NOTES ON RONG'S CYCLICITY THEOREM

ABSTRACT. Self-learning notes on Rong's $C(n)$-cyclicity theorem about fundamental groups of closed manifolds with positive sectional curvature and S^1 symmetry.

CONTENTS

Theorem A (Rong)**.** *Let M be a closed n-manifold of* $\sec \geq 1$. *Suppose that its universal cover $\widetilde{M}$ admits an isometric S^1-action, then $\pi_1(M)$ contains a cyclic subgroup of index at most $C(n)$.*

Rong's theorem in its original form also requires that S^1-action commutes with π_1-action. With a later result by Su-Wang (see Theorem 3.4), this assumption can be dropped because one can always find a subgroup of $\pi_1(M)$ with bounded index that commutes with S^1.

It is conjectured by Rong that the circle symmetry condition in Theorem A can be dropped.

Conjecture 0.1 (Rong)**.** *Let M be a closed n-manifold of* $\sec \geq 1$. *Then $\pi_1(M)$ contains a cyclic subgroup of index at most $C(n)$.*

1. SYNGE'S THEOREM

Theorem 1.1 (Synge)**.** *Let M^n be a closed manifold with* $\sec \geq 1$.
(1) If n is even, then M is either simply connected or not orientable.
(2) If n is odd, then M is orientable.

Lemma 1.2. *Let M^n be a closed manifold with* $\sec \geq 1$. *Given a minimal geodesic $\gamma : [0,l] \to M$ of unit speed and a normal parallel vector field $V(t)$ along γ. We consider the variation $\Gamma_s(t) = \exp_{\gamma(t)} sV(t)$, where $s \in (-\epsilon, \epsilon)$. Then there is $s \in (-\epsilon, \epsilon)$ such that* $\mathrm{length}(\Gamma_s) < \mathrm{length}(\gamma)$.

Proof. By the formula of second variation,

$$\frac{d^2}{ds^2}\Big|_{s=0} \mathrm{length}(\Gamma_s) = -\int_0^l Rm(V, \gamma', \gamma', V)\, dt < 0.$$

The result follows. □

Weinstein later rephrased the proof of Synge's theorem so as to establish fixed points of isometries.

Theorem 1.3 (Weinstein)**.** *Let M^n be a closed and orientable manifold with* $\sec \geq 1$.
(1) If n is even, then any orientation preserving isometry ϕ of M has a fixed point.
(2) If n is odd, then any orientation reversing isometry ϕ of M has a fixed point.

Proof. (1) Suppose that ϕ does not have any fixed points. We choose $z \in M$ such that

$$d(z, \phi(z)) = \min_{x \in M} d(x, \phi(x)) =: l > 0.$$

We join z and $\phi(z)$ by a minimal geodesic $\gamma : [0, l] \to M$. First note that $d\phi(\gamma'(0)) = \gamma'(l)$; otherwise, for any $t \in (0, l)$, we have

$$d(\gamma(t), \phi \circ \gamma(t)) < d(\gamma(t), \gamma(l)) + d(\gamma(l), \phi \circ \gamma(t)) = (l - t) + t = l,$$

which is a contradiction to the choice of z.

Let N_t be the orthogonal complement of $\gamma'(t)$ in $T_{\gamma(t)}M$. Note that we have two linear isometries

$$d\phi : N_0 \to N_l, \quad P_\gamma : N_0 \to N_l,$$

where P_γ is the parallel transport along γ. Then

$$P_\gamma^{-1} \circ d\phi : N_0 \to N_0$$

is an element of $SO(n-1)$, where $n - 1$ is odd. Thus it must have an eigenvector v with eigenvalue 1; in other words, we have a nonzero vector $v \in N_0$ such that $d\phi(v) = P_\gamma(v)$. We parallel transport v along γ to obtain a normal parallel vector field $V(t)$ such that $V(0) = v$ and $V(l) = d\phi(v)$.

We consider the variation $\Gamma_s(t) = \exp_{\gamma(t)} sV(t)$, where $s \in (-\epsilon, \epsilon)$. By Lemma 1.2, there is some s such that Γ_s is shorter than γ. Let $y = \Gamma_s(0)$. Note that

$$\Gamma_s(l) = \exp_{\gamma(l)} sV(l) = \exp_{\phi(z)} sd\phi(v) = \phi \circ \exp_z sv = \phi(y).$$

We end in a contradiction to the choice of z because

$$d(y, \phi(y)) \le \mathrm{length}(\Gamma_s) < \mathrm{length}(\gamma) = d(\phi(z), z).$$

This completes the proof of (1).

(2) The proof is similar. If n is odd and ϕ is orientation reversing, then the same argument in (1) leads to $P_\gamma^{-1} \circ d\phi \in O(n-1)$ with determinant -1 and $n - 1$ being even. Hence it has eigenvalue 1 and the remaining proof goes through. □

Theorem 1.4 (Berger)**.** *Let M^n be a closed manifold with* $\sec \ge 1$*, where n is even. Then any isometric S^1-action of M has a fixed point.*

Proof. We first assume that M is orientable. Let $\theta \in S^1$ such that $\langle \theta \rangle$ is dense in S^1. Because θ is orientation preserving and n is even, by Theorem 1.3, θ has a fixed point x_0. It follows that S^1-action fixes x_0.

If M is not orientable. Let $\hat{M}$ be its orientable double cover. We can lift the S^1-action on M to a S^1-action on $\hat{M}$. Then the fixed point on $\hat{M}$ projects to a fixed point on M. □

2. An equivariant version of Synge's theorem

Theorem 2.1 (Rong)**.** *Let M be a closed manifold with* $\sec \ge 1$ *and circle symmetry. If ϕ is an isometry of M without fixed points that commutes with S^1-action, then ϕ preserves a circle orbit.*

We remark that Theorem 2.1 also holds for ϕ that has fixed points. For the sake of proving Rong's cyclicity theorem, ϕ comes from $\pi_1(M)$-action on $\widetilde{M}$, so we can always assume that ϕ does not have fixed points.

For a G-action on M, below we denote the isotropy subgroup of G at x by

$$G_x = \{g \in G | g \cdot x = x\}.$$

Lemma 2.2 (Kleiner)**.** *Let M be a complete manifold with an isometric G-action. Let $\gamma : [0, l] \to M$ be a minimal geodesic between $G \cdot \gamma(0)$ and $G \cdot \gamma(l)$. Then $G_{\gamma(t)}$ is constant for $t \in (0, l)$ and is a subgroup of $G_{\gamma(0)} \cap G_{\gamma(l)}$.*

Proof. We first prove that $G_{\gamma(t)} \leq G_{\gamma(0)} \cap G_{\gamma(l)}$, where $t \in (0, l)$. Suppose that $g \in G - \{e\}$ fixes $\gamma(t)$ but moves $\gamma(l)$. Then $g \circ \gamma$ is also a minimal geodesic between $G \cdot \gamma(0)$ and $G \cdot \gamma(l)$. We note that

$$d(\gamma(0), g \cdot \gamma(l)) \geq l = d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(l)) = d(\gamma(0), \gamma(t)) + d(g\gamma(t), g\gamma(l)).$$

Thus $\gamma|_{[0,t]}$ joining $g \circ \gamma|_{[t,l]}$ is a minimal geodesic between $\gamma(0)$ and $g \cdot \gamma(l)$. We obtain a branching geodesic; a contradiction.

Next, we show that $G_{\gamma(t)}$ is constant for $t \in (0, l)$. Let $0 < t < s < l$. Observe that $\gamma|_{[0,s]}$ is a minimal geodesic between $G \cdot \gamma(0)$ and $G \cdot \gamma(s)$. Then we have $G_{\gamma(t)} \leq G_{\gamma(s)}$. The other direction similarly follows. □

Proof of Theorem 2.1. We consider the case that n is odd, then M is orientable and ϕ is orientation preserving. The proof for even dimensions is similar.

The proof is by induction on $n = \dim M$. We assume that the statement holds for odd dimension $\leq n-2$ and prove the inductive step first. We will visit the base case $n = 3$ afterwards.

Suppose that ϕ does not preserve any circle orbits. We choose $z \in M$ such that

$$d(z, \phi(S^1 z)) = \min_{x \in M} d(S^1 x, \phi(S^1 x)) = l > 0.$$

Let γ be a minimal geodesic from z to $\phi(S^1 z) = S^1(\phi(z))$. We write its end point as $\gamma(l) = \theta \cdot \phi(z)$, where $\theta \in S^1$.

Claim: $S^1_{\gamma(t)}$, the isotropy subgroup of S^1 at $\gamma(t)$, is constant for $t \in [0, l]$. We consider the curve $(\theta \circ \phi) \circ \gamma$. Similar to the proof of Theorem 1.3, we can show that $d(\theta \circ \phi)(\gamma'(0)) = \gamma'(1)$. Otherwise, for any $t \in (0, l)$ we would have

$$d(\gamma(t), (\theta \circ \phi) \circ \gamma(t)) < d(\gamma(t), \gamma(l)) + d(\gamma(l), (\theta \circ \phi) \circ \gamma(t)) = (l - t) + t = l,$$

which contradicts to the choice of z. This also shows that the curve constructed by joining $\gamma|_{[s,l]}$ and $(\theta \circ \phi) \circ \gamma|_{[0,s]}$ is minimal between $S^1 \cdot \gamma(s)$ and $S^1 \cdot (\phi \circ \gamma(s))$. By Lemma 2.2, we have $S^1_{\gamma(l)} \leq S^1_{\gamma(s)}$. Similarly, one can show the inclusion at $\gamma(0)$. This proves the Claim.

Below, we write $H = S^1_{\gamma(t)} = S^1_z$.

Case 1. $H = \{e\}$. Let M_0 be the set of all points where S^1 acts freely. M_0 is open in M. Let

$$\pi : M \to \overline{M} = M/S^1$$

be the quotient map and let $\overline{M}_0 = \pi(M_0)$. $\overline{M}_0$ is open in $\overline{M}$ and carries a Riemannian metric with $\sec \geq 1$. Because ϕ commutes with S^1-action, ϕ descends to $\overline{\phi} \in \mathrm{Isom}(\overline{M})$. Using the fact that ϕ does not have fixed points, it is direct to check that $\overline{\phi}$ maps to $\overline{M}_0$ to $\overline{M}_0$. Moreover, $\overline{\phi} \in \mathrm{Isom}(\overline{M}_0)$ is orientation preserving. By the assumption $H = \{e\}$, $\overline{\gamma} = \pi(\gamma)$ is a minimal geodesic contained in $\overline{M}_0$. These set up the conditions to run a variation argument as in the proof of Theorem 1.3, which leads to a contradiction to the choice of z.

Case 2. $H = \mathbb{Z}_h$. We write its fixed point set

$$M^H = \{x \in M | H \cdot x = x\} = \cup F_j$$

as a union of components with F_0 containing z. Because H-action preserves orientation, F_0 is a closed totally geodesic submanifold of even codimension and $\sec \geq 1$. Since ϕ commutes with H, ϕ permutates the components of M^H. Observe that γ is a minimal geodesic in F_0, then we see that ϕ preserves F_0 because

$$\phi(z) = \theta^{-1} \cdot \gamma(l) \in \theta^{-1}(F_0) = F_0.$$

Now we have a triple (F_0, ϕ, S^1) with the desired properties to complete the induction. Hence ϕ preserves some circle orbit in F_0. This completes Case 2.

Case 3. $H = S^1$. We will construct a suitable variation of γ. Let N_t be the orthogonal complement of $\gamma'(t)$ in $T_{\gamma(t)}M$. Because S^1-action fixes γ, it acts on N_t by differential, written as $d\theta$ for $\theta \in S^1$. We have two linear isometries

$$P_\gamma : N_0 \to N_l, \quad d\phi : N_0 \to N_l.$$

We note that $d\theta$ and P_γ commutes because both $P_\gamma(d\theta(v))$ and $d\theta(P_\gamma(v))$ are parallel fields along γ with the same initial condition.

Claim: There is a unit vector $v \in N_0$ and $\theta \in S^1$ such that

$$P_\gamma(v) = d(\phi \circ \theta)(v).$$

Let S^{n-2} be the unit sphere in N_0. The map

$$\psi = d\phi^{-1} \circ P_\gamma : S^{n-2} \to S^{n-2}$$

commutes with S^1-action on S^{n-2}. If ψ has a fixed point v, then this v and $\theta = e$ fulfill the property. If ψ does not have fixed points, then we apply the inductive assumption to (S^{n-2}, ψ, S^1) to obtain a circle orbit $S^1 \cdot v$ that is preserved by ψ. In other words, we have $v \in S^{n-2}$ and $\theta \in S^1$ such that $\psi(v) = d\theta(v)$. This proves the Claim.

We continue to deal with Case 3. We parallel transport v along γ to obtain $V(t)$ and then consider the variation $\Gamma_s(t) = \exp_{\gamma(t)} sV(t)$. By Lemma 1.2, there is some s such that Γ_s is shorter than γ. Set $y = \Gamma_s(0)$. We note that

$$\Gamma_s(l) = \exp_{\gamma(l)} sV(l) = \exp_{\phi(z)} sd(\phi \circ \theta)(v) = \phi \circ \theta(\exp_z sv) = \phi \circ \theta(y).$$

Then we end in a desired contradiction because

$$d(y, \phi(S^1 y)) \leq d(y, \phi \circ \theta(y)) \leq \text{length}(\Gamma_s) < \text{length}(\gamma) = d(z, \phi(S^1 z)).$$

We have completed the inductive step. For the base step $n = 3$. The above argument leads to the situation (S^1, ϕ, S^1), then it is trivial that ϕ preserves the circle orbit.

For the proof in even dimensions, by Theorems 1.1 and 1.3, M is simply connected and ϕ reverses the orientation. All three cases in the above proof go through with some clear modifications. In the base step $n = 2$, that is, (S^2, ϕ, S^1), case 1 leads to $M/S^1 = [-1, 1]$, then $\overline{\phi}$ clearly has a fixed point. Both cases 2 and 3 cannot occur on (S^2, ϕ, S^1). □

3. Proof of cyclicity

3.1. **Bounding the index of F_0-preserving subgroup.** Below we always write p as a prime number.

Lemma 3.1. *Let N be a closed manifold with* $\sec \geq 1$. *Suppose that N has two commuting isometric actions: a S^1-action and a free Γ-action such that*
(1) $\langle S^1, \Gamma \rangle$ has a $\mathbb{Z}_p$-subgroup whose action commutes with Γ-action;

(2) this $\mathbb{Z}_p$-action fixes a point $x_0 \in N$.
Let F_0 be the component of $N^{\mathbb{Z}_p}$ containing x_0, and let Λ be the subgroup of Γ that preserves F_0. Then $[\Gamma : \Lambda] \le b(n)$.

For application of Lemma 3.1 in the next subsection, N will be the universal cover or an intermediate cover of M.

The proof of Lemma 3.1 is rather short but it relies on two big theorems below.

Theorem 3.2 (Smith)**.** *Let M be a closed manifold with a $\mathbb{Z}_p$-action. Then*

$$\operatorname{rank} H_*(M^{\mathbb{Z}_p}, \mathbb{Z}_p) \le \operatorname{rank} H_*(M, \mathbb{Z}_p).$$

Theorem 3.3 (Gromov)**.** *Let M^n be a closed manifold with* $\sec \ge 0$*. Then for any field F,*

$$\operatorname{rank} H_*(M, F) \le b(n).$$

Proof of Lemma 3.1. Note that for any $\beta_1, \beta_2 \in \Gamma$,

$$\beta_1\Lambda = \beta_2\Lambda \iff \beta_2^{-1}\beta_1 \in \Lambda \iff \beta_2^{-1}\beta_1 F_0 = F_0 \iff \beta_1 F_0 = \beta_2 F_0.$$

Because β commutes with $\mathbb{Z}_p$ for all $\beta \in \Gamma$, βF_0 is a component of $N^{\mathbb{Z}_p}$. Hence we can define an injective map

$$\Gamma/\Lambda \to \text{ components of } N^{\mathbb{Z}_p}, \quad \beta\Lambda \mapsto \beta F_0.$$

Together with Theorems 3.2 and 3.3, we conclude

$$[\Gamma : \Lambda] \le \#\text{ components of } N^{\mathbb{Z}_p} \le \operatorname{rank} H_*(N^{\mathbb{Z}_p}, \mathbb{Z}_p) \le \operatorname{rank} H_*(N, \mathbb{Z}_p) \le b(n).$$

□

3.2. **Induction.**

Theorem 3.4 (Su-Wang)**.** *Let M be a closed manifold with a finite $\pi_1(M)$. If $\widetilde{M}$ has an isometric S^1-action, then $\pi_1(M)$ has a subgroup of index at most $C(n)$ whose action commutes with S^1-action.*

Theorem 3.5 (Kapovitch-Petrunin-Tuschmann)**.** *There are positive constants $\epsilon(n)$ and $C(n)$ such that for any closed n-manifold with*

$$\sec \ge -\epsilon(n), \quad \operatorname{diam}(M) = 1,$$

its fundamental group $\pi_1(M)$ must contain a nilpotent subgroup of index at most $C(n)$.

With the above two theorems, to prove Theorem A, we can always assume that $\pi_1(M)$ is nilpotent and commutes with S^1-action after passing to an intermediate cover of bounded index.

Proof of Theorem A. We prove Theorem A by induction. Suppose that the statement holds in dimension $\le n-2$, where n is odd. We shall prove the theorem for dimension n. The base step $n = 3$ will be discussed at the end.

Case 1. The S^1-action on $\widetilde{M}$ has a nontrivial finite isotropy subgroup at some point $x_0 \in \widetilde{M}$.

We pick a $\mathbb{Z}_p$-subgroup in the finite isotropy subgroup, where p is a prime. This $\mathbb{Z}_p$-subgroup satisfies the assumptions of Lemma 3.1 with $(N, \Gamma) = (\widetilde{M}, \pi_1(M))$. We denote F_0 the component of $\widetilde{M}^{\mathbb{Z}_p}$ containing x_0 and $\Lambda \le \pi_1(M)$ the subgroup preserving F_0. Then by Lemma 3.1, $[\pi_1(M) : \Lambda] \le b(n)$.

F_0 is a connected and totally geodesic submanifold of even codimension and S^1 symmetry. By inductive assumption, $\pi_1(F_0/\Lambda)$ has a cyclic subgroup $\mathbb{Z}_h$ of index at most $C(n-2)$. The covering map $F_0 \to F_0/\Lambda$ provides a short exact sequence

$$0 \to \pi_1(F_0) \to \pi_1(F_0/\Lambda) \xrightarrow{\psi} \Lambda \to 0.$$

Then

$$[\Lambda : \psi(\mathbb{Z}_h)] \le [\pi_1(F_0/\Lambda) : \mathbb{Z}_h] \le C(n-2).$$

Hence the cyclic subgroup $\psi(\mathbb{Z}_h)$ satisfies

$$[\pi_1(M) : \psi(\mathbb{Z}_h)] \le [\pi_1(M) : \Lambda] \cdot [\Lambda : \psi(\mathbb{Z}_h)] \le b(n)C(n-2).$$

This completes the proof of Case 1.

Case 2. Any isotropy subgroup from the S^1-action on $\widetilde{M}$ is trivial or S^1.

Let $H = \pi_1(M) \cap S^1 = \langle \alpha \rangle$. We consider the intermediate cover

$$(\hat{M}, \hat{\Gamma}, \hat{S}^1) = (\widetilde{M}/H, \pi_1(M)/H, S^1/H).$$

We remark that if H is trivial, then there is no need for this step and the proof below directly goes through on $\widetilde{M}$. On $\hat{M}$, $\hat{S}^1$-action and $\hat{\Gamma}$ action commutes. Also, $\hat{S}^1 \cap \hat{\Gamma}$ is trivial. By the nilpotency of $\hat{\Gamma}$, we can choose an element $\hat{\beta} \in Z(\hat{\Gamma})$ of prime order p. Applying Theorem 2.1, we see that $\hat{\beta}$ preserves a circle orbit $\hat{S}^1 \cdot \hat{x}_0$ in $\hat{M}$. Let $\hat{t}_0 \in \hat{S}^1$ such that $\hat{t}_0\hat{\beta}\hat{x}_0 = \hat{x}_0$. Because $\hat{S}^1 \cap \hat{\Gamma} = \{e\}$, the element $\hat{t}_0\hat{\beta}$ is non-identity.

It is not difficult to see that $\hat{t}_0\hat{\beta}$ also has order p. By construction, this $\mathbb{Z}_p$-subgroup $\langle \hat{t}_0\hat{\beta} \rangle$ satisfies the assumptions of Lemma 3.1 with $(N, \Gamma) = (\hat{M}, \hat{\Gamma})$. Under the similar notations

$$\hat{M}^{\mathbb{Z}_p} = \cup \hat{F}_j, \quad \hat{x}_0 \in \hat{F}_0, \quad \hat{\Lambda} = \{\hat{\gamma} \in \hat{\Gamma} | \hat{\gamma}\hat{F}_0 = \hat{F}_0\}.$$

It follows from Lemma 3.1 that $[\hat{\Gamma} : \hat{\Lambda}] \le b(n)$.

Then following the same proof in Case 1, we can obtain a dimension reduction on $\hat{F}_0/\hat{\Lambda}$ and find a cyclic subgroup $\langle \hat{\gamma} \rangle$ in $\hat{\Gamma}$ of index at most $C(n)$. Let $\gamma \in \pi_1(M)$ be a lift of this $\hat{\gamma} \in \hat{\Gamma} = \pi_1(M)/\langle \alpha \rangle$. By Theorem 2.1, γ preserves some circle orbit $S^1 x_0$ on $\widetilde{M}$. Note that this $S^1 x_0$ is a free circle orbit due to the assumption of Case 2. We choose the unique $\theta \in S^1$ such that $\gamma x_0 = \theta x_0$ and define a group homomorphism by

$$\psi : \langle \alpha, \gamma \rangle \to S^1 \text{ such that } \psi(\alpha) = \alpha, \psi(\gamma) = \theta.$$

If a word w of $\langle \alpha, \gamma \rangle$ is in the kernel of ψ, then $w \cdot x_0 = x_0$ and thus $w = e$. Hence ψ is injective and $\langle \alpha, \gamma \rangle$ must be cyclic. Now we complete the proof of Case 2 by

$$[\pi_1(M) : \langle \alpha, \gamma \rangle] \le [\pi_1(M)/H : \langle \alpha, \gamma \rangle / H] = [\hat{\Gamma} : \hat{\gamma}] \le C(n).$$

For the base step $n = 3$, in either case above, (F_0, Λ) or $(\hat{F}_0, \hat{\Lambda})$ is (S^1, Λ). Hence Λ is cyclic. □