event that the selected student is a junior;
C denotes the event that the selected student is a computer science student.

From the given information:

$$P(C \mid A) = 0.20 \qquad P(C \mid B) = 0.30$$

$$P(A) = \frac{1}{6} \qquad P(B) = \frac{5}{6}$$

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

(a) the probability that he/she is a senior computer science student

By the multiplicative rule,

$$P(A \cap C) = P(A) \cdot P(C \mid A) = \frac{1}{6} \times 0.20 = \cancel{0.3333}$$

=0.0333

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

(b) the probability that he/she is a computer science major student

By the law of total probability,

$$P(C) = P(A \cap C) + P(B \cap C)$$

$$= P(A) \cdot P(C \mid A) + P(B) \cdot P(C \mid B)$$

$$= \frac{1}{6} \times 0.20 + \frac{5}{6} \times 0.30 = 0.2833$$

Section 5 – Counting Rule
Section 5.8 Law of Total Probability

(c) the probability that he/she is a junior given that he/she is a computer science student

By the conditional probability,

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{P(B) \cdot P(C \mid B)}{P(C)} = 0.8825$$

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

Bayes' Theorem

Suppose that $B_1, B_2, \ldots, B_n$ are n mutually exclusive and exhaustive events, then

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)}$$
$$= \frac{P(B_k)P(A \mid B_k)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + \cdots + P(B_n)P(A \mid B_n)}$$

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

Definition: $P(B_k)$ is called the **prior probability** in the sense that it is assigned prior to the observation of any empirical information.

Definition: $P(B_k \mid A)$ is called the **posterior probability** or the revised probability. It is assigned after obtaining additional information.

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

Example:

In a certain assembly plant, three machines, B_1, B_2, and B_3, makes 30%, 45% and 25%, respectively, of the products.

It is known from past experience that 2%, 1% and 2% of the products made by each machine, respectively, are defective.

Now, suppose that a finished product is randomly selected. What is the probability that it is defective? If a product were chosen randomly and found to be defective, decide which of the three machines is more likely to have produced the defective product?

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

Solution:

Let
A denotes the event that the product is defective;
B_1 denotes the event that the product is made by machine B_1;
B_2 denotes the event that the product is made by machine B_2;
B_3 denotes the event that the product is made by machine B_3

From the given information:

$$P(B_1) = 0.30 \qquad P(B_2) = 0.45 \qquad P(B_3) = 0.25$$

$$P(A \mid B_1) = 0.02 \qquad P(A \mid B_2) = 0.01 \qquad P(A \mid B_3) = 0.02$$

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

The probability that the product is defective

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

$$= P(B_1) \cdot P(A \mid B_1) + P(B_2) \cdot P(A \mid B_2) + P(B_3) \cdot P(A \mid B_3)$$

$$= (0.30)(0.02) + (0.45)(0.01) + (0.25)(0.02) = 0.0155$$

Section 5 – Counting Rule
Section 5.9 Bayes' Theorem

Given the product is defective, the probability that it is made by machine B_1

$$P(B_1 \mid A)$$

$$= \frac{P(B_1) \cdot P(A \mid B_1)}{P(B_1) \cdot P(A \mid B_1) + P(B_2) \cdot P(A \mid B_2) + P(B_3) \cdot P(A \mid B_3)}$$

$$= \frac{(0.30)(0.02)}{(0.30)(0.02) + (0.45)(0.01) + (0.25)(0.02)} = 0.3871$$

Section 5 – Counting Rule

Section 5.9 Bayes' Theorem

Similarly,

$$P(B_2 \mid A) = \frac{(0.45)(0.01)}{(0.30)(0.02)+(0.45)(0.01)+(0.25)(0.02)} = 0.2903$$

$$P(B_3 \mid A) = \frac{(0.25)(0.02)}{(0.30)(0.02)+(0.45)(0.01)+(0.25)(0.02)} = 0.3226$$

→ If a product were chosen randomly and found to be defective, the machine B_1 is more likely to have produced the defective product.