Semi-local simple connectedness of non-collapsing Ricci limit spaces

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Gromov-Hausdorff distance

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$$\mathrm{Ric}_{M_i} \geq -(n-1),$$

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$(M_i, p_i) \xrightarrow{GH} (X, p)$...a useful tool to study a class of Riemannian manifolds with geometric conditions (curvature, volume, diameter, etc.)/to study infinitesimal or asymptotic geometry...important to understand the structure of the limit space X/relations between X and M_i for i large.

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$\mathrm{Ric} \geq -(n-1)$, X Ricci limit space (Cheeger-Colding)
Even when non-collapsing, X may have infinite local topological type (Menguy)

"Non-collapsing" means there is $v > 0$ such that $\mathrm{vol}(B_1(p_i)) \geq v$ for all i.

Topology of Ricci limit spaces

Since Ric controls fundamental groups, it is natural to ask:

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Any Ricci limit space has a universal cover.

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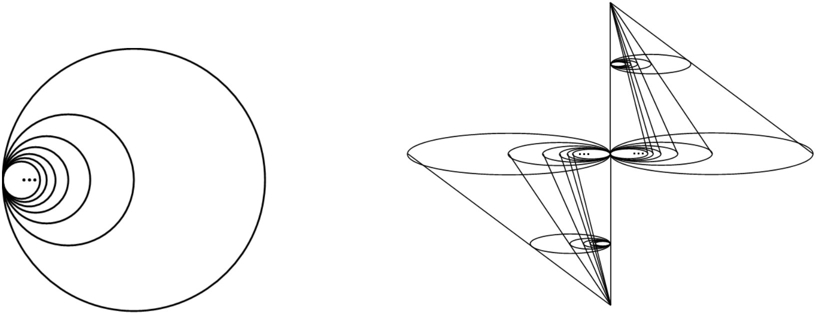
Theorem (Mondino-Wei, 2016)

Let (X, d, m) be an $\mathrm{RCD}^*(K, N)$-space for some $K \in \mathbb{R}$, $N \in (1, +\infty)$. Then (X, d, m) admits a universal cover.

A Hawaii ring does not have a universal cover.

The Griffiths twin cone is one-point-join of two “cutted” cones over a Hawaii ring. It is not semi-locally simply connected, but it has a universal cover (as itself); also, in this example, a non-contractible loop may have infinite length.



Main result

Main Theorem (Pan-Wei, 2019)

Any non-collapsing Ricci limit space is semi-locally simply connected.

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Not true for Ricci case on π_1-level; examples by Otsu.

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True (even without curvature conditions), but requires uniform control at all points.

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Theorem (Grove-Petersen)

Given $n, \kappa, v > 0$, there exist positive constants $\epsilon(n, \kappa, v)$ and $C(n, \kappa, v)$ such that for any complete n-manifold (M, p) of

$$\sec_M \geq -\kappa, \quad \mathrm{vol}(B_1(p)) \geq v,$$

$B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0, \epsilon)$.

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Otsu's example

$S^3 \times \mathbb{R}P^2$ admits a sequence of Riemannian metrics g_i of positive Ricci curvature with a non-collapsing limit space as a metric suspension over $S^2 \times \mathbb{R}P^2$. At the "tip" point, there are shorter and shorter non-contractible loops.

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Also, the Eguchi-Hanson metric on the tangent bundle of $\mathbb{R}P^2$ gives a Ricci flat example.

Approach

Definition (1-contractibility radius):

$\rho(t,x) = \inf\{\infty, \rho \geq t |$ any loop in $B_t(x)$ is contractible in $B_\rho(x)\}$.

Remark: X is semi-locally simply connected if for any $x \in X$, there is $T > 0$ such that $\rho(T,x) < \infty$; X is locally simply connected if for any $x \in X$, there is $t_i \to 0$ such that $\rho(t_i,x) = t_i$.

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Local version of the main theorem with an estimate on $\rho(t,x)$

Let (M_i, p_i) be a sequence of Riemannian n-manifolds (not necessarily complete) converging to (X,p) such that for all i,
(1) $B_2(p_i) \cap \partial M_i = \emptyset$ and the closure of $B_2(p_i)$ is compact;
(2) $\mathrm{Ric} \geq -(n-1)$ on $B_2(p_i)$, $\mathrm{vol}(B_1(p_i)) \geq v > 0$.
Then

$$\lim_{t\to 0} \frac{\rho(t,x)}{t} = 1$$

holds for any $x \in B_1(p)$.

Approach

Key: show $\lim_{t\to 0} \rho(t,x) = 0$.

After this, we can improve the result to $\lim_{t\to 0} \rho(t,x)/t = 1$ by using the structure of tangent cones and Sormani's uniform cut techniques.

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Classification: We classify the points in X according to the local 1-contractibility radius on manifolds: $x \in X$, let x_i in M_i to x;

- x is of type I, if there is $r > 0$ such that the family of functions $\{\rho(q,t) | q \in B_r(x_i), i \in \mathbb{N}\}$ is equi-continuous at $t = 0$;
- x is of type II, if $\{\rho(x_i,t)\}_{i\in\mathbb{N}}$ is not equi-continuous at $t = 0$;
- x is of type III, if it is not of type I nor type II. (In other words, $\{\rho(x_i,t)\}_{i\in\mathbb{N}}$ is equi-continuous at $t = 0$, but the family $\{\rho(q,t) | q \in B_r(x_i), i \in \mathbb{N}\}$ is not equi-continuous for any $r > 0$.)

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Remark: Type II point may exist due to Otsu's example.
Spoiler: Type III points are the most difficult ones to deal with.

Type I

For a family of functions $\{\rho_\alpha(t)\}_{\alpha\in A}$ with $\rho_\alpha(0)=0$, the family is equi-continuous at $t=0$ if and only if there is a continuous function $\lambda(t)$ defined on $[0,T)$ with $\lambda(0)=0$ such that $\rho_\alpha(t)\le\lambda(t)$ for all $t\in[0,T)$ and all $\alpha\in A$.

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Proposition (without curvature conditions)

Let (X_i,x_i) be a sequence of length metric spaces with the conditions below:
(1) the closure of $B_1(x_i)$ is compact;
(2) there is a nice function λ on $[0,T)$ such that for all i and all $q\in B_2(x_i)$, $\rho(t,q)\le\lambda(t)<1/2$ holds on $[0,T)$;
(3) $(X_i,x_i)\xrightarrow{GH}(Y,y)$.
Then $\rho(t,q)\le\lambda(t)$ for all $t\in[0,T)$ and all $q\in B_1(y)$.

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For c in a sufficiently small ball, we can contract c_i for a large i. Take a skeleton of this homotopy, then we transfer this skeleton to the next manifold in the sequence and fill in the skeleton there. This allows us to transfer the nullhomotopy of c_i along the sequence and pass it to the limit space by uniform convergence. In this construction, we need to control the distance between nullhomotopies to assure uniform convergence.

Type II

A point x is of **Type II** means there are $\epsilon > 0$ and $t_i \to 0$ such that $\rho(t_i, x_i) \geq \epsilon$. We consider the universal covering space of $B_\epsilon(x_i)$. The local fundamental group Γ_i has a subgroup generated by these non-contractible "small" loops H_i. In Otsu's example, $\Gamma_i = \mathbb{Z}_2$. We consider the equivariant Gromov-Hausdorff convergence:

$$\begin{tikzcd}
(U_i, y_i, \Gamma_i, H_i) \arrow[r, "GH"] \arrow[d, "\pi_i"] & (Y, y, G, H) \arrow[d, "\pi"] \\
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Roughly speaking, covering group increases the volume around y_i when compared with the volume around x_i. Together with volume convergence (Colding, CC), local volume of y is at least double of the local volume of x: $\mathcal{H}^n(B_s(y)) \geq 2 \cdot \mathcal{H}^n(B_s(x))$.

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Theorem (By-product for half-volume lower bounds)

Given $n \geq 2$, $\kappa \geq 0$, and $\omega > 1/2$, there exist positive constants $\epsilon(n, \kappa, \omega)$ and $C(n, \kappa, \omega)$ such that the following holds.
Let (M, p) be a complete n-manifold satisfying

$$\mathrm{Ric} \geq -(n-1)\kappa, \quad \mathrm{vol}(B_1(p)) \geq \omega \cdot \mathrm{vol}(B_1^n(-\kappa)).$$

Then every loop in $B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0, \epsilon)$.

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...We need a technical lemma to deal with type III points.

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Technical lemma (without curvature conditions)

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A trivial, but important, observation: for any x, either $\rho(t, x)$ is good (type II case, by assumption), or $\{\rho(t, x_i)\}_{i\in\mathbb{N}}$ is good (type III case); for type I points, both $\rho(t, x)$ and $\{\rho(t, x_i)\}_{i\in\mathbb{N}}$ are good.

A very brief sketch of the proof of technical lemma

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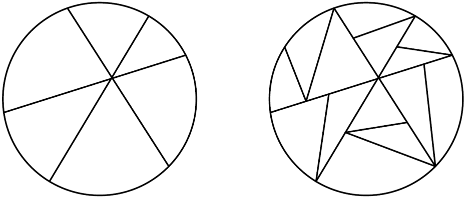
We construct the nullhomotopy gradually in the limit space through a sequence of refining skeletons. By controlling the extensions on the new skeletons at each step, these maps on the skeletons converge uniformly to a continuous map defined on the disk.



A very brief sketch of the proof of technical lemma

Sketch: Roughly speaking, if a sub-triangle is away from a point of type III, we can directly contract this sub-triangle; if not, we will use the local 1-contractibility from the sequence to extend the map on a finer 1-skeleton.