An Invitation to Gromov-Hausdorff convergence

Jiayin Pan

The Fields Institute for Research in Mathematical Science

In a nutshell

The *Gromov-Hausdorff distance* between two metric spaces measures how they look alike.
A sequence of metric spaces X_i *Gromov-Hausdorff converges* to some limit metric space Y, written as $X_i \stackrel{GH}{\to} Y$, means almost impossible to distinguish Y and X_i for large i.

In a nutshell

The *Gromov-Hausdorff distance* between two metric spaces measures how they look alike.
A sequence of metric spaces X_i *Gromov-Hausdorff converges* to some limit metric space Y, written as $X_i \overset{GH}{\to} Y$, means almost impossible to distinguish Y and X_i for large i.

Examples:

- Sphere (S^2, g_i) to football sphere
- Flat cylinder to a line
- Flat torus to a point
- Curved surface to a tangent plane
- Integers to a line

Principle

To undertand some *nice* spaces, we have to understand some *singular* spaces first.

Analog: To understand the *smooth* solutions of some differential equation, we have to understand its *weak* solutions first.

A note on its history

According to Wikipedia:

The Gromov–Hausdorff distance was introduced by David Edwards in 1975, and it was later rediscovered and generalized by Mikhail Gromov in 1981.

Hausdorff distance

Definition (Hausdorff distance)

Let Z be a metric space and let X and Y be two non-empty compact subsets of Z. The *Hausdorff distance* between X and Y in Z is defined as

$$d_H^Z(X, Y) = \inf\{r > 0 | X \subseteq B_r(Y), Y \subseteq B_r(X)\}.$$

Hausdorff distance

Definition (Hausdorff distance)

Let Z be a metric space and let X and Y be two non-empty compact subsets of Z. The *Hausdorff distance* between X and Y in Z is defined as

$$d_H^Z(X,Y) = \inf\{r > 0 | X \subseteq B_r(Y), Y \subseteq B_r(X)\}.$$

Facts:

- $d_H^Z(X,Y) = 0$ iff $X = Y$.
- d_H^Z is a distance function on $\mathcal{C}_Z$, the set of non-empty compact subsets of Z.
- If Z is a complete metric space, then $(\mathcal{C}_Z, d_H^Z)$ is complete.

Gromov-Hausdorff distance/convergence

Definition (Gromov-Hausdorff distance)

Let X and Y be two compact metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf\{d_H^Z(f(X), g(Y)) | f : X \to Z, g : Y \to Z \text{ are isom. embeddings to some metric space } Z\}.$$

$X_i \xrightarrow{GH} Y$, if $d_{GH}(X_i, Y) \to 0$.

Gromov-Hausdorff distance/convergence

Definition (Gromov-Hausdorff distance)

Let X and Y be two compact metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf\{d_H^Z(f(X), g(Y)) | f : X \to Z, g : Y \to Z \text{ are isom. embeddings to some metric space } Z\}.$$

$X_i \xrightarrow{GH} Y$, if $d_{GH}(X_i, Y) \to 0$.

Facts:

- $d_{GH}(X,Y) = 0$ iff X is isometric to Y.
- $(\mathcal{M}et_c, d_{GH})$ is a complete metric space, where $\mathcal{M}et_c$ is the isometry classes of all compact metric spaces.

Definition (Gromov-Hausdorff distance)

Let X and Y be two compact metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf\{d_H^Z(f(X),g(Y)) | f : X \to Z, g : Y \to Z \text{ are isom. embeddings to some metric space } Z\}.$$

$X_i \xrightarrow{GH} Y$, if $d_{GH}(X_i, Y) \to 0$.

Facts:

- $d_{GH}(X,Y) = 0$ iff X is isometric to Y.
- $(\mathcal{M}et_c, d_{GH})$ is a complete metric space, where $\mathcal{M}et_c$ is the isometry classes of all compact metric spaces.

Pointed Gromov-Hausdorff convergence for locally compact metric spaces: $(X_i, x_i) \xrightarrow{GH} (Y, y)$, if $B_r(x_i) \xrightarrow{GH} B_r(y)$ for all $r > 0$.

Wishlist: *Precompactness* to obtain convergent subsequences.

How to use GH convergence

Wishlist: *Precompactness* to obtain convergent subsequences.

If we do have certain precompactness, then (after passing to a subsequence) we can use GH convergence to

- study a family of metric spaces: take a sequence X_i, then $X_i \xrightarrow{GH} Y$.
- study the infinitesimal structure of a metric space at a point: $(r_i X, x) \xrightarrow{GH} (C_x X, v)$, where $r_i \to \infty$.
- study the large structure of a non-compact metric space at infinity: $(r_i^{-1} X, x) \xrightarrow{GH} (C_\infty X, v)$, where $r_i \to \infty$.

Wishlist: *Precompactness* to obtain convergent subsequences.

If we do have certain precompactness, then (after passing to a subsequence) we can use GH convergence to

- study a family of metric spaces: take a sequence X_i, then $X_i \xrightarrow{GH} Y$.
- study the infinitesimal structure of a metric space at a point: $(r_i X, x) \xrightarrow{GH} (C_x X, v)$, where $r_i \to \infty$.
- study the large structure of a non-compact metric space at infinity: $(r_i^{-1} X, x) \xrightarrow{GH} (C_\infty X, v)$, where $r_i \to \infty$.

We can also study addtional structures (functions, measures, group actions...) on metric spaces by passing them from the sequence to limit space.

Precompactness theorem (Gromov 1981)

Let X_i be a sequence of compact metric spaces. X_i has a convergent GH subsequence if it satisfies:

- (uniformly bounded diameter) $\mathrm{diam} X_i \leq D$ for all i;
- (uniformly bounded covering) for any $\epsilon > 0$, there is a number Q such that X_i can be covered by Q many ϵ-balls for all i.

Precompactness

Precompactness theorem (Gromov 1981)

Let X_i be a sequence of compact metric spaces. X_i has a convergent GH subsequence if it satisfies:

- (uniformly bounded diameter) $\mathrm{diam} X_i \leq D$ for all i;
- (uniformly bounded covering) for any $\epsilon > 0$, there is a number Q such that X_i can be covered by Q many ϵ-balls for all i.

As an example, a sequence that do not have any convergent subsequence: attaching segments to a common point.

Precompactness

Precompactness theorem (Gromov 1981)

Let X_i be a sequence of compact metric spaces. X_i has a convergent GH subsequence if it satisfies:

- (uniformly bounded diameter) $\mathrm{diam} X_i \leq D$ for all i;
- (uniformly bounded covering) for any $\epsilon > 0$, there is a number Q such that X_i can be covered by Q many ϵ-balls for all i.

As an example, a sequence that do not have any convergent subsequence: attaching segments to a common point.

Geometric objects with the uniformly bounded covering condition:

- Finitely generated groups with controlled growth
- Riemannian n-manifolds with Ricci curvature bounded below

Let S be a finite generating set of a group Γ.
Word metric:
$d_S(e,\gamma) = \min\{k|\gamma$ can be written as a word in S of length $k\}$.
$d_S(\gamma,\gamma') = d_S(e,\gamma^{-1}\gamma')$.

Growth of finitely generated groups

Let S be a finite generating set of a group Γ.
Word metric:
$d_S(e,\gamma) = \min\{k | \gamma \text{ can be written as a word in } S \text{ of length } k\}$.
$d_S(\gamma,\gamma') = d_S(e,\gamma^{-1}\gamma')$.
Growth function: $G_S(r) = \#\{\gamma \in \Gamma | d_S(e,\gamma) \leq r\}$.
Γ has *polynomial growth*, if $G_S(r) \leq Cr^k$. (This definition is indeed independent of the choice of S.)

Let S be a finite generating set of a group Γ.
Word metric:
$d_S(e,\gamma) = \min\{k|\gamma$ can be written as a word in S of length $k\}$.
$d_S(\gamma,\gamma') = d_S(e,\gamma^{-1}\gamma')$.
Growth function: $G_S(r) = \#\{\gamma \in \Gamma | d_S(e,\gamma) \leq r\}$.
Γ has *polynomial growth*, if $G_S(r) \leq Cr^k$. (This definition is indeed independent of the choice of S.)

Examples:

- $\mathbb{Z}^k$ has polynomial growth (of degree k).
- The free group generated by two elements does not have polynomial growth.
- If subgroup H has finite index in Γ, then Γ have the same growth type as H.

Nilpotent groups

Abelian: $[\Gamma, \Gamma] = \{e\}$.
Nilpotent: $[\Gamma, \Gamma] = \Gamma^1, [\Gamma, \Gamma^i] = \Gamma^{i+1}$; eventually, $\Gamma^m = \{e\}$.

Example: (Discrete) Heisenberg 3-group

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}.$$

$$N^1 = [N, N] = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c \in \mathbb{Z} \right\}.$$

$$N^2 = [N, N^1] = \{e\}.$$

Nilpotent groups

Abelian: $[\Gamma, \Gamma] = \{e\}$.
Nilpotent: $[\Gamma, \Gamma] = \Gamma^1, [\Gamma, \Gamma^i] = \Gamma^{i+1}$; eventually, $\Gamma^m = \{e\}$.

Example: (Discrete) Heisenberg 3-group

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}.$$

$$N^1 = [N, N] = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c \in \mathbb{Z} \right\}.$$

$$N^2 = [N, N^1] = \{e\}.$$

Theorem (Wolf 1968)

Any finitely generated (virtually) nilpotent group has polynomial growth.

Gromov's work on finitely generated groups

Theorem (Gromov 1981)

Any finitely generated group with polynomial growth is virtually nilpotent.

Theorem (Gromov 1981)

Any finitely generated group with polynomial growth is virtually nilpotent.

Gromov's approach:
If Γ has polynomial growth, then the sequence $(r_i^{-1}\Gamma, e, d_S)$ is precompact in GH, where $r_i \to \infty$. Passing to a subsequence, $(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d)$. Then use the limit space (G, d) to prove the theorem.

Gromov's work on finitely generated groups

Theorem (Gromov 1981)

Any finitely generated group with polynomial growth is virtually nilpotent.

Gromov's approach:
If Γ has polynomial growth, then the sequence $(r_i^{-1}\Gamma, e, d_S)$ is precompact in GH, where $r_i \to \infty$. Passing to a subsequence, $(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d)$. Then use the limit space (G, d) to prove the theorem.
(Simple) Examples:

- $\Gamma = \mathbb{Z}$, $S = \{1\}$, then $G = \mathbb{R}$ with standard metric.
- $\Gamma = \mathbb{Z}^2$, $S = \{(1,0),(0,1)\}$, then $G = \mathbb{R}^2$ with box metric.

Theorem (Gromov 1981)

Any finitely generated group with polynomial growth is virtually nilpotent.

Gromov's approach:
If Γ has polynomial growth, then the sequence $(r_i^{-1}\Gamma, e, d_S)$ is precompact in GH, where $r_i \to \infty$. Passing to a subsequence, $(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d)$. Then use the limit space (G, d) to prove the theorem.
(Simple) Examples:

- $\Gamma = \mathbb{Z}$, $S = \{1\}$, then $G = \mathbb{R}$ with standard metric.
- $\Gamma = \mathbb{Z}^2$, $S = \{(1,0),(0,1)\}$, then $G = \mathbb{R}^2$ with box metric.

(Pansu 1982) Detailed description of (G, d)...leads to the study of subRiemannian/subFinsler geometry.

Gromov's precompactness theorem 1981

Let (M_i, p_i) be a sequence of complete Riemannian n-manifolds of $\mathrm{Ric} \geq -(n-1)$, then (M_i, p_i) has a GH convergent subsequence.

GH convergence provides a platform to study a class of Riemannian manifolds with uniform geometric conditions.

GH convergence in Riemannian geometry

Gromov's precompactness theorem 1981

Let (M_i, p_i) be a sequence of complete Riemannian n-manifolds of $\mathrm{Ric} \geq -(n-1)$, then (M_i, p_i) has a GH convergent subsequence.

GH convergence provides a platform to study a class of Riemannian manifolds with uniform geometric conditions.

Various curvature bounds:

$$\lambda \leq \sec \leq \Lambda, \quad \sec \geq \lambda, \quad \lambda \leq \mathrm{Ric} \leq \Lambda, \quad \mathrm{Ric} \geq \lambda.$$

Non-collapsing/collapsing:

$$\mathrm{vol}(B_1(p_i)) \geq v > 0, \quad \mathrm{vol}(B_1(p_i)) \to 0.$$

Questions to be addressed

$M_i^n \xrightarrow{GH} X$ with various geometric conditions.

Questions:

- What can we say about the structure of X?
- What can we say about the relations between X and M_i for large i?

Questions to be addressed

$M_i^n \xrightarrow{GH} X$ with various geometric conditions.

Questions:

- What can we say about the structure of X?
- What can we say about the relations between X and M_i for large i?

Some geometric properties can be directly passed to X.
(e.g. X is a length metric space of Hausdorff dimension $\leq n$.)

In general, when the sequence satisfies $\mathrm{Ric} \geq -(n-1)$, the limit space X may not be a manifold, may not be locally contractible, may not have integer Hausdorff dimension.

Stability: $M_i \xrightarrow{GH} X$ non-collapsing, under certain curvature conditions, X is a manifold diffeo./homeo. to M_i for i large. In some cases, certain convergence of Riemannian metrics holds.

Stability/Finiteness

Stability: $M_i \xrightarrow{GH} X$ non-collapsing, under certain curvature conditions, X is a manifold diffeo./homeo. to M_i for i large. In some cases, certain convergence of Riemannian metrics holds.

Stability $\Rightarrow$ Finiteness: Suppose not...a sequence of mutually distinct M_i...subseqeunce GH converges...$\rightarrow\leftarrow$ stability.

Stability: $M_i \xrightarrow{GH} X$ non-collapsing, under certain curvature conditions, X is a manifold diffeo./homeo. to M_i for i large. In some cases, certain convergence of Riemannian metrics holds.

Stability $\Rightarrow$ Finiteness: Suppose not...a sequence of mutually distinct M_i...subseqeunce GH converges...$\rightarrow\leftarrow$ stability.

- (Cheeger 1970) Collection of closed n-manifolds M with $|\mathrm{sec}| \leq 1, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many diffeo. types.
- (Perelman 1991) Collection of closed n-manifolds M with $\mathrm{sec} \geq -1, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many homeo. types.
- (Cheeger-Naber 2015) Collection of closed 4-manifolds M with $|\mathrm{Ric}| \leq 3, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many diffeo. types.

General limit spaces

Curvature	*Limit spaces*
$\lvert\mathrm{sec}\rvert \leq 1$	orbit space (Cheeger,Gromov,Fukaya)
$\mathrm{sec} \geq -1$	Alexandrov space (Burago,Gromov,Perelman)
$\mathrm{Ric} \geq -(n-1)$	Ricci limit space (Cheeger,Colding,Naber)

General limit spaces

Curvature	*Limit spaces*
$\lvert\sec\rvert \leq 1$	orbit space (Cheeger,Gromov,Fukaya)
$\sec \geq -1$	Alexandrov space (Burago,Gromov,Perelman)
$\mathrm{Ric} \geq -(n-1)$	Ricci limit space (Cheeger,Colding,Naber)

...also leads to the study of non-smooth metric spaces with synthetic curvature conditions.

Maximally collapsed manifolds

$(M, g_i) \xrightarrow{GH}$ point while keeping the curvature bounds.
Example: any closed flat manifold.

- Almost flat: $|\sec| \le 1$.
- Almost nonnegative sectional curvature: $\sec \ge -1$.
- Almost nonnegative Ricci curvature: $\mathrm{Ric} \ge -(n-1)$.

Maximally collapsed manifolds

$(M, g_i) \xrightarrow{GH}$ point while keeping the curvature bounds.
Example: any closed flat manifold.

- Almost flat: $|\sec| \leq 1$.
- Almost nonnegative sectional curvature: $\sec \geq -1$.
- Almost nonnegative Ricci curvature: $\mathrm{Ric} \geq -(n-1)$.

Observation: Any closed flat manifold is finitely covered by a torus.

Theorem

If M^n is almost ..., then

- (Gromov 1978) M is finitely covered by a nilmanifold.
- (Kapovitch-Petrunin-Tuschmann 2010) M is finitely covered by a nilpotent space.
- (Kapovitch-Wilking 2012) $\pi_1(M)$ is $C(n)$-nilpotent.

Selected results derived from collapsing theory

- (Rong 1996) Closed n-manifold M with $0 < \delta \leq \sec \leq 1$. Then $\pi_1(M)$ is $C(n, \delta)$-cyclic.

Selected results derived from collapsing theory

- (Rong 1996) Closed n-manifold M with $0 < \delta \leq \sec \leq 1$. Then $\pi_1(M)$ is $C(n, \delta)$-cyclic.
- (Petrunin-Tuschmann 1999, Fang-Rong 2002) Collection of closed n-manifolds M with
$$\pi_1(M) = \pi_2(M) = 0, \quad |\sec| \leq 1, \quad \mathrm{diam}(M) \leq D$$
has finitely many diffeo. types.

- (Rong 1996) Closed n-manifold M with $0 < \delta \le \sec \le 1$. Then $\pi_1(M)$ is $C(n, \delta)$-cyclic.
- (Petrunin-Tuschmann 1999, Fang-Rong 2002) Collection of closed n-manifolds M with
  $\pi_1(M) = \pi_2(M) = 0, \quad |\sec| \le 1, \quad \mathrm{diam}(M) \le D$
  has finitely many diffeo. types.
- (Kapovitch-Wilking 2012) Closed n-manifold M with
  $\mathrm{Ric} \ge -(n-1), \quad \mathrm{diam}(M) \le D.$
  Then $\pi_1(M)$ is generated by at most $C(n, D)$ many elements.

Selected results derived from collapsing theory

- (Rong 1996) Closed n-manifold M with $0 < \delta \leq \sec \leq 1$. Then $\pi_1(M)$ is $C(n,\delta)$-cyclic.
- (Petrunin-Tuschmann 1999, Fang-Rong 2002) Collection of closed n-manifolds M with
  $\pi_1(M) = \pi_2(M) = 0, \quad |\sec| \leq 1, \quad \mathrm{diam}(M) \leq D$
  has finitely many diffeo. types.
- (Kapovitch-Wilking 2012) Closed n-manifold M with
  $\mathrm{Ric} \geq -(n-1), \quad \mathrm{diam}(M) \leq D$.
  Then $\pi_1(M)$ is generated by at most $C(n,D)$ many elements.
- (Pan 2017) Alternative proof of Milnor conjecture in dim 3: any open 3-manifold M with $\mathrm{Ric} \geq 0$ has a finitely generated $\pi_1(M)$. (First proved by Liu 2013, using minimal surface theory and Perelman's solution of Poincare conjecutre.)