在 $a = x_j$ 的 Taylor series $\quad f(x) = \dfrac{f(x_j)(x-x_j)^0}{0!} + \dfrac{f'(x_j)(x-x_j)^1}{1!} + \dfrac{f''(x_j)(x-x_j)^2}{2!} + \dfrac{f'''(x_j)(x-x_j)^3}{3!} + \cdots$

在 $x = x_{j+1}$ 時 $\quad f(x_{j+1}) = \dfrac{f(x_j)(x_{j+1}-x_j)^0}{0!} + \dfrac{f'(x_j)(x_{j+1}-x_j)^1}{1!} + \dfrac{f''(x_j)(x_{j+1}-x_j)^2}{2!} + \dfrac{f'''(x_j)(x_{j+1}-x_j)^3}{3!} + \cdots$

在 $x = x_{j+2}$ 時 $\quad f(x_{j+2}) = \dfrac{f(x_j)(x_{j+2}-x_j)^0}{0!} + \dfrac{f'(x_j)(x_{j+2}-x_j)^1}{1!} + \dfrac{f''(x_j)(x_{j+2}-x_j)^2}{2!} + \dfrac{f'''(x_j)(x_{j+2}-x_j)^3}{3!} + \cdots$

在 $x = x_{j-1}$ 時 $\quad f(x_{j-1}) = \dfrac{f(x_j)(x_{j-1}-x_j)^0}{0!} + \dfrac{f'(x_j)(x_{j-1}-x_j)^1}{1!} + \dfrac{f''(x_j)(x_{j-1}-x_j)^2}{2!} + \dfrac{f'''(x_j)(x_{j-1}-x_j)^3}{3!} + \cdots$

在 $x = x_{j-2}$ 時 $\quad f(x_{j-2}) = \dfrac{f(x_j)(x_{j-2}-x_j)^0}{0!} + \dfrac{f'(x_j)(x_{j-2}-x_j)^1}{1!} + \dfrac{f''(x_j)(x_{j-2}-x_j)^2}{2!} + \dfrac{f'''(x_j)(x_{j-2}-x_j)^3}{3!} + \cdots$

Let $h = x_{j+2} - x_{j+1} = x_{j+1} - x_j = x_j - x_{j-1} = x_{j-1} - x_{j-2}$

$$\Rightarrow f(x_{j-2}) = f(x_j) - 2f'(x_j)h + \frac{4}{2}f''(x_j)h^2 - \frac{8}{6}f'''(x_j)h^3 + \frac{16}{24}f^{(4)}(x_j)h^4 - \frac{32}{120}f^{(5)}(x_j)h^5 + \cdots \quad ※$$

$$f(x_{j-1}) = f(x_j) - f'(x_j)h + \frac{1}{2}f''(x_j)h^2 - \frac{1}{6}f'''(x_j)h^3 + \frac{1}{24}f^{(4)}(x_j)h^4 + \frac{1}{120}f^{(5)}(x_j)h^5 + \cdots \quad ※$$

$$f(x_{j+1}) = f(x_j) + f'(x_j)h + \frac{1}{2}f''(x_j)h^2 + \frac{1}{6}f'''(x_j)h^3 + \frac{16}{24}f^{(4)}(x_j)h^4 + \frac{32}{120}f^{(5)}(x_j)h^5 + \cdots \quad ※$$

$$f(x_{j+2}) = f(x_j) + 2f'(x_j)h + \frac{4}{2}f''(x_j)h^2 + \frac{8}{6}f'''(x_j)h^3 + \frac{1}{24}f^{(4)}(x_j)h^4 - \frac{1}{120}f^{(5)}(x_j)h^5 + \cdots \quad ※$$

To get h^2, h^3, h^4 terms to cancel out

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \\ 16 & 1 & 1 & 16 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & 7 & 9 & 16 \\ 0 & -15 & -15 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 9 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = -d \\ b = -c \\ c = -8d \end{cases} \Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} t \\ -8t \\ 8t \\ -t \end{pmatrix}$$

$$\Rightarrow f(x_{j-2}) - 8f(x_{j-1}) + 8f(x_{j+1}) - f(x_{j+2}) = 12f'(x_j)h - \frac{48}{120}f^{(5)}(x_j)h^5$$

$$\Rightarrow f'(x_j) = \frac{f(x_{j-2}) - 8f(x_{j-1}) + 8f(x_{j+1}) - f(x_{j+2})}{12h} + O(h^4)$$

$$\approx \frac{f(x_{j-2}) - 8f(x_{j-1}) + 8f(x_{j+1}) - f(x_{j+2})}{12h} \quad ※$$

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