An Invitation to Gromov-Hausdorff convergence

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In a nutshell

The *Gromov-Hausdorff distance* between two metric spaces measures how they look alike.
A sequence of metric spaces X_i *Gromov-Hausdorff converges* to some limit metric space Y, written as $X_i \stackrel{GH}{\to} Y$, means almost impossible to distinguish Y and X_i for large i.

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Examples:

- Sphere (S^2, g_i) to a football sphere
- Curved surface to a tangent plane $\mathbb{R}^2$
- A closed manifold to a point
- Flat cylinder to a line
- Integers to a line

Principle

To understand some *nice* spaces, we have to understand some *singular* spaces first.

Analog: To understand the *smooth* solutions of some differential equation, we have to understand its *weak* solutions first.

A note on its history

According to Wikipedia:

The Gromov–Hausdorff distance was introduced by David Edwards in 1975, and it was later rediscovered and generalized by Mikhail Gromov in 1981.

Hausdorff distance

Definition (Hausdorff distance)

Let Z be a metric space and let X and Y be two non-empty compact subsets of Z. The *Hausdorff distance* between X and Y in Z is defined as

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Facts:

- $d_H^Z(X,Y) = 0$ iff $X = Y$.
- d_H^Z is a distance function on $\mathcal{C}_Z$, the set of non-empty compact subsets of Z.
- If Z is a complete metric space, then $(\mathcal{C}_Z, d_H^Z)$ is complete.

Gromov-Hausdorff distance/convergence

Definition (Gromov-Hausdorff distance)

Let X and Y be two compact metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf\{d_H^Z(f(X),g(Y)) | f : X \to Z, g : Y \to Z \text{ are isom. embeddings to some metric space } Z\}.$$

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Pointed Gromov-Hausdorff convergence for locally compact metric spaces: $(X_i, x_i) \xrightarrow{GH} (Y, y)$, if $B_r(x_i) \xrightarrow{GH} B_r(y)$ for all $r > 0$.

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How to use GH convergence

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If we do have certain precompactness, then (after passing to a subsequence) we can use GH convergence to

- study a family of metric spaces: take a sequence X_i, then $X_i \xrightarrow{GH} Y$.
- study the infinitesimal structure of a metric space at a point: $(r_i X, x) \xrightarrow{GH} (C_x X, v)$, where $r_i \to \infty$.
- study the large structure of a non-compact metric space at infinity: $(r_i^{-1} X, x) \xrightarrow{GH} (C_\infty X, v)$, where $r_i \to \infty$.

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We can also study additional structures (functions, measures, group actions,...) on metric spaces by passing them from the sequence to limit space.

Precompactness

Precompactness theorem (Gromov 1981)

Let X_i be a sequence of compact metric spaces. X_i has a convergent GH subsequence if it satisfies:

- (uniformly bounded diameter) $\mathrm{diam} X_i \leq D$ for all i;
- (uniformly bounded covering) for any $\epsilon > 0$, there is a number Q such that X_i can be covered by Q many ϵ-balls for all i.

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Geometric objects with the uniformly bounded covering condition:

- Finitely generated groups with controlled growth
- Riemannian n-manifolds with Ricci curvature bounded below

Let S be a finite generating set of a group Γ.
Word metric:
$d_S(e,\gamma) = \min\{k | \gamma \text{ can be written as a word in } S \text{ of length } k\}$.
$d_S(\gamma,\gamma') = d_S(e,\gamma^{-1}\gamma')$.

Growth of finitely generated groups

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Examples:

- $\mathbb{Z}^k$ has polynomial growth (of degree k).
- The free group generated by two elements does not have polynomial growth.
- If a subgroup H has finite index in Γ, then Γ have the same growth type as H.

Nilpotent groups

Abelian: $[\Gamma,\Gamma]=\{e\}$.
Nilpotent: $[\Gamma,\Gamma]=\Gamma^1, [\Gamma,\Gamma^i]=\Gamma^{i+1}$; eventually, $\Gamma^m=\{e\}$.
Example: (Discrete) Heisenberg 3-group

$$N=\left\{\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\middle| a,b,c\in\mathbb{Z}\right\}.$$

$$N^1=[N,N]=\left\{\begin{pmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\middle| c\in\mathbb{Z}\right\}.$$

$N^2=[N,N^1]=\{e\}$.
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N has polynomial growth of degree 4.

Theorem (Wolf 1968)

Any finitely generated (virtually) nilpotent group has polynomial growth.

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Gromov's approach: If Γ has polynomial growth, then the sequence $(r_i^{-1}\Gamma, e, d_S)$ is precompact in GH, where $r_i \to \infty$. Passing to a subsequence, $(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d)$. Then use the limit space (G, d) to prove the theorem.

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(Simple) Examples:

- $\Gamma = \mathbb{Z}$, $S = \{1\}$, then $G = \mathbb{R}$ with standard metric.
- $\Gamma = \mathbb{Z}^2$, $S = \{(1,0),(0,1)\}$, then $G = \mathbb{R}^2$ with box metric.

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(Pansu 1982) Detailed description of (G, d)...leads to the study of Carnot groups and subRiemannian/subFinsler geometry.

GH convergence in Riemannian geometry

Gromov's precompactness theorem (1981)

Let (M_i, p_i) be a sequence of complete Riemannian n-manifolds of $\mathrm{Ric} \geq -(n-1)$, then (M_i, p_i) has a GH convergent subsequence.

GH convergence provides a platform to study a class of Riemannian manifolds with uniform geometric conditions.

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Various curvature bounds:

$$\lambda \leq \sec \leq \Lambda, \quad \sec \geq \lambda, \quad \lambda \leq \mathrm{Ric} \leq \Lambda, \quad \mathrm{Ric} \geq \lambda.$$

Non-collapsing/collapsing:

$$\mathrm{vol}(B_1(p_i)) \geq v > 0, \quad \mathrm{vol}(B_1(p_i)) \to 0.$$

Questions to be addressed

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Some geometric properties can be directly passed to X.
(e.g. X is a length metric space.)

In general, when the sequence satisfies $\mathrm{Ric} \geq -(n-1)$, the limit space X may not be a topological manifold, may not be locally contractible, may not have integer Hausdorff dimension.

Stability/Finiteness

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Stability $\Rightarrow$ Finiteness: Suppose not...a sequence of mutually distinct M_i...subseqeunce GH converges...$\rightarrow\leftarrow$ stability.

- (Cheeger 1970) Collection of closed n-manifolds M with $|\sec| \leq 1, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many diffeo. types.
- (Perelman 1991) Collection of closed n-manifolds M with $\sec \geq -1, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many homeo. types.
- (Cheeger-Naber 2015) Collection of closed 4-manifolds M with $|\mathrm{Ric}| \leq 3, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v > 0$ has finitely many diffeo. types.

General limit spaces

Curvature	*Limit spaces*
$\lvert\sec\rvert \leq 1$	orbit space (Cheeger,Gromov,Fukaya)
$\sec \geq -1$	Alexandrov space (Burago,Gromov,Perelman)
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...also leads to the study of non-smooth metric spaces with synthetic curvature conditions.

Maximally collapsed manifolds

$(M, g_i) \xrightarrow{GH}$ point while keeping curvature bounded.
Example: any closed flat manifold.

- Almost flat: $|\mathrm{sec}| \leq 1$.
- Almost nonnegative sectional curvature: $\mathrm{sec} \geq -1$.
- Almost nonnegative Ricci curvature: $\mathrm{Ric} \geq -(n-1)$.

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Recall: Any closed flat manifold is finitely covered by a torus.

Theorem

If M^n is almost ..., then

- (Gromov 1978) M is finitely covered by a nilmanifold.
- (Kapovitch-Petrunin-Tuschmann 2010) M is finitely covered by a nilpotent space.
- (Kapovitch-Wilking 2012) $\pi_1(M)$ is $C(n)$-nilpotent.

Selected results derived from collapsing theory

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  Then $\pi_1(M)$ is generated by at most $C(n, D)$ many elements.
- (Pan 2017) Alternative proof of Milnor conjecture in dim 3: any open 3-manifold M with $\mathrm{Ric} \ge 0$ has a finitely generated $\pi_1(M)$. (First proved by Liu 2013, using minimal surface theory.)