Portfolio IX - Inhomogeneous Arrival Rates in Event Simulation

Introduction

Building upon my previous work contracted by the FBI on simulating queues at polling stations, this report addresses a significant limitation in my previous model: the fluctuation of voter turnout throughout the day. By incorporating time-dependent (inhomogeneous) arrival rates, we recognise that voters don't arrive in a steady stream but tend to peak during meal times and breaks. This refinement will improve accuracy of my analysis of queue lengths, as well as determining if funding more polling stations are within the best interest of the FBI.

Methodology

The polling station is modelled as an M/M/1 queue with finite capacity (K), using a continuous-time, event-driven simulation:

- **Arrival Rate** ($\lambda(t)$): A time-varying arrival rate that captures changes in voter turnout throughout the day.
- **Service Rate** (μ): Voters are serviced with an exponential rate, assuming an i.i.d. service process.
- **Queue Capacity** (K): The maximum queue length; any voter arriving when the queue is full ($Q(t) = K$) is rejected.
- **Single Server**: One voting booth handles all arrivals and services.

Time-Varying Arrival Rate

The arrival rate at time t is defined as:

$$\lambda(t) = \lambda_0 + \sum_{i=1}^{N} A_i \exp\left(-\frac{(t - t_i)^2}{2\sigma_i^2}\right),$$

where A_i, t_i, and σ_i represent the amplitude, peak time, and width of each peak, respectively. This models fluctuations in voter turnout throughout the day.

Event Handling

In the M/M/1 queue model, both arrival times and service times follow exponential distributions. For the time-varying arrival rate $\lambda(t)$, we use the following method to simulate the arrival proccess:

- Determine the maximum arrival rate:

$$\lambda_{\max} = \max_t \lambda(t).$$

- Generate inter-arrival times Δt using an exponential random variable with rate $\lambda_{\max}$:

$$\Delta t \sim \mathrm{Exp}(\lambda_{\max}).$$

- For each potential arrival time t_{arrival}, compute the acceptance probability:

$$p = \frac{\lambda(t_{\mathrm{arrival}})}{\lambda_{\max}}.$$

- Generate a uniform random number $U \sim \mathrm{Uniform}(0, 1)$. If $U \leq p$, accept the arrival; otherwise, reject it.

The service times are generated as follows:

$$S \sim \mathrm{Exp}(\mu).$$

Simulation Process

The simulation involves handling two types of events:

- **Arrival Events**:
  - Generate a potential arrival time using:

$$\Delta t \sim \mathrm{Exp}(\lambda_{\max}).$$

  - Update the next arrival time:

$$t_{\mathrm{arrival}} = t_{\mathrm{current}} + \Delta t.$$

  - Compute the acceptance probability:

$$p = \frac{\lambda(t_{\mathrm{arrival}})}{\lambda_{\max}}.$$

  - Generate a uniform random number U. If $U \leq p$, accept the arrival:
    * If $Q(t_{\mathrm{arrival}}) < K$, the voter joins the queue.
    * If $Q(t_{\mathrm{arrival}}) = K$, the voter is rejected.

- If the server is idle (i.e., the queue length was zero before the arrival), schedule a service completion time.

- **Departure Events**:
  - When a voter begins service, generate a service time:

$$S \sim \text{Exp}(\mu).$$

  - Schedule the departure event at:

$$t_{\text{departure}} = t_{\text{current}} + S.$$

  - Upon departure at $t_{\text{departure}}$:
    * Decrease the queue length $Q(t_{\text{departure}})$ by one.
    * If $Q(t_{\text{departure}}) > 0$, schedule the next departure event by generating a new service time.

Key Statistics

- **Base Arrival Rate**: 5 voters/hour
- **Total Voters Served**: 369
- **Rejected Voters**: 57
- **Average Queue Length**: 11.2 voters (95% CI: [10.6, 11.8])
- **Max Queue Length**: 20 voters (at capacity K)

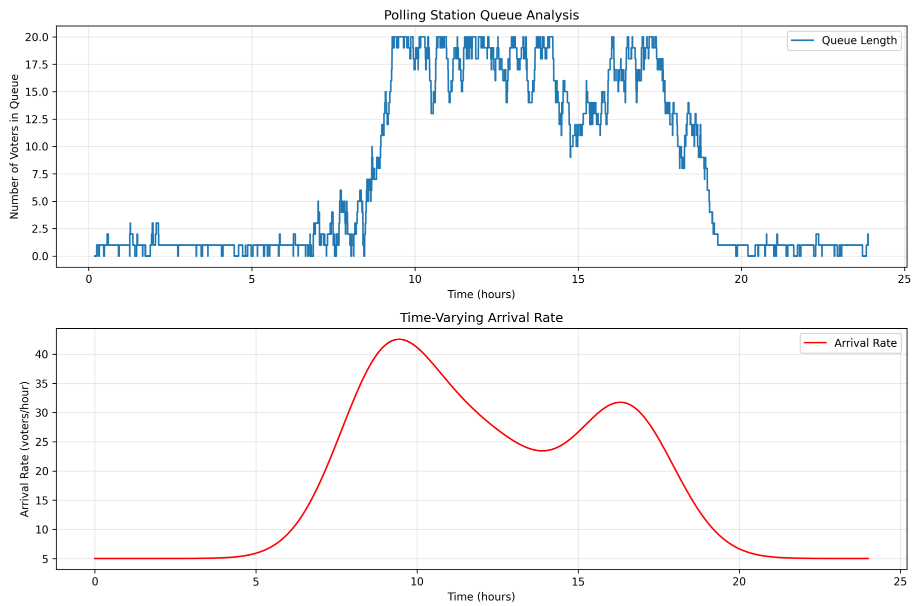

Figure 1: Top plot: Queue size over 24 hours, showing peak congestion during morning, lunch, and late afternoon. Bottom plot: Time-varying arrival rate with distinct peaks and declines.

Conclusion and Limitations

The continuous-time event driven simulation shows that one polling station is insufficient during peak times, leading to queues hitting capacity and 57 voters turned away. More polling stations are needed to avoid voter frustration. The FBI should work on increasing capacity, possibly more booths to increase servicing agents.

A key limitation is the use of a constant service rate, which ignores factors like voter impatience or staff shifts that could slow service. Future models could use time-varying service rates for more accuracy.

What I Learned

- Incorporating time-varying arrival rates captures daily fluctuations better and provides more realistic queue predictions.
- Simple simulations can reveal important system bottlenecks that aren't obvious with static homogeneous models.