Mini project 3 - Fixed-Point Iteration for Intersection of Curves

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OBJECTIVE: The objective of this project is to find the intersection point of the functions $y = xe^{x^2}$ and $y = \sqrt{1 - x^2}$ in the first quadrant using fixed-point iteration. Fixed-point iteration is a numerical method that finds solutions to equations of the form $x = g(x)$ by repeatedly applying the function $g(x)$ [1]. The iteration continues until the difference between successive approximations is smaller than a given tolerance.

SUMMARY OF PROCEDURE: To find the intersection of $y = xe^{x^2}$ and $y = \sqrt{1 - x^2}$ in the first quadrant, I first plotted the curves over the interval $[0, 1]$. Visual inspection of the plot suggested an initial guess, which was then refined using the fixed-point iteration method.

I reformulated the problem for fixed-point iteration as:

$$x = g(x) = \sqrt{\frac{1}{e^{2x^2} + 1}}, \quad x_{n+1} = g(x_n) \tag{1}$$

where x_{n+1} is the next approximation obtained from $g(x_n)$, as defined in Equation 1.

The iteration continued until the absolute difference between successive approximations was below 10^{-15}, as shown in Equation 2:

$$|x_{n+1} - x_n| < 10^{-15}. \tag{2}$$

After the final approximation was obtained, I calculated the absolute errors by comparing each approximation to the final approximation. These errors were then plotted on a log-log scale to analyse the convergence behavior.

Results and Discussion: Figure 1 shows the two functions, $y = xe^{x^2}$ (shown by blue curve) and $y = \sqrt{1 - x^2}$ (shown by orange curve), plotted over the interval $[0, 1]$. The intersection of these curves visually suggests the location of the fixed point, with an initial guess of $x_0 = 0.6$ being chosen based on this plot. This initial guess was further refined using the fixed-point iteration process.

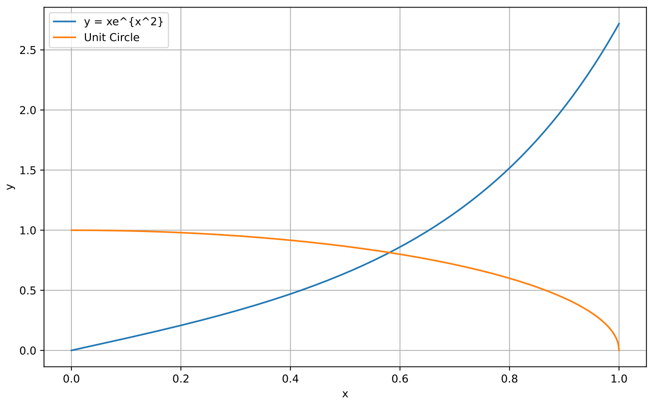

Figure 1: Intersection of $y = xe^{x^2}$ and $y = \sqrt{1 - x^2}$ in the first quadrant.

The final approximation of $x^* = 0.5808750357617376$ was obtained after 39 iterations. The errors between successive iterations decreased linearly, indicating first-order convergence typical of fixed-point iteration when $|g'(x^*)|$ is near 1. This explains the slow convergence and the need for many iterations to meet the tolerance 10^{-15}.

To further analyze the convergence behavior, Figure 2 presents a log-log plot of the error at iteration n versus the error at iteration $n - 1$. The plot shows a straight line with slope close to 1, confirming the linear convergence of the method. This behavior is expected due to the nature of the function $g(x)$, where $|g'(x^*)|$ is close to 1, slowing the convergence rate. In general, when $|g'(x^*)|$ approaches 1, fixed-point iteration converges more slowly, as observed in this case [2].

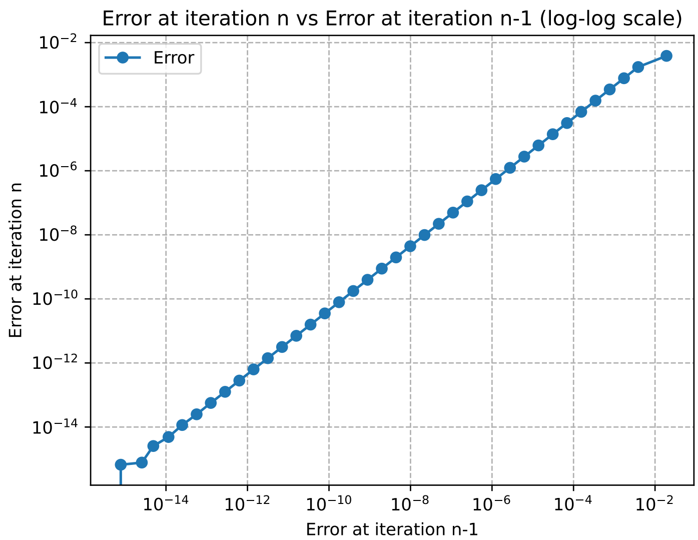

Figure 2: Log-log plot of error at iteration n versus error at iteration $n-1$, demonstrating linear convergence.

While the fixed-point iteration method successfully found the intersection, the slow linear convergence due to the derivative $g'(x^*)$ being close to 1 implies that further improvements could be made. One potential improvement is to use the Newton-Raphson method, which has quadratic convergence. This method would apply the iteration [3]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

where $f(x) = xe^{x^2} - \sqrt{1-x^2}$, as shown in Equation 3. This method typically converges much faster than fixed-point iteration when the initial guess is sufficiently close to the root, as the error decreases quadratically.

Conclusively, although the fixed-point iteration converged to the correct intersection point, it required 39 iterations due to the slow convergence. Using the Newton-Raphson method or using acceleration techniques such as Aitken's Δ^2 method could reduce the computational cost and achieve faster convergence.

References

[1] Wikipedia. *Fixed-point iteration*. https://en.wikipedia.org/wiki/Fixed-point_iteration. Accessed: 16th October 2024.

[2] Maths Stack Exchange. *Convergence of Fixed-point Iteration when $g'(p) = 1$*. https://math.stackexchange.com/questions/1463134/convergence-of-fixed-point-iteration-when-gp-1. Accessed: 17th October 2024.

[3] Wikipedia. *Newton's method*. https://en.wikipedia.org/wiki/Newton's_method. Accessed: 17th October 2024.