Nonnegative Ricci Curvature, Nilpotency, and Asymptotic Geometry

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SLMath, Recent Progress on
Geometric Analysis and Riemannian Geometry

Sectional curvature:
Let $p \in M$ and Σ be a plane in T_pM through 0. Roughly speaking, $\sec_p(\Sigma) \geq 0$ means geodesics with initial data in Σ cannot (locally) spread wider than the ones in $\mathbb{R}^2$.
$\sec \geq 0$ means $\sec_p(\Sigma) \geq 0$ for all p and all Σ.

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Ricci curvature:
Let $p \in M^n$ and v be a unit vector in T_pM. Extend v to an orthonormal basis $\{v, E_2, ..., E_n\}$ at p. $\mathrm{Ric}_p(v, v) = \sum_i \sec(v \wedge E_i)$.
$\mathrm{Ric} \geq 0$ means $\mathrm{Ric}_p(v, v) \geq 0$ for all p and all v.

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$\mathrm{Ric} \geq 0$ means $\mathrm{Ric}_p(v, v) \geq 0$ for all p and all v.

$\mathrm{Ric} \geq 0$ controls the volume growth:

Theorem (Bishop 1964, Gromov 1980)

$\mathrm{vol}(B_r(p)) \leq \omega_n r^n, \quad \dfrac{\mathrm{vol}(B_R(p))}{\mathrm{vol}(B_r(p))} \leq \dfrac{R^n}{r^n}$, where $0 < r < R$.

Nonnegative curvature and virtual abelianness

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Theorem (Cheeger–Gromoll 1972)

Let M be an open (complete and non-compact) manifold of $\sec \geq 0$. Then M contains a closed totally convex submanifold S (soul) s.t. M is diffeo to the normal bundle over S.
$\Rightarrow \pi_1(M) = \pi_1(S)$ is virtually abelian.

Abelian/nilpotent groups

How about open manifolds of $\mathrm{Ric} \geq 0$?
(Wei 1988) Examples of open M with $\mathrm{Ric} > 0$ and torsion-free nilpotent π_1. (In particular, not virtually abelian.)
(Wilking 2000) Every finitely generated virtually nilpotent group can be realized as π_1 of some open M with $\mathrm{Ric} > 0$.

Abelian: $[G, G] = \{e\}$.
Nilpotent: $[G, G] = G^1, [G, G^i] = G^{i+1}$; eventually, $G^k = \{e\}$.

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Example: Discrete Heisenberg 3-group $H^3(\mathbb{Z})$

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}.$$

$$N^1 = [N, N] = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c \in \mathbb{Z} \right\}.$$

$$N^2 = [N, N^1] = \{e\}.$$

Curvature and virtual abelianness/nilpotency

Let M be an open n-manifold of $\mathrm{Ric} \geq 0$.

Theorem

Any finitely generated subgroup of $\pi_1(M)$
(1) has polynomial growth with degree $\leq n$ (Milnor 1968),
(2) is virtually nilpotent (Gromov 1981).

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Counterexample to Milnor Conjecture (Brue–Naber–Semola 2023)

M^n ($n \geq 6$) with $\mathrm{Ric} \geq 0$ and $\pi_1(M) = \mathbb{Q}/\mathbb{Z}$.

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M^n ($n \geq 6$) with $\mathrm{Ric} \geq 0$ and $\pi_1(M) = \mathbb{Q}/\mathbb{Z}$.

Question: Relationships between the virtual abelianness/nilpotency of $\pi_1(M)$ and the geometry of M?

Escape phenomenon

sec $\geq 0 \Rightarrow$ (by soul structure and metric retractions) All minimal representing loops in $\pi_1(M, p)$ are contained in a bounded set.

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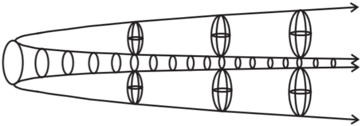
$\mathrm{Ric} > 0$, $\pi_1(M)$ is infinite $\Rightarrow$ (also by splitting theorem) These loops cannot be contained in any bounded set.

Example (Nabonnard 1980): a doubly warped product $\mathrm{Ric} > 0$
$M = [0, \infty) \times_f S^2 \times_h S^1, \quad g = dr^2 + f(r)^2 ds_2^2 + h(r)^2 ds_1^2.$
$h(r)$ is decreasing $\Rightarrow$ minimal representing loops of elements in $\pi_1(M, p)$ cannot be contained in any bounded sets.



Wei's examples (1988)

Let $\widetilde{N}$ be a simply connected (non-abelian) nilpotent Lie group and let Γ be a lattice in $\widetilde{N}$. The nilmanifold $N = \widetilde{N}/\Gamma$ admits a family of metrics $\{g_r\}_{r\geq 0}$ with

$$|\sec(g_r)| \to 0, \quad \mathrm{diam}(N, g_r) \to 0 \text{ as } r \to \infty.$$

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For suitable $f(r)$ and g_r,

$$M = [0, \infty) \times_f S^k \times N_r, \quad dr^2 + f^2(r) ds_k^2 + g_r$$

has $\mathrm{Ric} > 0$ when k is large.
$\pi_1(M) = \pi_1(N) = \Gamma$ is not virtually abelian.

The escape phenomenon is also clear since $\mathrm{diam}(N, g_r) \to 0$.

Escape rate

Quantify this escape phenomenon.

Definition (Pan 2020)

Let (M, p) be an open manifold with an infinite fundamental group. We define the escape rate of (M, p), a scaling invariant, as

$$E(M, p) = \limsup_{|\gamma| \to \infty} \frac{\mathrm{size}(c_\gamma)}{\mathrm{length}(c_\gamma)},$$

where c_γ is a minimal representing geodesic loop of $\gamma \in \pi_1(M, p)$.

Convention: if $\pi_1(M)$ is finite, then we set $E(M, p) = 0$.

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Convention: if $\pi_1(M)$ is finite, then we set $E(M, p) = 0$.

- $E(M, p) \leq 1/2$ always holds.
- (Sormani 2000) $E(M, p) < 1/2 \Rightarrow \pi_1(M)$ is finitely generated.
- The example by Brue–Naber–Semola has an infinitely generated $\pi_1(M) = \mathbb{Q}/\mathbb{Z}$, thus $E(M, p) = 1/2$.
- Wei's construction has $E(M, p) \geq c > 0$.

Small escape rate and virtual abelianness

Theorem (Pan 2020)

There is a constant $\epsilon(n) > 0$ s.t. if an open manifold M^n satisfies $\mathrm{Ric} \geq 0$ and $E(M,p) \leq \epsilon(n)$, then $\pi_1(M)$ is virtually abelian.

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Equivariant asymptotic cone: $(r_i^{-1}\widetilde{M}, \tilde{p}, \Gamma) \xrightarrow{GH} (Y, y, G)$.

$\Omega(\widetilde{M}, \Gamma)$: the set of all equivariant asymptotic cones of $(\widetilde{M}, \Gamma)$.

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$\sec \geq 0 \Rightarrow$ unique $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$; Y is a metric cone with vertex y; the orbit Gy is a Euclidean factor of Y.

$\mathrm{Ric} \geq 0$: in general, Y may not be unique and may not be metric cones.

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$\mathrm{Ric} \geq 0$: in general, Y may not be unique and may not be metric cones.

(Sketch) $E(M, p) = 0$
$\Leftrightarrow$ (1) Gy is convex for all $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$;
$\Leftrightarrow$ (2) Gy is a Euclidean factor of Y for all $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$.
Also, (2) $\Rightarrow$ virtual abelianness.

Equivariant asymptotic geometry

Question: If M has $\mathrm{Ric} \geq 0$ and $E(M, p) > 0$ (or $\pi_1(M)$ being torsion-free nilpotent), how is the equivariant asymptotic geometry?

It should be different from the ones with $\sec \geq 0$.

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Theorem (Pan–Wei 2021)

Examples of Ricci limits (as asymptotic cones) where Hausdorff dimension of the singular set exceeds that of the regular set.

Actually, they are isometric to the α-Grushin halfplane $\mathbb{G}^+(\alpha)$, a classical example from sub/almost Riemannian geometry.
$g = dr^2 + r^{-2\alpha} dv^2$ on $[0, \infty) \times \mathbb{R}$.
$\mathcal{S} = \{0\} \times \mathbb{R}$ has Hausdorff dimension $1 + \alpha$.

Asymptotic geometry of Wei's example

Open manifold M with $\mathrm{Ric} > 0$ and $\pi_1(M) = H^3(\mathbb{Z})$ (Wei's construction). Then unique $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$ satisfies:

- G is the 3-dimensional simply connected Heisenberg group.
- The orbit Gy is not convex.
- Y is not a metric cone.
- Let $L(\simeq \mathbb{R})$ be the center of G. Then $\dim_{\mathcal{H}}(Ly) \geq 2$.

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Questions

Does any of these conditions relate to abelianness/nilpotency?
(1) G is (virtually) abelian for all $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$;
(2) The orbit Gy is convex for all $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$;
(3) (Y, y) is a metric cone with vertex y for all $(Y, y) \in \Omega(\widetilde{M})$.
(4) Hausdorff dimension of (closed) $\mathbb{R}$-orbits at y?

Remark: sec $\geq 0 \Rightarrow$ (1,2,3) all hold; also, $\dim_{\mathcal{H}}(Ly) = 1$ for all $(Y, y) \in \Omega(\widetilde{M}, \Gamma)$ and all closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$.

Answers to (1), (2), and (3)

Remark: sec $\geq 0 \Rightarrow$ (1,2,3) all hold; also, $\dim_{\mathcal{H}}(Ly) = 1$ for all $(Y, y) \in \Omega(\widetilde{M}, \Gamma)$ and all closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$.

We have seen that: $E(M, p) = 0 \Leftrightarrow (2) \Rightarrow$ virtual abelianness.

(1) $\not\Rightarrow$ virtual abelianness.
In fact, one can modify Wei's example s.t. G is isomorphic to $\mathbb{R}^3$ for all asymptotic limits but $\Gamma = H^3(\mathbb{Z})$.

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We say that M is *conic at infinity*, if any asymptotic cone (Y, y) of M is a metric cone (may not be unique) with vertex y.

Theorem (Pan 2022)

Let (M, p) be an open n-manifold with $\mathrm{Ric} \geq 0$ and $E(M, p) \neq \frac{1}{2}$.
(1) If $\widetilde{M}$ is conic at infinity, then $\pi_1(M)$ is virtually abelian.
(2) If $\widetilde{M}$ has Euclidean volume growth of constant at least L, then $\pi_1(M)$ is $C(n, L)$-abelian.

Nilpotency step

Question (Pan–Wei 2021)

Is the nilpotency step of $\pi_1(M)$ related to the Hausdorff dimension of some isometric $\mathbb{R}$-action in some asymptotic cone of $\widetilde{M}$?

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Let N be a finitely generated torsion-free nilpotent group.
$N = N^0 \triangleright N^1 \triangleright ... \triangleright N^k = \{e\}$, where $N^{j+1} = [N, N^j]$
Nilpotency step: $\mathrm{step}(N) =$ smallest integer k s.t. $N^k = \{e\}$.

Example: $N = H^3(\mathbb{Z})$ has $\mathrm{step}(N) = 2$.

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Example: $N = H^3(\mathbb{Z})$ has $\mathrm{step}(N) = 2$.

Let Γ be a finitely generated and virtual nilpotent group.
Then Γ has a torsion-free nilpotent subgroup N of finite index.
Define $\mathrm{vir.step}(\Gamma) := \mathrm{step}(N)$. (It is well-defined.)
In particular, this definition applies to any finitely generated $\pi_1(M)$ with $\mathrm{Ric} \geq 0$.

Carnot group

Let M be a closed manifold with an infinite and virtually nilpotent fundamental group Γ (or equivalently, of polynomial growth).

(Gromov 1981, Pansu 1983) $(\widetilde{M}, \tilde{p})$ has a unique asymptotic cone (G, e, d), which is a simply connected stratified nilpotent Lie group G with a complete left-invariant subFinsler metric (Carnot group).

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It holds that:
$\mathrm{step}(G) = \mathrm{vir.step}(\Gamma)$;
$\dim_{\mathcal{H}}(L, d) = \mathrm{vir.step}(\Gamma)$, where L is any $\mathbb{R}$-subgroup in the last nontrivial subgroup of the lower central series of G.

Nilpotency step and Hausdorff dimension

Let (M, p) be an open n-manifold with $\mathrm{Ric} \geq 0$, an infinite fundamental group Γ, and $E(M, p) \neq 1/2$.

Theorem (Pan 2023)

There exists $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$ and a closed $\mathbb{R}$-subgroup L of G s.t. $\dim_{\mathcal{H}}(Ly) \geq \mathrm{vir.step}(\Gamma)$.

Theorem (Pan 2023)

For any $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$, the orbit Gy has a natural structure as a connected and simply connected nilpotent Lie group with $\mathrm{step}(Gy) \leq \mathrm{vir.step}(\Gamma)$.

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Cf. When $\pi_1(M) = \mathbb{Q}/\mathbb{Z}$ (Brue-Naber-Semola), $E = 1/2$ and asymptotic orbit Gy could be compact or disconnected.

Discrete Heisenberg 3-group $\Gamma = H^3(\mathbb{Z})$:
Let γ in the center of Γ. γ^m has word length
$C_1 m^{1/2} \leq d_W(e, \gamma^m) \leq C_2 m^{1/2}$.

Nilpotency step and growth of length (upper bound)

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$$|\gamma^m| := d(\gamma^m \tilde{p}, \tilde{p}) \leq C m^{1/2}.$$

To prove virtual abelianness, it suffices to find a lower bound for $|\gamma^m|$ that contradicts $|\gamma^m| \leq C m^{1/2}$.

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To prove virtual abelianness, it suffices to find a lower bound for $|\gamma^m|$ that contradicts $|\gamma^m| \leq C m^{1/2}$.

In general, let N be a torsion-free nilpotent group with nilpotency step k. Take $\gamma \in N^{k-1} - \{e\}$. Then

$$|\gamma^m| \leq C m^{1/k}.$$

$\gamma \in N^{k-1} - \{e\}$, where N torsion-free nilpotent of step k.

Small escape rate
$\Rightarrow$ Limit orbit is always "close" to a standard Euclidean subspace
$\Rightarrow$ almost translation estimate at large: $\forall \epsilon > 0$,
$|\gamma^2| \geq (1.99 - \epsilon)|\gamma|$ holds for all γ with sufficiently large $|\gamma|$.

Asymptotic geometry and growth of length (lower bound)

$\gamma \in N^{k-1} - \{e\}$, where N torsion-free nilpotent of step k.

Small escape rate
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$\widetilde{M}$ conic at infinity & $E \neq 1/2$
$\Rightarrow$ almost linear growth estimate: $\forall \epsilon > 0$, $\exists$ a constant C' s.t.
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$\Rightarrow$ almost linear growth estimate: $\forall \epsilon > 0$, $\exists$ a constant C' s.t.
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Define $\mathcal{D} := \sup\{\dim_{\mathcal{H}}(Ly) | (Y, y, L) \in \Omega(\widetilde{M}, \langle\gamma\rangle)\}$.
(Rem. As a crucial intermediate step, Ly is homeomorphic to $\mathbb{R}$.)

$E \neq 1/2$
$\Rightarrow \forall \epsilon > 0$, $\exists$ a constant C' s.t. $|\gamma^m| \geq C' m^{(1/\mathcal{D})-\epsilon}$ for all m large.
$\Rightarrow \mathcal{D} \geq l$.

This result generalizes the previous results on virtual abelianness.

Relation to metric cones

Let $Y = C(Z)$ be a metric cone with vertex y. Then $\dim_{\mathcal{H}}(Ly) = 1$ for any closed $\mathbb{R}$-subgroups $L \leq \mathrm{Isom}(Y)$.

$\Rightarrow$ If $E(M,p) \neq 1/2$ and $\widetilde{M}$ is conic at infinity, then $\mathrm{vir.step}(\pi_1(M)) \leq \sup \dim_{\mathcal{H}}(Ly) = 1$.

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Relation to small escape rate

Let M be an open n-manifold with $\mathrm{Ric} \geq 0$ and $E(M,p) \leq \epsilon$. Then $\dim_{\mathcal{H}}(Ly) \leq 1 + \delta(\epsilon|n)$, where $\delta(\epsilon|n) \to 0$ as $\epsilon \to 0$.

$\Rightarrow$ $\pi_1(M)$ is virtually abelian when ϵ is small.

Open questions

Question (Finite generation v.s. $E < 1/2$)

If an open M has $\mathrm{Ric} \geq 0$ and $\pi_1(M)$ is finitely generated, is it true that $E(M,p) < 1/2$?

Question (Escape rate hierarchy)

Given n, are there constants $0 < \epsilon(n,1) < ...\epsilon(n,l(n)) < 1/2$ such that if an open M^n has $\mathrm{Ric} \geq 0$ and $E(M,p) \leq \epsilon(n,k)$, then $\mathrm{step}(\pi_1(M)) \leq k$?

Question (Minimal dimension with nilpotent π_1)

Let M be an open n-manifold with $\mathrm{Ric} \geq 0$. Does $n \leq 12$ imply that any finitely generated subgroup of $\pi_1(M)$ is virtually abelian?

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Theorem (Navarro–Pan–Zhu 2024)

Let M be an open manifold with $\mathrm{Ric} \geq 0$ and linear volume growth. Then $\pi_1(M)$ is virtually abelian. (In fact, $E(M, p) = 0$.)

Theorem (Navarro–Pan–Zhu 2024)

Let M be an open manifold with $\mathrm{Ric} > 0$ and linear volume growth. Then $\pi_1(M)$ is finite.