On the escape rate of geodesic loops in an open manifold of nonnegative Ricci curvature

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Splitting theorem

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Let M be an open manifold of $\mathrm{Ric} \geq 0$. If M contains a line, then M splits isometrically as $\mathbb{R} \times N$.

Corollary

Let M be a closed manifold of $\mathrm{Ric} \geq 0$. Then $\pi_1(M)$ contains an abelian subgroup of finite index ($\pi_1(M)$ is virtually abelian).

Soul theorem (Cheeger-Gromoll)

Let M be an open manifold of $\sec \geq 0$. Then M contains a closed totally convex submanifold S such that M is diffeomorphic to the normal bundle over S.

Open manifolds of nonnegative sectional curvature

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Corollaries

(1) $\pi_1(M)$ is virtually abelian.
(2) Let $p \in S$. For any element $\gamma \in \pi_1(M, p)$, it has a minimal representing geodesic loop c_γ contained in S.

Open manifolds of nonnegative Ricci curvature

Wei's example

There is an open manifold M of $\mathrm{Ric} > 0$ such that $\pi_1(M)$ is not virtually abelian ($\pi_1(M)$ as any torsion-free nilpotent group).

Theorem (Wilking)

Any finitely generated virtually nilpotent group can be realized as the fundamental group of some open manifold of $\mathrm{Ric} > 0$.

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Theorem (Wilking)

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Escape phenomenon (Sormani-Wei)

The representing geodesic loops of $\pi_1(M)$ may not be contained in any bounded sets in M. In fact, if M has $\mathrm{Ric} > 0$ and an infinite fundamental group, then this escape phenomenon always occurs.

Open manifolds of nonnegative Ricci curvature

Theorem (Milnor, Gromov)

Any finitely generated subgroup of $\pi_1(M, p)$ has at most polynomial growth. Consequently, it is virtually nilpotent.

It is conjectured by Milnor that $\pi_1(M, p)$ is finitely generated.

Open manifolds of nonnegative Ricci curvature

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Theorem (Kapovitch-Wilking)

Given n, there exists a constant $C(n)$ such that the following holds. Let M be an open n-manifold of $\mathrm{Ric} \geq 0$. Then $\pi_1(M)$ contains a nilpotent subgroup N of index at most $C(n)$. Moreover, any finitely generated subgroup of N has nilpotency length at most n.

Summary (nonnegative sectional/Ricci curvature)

Let M be an open manifold of		$\sec \geq 0$	$\mathrm{Ric} \geq 0$
Then $\pi_1(M)$ is	finitely generated	True	Unknown
	virtually abelian	True	False
	virtually nilpotent	True	True

Question

Let M be an open manifold of $\mathrm{Ric} \geq 0$. On what additional conditions is $\pi_1(M)$ virtually abelian?

The following analog for open manifolds also follows from the Cheeger-Gromoll splitting theorem.

Analog

Let M be an open manifold of $\mathrm{Ric} \geq 0$. If there is $p \in M$ and $R > 0$ such that all representing geodesic loops of elements in $\pi_1(M, p)$ are contained in $B_R(p)$, then $\pi_1(M)$ is virtually abelian.

Virtual abelianness

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Analog

Let M be an open manifold of $\mathrm{Ric} \geq 0$. If there is $p \in M$ and $R > 0$ such that all representing geodesic loops of elements in $\pi_1(M, p)$ are contained in $B_R(p)$, then $\pi_1(M)$ is virtually abelian.

However, the above result is quite restricted: it cannot be applied to manifolds of $\mathrm{Ric} > 0$ due to the escape phenomenon.

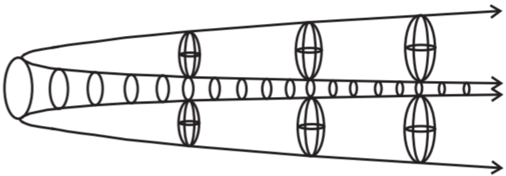
Question

Can we improve the above result?

Let (N, g_0) be a closed manifold of $\mathrm{Ric}(g_0) \geq 0$. Then there are functions $f(r)$ and $h(r)$ such that the double warped product

$$M = [0, \infty) \times_f S^{p-1} \times_h N, \quad dr^2 + f^2(r)ds^2 + h^2(r)g_0$$

has positive Ricci curvature when p is sufficiently large.
In this construction, $h(r)$ either $\to 0$ or $\to c > 0$ as $r \to \infty$.
Note that $\pi_1(M) = \pi_1(N)$ is virtually abelian.



Wei's examples

Let $\widetilde{N}$ be a simply connected nilpotent Lie group and let L be a lattice in $\widetilde{N}$. $N = \widetilde{N}/\Gamma$ admits a family of metrics g_r ($r \geq 0$) with

$$\mathrm{Ric}(g_r) \geq -\frac{c}{1+r^2}.$$

Then for a suitable function $f(r)$, the warped product

$$M = [0,\infty) \times_f S^{p-1} \times N_r, \quad dr^2 + f^2(r)ds^2 + g_r$$

has positive Ricci curvature when p is sufficiently large.
In this construction, $\mathrm{diam}(N_r)$ decays at a rate of $r^{-\alpha}$ ($\alpha > 0$).
(Different steps of N_r decays with different orders.)
Also note that $\pi_1(M) = \pi_1(N)$ is not virtually abelian when N has nilpotency length ≥ 2.

Examples with logarithm decay $h(r)$

For suitable warping functions $f(r)$ and $h(r)$, where $h(r)$ decays at a rate of $\ln^{-\alpha} r$ $(\alpha > 0)$, the double warped product

$$M = [0, \infty) \times_f S^{p-1} \times_h S^1, \quad dr^2 + f^2(r)ds^2 + h^2(r)g_0$$

has positive Ricci curvature when p is sufficiently large.

However, one cannot use this type of warping functions to construct $M = [0, \infty) \times_f S^{p-1} \times N_r$ of positive Ricci curvature as in Wei's examples. In fact, though N admits a family of metrics g_r with

$$\mathrm{Ric}(g_r) \geq -\frac{c}{\ln(1 + r^2)},$$

but this becomes the dominating term in $\mathrm{Ric}(M)$, which cannot be modified to be positive by raising other lower order terms.

Questions raised from these examples

Questions

Any deeper reason explaining why we cannot use a logarithm decay h to warp a nilpotent N?

For warped products, is the decay rate of warping functions related to the structure of fundamental groups?

Let (M, p) be an open manifold. For any element $\gamma \in \pi_1(M, p)$, we denote its representing geodesic loop as c_γ. For simplicity, we write $|\gamma| = \mathrm{length}(c_\gamma)$. We introduce a quantity to measure how fast c_γ escapes from any bounded balls as γ exhausts $\pi_1(M, p)$.

Escape rate

Let (M, p) be an open manifold. For any element $\gamma \in \pi_1(M, p)$, we denote its representing geodesic loop as c_γ. For simplicity, we write $|\gamma| = \mathrm{length}(c_\gamma)$. We introduce a quantity to measure how fast c_γ escapes from any bounded balls as γ exhausts $\pi_1(M, p)$.

Definition

Let (M, p) be an open manifold with an infinite fundamental group. We define the *escape rate* of (M, p), a scaling invariant, as

$$E(M, p) = \limsup_{|\gamma| \to \infty} \frac{d_H(p, c_\gamma)}{|\gamma|},$$

where $\gamma \in \pi_1(M, p)$ and d_H is the Hausdorff distance.

In short, size over length.

Some remarks on the escape rate

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- Suppose that $\mathrm{Ric} \geq 0$ and $E(M, p) = 0$ for some $p \in M$, then $E(M, q) = 0$ for all $q \in M$.

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- If all representing geodesic loops are contained in a bounded ball, or escape from any bounded ball at a sublinear rate with respect their lengths, then $E(M, p) = 0$.
- Suppose that $\mathrm{Ric} \geq 0$ and $E(M, p) = 0$ for some $p \in M$, then $E(M, q) = 0$ for all $q \in M$.
- For a double warped product $M = [0, \infty) \times_f S^{p-1} \times_h S^1$, as $h(r)$ decreases, a representing geodesic loop would take advantage the thin end to shorten its length, while this also increases its size. Hence the faster $h(r)$ decays, the larger escape rate (M, p) has.
  If $h(r) \to c > 0$ as $r \to \infty$, then $E(M, p) = 0$;
  If $h(r) \sim r^{-\alpha}$, then $E(M, p) > 0$;
  If $h(r) \sim \ln^{-\alpha}(r)$, then $E(M, p) = 0$.

Main theorems

Theorem (Pan, 2020, to appear in Geometry & Topology)

Let (M, p) be an open n-manifold of $\mathrm{Ric} \geq 0$. If $E(M, p) = 0$, then $\pi_1(M, p)$ is virtually abelian.

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Theorem (Pan, 2020)

Given n, there is a constant $\epsilon(n) > 0$ such that for any open n-manifold (M, p) of $\mathrm{Ric} \geq 0$, if $E(M, p) \leq \epsilon(n)$, then $\pi_1(M, p)$ is virtually abelian.

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The contra-positive gives a geometric characterization of any open manifold with $\mathrm{Ric} \geq 0$ and a non-virtually-abelian fundamental group: if $\pi_1(M,p)$ contains a torsion-free nilpotent subgroup of step ≥ 2, then $E(M,p) > \epsilon(n)$, that is, there is a sequence $\gamma_i \in \pi_1(M,p)$ such that $d_H(p, c_{\gamma_i})$ is at least $\epsilon(n) \cdot \mathrm{length}(c_{\gamma_i})$.

Asymptotic cones I

Let (M, p) be an open manifold of $\mathrm{Ric} \geq 0$. For any sequence $r_i \to \infty$, passing to a subsequence if necessary, we obtain the following pointed Gromov-Hausdorff convergence:

$$(r_i^{-1} M, p) \xrightarrow{GH} (Z, z).$$

We call (Z, z) an *asymptotic cone* of M.

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For the universal cover $(\widetilde{M}, \tilde{p})$ with $\Gamma = \pi_1(M, p)$-action, we can obtain the following equivariant pointed GH convergence:

$$\begin{array}{ccc}
(r_i^{-1}\widetilde{M}, \tilde{p}, \Gamma) & \xrightarrow{GH} & (Y, y, G) \\
\downarrow \pi & & \downarrow \pi \\
(r_i^{-1}M, p) & \xrightarrow{GH} & (Z, z).
\end{array}$$

We call (Y, y, G) an *equivariant asymptotic cone* of $(\widetilde{M}, \Gamma)$.

Asymptotic cones II

Theorem (Gromov)

Let Γ be a finitely generated group of polynomial growth, then Γ is virtually nilpotent.

A kep step in Gromov's proof: endowed with the word metric, for any sequence $r_i \to \infty$, $(r_i^{-1}\Gamma, e)$ converges in the point GH topology to a locally compact, homogeneous length metric space.

Later, Pansu showed that the asymptotic cone of Γ is unique and gave a detailed description of the asymptotic cone (Carnot group).

Asymptotic cones III

For a closed manifold M, we can endow a natural metric ρ on $\Gamma = \pi_1(M, p)$ by using the orbit $\Gamma\tilde{p}$ on the universal cover $\widetilde{M}$:

$$\rho(\gamma_1, \gamma_2) = d_{\widetilde{M}}(\gamma_1\tilde{p}, \gamma_2\tilde{p}).$$

ρ is bi-Lipschitz equivalent to any word metric on Γ.

When Γ has polynomial growth, both sequences $(r_i^{-1}\widetilde{M}, \tilde{p})$ and $(r_i^{-1}\Gamma, \rho)$ are pre-compact in the GH topology and converge to the same limit, which is a locally compact homogeneous length metric space. Note that the sequence $r_i^{-1}\widetilde{M}$ does not have a uniform curvature lower bound.

Asymptotically geodesic metric

(Γ, ρ) satisfies a condition called *asymptotically geodesic*.

Definition

Let Γ be a finitely generated group with a left invariant metric ρ. We say that (Γ, ρ) is *asymptotically geodesic*, if for any $\epsilon > 0$, there is $s = s(\epsilon) > 0$ such that for any $\gamma \in \Gamma$, we can find a word $\prod_{j=1}^{N} \gamma_j = \gamma$ satisfying

$$\sum_{j=1}^{N} \rho(e, \gamma_j) \leq (1+\epsilon)\rho(e, \gamma)$$

and $\rho(e, \gamma_j) \leq s$ for all j.

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and $\rho(e, \gamma_j) \leq s$ for all j.

This condition assures that the limit of $(r_i^{-1}\Gamma, \rho)$ is a length metric space, if the limit exists.

Weakly asymptotically geodesic metric

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Definition

Let Γ be a finitely generated group with a left invariant metric ρ. We say that (Γ, ρ) is *weakly asymptotically geodesic*, if there is a function $s(\epsilon, R)$ with the properties below:
(1) for every fixed ϵ, $R/s(\epsilon, R) \to \infty$ as $R \to \infty$;
(2) for any $\epsilon > 0$, $R > 0$ and any $\gamma \in \Gamma$ with $\rho(e, \gamma) = R$, we can find a word $\prod_{j=1}^{N} \gamma_j = \gamma$ such that

$$\sum_{j=1}^{N} \rho(e, \gamma_j) \leq (1+\epsilon)\rho(e, \gamma)$$

and $\rho(e, \gamma_j) \leq s(\epsilon, R)$ for all j.

Equivalent statements to $E(M,p)=0$

We write $\Omega(\widetilde{M},\Gamma)$ as the set of all equivariant asymptotic cones of $(\widetilde{M},\Gamma)$.

Theorem

Let (M,p) be an open manifold of $\mathrm{Ric}\geq 0$. Then the following statements are equivalent:
(1) $E(M,p)=0$;
(2) the orbit $\Gamma\tilde{p}$ is weakly asymptotically geodesic;
(3) for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$, the orbit Gy is geodesic in Y;
(4) for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$, the orbit Gy is geodesic in Y and is a metric product $\mathbb{R}^k\times Z$, where Z is compact;
(5) for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$, the orbit Gy is geodesic in Y and is isometric to a standard Euclidean space $\mathbb{R}^k$.

Sketch of (3)⇒(4)⇒(5)

Splitting theorem (Cheeger-Colding)

Let M be an open manifold with $\mathrm{Ric} \geq 0$ and let Y be an asymptotic cone of M. If Y contains a line, then Y splits off this line isometrically.

Geodesic subsets in a metric product

Let Y be a locally compact length metric space. Let N be a closed subset in the product metric space $\mathbb{R}^k \times Y$. Suppose that N is geodesic in $\mathbb{R}^k \times Y$ and N contains a slice $\mathbb{R}^k \times \{y\}$ for some $y \in Y$. Then N equals $\mathbb{R}^k \times Z$ as a subset of $\mathbb{R}^k \times Y$.

The above two imply that any limit orbit Gy is a metric product $\mathbb{R}^k \times Z$, where Z is compact.

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The above two imply that any limit orbit Gy is a metric product $\mathbb{R}^k \times Z$, where Z is compact.

(Technical part) Using a critical rescaling argument, we can further get rid of the compact factor Z.

Sketch of (5)⇒virtual abelianness

For any nilpotent subgroup N of Γ with a finite index, we consider the convergence

$$(r_i^{-1}\widetilde{M},\tilde{p},\Gamma,N)\xrightarrow{GH}(Y,y,G,H).$$

With (5), we can further show that H acts on $Hy=Gy=\mathbb{R}^k$ as translations.

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With (5), we can further show that H acts on $Hy = Gy = \mathbb{R}^k$ as translations.

Back to the geometry of $(\widetilde{M}, \tilde{p}, \Gamma)$, this implies that any element $\gamma \in N$ with large displacement at $\tilde{p}$ acts almost as a translation: there is $R > 0$ such that $d(\gamma^2\tilde{p}, \tilde{p}) \geq 1.9 \cdot d(\gamma\tilde{p}, \tilde{p})$ for all γ with $d(\gamma\tilde{p}, \tilde{p}) \geq R$.

On the other hand, a non-virtually-abelian nilpotent group does not admit such a left-invariant metric.

We now consider the case $E(M,p)\leq\epsilon$.

Recall that in the proof of case $E(M,p)=0$, we did:

$E(M,p)=0$

$\Rightarrow$ Gy is geodesic in Y for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$;

$\Rightarrow$ Gy is a metric product $\mathbb{R}^k\times Z$, where Z is compact, for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$;

$\Rightarrow$ Gy is a standard Euclidean space for any $(Y,y,G)\in\Omega(\widetilde{M},\Gamma)$.

Escape rate gap

We now consider the case $E(M,p) \leq \epsilon$.

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$\Rightarrow$ Gy is a standard Euclidean space for any $(Y,y,G) \in \Omega(\widetilde{M},\Gamma)$.

To prove the escape rate gap, we quantify this approach.

To quantify the first step above, we show that Gy is very close to being geodesic.

δ-geodesic subsets

Definition

Let X be a complete length space and let N be a closed subset of X. Let $\delta > 0$. We say that N is *δ-geodesic in X*, if for any two points $a, b \in N$, there is a chain of points $z_0 = a, ..., z_j, ..., z_k = b$ in N such that

$$\sum_{j=1}^{k} d(z_{j-1}, z_j) \leq (1+\delta) \cdot d(a,b),$$

$$d(z_{j-1}, z_j) \leq \delta \cdot d(a,b) \text{ for all } j = 1, ..., k.$$

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$$d(z_{j-1}, z_j) \leq \delta \cdot d(a,b) \text{ for all } j = 1, ..., k.$$

Proposition

Let (M, p) be an open manifold of $\mathrm{Ric} \geq 0$ and $E(M,p) \leq \epsilon$. For any $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$, the orbit Gy is δ_ϵ-geodesic in Y, where $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

How to quantify the second step?

Recall that when $E(M, p) = 0$, we show that Gy is a metric product $\mathbb{R}^k \times Z$, where Z is compact, then we rule out this compact factor. Here, we use pointed GH closeness to write the quantitative results; more precisely, one may expect that Gy is $\Phi(\epsilon|n)$-close to $\mathbb{R}^k \times Z$, where Z is compact, or $\mathbb{R}^k$ in the pointed GH sense, where $\Phi(\epsilon|n) \to 0$ as $\epsilon \to 0$.

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Execution: For each (Y, y, G), we consider an associated family of spaces $\{(sY, y, G)\}_{s>0}$. We apply Cheeger-Colding quantitative splitting theorem to (Y, y, Gy) only when all (sY, y, Gy) almost split.

Splitting of limit orbits under all scales

Let (M, p) be an open manifold of $\mathrm{Ric} \geq 0$ and $E(M, p) \leq \epsilon$.

Proposition

Let (Y, y, G) be an equivariant asymptotic cone of $(\widetilde{M}, \Gamma)$. Then there is an integer k such that for all $s > 0$,

$$d_{GH}((sY, y, Gy), (\mathbb{R}^k \times X_s, (0, x_s), \mathbb{R}^k \times Z_s)) \leq \Phi(\epsilon | n),$$

where (X_s, x_s) is a complete length space depending on sY, and Z_s is a closed geodesic subset of X_s. Moreover, one of the following holds:
(1) Z_s is a single point for all $s > 0$;
(2) $\mathrm{diam}(Z_s) \in [0.9, 1.1]$ for some $s > 0$.

Limit orbits as almost Euclidean spaces

Then we further rule out case (2) by a critical rescaling argument.

Theorem

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$$d_{GH}((Y, y, Gy), (\mathbb{R}^k \times X, (0, x), \mathbb{R}^k \times \{x\})) \leq \Phi(\epsilon | n),$$

where (X, x) is a complete length metric that depends on (Y, y).

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where (X, x) is a complete length metric that depends on (Y, y).

With this, it can be further shown that any $\gamma \in N$ with large displacement at $\tilde{p}$ has

$$d(\gamma^2 \tilde{p}, \tilde{p}) \geq 1.9 \cdot d(\gamma \tilde{p}, \tilde{p}),$$

where N is a nilpotent subgroup of Γ with finite index. Then virtual abelianness follows.

Open questions

(Pan) If $\mathrm{Ric} \geq 0$ and universal cover $\widetilde{M}$ satisfies certain geometric stability condition at infinity, (for example, $\widetilde{M}$ has a unique asymptotic cone as a metric cone,) then $E(M, p) = 0$.

Conjecture

Let (M, p) be an open manifold of $\mathrm{Ric} \geq 0$. If its Riemannian universal cover has Euclidean volume growth, then $E(M, p) = 0$.

If this conjecture is true, then the fundamental groups of the above class of manifolds are finitely generated and virtually abelian (moreover, the index of the abelian subgroup can be bounded by some constant $C(n, L)$, where L is the Euclidean volume growth rate of $\widetilde{M}$).

Open questions

We have seen that the escape rate can detect whether $\pi_1(M)$ is virtually abelian or not. We further ask whether escape rate can detect nilpotent groups of different steps.

Question

For each n, are there positive constants $\epsilon(n,k)$ strictly increasing in k such that the following holds?
For any open n-manifold (M,p) with $\mathrm{Ric} \geq 0$ and an infinite fundamental group, if $E(M,x) \leq \epsilon(n,k)$, then $\pi_1(M)$ contains a torsion-free nilpotent subgroup of nilpotency length at most k and finite index.