$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & -0.5 \\ 0 & 2 & 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & -0.5 \\ 0 & 0 & 0.75 \end{pmatrix}$$

$\Rightarrow$ 三條獨立的 rows

$\Rightarrow$ rank = 3 ※

$$\begin{cases} \det(A) = \lambda_1 \lambda_2 \lambda_3 = 6 \\ tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = 6 \end{cases}$$

$\Rightarrow \lambda = 1, 2, 3$ ※

Let $\underset{\sim}{v}$ = eigenvector

$\underset{\sim}{A}\underset{\sim}{v} = \lambda \underset{\sim}{v} \Rightarrow (\underset{\sim}{A} - \lambda \underset{\sim}{I})\underset{\sim}{v} = 0$

1. when $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} v_1 + v_3 = 0 \\ -v_1 + 3v_2 - v_3 = 0 \\ -v_1 + 2v_2 - v_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad c_1 \in \mathbb{R}$$ ※

2. when $\lambda = 2$

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} v_3 = 0 \\ -v_1 + 2v_2 - v_3 = 0 \\ -v_1 + 2v_2 - 2v_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad c_2 \in \mathbb{R}$$ ※

3. when $\lambda = 3$

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -v_1 + v_3 = 0 \\ -v_1 + v_2 - v_3 = 0 \\ -v_1 + 2v_2 - 3v_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad c_3 \in \mathbb{R}$$ ※