

# An Invitation to Gromov-Hausdorff convergence

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The *Gromov-Hausdorff distance* between two metric spaces measures how they look alike.

A sequence of metric spaces  $X_i$  *Gromov-Hausdorff converges* to some limit metric space  $Y$ , written as  $X_i \xrightarrow{GH} Y$ , means almost impossible to distinguish  $Y$  and  $X_i$  for large  $i$ .

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Examples:

- Sphere  $(S^2, g_i)$  to a football sphere
- A closed manifold to a point
- Flat cylinder to a line
- Curved surface to a tangent plane
- Integers to a line

To understand some *nice* spaces, we have to understand some *singular* spaces first.

Analog: To understand the *smooth* solutions of some differential equation, we have to understand its *weak* solutions first.

# A note on its history

According to Wikipedia:

*The Gromov–Hausdorff distance was introduced by David Edwards in 1975, and it was later rediscovered and generalized by Mikhail Gromov in 1981.*

## Definition (Hausdorff distance)

Let  $Z$  be a metric space and let  $X$  and  $Y$  be two non-empty compact subsets of  $Z$ . The *Hausdorff distance* between  $X$  and  $Y$  in  $Z$  is defined as

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Facts:

- $d_H^Z(X, Y) = 0$  iff  $X = Y$ .
- $d_H^Z$  is a distance function on  $\mathcal{C}_Z$ , the set of non-empty compact subsets of  $Z$ .
- If  $Z$  is a complete metric space, then  $(\mathcal{C}_Z, d_H^Z)$  is complete.

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$$d_{GH}(X, Y) = \inf\{d_H^Z(f(X), g(Y)) \mid f : X \rightarrow Z, g : Y \rightarrow Z \\ \text{are isom. embeddings to some metric space } Z\}.$$

$X_i \xrightarrow{GH} Y$ , if  $d_{GH}(X_i, Y) \rightarrow 0$ .



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- $d_{GH}(X, Y) = 0$  iff  $X$  is isometric to  $Y$ .
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Pointed Gromov-Hausdorff convergence for locally compact metric spaces:  $(X_i, x_i) \xrightarrow{GH} (Y, y)$ , if  $B_r(x_i) \xrightarrow{GH} B_r(y)$  for all  $r > 0$ .

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- study a family of metric spaces: take a sequence  $X_i$ , then  $X_i \xrightarrow{GH} Y$ .
- study the infinitesimal structure of a metric space at a point:  $(r_i X, x) \xrightarrow{GH} (C_x X, v)$ , where  $r_i \rightarrow \infty$ .
- study the large structure of a non-compact metric space at infinity:  $(r_i^{-1} X, x) \xrightarrow{GH} (C_\infty X, v)$ , where  $r_i \rightarrow \infty$ .

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We can also study additional structures (functions, measures, group actions...) on metric spaces by passing them from the sequence to limit space.

## Precompactness theorem (Gromov 1981)

Let  $X_i$  be a sequence of compact metric spaces.  $X_i$  has a convergent GH subsequence if it satisfies:

- (uniformly bounded diameter)  $\text{diam} X_i \leq D$  for all  $i$ ;
- (uniformly bounded covering) for any  $\epsilon > 0$ , there is a number  $Q$  such that  $X_i$  can be covered by  $Q$  many  $\epsilon$ -balls for all  $i$ .

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Geometric objects with the uniformly bounded covering condition:

- Finitely generated groups with controlled growth
- Riemannian  $n$ -manifolds with Ricci curvature bounded below



# Growth of finitely generated groups

Let  $S$  be a finite generating set of a group  $\Gamma$ .

*Word metric:*

$d_S(e, \gamma) = \min\{k \mid \gamma \text{ can be written as a word in } S \text{ of length } k\}.$

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$\Gamma$  has *polynomial growth*, if  $G_S(r) \leq Cr^k$ . (This definition is indeed independent of the choice of  $S$ .)

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Examples:

- $\mathbb{Z}^k$  has polynomial growth (of degree  $k$ ).
- The free group generated by two elements does not have polynomial growth.
- If a subgroup  $H$  has finite index in  $\Gamma$ , then  $\Gamma$  have the same growth type as  $H$ .

# Nilpotent groups

Abelian:  $[\Gamma, \Gamma] = \{e\}$ .

Nilpotent:  $[\Gamma, \Gamma] = \Gamma^1$ ,  $[\Gamma, \Gamma^i] = \Gamma^{i+1}$ ; eventually,  $\Gamma^m = \{e\}$ .

Example: (Discrete) Heisenberg 3-group

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

$$N^1 = [N, N] = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{Z} \right\}.$$

$$N^2 = [N, N^1] = \{e\}.$$

## Theorem (Wolf 1968)

Any finitely generated (virtually) nilpotent group has polynomial growth.

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Gromov's approach: If  $\Gamma$  has polynomial growth, then the sequence  $(r_i^{-1}\Gamma, e, d_S)$  is precompact in GH, where  $r_i \rightarrow \infty$ . Passing to a subsequence,  $(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d)$ . Then use the limit space  $(G, d)$  to prove the theorem.

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(Simple) Examples:

- $\Gamma = \mathbb{Z}$ ,  $S = \{1\}$ , then  $G = \mathbb{R}$  with standard metric.
- $\Gamma = \mathbb{Z}^2$ ,  $S = \{(1, 0), (0, 1)\}$ , then  $G = \mathbb{R}^2$  with box metric.

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(Pansu 1982) Detailed description of  $(G, d)$ ...leads to the study of subRiemannian/subFinsler geometry.



# GH convergence in Riemannian geometry

## Gromov's precompactness theorem (1981)

Let  $(M_i, p_i)$  be a sequence of complete Riemannian  $n$ -manifolds of  $\text{Ric} \geq -(n-1)$ , then  $(M_i, p_i)$  has a GH convergent subsequence.

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Various curvature bounds:

$$\lambda \leq \sec \leq \Lambda, \quad \sec \geq \lambda, \quad \lambda \leq \text{Ric} \leq \Lambda, \quad \text{Ric} \geq \lambda.$$

Non-collapsing/collapsing:

$$\text{vol}(B_1(p_i)) \geq \nu > 0, \quad \text{vol}(B_1(p_i)) \rightarrow 0.$$

# Questions to be addressed

$M_i^n \xrightarrow{GH} X$  with various geometric conditions.

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- What can we say about the structure of  $X$ ?
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Some geometric properties can be directly passed to  $X$ .  
(e.g.  $X$  is a length metric space.)

In general, when the sequence satisfies  $\text{Ric} \geq -(n-1)$ , the limit space  $X$  may not be a manifold, may not be locally contractible, may not have integer Hausdorff dimension.

Stability:  $M_i \xrightarrow{GH} X$  non-collapsing, under certain curvature conditions,  $X$  is a manifold diffeo./homeo. to  $M_i$  for  $i$  large. In some cases, certain convergence of Riemannian metrics holds.

# Stability/Finiteness

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- (Cheeger 1970) Collection of closed  $n$ -manifolds  $M$  with  $|\text{sec}| \leq 1$ ,  $\text{diam}(M) \leq D$ ,  $\text{vol}(M) \geq \nu > 0$  has finitely many diffeo. types.
- (Perelman 1991) Collection of closed  $n$ -manifolds  $M$  with  $\text{sec} \geq -1$ ,  $\text{diam}(M) \leq D$ ,  $\text{vol}(M) \geq \nu > 0$  has finitely many homeo. types.
- (Cheeger-Naber 2015) Collection of closed 4-manifolds  $M$  with  $|\text{Ric}| \leq 3$ ,  $\text{diam}(M) \leq D$ ,  $\text{vol}(M) \geq \nu > 0$  has finitely many diffeo. types.

## *Curvature    Limit spaces*

$|\sec| \leq 1$     orbit space (Cheeger, Gromov, Fukaya)

$\sec \geq -1$     Alexandrov space (Burago, Gromov, Perelman)

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...also leads to the study of non-smooth metric spaces with synthetic curvature conditions.

# Maximally collapsed manifolds

$(M, g_i) \xrightarrow{GH}$  point while keeping curvature bounded.

Example: any closed flat manifold.

- Almost flat:  $|\sec| \leq 1$ .
- Almost nonnegative sectional curvature:  $\sec \geq -1$ .
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Observation: Any closed flat manifold is finitely covered by a torus.

## Theorem

If  $M^n$  is almost ..., then

- (Gromov 1978)  $M$  is finitely covered by a nilmanifold.
- (Kapovitch-Petrunic-Tuschmann 2010)  $M$  is finitely covered by a nilpotent space.
- (Kapovitch-Wilking 2012)  $\pi_1(M)$  is  $C(n)$ -nilpotent.

# Selected results derived from collapsing theory

- (Rong 1996) Closed  $n$ -manifold  $M$  with  $0 < \delta \leq \sec \leq 1$ .  
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- (Pan 2017) Alternative proof of Milnor conjecture in dim 3:  
any open 3-manifold  $M$  with  $\text{Ric} \geq 0$  has a finitely generated  
 $\pi_1(M)$ . (First proved by Liu 2013, using minimal surface  
theory.)