NONNEGATIVE RICCI CURVATURE, NILPOTENCY, AND HAUSDORFF DIMENSION

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Abstract. Let M be an open (complete and non-compact) manifold with $\mathrm{Ric} \geq 0$ and escape rate not $1/2$. It is known that under these conditions, the fundamental group $\pi_1(M)$ has a finitely generated torsion-free nilpotent subgroup $\mathcal{N}$ of finite index, as long as $\pi_1(M)$ is an infinite group. We show that the nilpotency step of $\mathcal{N}$ must be reflected in the asymptotic geometry of the universal cover $\widetilde{M}$, in terms of the Hausdorff dimension of an isometric $\mathbb{R}$-orbit: there exist an asymptotic cone (Y, y) of $\widetilde{M}$ and a closed $\mathbb{R}$-subgroup L of the isometry group of Y such that its orbit Ly has Hausdorff dimension at least the nilpotency step of $\mathcal{N}$. This resolves a question raised by Wei and the author (see [27, Remark 1.7] and [24, Conjecture 0.2]).

Contents

1. Introduction

Collapsed Ricci limit spaces in general may admit isometric orbits whose Hausdorff dimension exceeds their topological dimension. The first examples with this feature are constructed by Wei and the author [27] as the asymptotic cone of the universal cover of an open (complete and non-compact) manifold M with $\mathrm{Ric} \geq 0$ and $\pi_1(M) = \mathbb{Z}$. More precisely, it is the equivariant Gromov-Hausdorff limit of

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \pi_1(M,p)) \xrightarrow{GH} (Y, y, L),$$

where $r_i \to \infty$ and $(\widetilde{M}, \tilde{p})$ is the universal cover of (M, p). In the limit space, L is a closed $\mathbb{R}$-subgroup of $\mathrm{Isom}(Y)$ and the orbit Ly has Hausdorff dimension $1 + \alpha$, where $\alpha \geq 0$ can be any large number by choosing a suitable metric and dimension of M. In the same paper, we asked whether the (non-abelian) nilpotency of $\pi_1(M)$ implies the existence of some asymptotic $\mathbb{R}$-orbit of large Hausdorff dimension [27, Remark 1.7]. This question was later formalized by the author in [24, Conjecture 0.2], relating the nilpotency step to Hausdorff dimension. More precisely, for any open manifold M with $\mathrm{Ric} \geq 0$ and escape rate not $1/2$, given that $\pi_1(M)$ contains a torsion-free nilpotent subgroup of nilpotency step l, is it true that $\dim_{\mathcal{H}}(Ly) \geq l$ for some asymptotic cone (Y, y) of $\widetilde{M}$ and some closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$?

Before proceeding further, we give more background on this problem.

Let M be an open manifold with $\mathrm{Ric} \geq 0$. By the work of Kapovitch-Wilking [17], $\pi_1(M)$ contains a nilpotent subgroup of index at most $C(n)$, a constant only depending on n. Also see [18, 12]. In the other direction, by the work of Wei [30] and Wilking [31], it is known that any finitely generated virtually nilpotent can be realized as the fundamental group of some open manifold with $\mathrm{Ric} \geq 0$. This is distinct from open manifolds with nonnegative sectional curvature, whose fundamental groups are always virtually abelian [6].

Therefore, it is natural to investigate on what additional conditions $\pi_1(M)$ is virtually abelian for nonnegative Ricci curvature; or equivalently, we can ask how virtual abelianness or nilpotency of $\pi_1(M)$ is related to the geometry of M. The author has studied this question in [22, 23, 24]. [22] introduced a geometric quantity, the escape rate $E(M, p)$, which measures how fast representing geodesic loops escape from bounded sets. The escape rate takes value within $[0, 1/2]$. It is known that $E(M, p) < 1/2$ implies the finite generation of $\pi_1(M)$ by Sormani's halfway lemma [29]. In [23], we proved that if the escape rate of M^n is smaller than some universal constant $\epsilon(n)$, then $\pi_1(M)$ is virtually abelian. In [24], we proved that if the escape rate of M is not $1/2$ and the universal cover is (metric) conic at infinity, then $\pi_1(M)$ is virtually abelian. The proofs in both [23] and [24] relate the equivariant asymptotic geometry to the structure of $\pi_1(M)$.

Given the above results, it is naturally to further study the equivariant asymptotic geometry without the smallness of escape rate and without the (metric) conic structure at infinity. Such understanding should have implications on virtual abelianness or nilpotency of $\pi_1(M)$. A close look at Wei's examples of open manifolds with $\mathrm{Ric} \geq 0$ and torsion-free nilpotent fundamental groups [30] indicates that the nilpotency step is reflected in a lower bound of the Hausdorff dimension of isometric $\mathbb{R}$-orbits. See Section 3 for these motivating examples.

This problem of nilpotency step and Hausdorff dimension is also related to the structure of Carnot groups. For a finitely generated virtually nilpotent group Γ, we define $\mathrm{step}(\Gamma)$ as the nilpotency step of a torsion-free nilpotent subgroup with finite index (see Definition 2.8). Any finite generating set S of Γ defines a word length metric d_S on Γ. The asymptotic structure of (Γ, d_S) was studied by Gromov [12] and Pansu [28]. For any sequence $r_i \to \infty$, Gromov-Hausdorff convergence holds:

$$(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d).$$

The unique limit space (G, d) is a Carnot group, that is, a simply connected stratified nilpotent Lie group G with nilpotency step l and a distance d induced by a left-invariant subFinsler metric, where $l = \mathrm{step}(\Gamma)$. Moreover, $\dim_{\mathcal{H}}(Ly) = l$ for any one-parameter subgroup L in $\zeta_{l-1}(G)$, the last nontrivial subgroup in the lower central series (see Definition 2.6). This structure also applies to closed manifolds. For a closed Riemannian manifold (M, g) with a virtually nilpotent fundamental group Γ, although g cannot have nonnegative Ricci curvature when $\mathrm{step}(\Gamma) \geq 2$, the blow-down sequence of the universal cover $(r_i^{-1}\widetilde{M}, \tilde{p}, \tilde{g})$ actually converges in the Gromov-Hausdorff topology to a limit space (G, e, d) as described above. Therefore, we can view the proposed problem as an extension of part of the Carnot group structure from closed manifolds to open ones.

The main result of this paper confirms this conjecture about nilpotency step and Hausdorff dimension [24, Conjecture 0.2], thus resolves the question raised in [27, Remark 1.7].

Theorem A. *Let (M, p) be an open manifold with $\mathrm{Ric} \geq 0$, an infinite $\pi_1(M)$, and $E(M, p) \neq 1/2$. Then there exists an asymptotic cone (Y, y) of $\widetilde{M}$, the universal cover of M, and a closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$ such that $\dim_{\mathcal{H}}(Ly) \geq \mathrm{step}(\pi_1(M))$.*

The condition $E(M, p) \neq 1/2$ assures that $\pi_1(M)$ is finitely generated. Thus $\mathrm{step}(\pi_1(M))$ is defined. $\mathrm{step}(\pi_1(M))$ is at least 1 when $\pi_1(M)$ is infinite.

The inequality in Theorem A is sharp. In fact, we have examples of open manifolds such that the equality holds on every asymptotic cone of $\widetilde{M}$ (see Section 3.2).

The $\mathbb{R}$-subgroup L in Theorem A indeed comes from $\langle\gamma\rangle$-action on $\widetilde{M}$, where γ belongs to $\zeta_{l-1}(N)$. In terms of the asymptotic limit of $\mathcal{N}$-action, we can show that its asymptotic orbit Gy always has a connected and simply connected nilpotent Lie group structure (Proposition 5.1). We emphasize that this nilpotent structure on Gy could be abelian when $\mathrm{step}(\mathcal{N}) > 1$ (see Remark 3.1 or [24, Appendix A]). In other words, the (non-abelian) nilpotency structure of $\pi_1(M)$ may not be preserved in the algebraic aspect of asymptotic limits. However, according to Theorem A, it must be reflected in the metric aspects of asymptotic limits.

It is unclear to the author whether the condition $E(M, p) \neq 1/2$ can be replaced by the finite generation of $\pi_1(M)$, mainly due to the lack of examples with $\mathrm{Ric} \geq 0$, finitely generated $\pi_1(M)$, and $E(M, p) = 1/2$. If such examples do exist, then we don't expect their equivariant asymptotic geometry to have good structures. Also see the recent examples by Bruè-Naber-Semola [3] with infinitely generated $\pi_1(M)$ (thus $E(M, p) = 1/2$); in particular, one may refer to [3, Section 2.2.4], where the equivariant asymptotic geometry of their examples are described.

As applications of Theorem A, we can use the asymptotic geometry of $\widetilde{M}$ to control the nilpotency step of $\pi_1(M)$. For convenience, we write $\Omega(\widetilde{M})$ as the set of all asymptotic cones of $\widetilde{M}$ and define

$$\mathcal{D}_\infty(\widetilde{M}) = \sup\{\dim_{\mathcal{H}}(Ly) | (Y,y) \in \Omega(\widetilde{M}),\ L \text{ is a closed } \mathbb{R}\text{-subgroup of } \mathrm{Isom}(Y)\}.$$

Using this quantity, we can reformulate Theorem A as follows.

Corollary B. *Let (M,p) be an open n-manifold with $\mathrm{Ric} \geq 0$ and $E(M,p) \neq 1/2$. Then*

$$\mathrm{step}(\pi_1(M)) \leq \mathcal{D}_\infty(\widetilde{M}).$$

In particular, if $\mathcal{D}_\infty(\widetilde{M}) < 2$, then $\pi_1(M)$ is virtually abelian.

Corollary B generalizes the main result in [24], where the universal cover $\widetilde{M}$ is assumed to be (metric) conic at infinity. When an asymptotic cone (Y,y) is a metric cone with vertex y, the orbit Ly always has Hausdorff dimension 1 for any closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$ (see proof of Corollary 6.8 for details); in other words, $\mathcal{D}_\infty(\widetilde{M}) = 1$ if $\widetilde{M}$ is conic at infinity. Then it follows from Corollary B that $\pi_1(M)$ is virtually abelian.

Besides metric cones, Corollary B applies to other asymptotic cones that are not covered by previous results. For instance, following the methods in [27] and [24, Appendix A], we can construct the Grushin halfspace below as the asymptotic cone of the universal cover of some open manifold with $\mathrm{Ric} \geq 0$ (also see [8, Remark 3.9], where the metric is clarified as a Grushin-type almost Riemannian metric). Given $0 \leq \alpha_1 \leq ... \leq \alpha_k$, we define an incomplete Riemannian metric g on $\mathbb{R}^k \times (0,\infty)$ by

$$g = dr^2 + \sum_{j=1}^{k} r^{-2\alpha_j} dx_j^2.$$

By taking its metric completion, g defines a distance d on $Y = \mathbb{R}^k \times [0,\infty)$. We denote this Grushin halfspace $(Y, 0, d)$ by $\mathbb{G}^+(\alpha_1, ..., \alpha_k)$. Each x_j-curve through $0 \in Y$ is the orbit of some isometric $\mathbb{R}$-action with Hausdorff dimension $1 + \alpha_j$. Suppose that every asymptotic cone of $\widetilde{M}$ is isometric to a Grushin halfplane $\mathbb{G}^+(\alpha_1, .., \alpha_k)$ for some $0 \leq \alpha_1 \leq ... \leq \alpha_k$, then by Corollary B,

$$\mathrm{step}(\pi_1(M)) \leq 1 + \alpha_k$$

for some $Y \in \Omega(\widetilde{M})$; in particular, if $\alpha_k < 1$ holds for all $Y \in \Omega(\widetilde{M})$, then $\pi_1(M)$ is virtually abelian.

Corollary B also extends the main result in [23] about small escape rate and virtual abelianness. In fact, when $E(M,p) \leq \epsilon$, we can show that the orbit of all asymptotic $\mathbb{R}$-orbits are close to 1. See Proposition 6.9 for the precise statement.

Outline of the proof. To illustrate our approach to Theorem A, we break its proof into two parts, as Proposition C(1) and C(2) below. We write $\Omega(\widetilde{M}, \langle\gamma\rangle)$ as the set of equivariant asymptotic cones of $(\widetilde{M}, \langle\gamma\rangle)$, where $\gamma \in \pi_1(M,p)$ and $\langle\gamma\rangle$ is the subgroup generated by γ.

Proposition C. *Let (M,p) be an open n-manifold with $\mathrm{Ric}\geq 0$ and $E(M,p)\neq 1/2$. Let $\mathcal{N}$ be a torsion-free nilpotent subgroup of $\pi_1(M,p)$ with finite index and let l be the nilpotency step of $\mathcal{N}$. Then the followings holds for any $\gamma\in\zeta_{l-1}(\mathcal{N})-\{\mathrm{id}\}$.*
(1) For every $(Y,y,H)\in\Omega(\widetilde{M},\langle\gamma\rangle)$, the orbit Hy is homeomorphic to $\mathbb{R}$.
(2) There exists $(Y,y,H)\in\Omega(\widetilde{M},\langle\gamma\rangle)$ such that $\dim_{\mathcal{H}}(Hy)\geq l$.

Below are the major steps in the proof of Proposition C(1):
$E(M,p)\neq 1/2$;
$\Rightarrow$ Equivariant GH distance gaps between different types of possible equivariant asymptotic cones of $(\widetilde{M},\mathcal{N})$ (Propositions 4.8 and 4.9);
$\Rightarrow$ There is an integer k_0 such that for every $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$, the orbit Gy has a natural simply connected nilpotent Lie group structure of dimension k_0 (Proposition 5.1);
$\Rightarrow$ Proposition C(1).

Though some intermediate results (for example, Propositions 5.1 and C(1)) have their counterparts in [24], where $\widetilde{M}$ is assumed to be conic at infinity, many new techniques are developed in this paper to study the equivariant asymptotic geometry without the metric cone structures. Compared to the conic at infinity case, the new difficulties are mainly due to the lack of structure results on singular sets with large Hausdorff dimension. When an asymptotic cone (Y,y) is a metric cone with vertex y, it follows from Cheeger-Colding splitting theorem [4] and the cone structure that the orbit Gy of an isometric G-action must stay in an Euclidean factor of Y. This fact about metric cones is used extensively in [24]. In contrast, in the proof of Theorem A here, we are essentially considering orbits of large Hausdorff dimension in collapsed Ricci limits. Such examples are first constructed in [27] and no general structure results are known so far.

Among the new techniques, we highlight the proof of distance gaps (Section 4). We develop convergence of tunnels (continuous curves in the orbit) and apply the large fiber lemma (Lemma 4.10) from topological dimension theory to derive equivariant GH distance gaps between equivariant asymptotic cones of different types. We shall describe more about this novel method in Section 4.2.

Proposition 5.1 depends on the above mentioned distance gaps and a critical rescaling argument. This kind of arguments was first developed in [20] and applied in different contexts to study the equivariant asymptotic geometry [21, 22, 23, 24].

The proof of Proposition C(2) relies on many structure results of spaces in $\Omega(\widetilde{M},\langle\gamma\rangle)$, including Proposition C(1). Below we give some indications how Hausdorff dimension of Hy and nilpotency step are related in the proof of Proposition C(2).

For $\mathcal{N}$ and γ in Proposition C, it is known that any word length metric d_S on $\mathcal{N}$ satisfies

$$C_1\cdot b^{1/l}\leq d_S(\gamma^b\tilde{p},\tilde{p})\leq C_2\cdot b^{1/l}$$

In fact, this two-side inequality implies $\dim_{\mathcal{H}}(Ly)=l$ in a Carnot group G, where L is the asymptotic limit of $\langle\gamma\rangle$. For an isometric-$\mathcal{N}$ orbit on $\widetilde{M}$, the nilpotency step only offers an upper bound:

$$d(\gamma^b \tilde{p}, \tilde{p}) \leq C \cdot b^{1/l}$$

for all $b \in \mathbb{Z}_+$ (see Corollary 2.10).

We approach Proposition C(2) by contradiction. We define

$$\mathcal{D} := \sup\{\dim_{\mathcal{H}}(Hy) | (Y, y, H) \in \Omega(\widetilde{M}, \langle \gamma \rangle)\}$$

and seek a contradiction if $\mathcal{D} < l$. We remark that $\mathcal{D}$ as a supremum indeed can be obtained (Proposition 6.4). The idea is to relate the Hausdorff dimension to a lower bound for orbit length: for each $s > \mathcal{D}$, there is a constant C' such that

$$d(\gamma^b \tilde{p}, \tilde{p}) \geq C' \cdot b^{1/s}$$

for all $b \in \mathbb{Z}_+$ large (Proposition 6.7). Then a contradiction to the orbit length upper bound would arise if $\mathcal{D} < l$.

To close the introduction, we point out that the proof of Theorem A naturally extends to nilpotent isometric actions on open manifolds with $\mathrm{Ric} \geq 0$ as well.

Theorem D. *Let M be an open n-manifold with* $\mathrm{Ric} \geq 0$ *and let $\mathcal{N}$ be a simply connected nilpotent Lie group with nilpotency step l. Suppose that $\mathcal{N}$ acts effectively and isometrically on M and (M, p, G) has escape rate less than $1/2$ (see Definition 2.17). Then there is an equivariant asymptotic cone (Y, y, G) of $(M, \mathcal{N})$ and a closed $\mathbb{R}$-subgroup L of G such that* $\dim_{\mathcal{H}}(Ly) \geq l$.

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2. Preliminaries

2.1. **Equivariant Gromov-Hausdorff convergence.** Throughout the paper, we use a tuple (Y, y, G) to denote a pointed complete length metric space (Y, y) with a closed subgroup G of the isometry group $\mathrm{Isom}(Y)$. We recall the basics of (pointed) equivariant Gromov-Hausdorff convergence from [9, 11]. For $R > 0$, we put

$$G(R) = \{g \in G \mid d(gy, y) \leq R\}.$$

Definition 2.1. [9, 11] Let (Y, y, G) and (Z, z, H) be two spaces. We say that

$$d_{GH}((Y, y, G), (Z, z, H)) \leq \epsilon,$$

where $\epsilon > 0$, if there are ϵ-approximation maps (f, ψ, ϕ), that is,

$$f : B_{1/\epsilon}(y) \to Z, \quad \psi : G(1/\epsilon) \to H(1/\epsilon), \quad \phi : H(1/\epsilon) \to G(1/\epsilon)$$

with the following properties:

(1) $f(y) = z$;

(2) the ϵ-neighborhood $f(B_{1/\epsilon}(y))$ contains $B_{1/\epsilon}(z)$;
(3) $|d(f(x), f(x')) - d(x, x')| \leq \epsilon$ for all $x, x' \in B_{1/\epsilon}(y)$;
(4) if $g \in G(1/\epsilon)$ and $x, gx \in B_{1/\epsilon}(y)$, then $d(f(gx), \psi(g)f(x)) \leq \epsilon$;
(5) if $h \in H(1/\epsilon)$ and $x, \phi(h)x \in B_{1/\epsilon}(y)$, then $d(f(\phi(h)x), hf(x)) \leq \epsilon$.

Theorem 2.2. [9, 11] *Let*

$$(Y_i, y_i) \xrightarrow{GH} (Z, z)$$

be a Gromov-Hausdorff convergence sequence and let G_i be closed subgroups of $\mathrm{Isom}(Y_i)$*. Then the followings hold.*
(1) After passing to a subsequence, we have an equivariant Gromov-Hausdorff convergence sequence

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H),$$

where H is closed subgroup of $\mathrm{Isom}(Z)$*.*
(2) The sequence of quotient spaces converges

$$(Y_i/G_i, \bar{y}_i) \xrightarrow{GH} (Z/H, \bar{z}).$$

Later in Sections 5.2 and 5.3, we shall use the convergence of symmetric subsets $S_i \subseteq G_i$. Recall that a subset S in a group G is *symmetric*, if $e \in S$ and $g^{-1} \in S$ for every $g \in S$.

Definition 2.3. Let

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$

be an equivariant Gromov-Hausdorff convergent sequence and let S_i be a sequence of closed symmetric subsets in G_i. We say that

$$(Y_i, y_i, S_i) \xrightarrow{GH} (Z, z, S),$$

for a closed symmetric subset S of H, if
(1) for every $h \in S$, there is a sequence $g_i \in S_i$ converging to h,
(2) every convergent sequence $g_i \in S_i$ has the limit h in S.

A corresponding precompactness result follows directly from the proof of [11, Proposition 3.6].

Proposition 2.4. *Let*

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$

be an equivariant Gromov-Hausdorff convergent sequence and let S_i be a sequence of closed symmetric subsets in G_i. Then after passing to a subsequence, we have convergence

$$(Y_i, y_i, S_i) \xrightarrow{GH} (Z, z, S),$$

where S is a closed symmetric subset of H.

Let M be an open manifold with $\mathrm{Ric} \geq 0$. Let $\widetilde{M}$ be its Riemannian universal cover and let $\Gamma = \pi_1(M, p)$. For a sequence $r_i \to \infty$, after passing to a subsequence, we have convergence

$$(r_i^{-1}\widetilde{M}, p, \Gamma) \xrightarrow{GH} (Y, y, G).$$

The limit space (Y, y, G) is called an *equivariant asymptotic cone* of $(\widetilde{M}, \Gamma)$. By the work of Colding-Naber [7], G is a Lie group.

In general, the above limit space (Y, y, G) is not unique and may depend on the choice of $r_i \to \infty$. We denote $\Omega(\widetilde{M}, \Gamma)$ as the set of all equivariant asymptotic cones of $(\widetilde{M}, \Gamma)$.

Proposition 2.5. *The set $\Omega(\widetilde{M}, \Gamma)$ is compact and connected in the pointed equivariant Gromov-Hausdorff topology.*

See [20, Proposition 2.1] for a proof of Proposition 2.5.

2.2. **Nilpotent groups.** We recall some basics results about nilpotent groups.

Definition 2.6. A group $\mathcal{N}$ is *nilpotent*, if the following central series terminates

$$\mathcal{N} = \zeta_0(\mathcal{N}) \triangleright \zeta_1(\mathcal{N}) \triangleright ... \triangleright \zeta_l(\mathcal{N}) = \{\mathrm{id}\},$$

where $\zeta_{j+1}(\mathcal{N}) = [\mathcal{N}, \zeta_j(\mathcal{N})]$. We define the *nilpotency step or class* of $\mathcal{N}$, denoted by $\mathrm{step}(\mathcal{N})$, as the smallest integer l such that $\zeta_l(\mathcal{N}) = \{\mathrm{id}\}$.

It is clear that $\zeta_{l-1}(\mathcal{N})$ is contained in $Z(\mathcal{N})$, the center of $\mathcal{N}$.

The following result is standard for finitely generated nilpotent groups; see [16].

Proposition 2.7. *Let Γ be a finitely generated nilpotent group. Then*
(1) Γ has a torsion-free nilpotent subgroup $\mathcal{N}$ of finite index;
(2) if $\mathcal{N}'$ is another torsion-free nilpotent subgroup of finite index, then $\mathrm{step}(\mathcal{N}) = \mathrm{step}(\mathcal{N}')$.

By Proposition 2.7, we can naturally define nilpotency steps for finitely generated virtually nilpotent groups.

Definition 2.8. Let Γ be a finitely generated virtually nilpotent group. We define

$$\mathrm{step}(\Gamma) := \mathrm{step}(\mathcal{N}),$$

where $\mathcal{N}$ is a torsion-free nilpotent subgroup of Γ with finite index.

The growth rate of finitely generated nilpotent groups are well understood by the work of Bass [1] and Guivarc'h [10]. In particular, the following word length estimate holds.

Proposition 2.9. [1, 10] *Let $\mathcal{N}$ be a finitely generated torsion-free nilpotent group of nilpotency step l and let γ be an element in $\zeta_{l-1}(\mathcal{N}) - \{\mathrm{id}\}$. Let S be a finite generating set of $\mathcal{N}$ and let d_S be the corresponding word length metric on $\mathcal{N}$. Then there are positive constants C_1 and C_2 such that*

$$C_1 \cdot b^{1/l} \leq d_S(\gamma^b, e) \leq C_2 \cdot b^{1/l}$$

for all $b \in \mathbb{Z}_+$.

Proposition 2.9 immediately implies an upper bound on the orbit length for $\mathcal{N}$-action on a metric space.

Corollary 2.10. *Let $\mathcal{N}$ and γ as in Proposition 2.9. Suppose that $\mathcal{N}$ acts freely and discretely on a metric space (X,d) by isometries. Then for every $x\in X$, there is a constant C such that*

$$d(\gamma^b x, x)\le C\cdot b^{1/l}$$

for all $b\in\mathbb{Z}_+$.

Proof. Let S be a finite generating set of $\mathcal{N}$ and let

$$R:=\max_{g\in S} d(gx,x).$$

By Proposition 2.9, we can express γ^b by at most $C_2 b^{1/l}$ many elements in S. Thus by triangle inequality, it is clear that

$$d(\gamma^b x,x)\le R\cdot C_2 b^{1/l}.$$

□

In general, the lower bound in Proposition 2.9 cannot be transferred to orbit length on a non-compact metric space X.

Lastly, below are some results about nilpotent Lie groups. See [15, Section 11.2] for references.

Lemma 2.11. *Let G be a connected nilpotent Lie group. Then any maximal torus of G is central in G; in particular, G has a unique maximal torus.*

Lemma 2.12. *Let G be a connected nilpotent Lie group. Then the exponential map* $\exp:\mathrm{Lie}(G)\to G$ *is a smooth covering map. If in addition that G is simply connected, then* $\exp$ *is a diffeomorphism.*

For an element $g\neq e$ in a connected and simply connected nilpotent Lie group G, according to Lemma 2.12, g uniquely determines a parameter subgroup $\{\exp(tv)|t\in\mathbb{R}\}$ of G, where $v\in\mathrm{Lie}(G)$ such that $\exp(v)=g$.

Lemma 2.13. *Let G be nilpotent Lie group. Then G_0 commutes with any compact subgroup K of G.*

Proof. For readers' convenience, we give a proof of this standard result in nilpotent Lie groups. Let $\mathfrak{g}$ be the Lie algebra of G and let

$$\mathrm{Ad}:G\to\mathrm{Aut}(\mathfrak{g})$$

be the adjoint representation. For any $g\in G$ and any $X\in\mathfrak{g}$, by nilpotency we have

$$[g,[...,[g,\exp(tX)]]...]=\mathrm{id}.$$

Differentiating both sides, we have $(\mathrm{Ad}(g)-I)^k=0$, that is, $\mathrm{Ad}(g)$ is unipotent. Hence under a suitable basis, $\mathrm{Ad}(g)$ is upper triangular with eigenvalues 1. In particular, for any $\mathrm{Ad}(g)\neq I$, it must generate a non-compact subgroup. Therefore, if $g\in K$, a compact subgroup of G, then $\mathrm{Ad}(g)=I$. It follows that $g\in K$ commutes with any element $\exp(tX)\in G_0$. □

2.3. **Escape rate.** [22] introduces the notion of escape rate by comparing the size of representing geodesic loops to their lengths. Let M be an open manifold with an infinite $\pi_1(M)$. For every element $\gamma \in \pi_1(M,p)$, we draw a representing loop c_γ at p such that it has the minimal length among all loops at p in the homotopy class γ. c_γ is always a geodesic loop at p. We denote

$$|\gamma| = d(\gamma\tilde{p}, \tilde{p}) = \text{length}(c_\gamma).$$

The escape rate of (M,p) is defined as

$$E(M,p) = \limsup_{|\gamma|\to\infty} \frac{d_H(p, c_\gamma)}{|\gamma|},$$

where $d_H(p, c_\gamma)$ is the Hausdorff distance between c_γ and p (in other words, the smallest radius R such that the closed ball $\overline{B_R(p)}$ covers c_γ.) If $\pi_1(M)$ is finite, then we set $E(M,p) = 0$ as a convention.

It is clear that $E(M,p)$ takes its value in $[0, 1/2]$. In [29], Sormani proved that if $\pi_1(M,p)$ is not finitely generated, then there is a sequence of element γ_i (the short generators) such that their representing loops c_{γ_i} are minimal up to halfway. Therefore, in the terminology of escape rate, we have

Theorem 2.14. [29] *Let M be an open manifold with $E(M,p) < 1/2$, then $\pi_1(M)$ is finitely generated.*

It is unclear to the author whether the converse of Theorem 2.14 holds for nonnegative Ricci curvature.

Throughout the paper, we always assume that M is an open (non-compact and complete) Riemannian manifold with $\text{Ric} \geq 0$ and $E(M,p) \neq 1/2$; in particular, $\pi_1(M)$ is finitely generated. When $\pi_1(M)$ is an infinite group, by [18, 12] and Proposition 2.7, $\pi_1(M)$ has a finitely generated torsion-free nilpotent subgroup $\mathcal{N}$ of finite index.

We gather some elementary lemmas from [24] about the basics of escape rate.

Lemma 2.15. [24, Lemma 1.5] *Let (M,p) be an open manifold with $\text{Ric} \geq 0$ and let $F : (\hat{M}, \hat{p}) \to (M,p)$ be a finite cover. Then $E(\hat{M}, \hat{p}) \leq E(M,p)$.*

In practice, we choose a torsion-free nilpotent subgroup $\mathcal{N}$ of $\pi_1(M,p)$ with finite index and set $\hat{M} = \widetilde{M}/\mathcal{N}$ as a finite cover of M. By Lemma 2.15, we have

$$E(\hat{M}, \hat{p}) \leq E(M,p) < 1/2, \quad \pi_1(\hat{M}, \hat{p}) = \mathcal{N}.$$

Therefore, without loss of generality, we can replace (M,p) by $(\hat{M}, \hat{p})$ and then assume that $\pi_1(M,p) = \mathcal{N}$ is a finitely generated torsion-free nilpotent group.

Lemma 2.16. [24, Lemma 2.1 and Proposition 2.2] *Let $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$. For any point $gy \in Gy$ that is not y, there is a minimal geodesic σ from y to gy and an orbit point $g'y \in Gy$ such that*

$$d(m, g'y) \leq E \cdot d(y, gy),$$

where m is the midpoint of σ. As a consequence, the orbit Gy is connected for all $(Y,y,G) \in \Omega(\widetilde{M},\Gamma)$.

The inequality in [24, Lemma 2.1] states $d(m,g'y) < (1/2)\cdot d(y,gy)$. However, inspecting its proof, it is clear that a stronger inequality as stated above holds.

Lastly, we mention that the notion of escape rate and its properties can be naturally extended to group actions on non-compact length metric spaces.

Definition 2.17. Let (X,p) be a complete non-compact length metric spaces and let G be a group that acts effectively and isometrically on X. For each $g \in G$, we denote c_g as a minimal geodesic from p to $g\cdot p$ and we put

$$|g| = d(p, g\cdot p) = \mathrm{length}(c_g).$$

We define $E(X,p,G)$, the *escape rate* of (X,p,G), by

$$E(X,p,G) := \limsup_{|g|\to\infty} \frac{R(c_g)}{|g|},$$

where $R(c_g)$ is the infimum of all $R>0$ such that c_g is contained in the R-tubular neighborhood of $G\cdot p$. If the orbit $G\cdot p$ is bounded, then we set $E(X,p,G)=0$ as a convention.

3. Motivating examples

3.1. **Wei's examples.** To better motivate Theorem A, we have a brief review of Wei's construction [30] of open manifolds with $\mathrm{Ric}>0$ and torsion-free nilpotent fundamental groups. We explain the choice of warping functions and its relation to Hausdorff dimension of asymptotic orbits. Also see [27], [22, Appendix B], and [24, Appendix A].

For simplicity, we use the discrete Heisenberg 3-group

$$\Gamma = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c\in\mathbb{Z} \right\} \subseteq \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c\in\mathbb{R} \right\} =: \widetilde{N}.$$

Γ is torsion-free nilpotent with nilpotency step 2. Let $\mathfrak{n}$ be the Lie algebra of $\widetilde{N}$. $\mathfrak{n}$ has a basis

$$X_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $[X_0,X_1]=X_2$ as the only nontrivial Lie bracket relation.

Let

$$h_0(r) = h_1(r) = (1+r^2)^{-\alpha}, \quad h_2(r) = (1+r^2)^{-\beta},$$

where $\alpha,\beta>0$ to be specified later. We define a family of inner products with parameter $r\in[0,\infty)$ on $\mathfrak{n}$ by

$$||X_j||_r := h_j(r)$$

and setting X_i and X_j orthogonal when $i \neq j$. This naturally defines a family of left-invariant metrics g_r on $\widetilde{N}$. Note

$$||[X_0, X_1]|| = ||X_2|| = (1+r^2)^{-\beta} = (1+r^2)^{-\beta+2\alpha}||X_0||_r||X_1||_r.$$

It follows from basic curvature formulas for left-invariant metrics on Lie group that

$$|\sec(g_r)| \leq C \cdot (1+r^2)^{-2\beta+4\alpha},$$

where C is a constant. Therefore, for g_r to be almost flat as $r \to \infty$, we should require $\beta > 2\alpha$.

Next, we define a doubly warped product

$$\widetilde{M} = [0, \infty) \times_{f(r)} S^{k-1} \times (\widetilde{N}, g_r), \quad g = dr^2 + f(r)^2 ds^2_{k-1} + g_r,$$

where $f(r) = r(1+r^2)^{-1/4}$. $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^k \times \widetilde{N}$. Following a similar computation as in [30, 2], for g to have $\mathrm{Ric} > 0$ when k is sufficiently large, the term $|\mathrm{Ric}(g_r)|$ should be comparable to or smaller than $(1+r^2)^{-1}$; equivalently, we require that

$$\beta \geq 2\alpha + 1/2.$$

To construct M, noting that $\widetilde{N}$ naturally acts on $\widetilde{M}$ freely by isometries, we can take the quotient Riemannian manifold $M = \widetilde{M}/\Gamma$. Then it is clear that M satisfies $\mathrm{Ric} > 0$ and $\pi_1(M) = \Gamma$.

We check the Hausdorff dimension of asymptotic orbits for the above example. Let $\gamma_j = \exp(X_j) \in \Gamma$ and let $\tilde{p} = (0, \star, \mathrm{id}) \in \widetilde{M}$. We use $|\cdot|$ to denote the displacement of an isometry at $\tilde{p}$. Following the length estimate in [27, Section 1.2], as $b \to \infty$ we have

$$|\gamma_0^b| = |\gamma_1^b| \sim b^{\frac{1}{1+2\alpha}}, \quad |\gamma_2^{(b^2)}| = |[\gamma_0^b, \gamma_1^b]| \leq C \cdot b^{\frac{1}{1+2\alpha}}.$$

$|\gamma_2^b|$ can also be estimated by $h_2(r)$. Since

$$\frac{1}{1+2\beta} \leq \frac{1}{2(1+2\alpha)},$$

we have

$$|\gamma_2^b| \sim b^{\frac{1}{1+2\beta}}.$$

After blowing-down

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \langle \gamma_j \rangle, \Gamma) \xrightarrow{GH} (Y, y, H_j, G),$$

each H_j is a closed $\mathbb{R}$-subgroup of G. It follows from the length estimate that the asymptotic orbits have Hausdorff dimension

$$\dim_{\mathcal{H}}(H_0 y) = \dim_{\mathcal{H}}(H_1 y) = 1 + 2\alpha, \quad \dim_{\mathcal{H}}(H_2 y) = 1 + 2\beta > 2.$$

Remark 3.1. As observed in [24, Appendix A], we also mention that the algebraic structure of G depends on β. In fact, if $\beta = 2\alpha + 1/2$, then G is isomorphic to the 3-dimensional Heisenberg group $\widetilde{N}$; if $\beta > 2\alpha + 1/2$, then G is isomorphic to the abelian group $\mathbb{R}^3$.

3.2. **Examples with minimal Hausdorff dimension.** In this subsection, we give examples of open manifolds M with $\mathrm{Ric} > 0$, $E(M,p) \neq 1/2$, and the following properties:
(1) $\pi_1(M,p) = \mathcal{N}$ is a torsion-free nilpotent group of nilpotency step $l \geq 2$;
(2) for every $(Y,y,H) \in \Omega(\widetilde{M},\langle\gamma\rangle)$, where $\gamma \in \zeta_{l-1}(\mathcal{N}) - \{\mathrm{id}\}$, H is a closed $\mathbb{R}$-subgroup of $\mathrm{Isom}(Y)$ and the orbit Hy has Hausdorff dimension exactly l.

These examples show that the inequality in Proposition C(2) is sharp.

The construction is a slight modification of Wei's examples [30]. We remark that the choice of functions in [30] always yield examples with $\dim_{\mathcal{H}}(Hy) > l$, a strict inequality, for all $(Y,y,H) \in \Omega(\widetilde{M},\langle\gamma\rangle)$. Hence some modifications are required.

Let $\mathfrak{n}$ be $(l+1)$-dimensional filiform Lie algebra spanned by $\{X_0, X_1, ..., X_l\}$ with only nontrivial Lie bracket relations

$$[X_0, X_j] = X_{j+1} \quad \text{for } j = 1, ..., l-1.$$

$\mathfrak{n}$ is nilpotent with nilpotency step l. Let $\widetilde{N}$ be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let

$$h_0(r) = h_1(r) = 1 + (1+r^2)^{-1/4}, \quad h_j(r) = (1+r^2)^{-(j-1)/2} \text{ for } j = 2, ..., l.$$

Note that h_0 and h_1 decays to 1 as $r \to \infty$; this is the main difference compared to [30]. Then similar to the construction in the last section, we define a family of inner products on $\mathfrak{n}$ by $||X_j||_r := h_j(r)$ and setting $\{X_j\}$ orthogonal to each other, thus define a family of left-invariant Riemannian metrics $\{g_r\}$ on $\widetilde{N}$. Given the Lie bracket relations and the choice of h_j, we have

$$||[X_0, X_j]||_r = ||X_{j+1}||_r = h_{j+1}(r) \leq (1+r^2)^{-1/2}||X_0||_r||X_j||_r,$$

where $j = 1, ..., l-1$. Hence g_r satisfies

$$|\mathrm{Ric}(g_r)| \leq C \cdot (1+r^2)^{-1}.$$

Then we similarly define a doubly warped product

$$\widetilde{M} = [0,\infty) \times_{f(r)} S^{k-1} \times (\widetilde{N}, g_r), \quad g = dr^2 + f(r)^2 ds^2_{k-1} + g_r,$$

where $f(r) = r(1+r^2)^{-1/4}$. Then g satisfies $\mathrm{Ric}(g) > 0$ when k is sufficiently large.

Let $\gamma = \exp(X_l) \in \zeta_{l-1}(\widetilde{N})$. we choose a lattice $\mathcal{N}$ in $\widetilde{N}$ such that $\zeta_{l-1}(\mathcal{N}) = \langle\gamma\rangle$ and set M as the quotient manifold $\widetilde{M}/\mathcal{N}$. Then M is our desired example. In fact, at $\tilde{p} = (0, \star, e) \in \widetilde{M}$, by the choice of $\{h_j\}$ and the estimate in [27], we can similarly show that

$$|\gamma^b| \sim b^{1/l}.$$

Therefore, $\dim_{\mathcal{H}}(Hy) = l$ for any $(Y,y,H) \in \Omega(\widetilde{M},\langle\gamma\rangle)$.

4. Equivariant Gromov-Hausdorff distance gaps

The main goal of this section is to establish equivariant Gromov-Hausdorff distance gaps between different types of equivariant asymptotic cones of $(\widetilde{M}, \mathcal{N})$ (Propositions 4.8 and 4.9).

In subsection 4.1, we introduce the notion of tunnels, that is, continuous curves inside the orbit Gy. Using the condition $E(M,p) < 1/2$, we study controls on the size of tunnels in the asymptotic cones and the convergence of tunnels. In subsection 4.2, we give the statements of distance gaps and a rough idea of the proof by using the convergence of tunnels and large fiber lemma. Subsection 4.3 is a technical part that we introduce some tools so that we can carry out the rough idea in general nilpotent group actions; in particular, we introduce the notion of adapted bases and adapted maps. We prove the distance gaps Propositions 4.8 and 4.9 in subsection 4.4.

4.1. **Tunnels.** In this subsection, we use E exclusively to denote the value of $E(M,p) < 1/2$.

Definition 4.1. Let (Y, y, G) be a space. A *tunnel* is a continuous path $\sigma : [0,1] \to Gy$. We say that (Y, y, G) is *C-tunneled* for some constant $C \in [1, \infty)$, if for every orbit point $gy \in Gy$ with $d(gy, y) =: d$, there is a tunnel σ from y to gy that is contained in $\overline{B_{Cd}}(y)$.

Remark 4.2. We remark that, in general, it is possible that every nontrivial curve in Gy is has infinite length. For example, as constructed in [27] and clarified in [8, Remark 3.9], the Grushin halfplane $\mathbb{G}(\alpha)$, or more generally the Grushin halfplane $\mathbb{G}(\alpha_1, ..., \alpha_k)$ mentioned in the introduction, are asymptotic cones of open manifolds with $\mathrm{Ric} \geq 0$. The orbit Gy is exactly $\mathbb{R}^k \times \{0\}$ under the $\mathbb{R}^k \times [0,\infty)$-coordinate of $\mathbb{G}(\alpha_1, ..., \alpha_k)$. When all α_i are positive, every nontrivial curve in Gy is not rectifiable. Therefore, we use size, instead of length, to measure tunnels in Definition 4.1.

Proposition 4.3. *Given $E \in [0, 1/2)$, there is a constant $C_0(E)$ such that the following holds.*

Let M be an open manifold with $\mathrm{Ric} \geq 0$ *and $E(M,p) = E$. Then any $(Y, y, G) \in \Omega(\widetilde{M}, \Gamma)$ is $C_0(E)$-tunneled.*

Proof. We put
$$C_0(E) = \sum_{j=1}^{\infty} (E + 1/2)^j < \infty.$$
Let $gy \in Gy - \{y\}$ with $d := d(y, gy)$. We use Lemma 2.16 to construct a desired tunnel σ from y to gy as follows. We define $\sigma(0) = y$ and $\sigma(1) = gy$. By Lemma 2.16, there is an orbit point $g_{(1,1)}y \in Gy$ such that
$$d(g_{(1,1)}y, y) \leq (E + 1/2)\, d, \quad d(g_{(1,1)}y, gy) \leq (E + 1/2)\, d.$$
We define $\sigma(1/2) = g_{(1,1)}y$. Next, we set
$$P_j = \{k/2^j | k = 0, 1, ..., 2^j\}.$$
Inductively, suppose that we have defined σ on P_j with $\sigma(k/2^j) = \sigma(g_{(k,j)}y)$ for some $g_{(k,j)} \in G$ such that
$$d(g_{(k-1,j)}y, g_{(k,j)}y) \leq (E + 1/2)^j\, d.$$

for all $k = 1, ..., 2^j$. Then we use Lemma 2.16 again to define σ on P_{j+1} as follows. When k is even, then $k/2^{j+1} \in P_j$; hence $\sigma(k/2^{j+1})$ has been defined in the previous steps. When k is odd, we assign $\sigma(k/2^{j+1})$ as an orbit point $g_{(k,j+1)}y$ with

$$d(g_{(k,j+1)}y, g_{(\frac{k-1}{2},j)}y) \leq (E+1/2)\, d(g_{(\frac{k-1}{2},j)}y, g_{(\frac{k+1}{2},j)}y) \leq (E+1/2)^{j+1}\, d,$$

$$d(g_{(k,j+1)}y, g_{(\frac{k+1}{2},j)}y) \leq (E+1/2)^{j+1}\, d.$$

So far, we have defined σ on $\cup_j P_j$ with image in Gy. We show that σ is uniform continuous on $\cup_j P_j$. In fact, for any $\epsilon > 0$, let $J \in \mathbb{N}$ large with

$$\sum_{j=J}^{\infty} (1/2+E)^j \leq \epsilon/2,$$

and we choose $\delta = 1/2^{J+1}$. Then for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leq \delta$, there is some $k_0/2^J \in P_J$ such that

$$\max\{|k_0/2^J - t_1|, |k_0/2^J - t_2|\}| \leq 1/2^J.$$

Then by construction of σ,

$$\begin{aligned} d(\sigma(t_1), \sigma(t_2)) \leq & d(\sigma(t_1), \sigma(k_0/2^J)) + d(\sigma(k_0/2^J), \sigma(t_2)) \\ \leq & 2\sum_{j=J}^{\infty} (1/2+E)^j \leq \epsilon. \end{aligned}$$

This verifies that σ is uniform continuous on $\cup_j P_j$, thus σ extends to a continuous path from y to gy in Gy.

Lastly, observe that by construction,

$$d(y, \sigma(t)) \leq \sum_{j=1}^{\infty} (1/2+E)^j d = C_0(E) d$$

for all $t \in [0,1]$. Therefore, the image of σ is contained in $\overline{B_{C_0(E)d}}(y)$. This proves that (Y, y, G) is $C_0(E)$-tunneled. □

Remark 4.4. Regarding Proposition 4.3, we mention that a stronger result holds: (Y, y, G) is 3-tunneled. Hence the size of the tunnel actually can be uniformly controlled regardless of the value of E, as long as $E \neq 1/2$. Because Proposition 4.3 is sufficient for this paper, we omit the proof of this stronger result.

Lemma 4.5. *Let (Y_i, y_i, G_i) be a convergent sequence of spaces*

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H),$$

where the limit group H is nilpotent. Suppose that there is C_0 such that (Y_i, y_i, G_i) is C_0-tunneled for all i. Then for any tunnel $\sigma : [0,1] \to Hz$ from z, there is a sequence of tunnels $\sigma_i : [0,1] \to G_i y_i$ from y_i such that σ_i converges uniformly to σ.

Proof. Let
$$\epsilon_i = d_{GH}((Y_i, y_i, G_i), (Z, z, H)) \to 0.$$
We shall construct σ_i on (Y_i, y_i, G_i) for each large i. By the uniform continuity of σ, we can choose a large integer N such that
$$\mathrm{diam}(\sigma|_{[(j-1)/N, j/N]}) \leq \epsilon_i$$
for all $j = 1, 2, .., N$. Let $z_j = \sigma(j/\mathcal{N}) \in Hz$. For each j, we choose $y_{i,j} \in G_i y_i$ that is ϵ_i-close to z_j; for $j = 0$, we use $y_{i,0} = y$. By triangle inequality, it is clear that
$$d(y_{i,j}, y_{i,j+1}) \leq 3\epsilon$$
for all j. Next, for two adjacent $y_{i,j}$ and $y_{i,j+1}$, we join them by a tunnel
$$\sigma_{i,j} : [(j-1)/N, j/N] \to G_i y_i$$
such that it is contained in $\overline{B_{3C_0\epsilon}}(y_{i,j})$. Let $\sigma_i : [0, 1] \to Gy$ be the concatenation of all $\sigma_{i,j}$, where $j = 1, ..., N$. For any $t \in [0, 1]$, let $j \in \{1, ..., N\}$ such that $t \in [(j-1)/N, j/N]$, then by construction we have
$$\begin{aligned} d(\sigma_i(t), \sigma(t)) &\leq d(\sigma_i(t), y_{i,j}) + d(y_{i,j}, z_j) + d(z_j, \sigma(t)) \\ &\leq 3C_0\epsilon_i + \epsilon_i + \epsilon_i = (3C_0 + 2)\epsilon_i \to 0. \end{aligned}$$
□

Let G be a Lie group. We use G_0 to denote the identity component subgroup of G. In a space (Y, y, G), where G is a Lie group, if Gy is connected, then $Gy = G_0 y$; consequently, every orbit point $z \in Gy$ can be represented as gy for some $g \in G_0$.

Lemma 4.6. *Let*
$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$
be a convergent sequence such that each (Y_i, y_i, G_i) is C_0-tunneled and H is nilpotent. Suppose that a sequence of orbit points $g_i y_i \in G_i y_i$ converges to a limit orbit point $gz \in Hz$, where $g_i \in (G_i)_0$ and $g \in H_0$. Then after passing to a subsequence if necessary, g_i converges to $gh \in H_0$ with $h \in \mathrm{Iso}(z, H_0)$, the isotropy subgroup of H_0 at z.

Proof. Let $d = d(z, gz)$. Passing to a subsequence, we have convergence
$$(Y_i, y_i, G_i, g_i) \xrightarrow{GH} (Z, z, H, g_\infty),$$
where $g_\infty \in H$.

We claim that $g_\infty \in H_0$. We argue by contradiction. Suppose that $g_\infty \notin H_0$, then by Proposition A.1 there is a point $q \in Z$ such that the orbit Hq has multiple components and $g_\infty q$ is not in the component containing q. On (Y_i, y_i, G_i), let $\sigma_i : [0, 1] \to G_i y_i$ be a tunnel from y_i to $g_i y_i$ such that $\mathrm{im}\sigma_i \subseteq \overline{B_{2C_0 d}}(y_i)$. Let $\widetilde{\sigma}_i : [0, 1] \to (G_i)_0$ be a continuous path from id to g_i such that $\sigma_i(t) = \widetilde{\sigma}_i(t) \cdot y_i$. Let $q_i \in Y_i$ converging to q. We consider the continuous path $\tau_i(t) := \widetilde{\sigma}_i(t) \cdot q_i$ in $G_i q_i$ from q_i to $g_i q_i$. Then
$$d(\tau_i(t), y_i) \leq d(\widetilde{\sigma}_i(t) q_i, \widetilde{\sigma}_i(t) y_i) + d(\widetilde{\sigma}_i(t) y_i, y_i) \leq d(q_i, y_i) + 2C_0 d.$$

We write $l = d(q, y)$. After passing to a subsequence, connected and closed subsets $\mathrm{im}\tau_i$ converges to a limit connected and closed subset $S \subseteq \overline{B_{l+2C_0 d}(y)} \cap Hq$. Since S contain both q and $g_\infty q$, we conclude that q and $g_\infty q$ belong to the same connected component of Hq. This proves the claim.

We have $g_i \to g_\infty$ and $g_i y_i \to gz$, thus $g_\infty z = gz$. Let $h = g^{-1} g_\infty$, then h fixes z. Together with $g \in H_0$ and the claim $g_\infty \in H_0$, we conclude that $h \in \mathrm{Iso}(z, H_0)$. □

4.2. **Statements of distance gaps and a rough idea of the proof.** As explained in Lemma 2.15, without loss generality, we assume that $\pi_1(M, p) = \mathcal{N}$ is a finitely generated torsion-free nilpotent group. Then for any $(Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N})$, G is a nilpotent Lie group.

Definition 4.7. Let (Y, y, G) be a space, where G is a nilpotent Lie group. Let T be the maximal torus of G_0. We say that (Y, y, G) is of type (k, d), if

$$\dim G - \dim T = k, \quad \mathrm{diam}(Ty) = d.$$

Using Definition 4.7, we state the equivariant Gromov-Hausdorff distance gaps.

Proposition 4.8. *There is a constant $\delta_1 = \delta_1(\widetilde{M}, \mathcal{N}) > 0$ such that the following holds.*

Let (Y, y, G) and $(Y', y', G') \in \Omega(\widetilde{M}, \mathcal{N})$ of type (k, d) and (k', d'), respectively. Suppose that $k < k'$ and $d = 0$, then

$$d_{GH}((Y, y, G), (Y', y', G')) \geq \delta_1.$$

Proposition 4.9. *There is a constant $\delta_2 = \delta_2(\widetilde{M}, \mathcal{N}) > 0$ such that the following holds.*

Let (Y, y, G) and $(Y', y', G') \in \Omega(\widetilde{M}, \mathcal{N})$ of type (k, d) and (k', d'), respectively. Suppose that $k = k'$, $d \leq 1$, and $d' \geq 10$, then

$$d_{GH}((Y, y, G), (Y', y', G')) \geq \delta_2.$$

Besides using tunnels, a key ingredient in the proof is the large fiber lemma from topological dimension theory.

Lemma 4.10 (Large Fiber Lemma)**.** *Let $F : [0, 1]^{k+1} \to \mathbb{R}^k$ be a continuous map. Then there are $a, b \in [0, 1]^{k+1}$ such that $F(a) = F(b)$ and $|a - b| \geq 1$.*

Remark 4.11. The large fiber lemma is a corollary of the Lebesgue covering lemma in topological dimension theory (see [13, Section 6]). It also follows from the Borsuk-Ulam theorem in algebraic topology (see [14, Corollary 2B.7]): we take a k-sphere of radius $1/2$ in $[0, 1]^{k+1}$, then by Borsuk-Ulam theorem, there exists a pair of antipodal points on the sphere such that they have the same image under F.

To illustrate how the large fiber lemma and the C-tunneled property can be applied to prove equivariant Gromov-Hausdorff distance gaps between two spaces, we rule out the following scenario. Suppose that there is a sequence

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$

such that
(1) for each i, (Y_i, y_i, G_i) is C_0-tunneled and G_i is isomorphic $\mathbb{R}^k$;
(2) H is isomorphic to $\mathbb{R}^{k+1}$.

Let $\{e_1, ..., e_{k+1}\} \subseteq H$ be an $\mathbb{R}$-basis of $H = \mathbb{R}^{k+1}$. Note that H-action does not have isotropy subgroups at z (otherwise, H would have a nontrivial compact subgroup). For each $j = 1, ..., k+1$, we write $\{te_j\}_{t\in\mathbb{R}}$ as the one-parameter subgroup through e_j. We consider an embedding

$$F : [0,1]^{k+1} \to Hz, \quad (t_1, ..., t_{k+1}) \mapsto \prod_{j=1}^{k+1} (t_j e_j) \cdot z.$$

$\sigma_j(t) := (te_j)z$, where $t \in [0,1]$, gives a tunnel in Hz from z to $e_j z$. We apply Lemma 4.5 to construct tunnels $\sigma_{i,j} : [0,1] \to G_i y_i$ from y_i that converges uniformly to σ_j as $i \to \infty$. Because G_i-action does not have isotropy subgroups at y_i, each $\sigma_{i,j}$ uniquely defines a continuous curve $\widetilde{\sigma_{i,j}} : [0,1] \to G_i$ such that $\widetilde{\sigma_{i,j}}(t) \cdot y_i = \sigma_{i,j}(t)$. This allows us to define a continuous map

$$F_i : [0,1]^{k+1} \to G_i y_i, \quad (t_1, ..., t_{k+1}) \mapsto \prod_{j=1}^{k+1} \widetilde{\sigma_{i,j}}(t_j) \cdot y_i.$$

By construction, it is not difficult to show that F_i converges uniformly to F. Since $G_i y_i$ is homeomorphic to $\mathbb{R}^k$, we can apply the large fiber lemma to F_i. It follows that there are $a_i, b_i \in [0,1]^{k+1}$ such that $F_i(a_i) = F_i(b_i)$ and $|a_i - b_i| \geq 1$. Passing to a subsequence, we have limit points $a', b' \in [0,1]^{k+1}$ such that $F(a') = F(b')$ and $|a' - b'| \geq 1$ by the uniform convergence of F_i to F. However, this contradicts the injectivity of F.

Remark 4.12. We remark that in general, it is possible for a sequence of $\mathbb{R}^k$-actions converges to a limit $\mathbb{R}^{k+1}$-action, as shown in the example below. Hence the C-tunneled property is crucial here.

We consider a sequence of $\mathbb{R}$-actions on the standard Euclidean space $\mathbb{R}^3$ as follows. Set $y = (0,0,0)$ as our base point. For each $i \in \mathbb{Z}_+$, $G_i = \mathbb{R}$ acts on $\mathbb{R}^3$ by rotating xy-plane by angle $2\pi t$ with center $(i,0,0)$ while translating along z-axis by t/i, where $t \in \mathbb{R}$. Then

$$(\mathbb{R}^3, p, G_i) \xrightarrow{GH} (\mathbb{R}^3, p, \mathbb{R}^2).$$

The limit $\mathbb{R}^2$-action are translations in yz-plane. Note that these spaces $(\mathbb{R}^3, p, G_i)$ are not C-tunneled for a uniform C.

4.3. **Adapted bases and adapted maps.** In general, the group actions involved are nilpotent and may have nontrivial torus subgroups that move the base point. Hence we need more preparations to carry out the strategy in subsection 4.2.

For convenience, we define

$$\Omega_Q(\widetilde{M}, \mathcal{N}) = \{(Y/H, \bar{y}, G/H) | (Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N}),$$
$$H \text{ is a closed normal subgroup of } G\}.$$

Note that $\Omega_Q(\widetilde{M}, \mathcal{N})$ includes $\Omega(\widetilde{M}, \mathcal{N})$ since we can take $H = \{\mathrm{id}\}$.

Definition 4.13. We say that a space (Y, y, G) is *good*, if the followings hold:
(1) $(Y, y) \in \Omega_Q(\widetilde{M}, \mathcal{N})$;
(2) $G \subseteq \mathrm{Isom}(Y)$ is closed nilpotent subgroup;
(3) (Y, y, G) is C_0-tunneled, where C_0 is the constant in Proposition 4.3;
(4) The isotropy subgroup of G at y is finite;
(5) G_0, the identity component subgroup of G, is simply connected.

Lemma 4.14. *Let $(Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N})$ and let T be the maximal torus subgroup of G_0. Then the quotient space $(Y/T, \bar{y}, G/T)$ is good in the sense of Definition 4.13.*

Proof. We first remark that by Lemma 2.11 T is normal in G, thus the quotient group G/T is defined. It is clear that (1,2) in Definition 4.13 are fulfilled. (4,5) are also straightforward since $(G/T)_0 = G_0/T$ does not have any nontrivial torus subgroup.

It remains to show (3). Let $\pi : Y \to Y/T$ be the quotient map. Since π maps Gy to $(G/T)\bar{y}$ and Gy is connected by Lemma 2.16, we see that $(G/T)\bar{y}$ is also connected. For any orbit point $\bar{z} \in (G/T)\bar{y}$, because $(G/T)\bar{y}$ is connected, we can write $\bar{z} = \bar{g}\bar{y}$, where $\bar{g} \in (G/T)_0 = G_0/T$. We choose $g \in G_0$ such that g projects to $\bar{g}$ and

$$d_Y(gy, y) = d_{Y/T}(\bar{g}\bar{y}, \bar{y}) =: d.$$

By Lemma 4.3, there is a tunnel $\sigma : [0, 1] \to Gy$ from y to gy that is contained in $\overline{B_{C_0 d}}(y)$. Then $\pi \circ \sigma$ is a desired tunnel from $\bar{y}$ to $\bar{g}\bar{y}$ in $(G/T)\bar{y}$. □

We first construct adapted bases and maps for good spaces in the sense of Definition 4.13. In this case, the maximal torus of G_0 is trivial.

Definition 4.15. Let (Y, y, G) be a space, where G_0 is a simply connected nilpotent Lie group. Let

$$G_0 = \zeta_0(G_0) \rhd \zeta_1(G_0) \rhd ... \rhd \zeta_{l-1}(G_0) \rhd \zeta_l(G_0) = \{\mathrm{id}\}$$

be the lower central series of G_0, where $\zeta_{l-1}(G_0) \neq \{\mathrm{id}\}$. We say an element $e_1 \in G_0$ is *initial*, if $e_1 \in \zeta_{l-1}(G_0)$ and $d(e_1 y, y) = 1$.

Note that every one-parameter subgroup of $\zeta_{l-1}(G_0)$ has an unbounded orbit at y. Thus the initial element defined above always exists.

This initial element e_1 is the first element of an adapted basis $\{e_1, ..., e_k\}$ with respect to (Y, y, G), where k is the dimension of G. We choose the remaining elements by induction. Let H_1 be the unique one-parameter subgroup through e_1 (see Lemma 2.12). By construction, H_1 is normal in G_0. We choose $\bar{e}_2 \in G_0/H_1$ as the initial element in $(Y/H_1, \bar{y}, G_0/H_1)$. Let $e_2 \in G_0$ such that e_2 projects to $\bar{e}_2 \in G_0/H_1$ and

$$d_Y(e_2 y, y) = d_{Y/H_1}(\bar{e}_2\bar{y}, \bar{y}) = 1.$$

Note that because $\bar{e}_2$ belongs to the last nontrivial subgroup of the lower central series of G_0/H_1, we have

$$[v, v_2] \in \mathrm{span}_{\mathbb{R}}\{v_1\}$$

for all $v \in \mathrm{Lie}(G_0)$, where $v_j \in \mathrm{Lie}(G_0)$ such that $\exp(v_j) = e_j$. Inductively, we choose $\{e_1, ..., e_k\}$ such that
(1) $[v, v_{j+1}] \in \mathrm{span}_{\mathbb{R}}\{v_1, ..., v_j\}$ for all $j = 1, 2, ..., k-1$, where $v_j \in \mathrm{Lie}(G_0)$ such that $\exp(v_j) = e_j$;
(2) $\bar{e}_{j+1}$ is an initial element of $(Y/H_j, \bar{y}, G_0/H_j)$, where H_j is the Lie subgroup with Lie algebra as $\mathrm{span}\{v_1, ..., v_j\}$, and $e_{j+1} \in G_0$ such that e_{j+1} projects to $\bar{e}_{j+1}$ and

$$d_Y(e_{j+1}y, y) = d_{Y/H_j}(\bar{e}_{j+1}\bar{y}, \bar{y}) = 1.$$

Definition 4.16. Let (Y, y, G) be a good space in the sense of Definition 4.13. We call the above constructed $\{e_1, ..., e_k\} \subseteq G_0$ an *adapted basis* with respect to (Y, y, G), where k is the dimension of G.

As a convention, $\prod$ means a product multiplying on the left $\prod_{j=1}^k g_j = g_k ... g_2 g_1$.

Definition 4.17. Let (Y, y, G) be a good space in the sense of Definition 4.13. Let $\mathcal{E} = \{e_1, ..., e_k\} \subseteq G_0$ be an adapted basis with respect to (Y, y, G), where k is the dimension of G. We define an *adapted map* for $\mathcal{E}$:

$$F : [0,1]^k \to Gy, \quad (t_1, ..., t_k) \mapsto \prod_{j=1}^k (t_j e_j) \cdot y,$$

where te_j denotes the elements on the unique one-parameter subgroup through e_j.

Lemma 4.18. *Let (Y, y, G) be a good space in the sense of Definition 4.13. Let $F : [0,1]^k \to Gy$ be an adapted map for an adapted basis $\mathcal{E}$ as in Definition 4.17. Then F is a continuous injection.*

Proof. Note that G_0-action is free at y; otherwise G_0 would have a compact torus subgroup as the isotropy subgroup at y. Because G_0-action is continuous and free at y, it suffices to show that the map

$$\widetilde{F} : [0,1]^k \to G_0, \quad (t_1, ..., t_k) \mapsto \prod_{j=1}^k (t_j e_j)$$

is a continuous injection. It is clear that $\widetilde{F}$ is continuous. We prove its injectivity by induction on the nilpotency step of G_0.

When G_0 is abelian, it is clear that $\widetilde{F}$ is injective. Assuming that $\widetilde{F}$ is injective when G_0 has nilpotency step $\leq l$, we consider the case that G_0 has nilpotency step $l+1$. For each j, let $v_j \in \mathrm{Lie}(G_0)$ such that $\exp(v_j) = e_j$. Suppose that

$$\prod_{j=1}^k \exp(t_j v_j) = \prod_{j=1}^k \exp(s_j v_j).$$

By the construction of the adapted basis, there is an integer $m \in [1, k)$ such that $\zeta_{l-1}(G_0) = \exp(V_m)$, where V_m is the span of $\{v_1, ..., v_m\}$. The quotient group $G_0/\zeta_{l-1}(G_0)$ has nilpotency step l. Let $\bar{v_j}$, where $j > m$, be the projection of v_j in the quotient Lie algebra $\mathrm{Lie}(G_0)/V_m = \mathrm{Lie}(G_0/\zeta_{l-1}(G_0))$. Then in $G_0/\zeta_{l-1}(G_0)$ we have

$$\prod_{j=m+1}^{k} \exp(t_j \bar{v_j}) = \prod_{j=m+1}^{k} \exp(s_j \bar{v_j}).$$

By the induction assumption, we have $t_j = s_j$ for all $j > m$. By Lemma 2.12, there is an element $Z \in \mathrm{Lie}(G_0)$ such that

$$\exp(Z) = \prod_{j=m+1}^{k} \exp(t_j v_j).$$

Because $v_1, .., v_m$ are in the center of $\mathrm{Lie}(G_0)$, by the Baker–Campbell–Hausdorff formula, it follows that

$$\exp\left(\sum_{j=1}^{m} t_j v_j + Z\right) = \exp\left(\sum_{j=1}^{m} s_j v_j + Z\right).$$

By Lemma 2.12 again, we see that

$$\sum_{j=1}^{m} t_j v_j + Z = \sum_{j=1}^{m} s_j v_j + Z.$$

We conclude that $t_j = s_j$ also holds for $j = 1, ..., m$. This completes the inductive step and thus $\widetilde{F}$ is injective. □

In general, we will also consider spaces that do not satisfy Definition 4.13. There are mainly two cases, corresponding to Propositions 4.8 and 4.9 respectively. We shall similarly construct adapted bases and adapted maps in each case.

For Proposition 4.8, we consider a space $(Z, z, H) \in \Omega_Q(\widetilde{M}, \mathcal{N})$ of type (k, d). Let T_H be the maximal torus subgroup of H_0. The quotient space $(Z/T_H, \bar{z}, H/T_H)$ is a good space in the sense of Definition 4.13 by Lemma 4.14. Thus we can follow Definitions 4.16 and 4.17 to construct an adapted basis $\{\bar{e}_1, ..., \bar{e}_k\} \subseteq (H/T_H)_0 = H_0/T_H$. For each $j = 1, ..., k$, we choose $e_j \in H_0$ such that e_j projects to $\bar{e_j} \in H/T_H$ and

$$d_Y(e_j z, z) = d_{Y/T_H}(\bar{e_j}\bar{z}, \bar{z}) = 1.$$

Definition 4.19. Let $(Z, z, H) \in \Omega_Q(\widetilde{M}, \mathcal{N})$ be a space of type (k, d). We call the above constructed $\mathcal{E} = \{e_1, ...e_k\} \subseteq H_0$ an *adapted basis* with respect to (Z, z, H). To further construct an adapted map, for each e_k, we choose a parameter subgroup $\tau_j : \mathbb{R} \to H_0$ such that $\tau_j(1) = e_j$; note that the choice of τ_j may not be unique. We define an *adapted map* for $\mathcal{E}$ as follows:

$$F : [0,1]^k \to Hz, \quad (t_1, ..., t_k) \mapsto \prod_{j=1}^{k} \tau_j(t_j) \cdot z.$$

Lemma 4.20. *Let $(Z, z, H) \in \Omega_Q(\widetilde{M}, \mathcal{N})$ be a space of type (k, d) and let T_H be the maximal torus subgroup of H. Let $\mathcal{E} = \{e_1, ...e_k\} \subseteq H_0$ be an adapted basis with respect to (Z, z, H, T_H) and let $F : [0,1]^k \to Hz$ an adapted map for $\mathcal{E}$. Then F is a continuous injection.*

Proof. The continuity of F is clear. We prove its injectivity. Recall that $\mathcal{E} = \{e_1, ..., e_k\}$ is the lift of an adapted basis $\overline{\mathcal{E}} = \{\bar{e_1}, ..., \bar{e_k}\}$ with respect to the quotient space $(Z/T_H, \bar{z}, H/T_H)$. Note that a one-parameter subgroup τ_j through e_j projects to the unique one-parameter subgroup through $\bar{e_j}$. Let $\pi : Z \to Z/T_H$ be the quotient map. By construction,

$$(\pi \circ F)(t_1, ..., t_k) = \prod_{j=1}^{k} \pi \circ \tau_j(t_j) \cdot \bar{z} = \prod_{j=1}^{k} t_j \bar{e}_j \cdot \bar{z}.$$

Thus $\pi \circ F$ is the adapted map for $\overline{\mathcal{E}}$. By Lemma 4.18, $\pi \circ F$ is injective, thus F is injective as well. $\square$

Next, we consider the scenario for Proposition 4.9. Let $(Z, z, H) \in \Omega(\widetilde{M}, \mathcal{N})$ be a space of type (k, d) with $d \geq 5$. Let T_H be the maximal torus subgroup of H_0 and let $\{e_1, ..., e_k\}$ be an adapted basis with respect to (Z, z, H) as constructed in Definition 4.19. For the sake of a dimensional argument, we need an additional element from T_H. We choose an element $e_0 \in T_H$ such that
(1) $d(e_0 z, z) = 1$,
(2) there is a piece of one-parameter subgroup τ_0 from id to e_0 such that $\tau_0|_{(0,1]}$ is outside $\mathrm{Iso}(z, H_0)$.

Definition 4.21. Let $(Z, z, H) \in \Omega_Q(\widetilde{M}, \mathcal{N})$ be a space of type (k, d), where $d \geq 5$, and let T_H be the normal torus subgroup of H. We call the above constructed $\{e_0, e_1 ..., e_k\} \subseteq H_0$ an *adapted basis* with respect to (Y, y, H, T_H). Next, we construct an adapted map. For e_0, we have already chosen a piece of one-parameter subgroup τ_0 from id to e_0. For each e_j, where $j = 1, ..., k$, let τ_j be a piece of a one-parameter subgroup from id to e_j. We define an *adapted map* for $\mathcal{E}$:

$$F : [0, 1]^{k+1} \to Hz, \quad (t_0, t_1, ..., t_k) \mapsto \prod_{j=0}^{k} \tau_j(t_j) \cdot z.$$

Lemma 4.22. *Let $(Z, z, H) \in \Omega_Q(\widetilde{M}, \mathcal{N})$ be a space of type (k, d), where $d \geq 5$. Let $\mathcal{E} = \{e_0, e_1, ...e_k\} \subseteq H_0$ be an adapted basis with respect to (Z, z, H, T_H) and let $F : [0, 1]^{k+1} \to Hz$ be an adapted map for $\mathcal{E}$. Then F is a continuous injection.*

Proof. It is clear that F is continuous. Suppose that

$$F(t_0, t_1, ..., t_k) = F(t_0', t_1', ..., t_k').$$

Let $\pi : Z \to Z/T_H$ be the quotient map. By the proof of Lemma 4.20, $\pi \circ F$ is an adapted map and thus is injective. This shows that $t_j' = t_j$ for $j = 1, ..., k$. Now we have

$$\left(\prod_{j=1}^{k} \tau_j(t_j)\right) \tau_0(t_0) z = \left(\prod_{j=1}^{k} \tau_j(t_j)\right) \tau_0(t_0') z.$$

Thus $\tau_0(t_0) z = \tau_0(t_0') z$. Because τ_0 is constructed from a one-parameter subgroup, we have $\tau_0(t_0 - t_0') z = z$. Recall that $\tau|_{(0,1]}$ is outside $\mathrm{Iso}(z, H_0)$. Hence we must have $t_0 = t_0'$. $\square$

To complete this subsection, we use the properties of tunnels (Lemmas 4.5 and 4.6) to construct maps converging uniformly to an adapted map.

Lemma 4.23. *Let*

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$

be a convergent sequence of spaces in $\Omega_Q(\widetilde{M}, \mathcal{N})$. *Suppose that*
(1) each (Y_i, y_i, G_i) *is good in the sense of Definition 4.13;*
(2) on the limit space (Z, z, H)*, there is an adapted map* F *defined in either Definition 4.19 or 4.21 with domain* $[0,1]^k$ *or* $[0,1]^{k+1}$*, respectively.*
Then there is a sequence of continuous maps

$$F_i : [0,1]^k \text{ or } [0,1]^{k+1} \to G_i y_i \subseteq Y_i$$

that converges uniformly to F.

Proof. Before starting the proof, we remark that the limit space (Z, z, H) may not be good in the sense of Definition 4.13.

Let $J = \{1, ..., k\}$ or $\{0, 1, ..., k\}$. We use $\vec{t}$ to denote

$$\vec{t} = (t_1, ..., t_k) \in [0,1]^k \quad \text{or} \quad \vec{t} = (t_0, t_1, ..., t_k) \in [0,1]^{k+1}.$$

In both Definitions 4.19 and 4.21, the adapted map has the form

$$F : [0,1]^k \text{ or } [0,1]^{k+1} \to Hz, \quad \vec{t} \mapsto \prod_{j \in J} \tau_j(t_j) \cdot z,$$

where $\tau_j : [0,1] \to H_0$ is a piece of one-parameter subgroup from id to e_j, an element in the adapted basis $\mathcal{E}$. For each $j \in J$, let $\sigma_j(t) = \tau_j(t) \cdot z$, which is a tunnel from z to $e_j z$. By Lemma 4.5, there is a sequence of tunnels

$$\sigma_{i,j} : [0,1] \to G_i y_i$$

from y_i that converges uniformly to σ_j as $i \to \infty$. Because each (Y_i, y_i, G_i) is good, $(G_i)_0$ acts freely at y_i. Hence $\sigma_{i,j}$ uniquely determines a continuous path

$$\widetilde{\sigma_{i,j}} : [0,1] \to (G_i)_0$$

such that $\widetilde{\sigma_{i,j}}(t) \cdot y_i = \sigma_{i,j}(t)$. We construct F_i as

$$F_i : [0,1]^k \text{ or } [0,1]^{k+1} \to G_i y_i, \quad \vec{t} \mapsto \prod_{j \in J} \widetilde{\sigma_{i,j}}(t_j) \cdot y_i.$$

We prove that F_i converges uniformly to F. It suffices to show that for every convergent sequence

$$(\vec{t})_i = (t_{i,j})_{j \in J} \to \vec{t} = (t_j)_{j \in J},$$

it holds that $F_i((\vec{t})_i) \to F(\vec{t})$ as $i \to \infty$, that is,

$$\prod_{j \in J} \widetilde{\sigma_{i,j}}(t_j) \cdot y_i \to \prod_{j \in J} \tau_j(t_j) \cdot z$$

given $t_{i,j} \to t_j$ for each $j \in J$. By construction of $\widetilde{\sigma_{i,j}}$, we have $\widetilde{\sigma_{i,j}}(t_j)y_i$ converges to $\tau_j(t_j)z$. After passing to a subsequence, we assume that for each $j \in J$, $\widetilde{\sigma_{i,j}}(t_j)$ converges to some element in H as $i \to \infty$. By Lemma 4.6, we have

$$\widetilde{\sigma_{i,j}}(t_j) \overset{GH}{\to} \tau_j(t_j)h_j,$$

where $h_j \in \mathrm{Iso}(z, H_0)$. The compact subgroup $\mathrm{Iso}(z, H_0)$ must be contained in the maximal torus subgroup of H_0, thus each h_j is central in H_0 by Lemma 2.11. It follows that

$$\prod_{j\in J} \widetilde{\sigma_{i,j}}(t_j) \cdot y_i \to \prod_{j\in J} (\tau_j(t_j)h_j) \cdot z = \prod_{j\in J} \tau_j(t_j) \cdot \prod_{j\in J} h_j \cdot z = \prod_{j\in J} \tau_j(t_j) \cdot z.$$

This verifies the uniform convergence of F_i to F. □

4.4. **Proof of the distance gaps.** We prove Propositions 4.8 and 4.9 in this subsection.

Lemma 4.24. *Let (Y_i, y_i, G_i) be a sequence of spaces in $\Omega(\widetilde{M}, \mathcal{N})$ and let T_i is the maximal torus subgroup of G_i. Suppose that there is $D > 0$ such that $\mathrm{diam}(T_i y_i) \leq D$ for all i and*

$$(Y_i, y_i, G_i, T_i) \xrightarrow{GH} (Z, z, H, K).$$

Then

$$(Y_i/T_i, \bar{y}_i, G_i/T_i) \xrightarrow{GH} (Z/K, \bar{z}, H/K).$$

Proof. The proof is standard by approximation maps. We give some details below for readers' convenience.

Let

$$\epsilon_i = 10 \cdot d_{GH}((Y_i, y_i, G_i), (Z, z, H)) \to 0.$$

It is clear that $\mathrm{diam}(Kz) \leq D$. When $1/\epsilon_i \gg D$, we have a tuple of ϵ_i-approximation maps (f_i, ψ_i, ϕ_i), that is,

$$f_i : B_{1/\epsilon_i}(y_i) \to Z, \quad \psi_i : G_i(1/\epsilon_i) \to H(1/\epsilon_i), \quad \phi_i : H(1/\epsilon_i) \to G_i(1/\epsilon_i)$$

with the properties (1)-(5) in Definition 2.1 and
(6) $\psi_i(T_i) \subseteq K$, $\phi_i(K) \subseteq T_i$.
By Theorem 2.2(2), we have

$$(Y_i/T_i, \bar{y}_i) \xrightarrow{GH} (Z/K, \bar{z}).$$

Moreover, the approximation map $\bar{f}_i$ from $B_{1/(5\epsilon_i)}(\bar{y}_i) \subseteq Y_i/T_i$ to Z/K can be chosen as an quotient of f_i; more precisely, for each $\bar{x} \in B_{1/(5\epsilon_i)}(\bar{y}_i)$, we define

$$\bar{f}_i(\bar{x}) := \overline{f_i(x)} \in Z/K,$$

where $x \in B_{1/(5\epsilon_i)}(y)$ is a point projecting to $\bar{x} \in Y_i/T_i$.

Let $\bar{g} \in \frac{G_i}{T_i}(\frac{1}{5\epsilon_i})$, then there are $t_1, t_2 \in T_i$ and $g \in G_i$ projecting to $\bar{g}$ such that

$$d(t_1 g y_i, t_2 y_i) = d(Tgy, Ty) = d(\bar{g}\bar{y}_i, \bar{y}_i) \leq \frac{1}{5\epsilon_i}.$$

Thus

$$d(gy_i, y_i) \leq d(gy_i, t_1 g y_i) + d(t_1 g y_i, t_2 y_i) + d(t_2 y_i, y_i) \leq D + \frac{1}{5\epsilon_i} + D.$$

We define

$$\bar{\psi}_i : \frac{G_i}{T_i}\left(\frac{1}{5\epsilon_i}\right) \to \frac{H}{K}, \quad \bar{g} \mapsto \overline{\psi_i(g)}.$$

We estimate

$$\begin{aligned} d(\overline{\psi_i(g)}\bar{z}, \bar{z}) &= d(K\psi_i(g)z, Kz) \\ &= d(k_1\psi_i(g)z, k_2 z) \ \text{ for some } k_1, k_2 \in K \\ &\leq d(\psi_i(g)z, z) + d(k_1 z, z) + d(k_2 z, z) \\ &\leq d(\psi_i(g)z, gy_i) + d(gy_i, y_i) + d(y_i, z) + 2D \\ &\leq \epsilon_i + [2D + 1/(5\epsilon_i)] + \epsilon_i + 2D \\ &\leq 1/(10\epsilon_i). \end{aligned}$$

Thus $\mathrm{im}(\bar{\psi}_i) \subseteq \frac{H}{K}(\frac{1}{10\epsilon_i})$. For any $g \in \frac{G_i}{T_i}(\frac{1}{5\epsilon_i})$ and $\bar{x}, \bar{g}\bar{x} \in B_{1/\epsilon_i}(\bar{y}) \subset Y/T$,

$$\begin{aligned} d(\bar{f}_i(\bar{g}\bar{x}), \bar{\psi}_i \bar{f}_i(\bar{x})) &= d(\overline{f_i(gx)}, \overline{\psi_i(g)} \cdot \overline{f_i(x)}) \\ &= d(K \cdot f_i(gx), K \cdot \psi_i(g) \cdot f_i(x)) \\ &\leq d(f_i(gx), \psi_i(g) \cdot f_i(x)) \\ &\leq \epsilon_i. \end{aligned}$$

Similarly, we can construct

$$\bar{\phi}_i : \frac{H}{K}\left(\frac{1}{5\epsilon_i}\right) \to \frac{G_i}{T_i}, \quad \bar{h} \mapsto \overline{\phi_i(h)}$$

with the desired estimates. Therefore, $(\bar{f}_i, \bar{\psi}_i, \bar{\phi}_i)$ gives $(10\epsilon_i)$-approximation maps between $(Y_i/T_i, \bar{y}_i, G_i/T_i)$ and $(Z/K, \bar{z}, H/K)$. This completes the proof. □

Lemma 4.25. *Let*

$$(Y_i, y_i, G_i) \xrightarrow{GH} (Z, z, H)$$

be a convergent sequence of spaces in $\Omega(\widetilde{M}, \mathcal{N})$. *Let* (k_i, d_i) *be the type of* (Y_i, y_i, G_i) *and let* (k_∞, d_∞) *be the type of* (Z, z, H).
(1) Suppose that $k_i \geq k$ *for all* i, *then* $k_\infty \geq k$.
(2) Suppose that $k_i = k$ *and* $d_i \geq 10$ *for all* i, *then either* $k_\infty > k$ *holds, or* $k_\infty = k$ *and* $d_\infty \geq 10$ *hold.*

Proof. (1) For each i, let $\{e_{i,1}, ..., e_{i,k_i}\}$ be an adapted basis for (Y_i, y_i, G_i) as defined in Definition 4.19, where $k_i \geq k$. Let T_i be the maximal torus subgroup of G_i and let $L_{i,j}$ be the subgroup

$$L_{i,j} = \langle T_i, \mathbb{R}e_{i,1}, ..., \mathbb{R}e_{i,j} \rangle,$$

where $j = 1, ..., k$. We remark that although the one-parameter subgroup $\mathbb{R}e_{i,j}$ is not unique, the above defined subgroup $L_{i,j}$ is uniquely defined and independent of the choice of the one-parameter subgroup through e_j. We consider the convergence

$$(Y_i, y_i, G_i, T_i, L_{i,j}) \xrightarrow{GH} (Z, z, H, T_\infty, L_{\infty,j}).$$

Let (l_j, c_j) be the type of $(Z, z, L_{\infty,j})$. We prove $l_j \geq j$ by induction on j, then (1) follows by setting $j = k$.

Let $j = 1$. Note that the quotient group $L_{i,1}/T_i$ is a closed $\mathbb{R}$-subgroup of $\mathrm{Isom}(Y_i/T_i)$. Thus for each $\delta > 0$, there is an element $g_i(\delta) \in L_{i,1} - T_i$ such that

$$\delta = d(g_i(\delta) \cdot T_i y_i, T_i y_i) = d(g_i(\delta) \cdot y_i, T_i y_i).$$

Passing this property to the limit, then for any $\delta > 0$, we have an element $g_\infty(\delta) \in L_{\infty,1} - T_\infty$ such that

$$\delta = d(g_\infty(\delta) \cdot z, T_\infty z).$$

In particular, there is a closed $\mathbb{R}$-subgroup of $L_{\infty,1}$ that is outside T_∞. Thus $l_1 \geq 1$.

Suppose that the statement holds for j. We consider $j + 1$ next. The argument is similar to the case $j = 1$. Since $L_{i,j+1}/L_{i,j}$ is a closed $\mathbb{R}$-subgroup of $\mathrm{Isom}(Y_i/L_{i,j})$, for each $\delta > 0$ there is an element $g_i(\delta) \in L_{i,j+1} - L_{i,j}$ such that

$$\delta = d(g_i(\delta) \cdot y_i, L_{i,j} y_i).$$

We pass this property to the limit. Hence there is $g_\infty(\delta) \in L_{\infty,j+1} - L_{\infty,j}$ such that

$$\delta = d(g_\infty(\delta) \cdot z, L_{\infty,j} z).$$

This shows that there is a closed $\mathbb{R}$-subgroup of $L_{\infty,j+1}$ that is outside $L_{\infty,j}$. Together with the inductive assumption, we conclude that $l_{j+1} \geq j + 1$. This completes the induction.

(2) Assuming that $k_i = k$, $d_i \geq 10$ for all i, and $k_\infty = k$, we shall prove that $d_\infty \geq 10$. We follow the same notations as in the proof of (1); in particular, we have

$$(Y_i, y_i, G_i, T_i, L_{i,j}) \xrightarrow{GH} (Z, z, H, T_\infty, L_{\infty,j}).$$

We set $L_{\infty,0} := T_\infty$ for convenience. From the proof of (1), we know that for each $j = 1, ..., k$, there is a closed $\mathbb{R}$-subgroup in $L_{\infty,j} - L_{\infty,j-1}$. This implies that T_∞ must be compact; otherwise, T_∞ would contain a closed $\mathbb{R}$-subgroup and thus $k_\infty \geq k + 1$, which violates with $k_\infty = k$. It follows that there is $D > 0$ such that $d_i \leq D$ for all i. Thus

$$d_\infty = \lim_{i \to \infty} d_i \geq 10.$$

□

We are ready to prove the distance gaps.

Proof of Propositions 4.8 and 4.9. The proofs of these two statements are similar. We argue by contradiction to prove them.

For Proposition 4.8, suppose that there are two sequences $\{(Y_i, y_i, G_i)\}_i$ and $\{(Y_i', y_i', G_i')\}_i$ of spaces in $\Omega(\widetilde{M}, \mathcal{N})$ with the conditions below:
(1) each (Y_i, y_i, G_i) has type $(k, 0)$;
(2) each (Y_i', y_i', G_i') has type (k', d_i') with $k' > k$;
(3) $d_{GH}((Y_i, y_i, G_i), (Y_i', y_i', G_i')) \to 0$ as $i \to \infty$.
After passing to some subsequences, we can assume that two sequences converge to the same limit $(Z, z, H) \in \Omega(\widetilde{M}, \mathcal{N})$. We write (k_∞, d_∞) as the type of (Z, z, H). Condition (2) above

and Lemma 4.25(1) imply $k_\infty \geq k' > k$. Let T_i be the maximal torus subgroup of G_i and let $K \subseteq H$ be its limit. Since T_i fixes y_i, K must fix z as well. Thus the quotient space $(Z/K, \bar{z}, H/K)$ also has type (k_∞, d_∞). By Lemma 4.24, we have convergence

$$(Y_i/T_i, \bar{y}_i, G_i/T_i) \xrightarrow{GH} (Z/T_\infty, \bar{z}, H/T_\infty).$$

Let $F : [0,1]^{k_\infty} \to (H/T_\infty)\bar{z}$ be an adapted map constructed in Definition 4.19. F is continuous and injective by Lemma 4.20. According to Lemma 4.23, there is a sequence of continuous maps $F_i : [0,1]^{k_\infty} \to (G_i/T_i)\bar{y}_i$ converges uniformly to F. Note that each $(G_i/T_i)\bar{y}_i$ is homeomorphic to $\mathbb{R}^k$ with $k < k_\infty$. By large fiber lemma, there are two sequences $\{a_i\}$ and $\{b_i\}$ in $[0,1]^{k_\infty}$ such that

$$F_i(a_i) = F_i(b_i), \quad |a_i - b_i| \geq 1.$$

By the uniform convergence of F_i to F, we can find $a, b \in [0,1]^{k_\infty}$ such that

$$F(a) = F(b), \quad |a - b| \geq 1.$$

This contradicts the injectivity of F.

For Proposition 4.9, suppose that we have contradicting convergent sequences $\{(Y_i, y_i, G_i)\}_i$ and $\{(Y_i', y_i', G_i')\}_i$ such that
(1) each (Y_i, y_i, G_i) has type (k, d_i) with $d_i \leq 1$;
(2) each (Y_i', y_i', G_i') has type (k, d_i') with $d_i' \geq 10$;
(3) $d_{GH}((Y_i, y_i, G_i), (Y_i', y_i', G_i')) \to 0$ as $i \to \infty$.
After passing to some subsequences, we let (Z, z, H) be their common limit, whose type is denoted as (k_∞, d_∞). By Condition (2) above and Lemma 4.25(2), either
Case I. $k_\infty > k$, or
Case II. $k_\infty = k$ and $d_\infty \geq 10$.
Let T_i be the maximal torus subgroup of G_i and let $T_\infty \subseteq H$ be its limit. By Lemma 4.24, we have convergence

$$(Y_i/T_i, \bar{y}_i, G_i/T_i) \xrightarrow{GH} (Z/T_\infty, \bar{z}, H/T_\infty).$$

In Case I, we construct an adapted map $F : [0,1]^{k_\infty} \to (H/T_\infty)\bar{z}$ as in Definition 4.19. Then we use Lemma 4.23 to obtain $F_i : [0,1]^{k_\infty} \to (G_i/T_i)\bar{y}_i$ that converges uniformly to F, where $(G_i/T_i)\bar{y}_i$ is homeomorphic to $\mathbb{R}^k$ with $k < k_\infty$. Then a similar contradiction follows from the large fiber lemma and injectivity of F.

In Case II, because $\mathrm{diam}(T_i y_i) \leq 1$ for all i, the limit space $(Z/T_\infty, \bar{z}, H/T_\infty)$ has type $(k, \bar{d})$ with $\bar{d} \geq 9$. Then we construct an adapted map $F : [0,1]^{k+1} \to (H/T_\infty)\bar{z}$ as in Definition 4.21. A similar contradiction arises since the targets of the approximated maps $F_i : [0,1]^{k+1} \to (G_i/T_i)\bar{y}_i$ are homeomorphic to $\mathbb{R}^k$. □

5. Asymptotic orbits

This section studies the geometry of spaces in $\Omega(\widetilde{M}, \mathcal{N})$ and $\Omega(\widetilde{M}, \langle\gamma\rangle)$, where $\gamma \in \zeta_{l-1}(\mathcal{N})$. One of the main goals of this section is Proposition C(1), which states that Hy is homeomorphic to $\mathbb{R}$ for all $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$.

Some understandings of $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$ are required beforehand to prove Proposition C(1). In subsection 5.1, we show that every $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$ is of type $(k_0,0)$ for some uniform k_0 (Proposition 5.1). As mentioned in the introduction, the proof of Proposition 5.1 uses the distance gaps in Section 4 and a critical rescaling argument. Subsection 5.2 studies the one-parameter orbits of Gy and makes the preparation for Proposition C(1). Lastly, we prove Proposition C(1) in subsection 5.3. The proof of Proposition C(1) follow a similar strategy as [24, Section 3]: suppose that the statement fails, then we would find a suitable rescaling such that its limit space violates Proposition 5.1.

5.1. **Uniform type of asymptotic orbits.**

Proposition 5.1. *Let (M,p) be an open n-manifold with $\mathrm{Ric}\geq 0$ and $E(M,p)\neq 1/2$. Let $\mathcal{N}$ be a torsion-free nilpotent subgroup of finite index. Then there is an integer k_0 such that every $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$ is of type $(k_0,0)$. Consequently,*
(1) the orbit Gy has a natural simply connected nilpotent group structure of dimension k_0;
(2) any compact subgroup of G fixes y;
(3) G has at most finitely many components.

We first prove a weaker statement without a uniform k_0.

Lemma 5.2. *Under the assumptions of Proposition 5.1, let $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$. Then (Y,y,G) is of type $(k,0)$ for some integer k.*

Proof. We argue by contradiction. Suppose that the statement fails. Then we can choose $(Y,y,G)\in\Omega(\widetilde{M},\mathcal{N})$ with type (k_0,d) such that
(1) $d>0$, and
(2) if another space $(Y',y',G')\in\Omega(\widetilde{M},\mathcal{N})$ is of type (k',d') with $d'>0$, then $k'\geq k_0$.

Let
$$(Y_1,y_1,G_1)=(10d^{-1}Y,y,G),\quad (Y_2,y_2,G_2)=(d^{-1}Y,y,G)$$
be two spaces in $\Omega(\widetilde{M},\mathcal{N})$. It is clear that they are of type $(k_0,10)$ and $(k_0,1)$, respectively. Let $r_i,s_i\to\infty$ such that
$$(r_i^{-1}\widetilde{M},\tilde{p},\mathcal{N})\xrightarrow{GH}(Y_1,y_1,G_1),\quad (s_i^{-1}\widetilde{M},\tilde{p},\mathcal{N})\xrightarrow{GH}(Y_2,y_2,G_2).$$
Passing to some subsequence, we can assume that $t_i:=r_i/s_i\to\infty$. Let
$$(N_i,q_i,\Gamma_i)=(r_i^{-1}\widetilde{M},\tilde{p},\mathcal{N}),$$
then
$$(N_i,q_i,\Gamma_i)\xrightarrow{GH}(Y_1,y_1,G_1),\quad (t_iN_i,q_i,\Gamma_i)\xrightarrow{GH}(Y_2,y_2,G_2).$$

Let $\delta=\min\{\delta_1,\delta_2\}>0$, where δ_1 and δ_2 are the constants in Propositions 4.8 and 4.9, respectively. For each i, we define a set of scales
$$\begin{aligned}L_i=\{l\geq 1| d_{GH}((lN_i,q_i,\Gamma),(W,w,H))\leq\delta/10 \text{ for some }\\ (W,w,H)\in\Omega(\widetilde{M},\mathcal{N})\text{ such that }(W,w,H)\text{ has}\\ \text{type }(k,d)\text{ with }k<k_0,\text{ or with }k=k_0\text{ and }d\leq 1\}.\end{aligned}$$

Recall that (Y_2, y_2, G_2) is of type $(k_0, 1)$, thus $t_i \in L_i$ for all i large. We choose $l_i \in L_i$ with $\inf L_i \le l_i \le \inf L_i + 1/i$.

Claim 1: $l_i \to \infty$. Suppose that $l_i \to l_\infty < \infty$ for a subsequence, then

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (l_\infty Y_1, y_1, G_1).$$

Since $l_i \in L_i$, for each i there is some (W_i, w_i, H_i) with the properties in the definition of L_i such that

$$d_{GH}((l_i N_i, q_i, \Gamma_i), (W_i, w_i, H_i)) \le \delta/10.$$

Hence for i large,

$$d_{GH}((W_i, w_i, H_i), (l_\infty Y_1, y_1, G_1)) \le \delta/2,$$

where $(l_\infty Y_1, y_1, G_1)$ is of type $(k_0, 10 l_\infty)$ with $10 l_\infty \ge 10$. Let (k_i, d_i) be the type of (W_i, w_i, H_i). If $k_i < k_0$, then $d_i = 0$ by our choice of k_0, and the above Gromov-Hausdorff distance estimate cannot hold due to Proposition 4.8 and the choice of δ. If $k_i = k_0$ and $d_i \le 1$, then it also leads to a contradiction due to Proposition 4.9. Therefore, $l_i \to \infty$.

Next, we consider the convergence

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (Y', y', G') \in \Omega(\widetilde{M}, \mathcal{N}).$$

Let (k', d') be the type of (Y', y', G').

Claim 2: $k' \le k_0$; moreover, $d' < 10$ when $k' = k_0$. For each i, there is some (W_i, w_i, H_i) with the properties in the definition of L_i and

$$d_{GH}((l_i N_i, q_i, \Gamma_i), (W_i, w_i, H_i)) \le \delta/10.$$

It follows that for i large,

$$d_{GH}((W_i, w_i, H_i), (Y', y', G')) \le \delta/2.$$

If $k' > k_0$, then we end in a contradiction to Proposition 4.8. If $k' = k_0$ and $d' > 10$, then this contradicts with Proposition 4.9. This proves Claim 2.

By Claim 2 and our choice of k_0, (Y', y', G') has type (k', d') with one of the following cases:
Case 1: $k' < k_0$ and $d' = 0$;
Case 2: $k' = k_0$ and $d' < 10$.
For Case 1, we consider the convergence sequence

$$(\frac{1}{2} l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\frac{1}{2} Y', y', G') \in \Omega(\widetilde{M}, \mathcal{N}).$$

The limit space $(\frac{1}{2} Y', y', G')$ is of type $(k', 0)$, where $k' < k_0$. This implies that $l_i/2 \in L_i$, a contradiction to $\inf L_i \le l_i \le \inf L_i + 1/i$. For Case 2, we consider

$$(\frac{1}{10} l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\frac{1}{10} Y', y', G') \in \Omega(\widetilde{M}, \mathcal{N}),$$

where $(\frac{1}{10} Y', y', G')$ is of type $(k', d'/10)$ with $d'/10 < 1$. We result in $l_i/10 \in L_i$ for i large and thus a desired contradiction.

With all possibilities of (Y', y', G') being ruled out, we complete the proof. $\square$

Next, we prove Proposition 5.1.

Proof of Proposition 5.1. Let (Y, y, G) and (Y', y', G') in $\Omega(\widetilde{M}, \mathcal{N})$ having type $(k, 0)$ and $(k', 0)$, respectively. We show that $k = k'$. Let δ_1 be the constant in Proposition 4.8 and let $\epsilon = \delta_1/2$. Because the set $\Omega(\widetilde{M}, \mathcal{N})$ is connected in the pointed equivariant Gromov-Hausdorff topology (Proposition 2.5), for the above $\epsilon > 0$, there is a chain of elements $\{(W_j, w_j, H_j)\}_{j=1}^{J}$ in $\Omega(\widetilde{M}, \mathcal{N})$ such that

$$(W_1, w_1, H_1) = (Y, y, G), \quad (W_J, w_J, H_J) = (Y', y', G'),$$

and

$$d_{GH}((W_j, w_j, H_j), (W_{j+1}, w_{j+1}, H_{j+1})) \leq \epsilon$$

for all $j = 1, ..., J-1$. Applying Lemma 5.2, we know that each (W_j, w_j, H_j) is of type $(k_j, 0)$ for some k_j. By Proposition 4.8 and our choice of ϵ, all k_j must be the same. In particular, we conclude $k = k'$.

Now we proceed to prove the consequences (1)-(3).

To prove (1), let T be the maximal torus subgroup of G_0. Because (Y, y, G) is of type $(k_0, 0)$, T-action fixes y. Thus the orbit $Gy = G_0 y$ can be naturally identified with the quotient group G_0/T, which is a simply connected nilpotent Lie group of dimension k_0.

Next, we prove (2). Let K be a compact subgroup of G. If K is contained in a torus subgroup of G, then it is clearly that $Ky = y$ because (Y, y, G) is of type $(k, 0)$. In general, suppose that (2) fails, then we can find a finite cyclic subgroup $\langle h \rangle \subseteq K$ such that $hy \neq y$. Let $g \in G_0$ such that $hy = gy$. Note that g is outside the maximal torus T of G_0 because T fixes y; in particular, $\langle g \rangle y$ is unbounded in Y. On the other hand, because $hy = gy$ and Lemma 2.13, we have $h^k y = g^k y$ for all $k \in \mathbb{Z}$. Thus the set $\{g^k y | k \in \mathbb{Z}\}$ is bounded because $\langle h \rangle$ is a finite group. A contradiction.

Lastly, we prove (3). Suppose that G has infinitely many connected components. We consider the continuous map

$$A : G \to Gy, \quad g \mapsto gy.$$

Let $\mathcal{C}$ be any connected component of G. Because the orbit Gy is connected, we have $A(\mathcal{C}) = Gy$. Thus we can choose $g_{\mathcal{C}} \in \mathcal{C}$ such that $g_{\mathcal{C}} \cdot y = y$. By hypothesis, this gives a set of infinitely many elements

$$\mathcal{F} = \{g_{\mathcal{C}} \mid \mathcal{C} \text{ is a connected component of } G\}$$

that fixes y. Since the isotropy subgroup of G at y is compact, we can find a convergence subsequence from $\mathcal{F}$. This contradicts with the fact that elements in $\mathcal{F}$ belong to distinct connected components of G. $\square$

Remark 5.3. For any convergent sequence

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \mathcal{N}) \xrightarrow{GH} (Y, y, G),$$

it is clear that

$$\text{step}(Gy) \leq \text{step}(G) \leq \text{step}(\mathcal{N}) = \text{step}(\pi_1(M)),$$

where the nilpotent structure of Gy comes from Proposition 5.1(1).

5.2. **One-parameter orbits.**

Definition 5.4. Let $(Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N})$. Given an orbit point $z \in Gy - \{y\}$, we define the *one-parameter orbit* through z as the one-parameter subgroup of Gy through z, where the simply connected nilpotent Lie group structure of Gy is given in Proposition 5.1(1). By Lemma 2.12, this is well-defined. We denote this one-parameter orbit through z as $\mathbb{R}z$, and the points on it as tz, where $t \in \mathbb{R}$.

As indicated in its name, a one-parameter orbit is indeed the orbit of a one-parameter subgroup. More precisely, we have the description below.

Lemma 5.5. *Let $(Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N})$ and let $z \in Gy - \{y\}$ be an orbit point. We write $z = gy$, where $g \in G_0$. Let $\sigma : \mathbb{R} \to G_0$ be one-parameter subgroup with $\sigma(1) = g$. Then $\sigma(t)y = tz$ for all $t \in \mathbb{R}$.*

Proof. Let T be the maximal torus subgroup of G_0. We naturally identify the orbit Gy with the quotient group G_0/T as in the proof of Proposition 5.1(1). Let $\pi : G_0 \to G_0/T$ be the quotient map and let $\bar{g} = \pi(g)$. Because G_0/T is connected and simply connected, by Lemma 2.12, there is a unique one-parameter subgroup $\bar{\sigma} : \mathbb{R} \to G_0/T$ such that $\bar{\sigma}(1) = \bar{g}$. Since π is a group homomorphism, we have

$$\sigma(t)y = \pi \circ \sigma(t) = \bar{\sigma}(t).$$

Because the choice of $\bar{g} \in G_0/T$ and $\bar{\sigma}$ are uniquely determined by the orbit point $z = gy$, the result follows. □

Corollary 5.6. *Let $(Y, y, G) \in \Omega(\widetilde{M}, \mathcal{N})$ and let $z \in Gy - \{y\}$ be an orbit point. Let $g \in G_0$ such that $z = gy$. Then $g \cdot (tz) = (1 + t)z \in \mathbb{R}z$ for all $t \in \mathbb{R}$.*

Proof. Let $\sigma : \mathbb{R} \to G_0$ be a one-parameter subgroup of G_0 with $\sigma(1) = g$. By Lemma 5.5, for any $t \in \mathbb{R}$, we have

$$g \cdot (tz) = g \cdot \sigma(t)y = \sigma(1 + t)y = (1 + t)z.$$

The result follows. □

Given $\gamma \in \mathcal{N} - \{\mathrm{id}\}$ and $k \in \mathbb{Z}$, we denote

$$S(\gamma, k) := \{\gamma^m \mid m = 0, \pm 1, ..., \pm k\},$$

which is a symmetric subset of $\langle \gamma \rangle$. Recall that $\mathcal{N}$ is torsion-free, thus γ has infinite order. Below, we will use the notion of convergence of symmetric subsets; see Definition 2.3.

Lemma 5.7. *Let $\gamma \in \mathcal{N} - \{\mathrm{id}\}$. We consider the convergence*

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \gamma) \xrightarrow{GH} (Y, y, g).$$

Then there is a non-decreasing sequence $\{t_i\}$ in $\mathbb{Z}_+$ such that after passing to a subsequence, we have

$$(r_i^{-1}\widetilde{M},\tilde{p},\gamma^{t_i},S(\gamma,t_i))\xrightarrow{GH}(Y,y,\mathrm{id},S)$$

with $Sy=y$.

Proof. Note that g fixes the base point y. If g has finite order, we let $t\in\mathbb{Z}_+$ such that $g^t=\mathrm{id}$. Then it holds that

$$(r_i^{-1}\widetilde{M},\tilde{p},\gamma^{t},S(\gamma,t))\xrightarrow{GH}(Y,y,\mathrm{id},S)$$

with S fixing y.

It remains to consider the case that g has infinite order. Since g fixes y, $\langle g\rangle$ is a precompact subgroup of the isotropy subgroup at y. Let $t_j\to\infty$ be a sequence such that $g^{t_j}\to\mathrm{id}$ as $j\to\infty$. By a standard diagonal argument, we can choose a subsequence and $t_{i(j)}\to\infty$ such that

$$(r_{i(j)}^{-1}\widetilde{M},\tilde{p},\gamma^{t_{i(j)}})\xrightarrow{GH}(Y,y,\mathrm{id}).$$

Also, by construction, the limit of $S(\gamma,t_{i(j)})$ fixes y with respect to the above convergence. □

Lemma 5.8. *Let $\gamma\in\mathcal{N}-\{\mathrm{id}\}$ and $r_i\to\infty$. We choose t_i as in Lemma 5.7 and*

$$k_i:=\min\{l\in\mathbb{Z}_+\mid d(\gamma^{t_il}\tilde{p},\tilde{p})\geq r_i\}\to\infty.$$

Passing to a subsequence, we consider the convergence

$$(r_i^{-1}\widetilde{M},\tilde{p},\gamma^{t_ik_i},S(\gamma^{t_i},k_i),\langle\gamma\rangle,\mathcal{N})\xrightarrow{GH}(Y,y,g,A,H,G).$$

We denote $z=gy$. Then
(1) A is connected, in particular $A\subseteq G_0$;
(2) Ay contains the set $\{tz|t\in[0,1]\}$;
(3) Hy contains the one-parameter orbit $\mathbb{R}z$; moreover, there is a one-parameter subgroup L of H_0 such that $Ly=\mathbb{R}z$.

Proof. (1) First note that by the choice of k_i and the construction of A, it is clear that $Ay\subseteq\overline{B_1}(y)$.

We argue by contradiction to prove that A is connected. Suppose otherwise, then there is a point $w\in Y$ such that Aw has multiple connected components. Let $\alpha w\in Aw$ be a point outside the connected component of Aw that contains w. We choose a sequence of point $q_i\in\widetilde{M}$ and a sequence of integers m_i within $[-k_i,k_i]$ such that

$$(r_i^{-1}\widetilde{M},\tilde{p},q_i,\gamma^{t_im_i})\xrightarrow{GH}(Y,y,w,\alpha).$$

Replacing α by α^{-1} if necessary, we can assume that $m_i>0$. Recall that t_i is chosen in Lemma 5.7 such that γ^{t_i} converges to identity under the convergence; in particular, we have

$$r_i^{-1}d(\gamma^{t_i}q_i,q_i)\to 0$$

as $i\to\infty$. Since α moves w, we must have $m_i\to\infty$.

Let $\epsilon \ll d(w,\alpha w)$. For each sufficiently large i, we choose an integer s_i within $[1,m_i]$ such that
$$\epsilon/2 \le r_i^{-1} d(\gamma^{t_i s_i} q_i, q_i) \le \epsilon.$$
We consider the symmetric subset
$$T_i = \langle \gamma^{t_i s_i} \rangle \cap S(\gamma^{t_i}, m_i)$$
and its convergence
$$(r_i^{-1}\widetilde{M}, \tilde{p}, q_i, T_i) \xrightarrow{GH} (Y, y, q, T).$$
By construction and the compactness of $Tw \subseteq Aw \subseteq \overline{B_{1+2D}}(y)$, where $D = d(w,y)$, we can find an ϵ-chain $\{y_0 = y, y_1, ..., y_N = \alpha w\}$ in $Tw \cup \{\alpha w\} \subseteq Aw$ such that
$$d(y_j, y_{j+1}) \le \epsilon$$
for all j. Because the small $\epsilon > 0$ is arbitrary, this shows that αw belongs to the same connected component of Aw as w; a contradiction.

(2) Note that by the choice of k_i, it holds that
$$r_i \le d(\gamma^{t_i k_i}\tilde{p}, \tilde{p}) \le r_i + d(\gamma^{t_i}\tilde{p}, \tilde{p})$$
with $r_i^{-1} d(\gamma^{t_i}\tilde{p}, \tilde{p}) \to 0$. Thus $d(gy, y) = 1$. By (1), we have $g \in A \subseteq G_0$.

Below we write $\delta_i = \gamma^{t_i}$ for convenience. Let $b \ge 2$ be an integer. Due to the choice of k_i, we have
$$d(\delta_i^{\lceil k_i/b \rceil}\tilde{p}, \tilde{p}) < r_i,$$
where $\lceil \cdot \rceil$ is the ceiling function. After passing to a subsequence, we can assume that
$$b \cdot \lceil k_i/b \rceil = k_i + b_0$$
for all i, where $b_0 \in \{0, 1, ..., b-1\}$, and the convergence
$$(r_i^{-1}\widetilde{M}, \tilde{p}, \delta_i^{\lceil k_i/b \rceil}, \delta_i^{b_0}, \delta_i^{b\cdot\lceil k_i/b \rceil}) \xrightarrow{GH} (Y, y, \beta, \mathrm{id}, g).$$
By construction, we have $\beta^b = g$; moreover, $\beta \in A \subseteq G_0$. Let $\sigma : \mathbb{R} \to G_0$ be a one-parameter subgroup of G_0 with $\sigma(1) = \beta$. Then
$$\eta : \mathbb{R} \to G_0 \quad t \mapsto \sigma(bt)$$
defines a one-parameter subgroup of G_0 with $\eta(1) = \sigma(b) = g$. By Lemma 5.5, we have
$$\sigma(bt)y = \eta(t)y = tz$$
for all $t \in \mathbb{R}$, where $z = gy$.

We set $t = m/b$, where $m \in \{0, \pm 1, ..., \pm(b-1)\}$. Then we have
$$\frac{m}{b} z = \sigma(m)y = \beta^m y.$$
Note that β^m is the limit of $\delta_i^{m\lceil k_i/b \rceil} \in S(\delta_i, k_i)$. Thus $\frac{m}{b}z \in Ay$ for all $b \ge 2$ and all $m \in \{0, \pm 1, ..., \pm(b-1)\}$. We conclude that Ay must contain $\{tz | t \in [0,1]\}$.

(3) Using the notations in the proof of (2), we choose $t = m/b$, where $m \in \mathbb{Z}$. Then similarly, we have

$$\frac{m}{b}z = \beta^m y.$$

Since $\langle \beta \rangle \in H$, we conclude that the orbit Hy contains the points

$$\left\{ \frac{m}{b}z | m \in \mathbb{Z} \right\}.$$

Because the integer $b \geq 2$ is arbitrary, Hy must contain $\mathbb{R}z$.

Lastly, we show that $\mathbb{R}z = Ly$ for a one-parameter subgroup L of H_0. In fact, let $\mathcal{C}$ be the connected component of Hy that contains y. Note that

$$\mathbb{R}z \subseteq \mathcal{C} \subseteq H_0 y.$$

Now the result follows from Lemma 5.5. □

Remark 5.9. In the context of Lemma 5.8, if a sequence n_i satisfies $n_i \gg t_i k_i$, then with respect to the convergence

$$(r_i^{-1}\widetilde{M}, \tilde{p}, S(\gamma, n_i)) \xrightarrow{GH} (Y, y, B),$$

By must contain $\mathbb{R}z$. In fact, from the proof of Lemma 5.8(2), we have

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \delta_i^{\lceil k_i/b \rceil}) \xrightarrow{GH} (Y, y, \beta), \quad \frac{m}{b}z = \beta^m y,$$

where $b \in \mathbb{Z} \cap [2, \infty)$ and $m \in \mathbb{Z}$. Together with $n_i \gg t_i k_i$, $\frac{m}{b}z$ is the limit of $\delta_i^{m\lceil k_i/b \rceil} \cdot \tilde{p} \in S(\gamma, n_i) \cdot \tilde{p}$. It follows that By contains $\mathbb{R}z$.

5.3. **Proof of Proposition C(1).** We prove Proposition C(1) in this subsection. For convenience, we restate it as follows by using the terminology from Section 5.2.

Proposition 5.10. *Let (M, p) be an open n-manifold with* Ric ≥ 0 *and $E(M, p) \neq 1/2$. Let $\mathcal{N}$ be a torsion-free nilpotent subgroup of $\pi_1(M, p)$ with finite index and let l be the nilpotency step of $\mathcal{N}$.*

Then for every $\gamma \in \zeta_{l-1}(\mathcal{N}) - \{\mathrm{id}\}$ and every $(Y, y, H, G) \in \Omega(\widetilde{M}, \langle \gamma \rangle, \mathcal{N})$, the orbit Hy is exactly the one-parameter orbit $\mathbb{R}z$, where $z \in Hy - \{y\}$. Moreover, the orbit Hy can be represented by $\{\sigma(t)y | t \in \mathbb{R}\}$, where $\sigma : \mathbb{R} \to H_0 \subseteq Z(G)$ is a one-parameter subgroup through g with $gy \in Hy$.

Two corollaries below follow directly from Proposition 5.1(2).

Corollary 5.11. *Let $(Y, y, H) \in \Omega(\widetilde{M}, \langle \gamma \rangle)$ and let $h_1, h_2 \in H$. Suppose that there is an integer $m \geq 2$ such that $h_1^m y = h_2^m y$, then $h_1 y = h_2 y$.*

Proof. Let $r_i \to \infty$ such that

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \langle \gamma \rangle, \mathcal{N}) \xrightarrow{GH} (Y, y, H, G).$$

We first claim that if an element $h \in H$ satisfies $h^m y = y$ for some integer $m \geq 2$, then $hy = y$. In fact, let $H \leq G$ be the closure of the subgroup generated by h. By assumption, its orbit Hy consists of at most m many elements. In particular, H is compact. By Proposition 5.1(2), we conclude $Hy = y$.

Now let $h_1, h_2 \in H$ such that $h_1^m y = h_2^m y$. Because H is abelian, we have

$$(h_2^{-1}h_1)^m y = h_2^{-m}h_1^m y = y.$$

It follows from the Claim that $h_2^{-1}h_1 y = y$. □

Corollary 5.12. *Let $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$ and let S be a closed symmetric subset of H. Suppose that the set $Sy = \{sy | s \in S\}$ satisfies the following:*
(1) Sy is closed under multiplication, that is, if $s_1, s_2 \in S$, then $s_1 s_2 y \in Sy$;
(2) Sy is bounded.
Then $Sy = \{y\}$.

Proof. Let $L = \langle S\rangle$, the subgroup generated by S. By assumptions, $Ly = Sy$ is bounded. Thus the closure $\overline{L}$ is a compact subgroup of H. It follows from Proposition 5.1(2) that $\overline{L}$ must fix y. Thus $Sy = Ly = \{y\}$. □

We are ready to prove Proposition 5.10.

Proof of Proposition 5.10. From Lemma 5.8, we know that the limit orbit Hy from

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \langle\gamma\rangle, \mathcal{N}) \xrightarrow{GH} (Y, y, H, G)$$

contains a one-parameter orbit $\mathbb{R}z$, where $z = gy$ is constructed in Lemma 5.8. Below, we continue to use the notations from Lemma 5.8; in particular, we have

$$(r_i^{-1}\widetilde{M}, \tilde{p}), \gamma^{t_i k_i}, S(\gamma^{t_i}, k_i)) \xrightarrow{GH} (Y, y, g, A)$$

with

$$g \in A \subseteq G_0, \quad d(gy, y) = 1, \quad Ay \subseteq \overline{B_1}(y),$$

where the sequences t_i and k_i are described in Lemmas 5.7 and 5.8.

We argue by contradiction to show that $Hy = \mathbb{R}z$. Suppose that there is $h \in H$ such that $hy \notin \mathbb{R}z$.

Claim 1: Without lose of generality, we can assume that $d(hy, \mathbb{R}z) \geq 2$.

The element h may not be in G_0. However, because G has at most finitely many components (see Proposition 5.1(3)), we can find a power $n \in \mathbb{Z}_+$ such that $h^n \in G_0$. For this h^n, we still have the property that $h^n y \notin \mathbb{R}z$. In fact, suppose that $h^n y \in \mathbb{R}z$. By Lemma 5.8(3), we have a one parameter subgroup $\eta : \mathbb{R} \to H_0$ such that $\eta(1) = g_0 \in H_0$ and $g_0 y = h^n y$. Then by Lemma 5.11, we see that $hy = \eta(1/n)y \in \mathbb{R}z$; a contradiction. Now that we have $h^n \in H \cap G_0$ with $h^n y \notin \mathbb{R}z$; next, we show that $d((h^n)^m y, \mathbb{R}z)$ is unbounded as $m \to \infty$. In fact, let $\sigma : \mathbb{R} \to G_0$ be a one-parameter subgroup such that $\sigma(1)y = z$. Let $\overline{L}$ be the subgroup generated by elements in σ and T, the maximal torus subgroup of G_0. Because T is central in G_0, each element $\overline{L}$ can be expressed as $\sigma(t) \cdot \theta$, where $\theta \in T$. Moreover, $\overline{L}y = \mathbb{R}z$ because T fixes y according to Proposition 5.1; in particular, $h^n \notin \overline{L}$. By construction, the quotient group $G_0/\overline{L}$ is a connected and simply connected nilpotent Lie group and $G_0/\overline{L}$ acts on the quotient space $(Y/\overline{L}, \bar{y})$. Let $q : G_0 \to G_0/\overline{L}$ be the quotient homomorphism, then $q(h^n)$ generates a discrete $\mathbb{Z}$-subgroup in $G_0/\overline{L}$. Thus $d(q(h^n)^m \bar{y}, \bar{y})$ is unbounded as $m \to \infty$. As a consequence, $d((h^n)^m y, \mathbb{R}z)$ is unbounded as $m \to \infty$. To this end, we choose m such that $d((h^n)^m y, \mathbb{R}z) \geq 2$. Replacing h by h^{nm}, we complete Claim 1.

Let $m_i \in \mathbb{Z}$ such that

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{m_i}) \xrightarrow{GH} (Y, y, h).$$

Replacing h by h^{-1} if necessary, we can assume that $m_i > 0$.

Claim 2: $m_i \gg t_i k_i$.

By $d(hy, y) \geq 2$ and the choice of k_i, we clearly have $m_i > t_i k_i$. To prove Claim 2, suppose that $m_i/(t_i k_i) \to C < \infty$ for a subsequence, then we can write

$$m_i = \lfloor C \rfloor t_i k_i + o_i,$$

where $\lfloor \cdot \rfloor$ is the floor function and $o_i \in \mathbb{Z} \cap [0, t_i k_i]$. Passing to a subsequence, we have convergence

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{\lfloor C \rfloor t_i k_i}, \gamma^{o_i}, \gamma^{m_i}) \xrightarrow{GH} (Y, y, g^{\lfloor C \rfloor}, \delta, h),$$

where $\delta \in A$. Since $g \in G_0$, by Corollary 5.6, we have

$$g^{\lfloor C \rfloor} \cdot (tz) = (\lfloor C \rfloor + t)z \in \mathbb{R}z$$

for all $t \in \mathbb{R}$. Consequently,

$$d(hy, \mathbb{R}z) = d(g^{\lfloor C \rfloor}\delta y, \mathbb{R}z) = d(\delta y, \mathbb{R}z) \leq 1;$$

A contradiction to $d(hy, \mathbb{R}z) \geq 2$. This proves Claim 2.

For each i, we define

$$d_i = \max\{d(\gamma^k \tilde{p}, \tilde{p}) \mid k \in \mathbb{Z} \cap [t_i k_i, m_i]\}.$$

Claim 3: $d_i \gg r_i$.

It is clear that $d_i \geq r_i$. Suppose that $d_i/r_i \to C < \infty$ for a subsequence. Then we consider the convergence

$$(d_i^{-1}\widetilde{M}, \tilde{p}, S(\gamma, m_i)) \xrightarrow{GH} (C^{-1}Y, y, B).$$

By the proof of Lemma 5.8 and $m_i \gg t_i k_i$, By must contain $\mathbb{R}z$ (see Remark 5.9). Hence By is unbounded. On the other hand, by the choice of d_i, we should have $By \subseteq \overline{B_1}(y)$; a contradiction. This proves Claim 3.

Next, we consider the blow-down under $d_i \to \infty$:

$$(d_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{m_i}, S(\gamma, m_i), \langle \gamma \rangle) \xrightarrow{GH} (Y', y', h', B', H').$$

By the choice of d_i, we have $B'y' \subseteq \overline{B_1}(y')$. Also, note that Claim 3 implies that $h'y' = y'$.

Claim 4: $B'y'$ is closed under multiplication.

Let $\beta_1, \beta_2 \in B'$. We shall show $\beta_1\beta_2 y' \in B'y'$. We choose $b_{1,i}, b_{2,i} \in \mathbb{Z} \cap [-m_i, m_i]$ such that

$$(d_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{b_{1,i}}, \gamma^{b_{2,i}}) \xrightarrow{GH} (Y', y', \beta_1, \beta_2).$$

Then $\beta_1\beta_2$ is the limit of $\gamma^{b_{1,i}+b_{2,i}}$. If $b_{1,i}+b_{2,i}\in[-m_i,m_i]$, then $\beta_1\beta_2\in B'$ clearly holds. If not, we write

$$b_{1,i}+b_{2,i}=\pm m_i+o_i,$$

where $o_i\in\mathbb{Z}\cap[-m_i,m_i]$. Passing to a subsequence if necessary, we have

$$(d_i^{-1}\widetilde{M},\tilde{p},\gamma^{m_i},\gamma^{o_i})\overset{GH}{\longrightarrow}(Y',y',h',\beta_0),$$

where $\beta_0\in B'$. Then with respect to the blow-down sequence of $\widetilde{M}$ by d_i^{-1}, we have

$$\beta_1\beta_2 y'=\lim\gamma^{o_i}\gamma^{\pm m_i}\tilde{p}=\beta_0(h')^{\pm 1}y'=\beta_0 y'\in B'y'.$$

This proves Claim 4.

Lastly, by Claim 4 and Corollary 5.12, we conclude that $B'y'=\{y'\}$. On the other hand, the choice of d_i implies that $d_H(B'y',y')=1$. This contradiction shows that $Hy=\mathbb{R}z$ and thus completes the proof of Proposition 5.10. □

We complete this section by proving a distance control on the one-parameter orbit.

Lemma 5.13. *Under the assumptions of Proposition C, there is a constant $C_1=C_1(\widetilde{M},\gamma)$ such that for any $(Y,y,H)\in\Omega(\widetilde{M},\langle\gamma\rangle)$ and any orbit point $z\in Hy-\{y\}$, we have*

$$d(tz,y)\leq C_1\cdot d(z,y)$$

for all $t\in[0,1]$.

Proof. Scaling (Y,y,H) by a constant, we may assume that $d(z,y)=1$. Recall that by Proposition 5.10 we can choose $h\in H_0$ such that $z=hy$. We argue by contradiction and suppose that there are contradicting sequences $(Y_j,y_j,H_j)\in\Omega(\widetilde{M},\langle\gamma\rangle)$ and $h_j\in(H_j)_0$ with $d(h_jy_j,y_j)=1$ but

$$R_j:=\max_{t\in[0,1]}d(tz_j,y_j)\to\infty$$

as $j\to\infty$, where $z_j=h_jy_j\in Y_j$. For each j, we choose a one-parameter subgroup of $(H_j)_0$ through h_j and use th_j to denote elements in the subgroup, where $t\in\mathbb{R}$. By Lemma 5.5, $(th_j)y_j=tz_j$ for all $t\in\mathbb{R}$. We consider symmetric subsets

$$S_j=\{th_j|t\in[-1,1]\}$$

and the convergence

$$(R_j^{-1}Y_j,y_j,H_j,S_j)\overset{GH}{\longrightarrow}(Y',y',H',S'),$$

where $(Y',y',H')\in\Omega(\widetilde{M},\langle\gamma\rangle)$. Since $R_j\to\infty$ and $d(h_jy_j,y_j)=1$, we have $h_jy_j\overset{GH}{\to}y'$ and $S'y'\subseteq\bar{B}_1(y')$ with respect to the above convergence.

Claim: The set $S'y'$ is closed under multiplication. The proof is similar to Claim 4 in the proof of Proposition 5.10. Let $\beta_1,\beta_2\in S'$ and let $b_{j,1},b_{j,2}\in[-1,1]$ such that

$$(R_j^{-1}Y_j,y_j,b_{j,1}h_j,b_{j,2}h_j)\overset{GH}{\longrightarrow}(Y',y',\beta_1,\beta_2).$$

If $b_{j,1}+b_{j,2}\in[-1,1]$, then clearly $\beta_1\beta_2\in S'$. If not, we write

$$b_{j,1}+b_{j,2}=\pm 1+o_j,$$

where $o_j \in [-1, 1]$. In a convergent subsequence, we have

$$(R_j^{-1} Y_j, y_j, h_j, o_j h_j) \xrightarrow{GH} (Y', y', h', \beta_0)$$

with $\beta_0 \in S'$. Then with respect to this convergence,

$$\beta_1 \beta_2 y' = \lim_{j\to\infty} (o_j h_j) \cdot (\pm h_j) y_j = \beta_0 y' \in S' y'.$$

This proves the Claim.

Together with Corollary 5.12, we conclude $S'y' = y'$. On the other hand, by the construction of S_j and R_j, the limit $S'y'$ should have a point with distance 1 to y'. A contradiction. □

Remark 5.14. Let $z \in Hy \subseteq Gy$ as in Lemma 5.13 and let $d = d(z, y)$. Recall that by Proposition 4.3, there is a tunnel $\sigma : [0, 1] \to Gy$ from y to z that is contained in $\overline{B_{C_0 d}}(y)$. Lemma 5.13 shows that we can follow a specific tunnel, the one-parameter orbit, such that it is contained in $\overline{B_{C_1 d}}(y)$.

6. Hausdorff dimension and orbit distance estimates

This section studies the Hausdorff dimension of the orbit Hy in $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$ and proves Proposition C(2).

In subsection 6.1, we prove distance controls on the orbit Hy and show that the supremum of $\dim_{\mathcal{H}}(Hy)$ among all $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$ can be obtained (Proposition 6.4). In subsection 6.2, we relate the Hausdorff dimension of Hy to a lower bound on the orbit length (Proposition 6.7) and then complete the proof of Proposition C(2). Lastly, we have a short subsection 6.3 that relates Proposition C(2) to previous results on virtual abelianness [23, 24].

6.1. **Hausdorff dimension of one-parameter orbits.** We fix an element $\gamma \in \zeta_{l-1}(\mathcal{N}) - \{\mathrm{id}\}$. From Proposition C(1), we know that for all $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$, the orbit Hy is homeomorphic to $\mathbb{R}$. For each $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$, we choose an orbit point $z \in Hy$ with $d(z, y) = 1$. The choice of such a point z may not be unique since the orbit Hy may cross $\partial B_1(y)$ multiple times. By Lemma 5.13, we always have distance control

$$d(tz, y) \le C_1$$

for all $t \in [0, 1]$. For convenience, we denote

$$\Omega(\widetilde{M}, \langle\gamma\rangle, 1) = \{(Y, y, H, z) | (Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle), z \in Hy \cap \partial B_1(y)\}.$$

Given $(Y, y, H, z) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$, for each $L \in \mathbb{Z}_+$, we define

$$\mathcal{O}^L_{(Y,y,H,z)} = \{tz \mid t \in [0, 1/L]\} \subseteq Hy.$$

If the space (Y, y, H, z) is clear, we shall write $\mathcal{O}^L$ for simplicity.

Lemma 6.1. *Let $C_1 = C_1(\widetilde{M}, \gamma)$ be the constant in Lemma 5.13. Then the followings hold for all $(Y, y, H, z) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$ and all $L \in \mathbb{Z}_+$:*
(1) $\mathrm{diam}(\mathcal{O}^L) \le 2C_1^3 L^{-1/n}$*;*
(2) $(L + 1) \cdot \mathrm{diam}(\mathcal{O}^{L+1})^s \le L \cdot \mathrm{diam}(\mathcal{O}^L)^s + C_1^s$*, where* $s \ge 1$*.*

Proof. (1) We write $r = d(\frac{1}{L}z, y)$. First note that by Lemma 5.13, the points $\{\frac{j}{L}z\}_{j=0}^{L}$ are all contained in $\overline{B_{C_1}}(y)$.

We claim that the points $\{\frac{j}{L}z\}_{j=0}^{L}$ are pairwise $C_1^{-1}r$-disjoint. In fact, suppose that there are $j_1 < j_2$ in $\{0, 1, ..., L\}$ with

$$d\left(\frac{j_1}{L}z, \frac{j_2}{L}z\right) < C_1^{-1}r.$$

Since $\mathbb{R}z$ is represented by the orbit of a one-parameter subgroup at y, we have

$$d\left(\frac{j_1 - j_2}{L}z, y\right) < C_1^{-1}r.$$

However, by Lemma 5.13,

$$r = d((1/L)z, y) \leq C_1 \cdot d\left(\frac{j_1 - j_2}{L}z, y\right) < r.$$

A contradiction. This verifies the claim.

Next, by a standard packing argument with respect to a limit renormalized measure on Y, we have

$$L \leq \left(\frac{C_1}{C_1^{-1}r/2}\right)^n = (2C_1^2)^n \cdot r^{-n}.$$

Thus

$$\mathrm{diam}(\mathcal{O}^L) \leq C_1 \cdot r \leq 2C_1^3 \cdot L^{-1/n}.$$

(2) It is clear that

$$\mathrm{diam}(\mathcal{O}^{L+1}) \leq \mathrm{diam}(\mathcal{O}^L)$$

because $\mathcal{O}^{L+1} \subseteq \mathcal{O}^L$. Thus

$$\begin{aligned}(L+1) \cdot \mathrm{diam}(\mathcal{O}^{L+1})^s \leq & L \cdot \mathrm{diam}(\mathcal{O}^L)^s + \mathrm{diam}(\mathcal{O}^{L+1})^s \\ \leq & L \cdot \mathrm{diam}(\mathcal{O}^L)^s + C_1^s,\end{aligned}$$

where the last inequality holds because $\mathrm{diam}(\mathcal{O}^{L+1}) \leq C_1$. $\square$

Recall that for a metric space (X, d), we have definition

$$\mathcal{H}_\delta^s(X) = \inf\left\{\sum_{j=1}^{\infty} r_j^s \;\middle|\; X \subseteq \cup_{j=1}^{\infty} B_j, \text{ where each } B_j \text{ has diameter } r_j \leq \delta\right\},$$

then s-dimensional Hausdorff measure and Hausdorff dimension of X are defined by

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(X),$$

$$\dim_{\mathcal{H}}(X) = \inf\{s > 0 | \mathcal{H}^s(X) = 0\} = \sup\{s > 0 | \mathcal{H}^s(X) = \infty\}.$$

When X is compact, we can use finite covers $\{B_j\}$ instead of countable ones to define $\mathcal{H}_\delta^s(X)$.

Next, we use equal partitions of $\mathcal{O}^1$ to give an alternative way to calculate its Hausdorff dimension.

Definition 6.2. We define

$$\mathcal{E}^s(\mathcal{O}^1) = \liminf_{L\to\infty} L \cdot \mathrm{diam}(\mathcal{O}^L)^s,$$

where L takes values in $\mathbb{Z}_+$.

Lemma 6.3. *Given $s \geq 1$, there is a constant $C_2 = C_2(\widetilde{M}, \gamma, s) > 1$ such that*

$$\mathcal{H}^s(\mathcal{O}^1_{(Y,y,H,z)}) \leq \mathcal{E}^s(\mathcal{O}^1_{(Y,y,H,z)}) \leq C_2 \cdot \mathcal{H}^s(\mathcal{O}^1_{(Y,y,H,z)})$$

holds for any $(Y, y, H, z) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$. As a consequence,

$$\dim_{\mathcal{H}}(Hy) = \inf\{s > 0 | \mathcal{E}^s(\mathcal{O}^1) = 0\} = \sup\{s > 0 | \mathcal{E}^s(\mathcal{O}^1) = \infty\}.$$

Proof. We first show that $\mathcal{H}^s(\mathcal{O}^1) \leq \mathcal{E}^s(\mathcal{O}^1)$, which is straightforward. Let $L \in \mathbb{Z}_+$. Note that $\mathcal{O}^L$ has diameter at most $2C_1^3 L^{-1/n}$ by Lemma 6.1(1). For each $j = 1, ..., L$, we define

$$\mathcal{B}_j = \{tz \mid t \in [(j-1)/L, j/L]\}.$$

Then $\{\mathcal{B}_j\}_{j=1}^L$ covers $\mathcal{O}^1$ and each $\mathcal{B}_j$ has diameter at most $2C_1^3 L^{-1/n}$. This shows that for $\delta = 2C_1^3 L^{-1/n}$, we have

$$\mathcal{H}^s_\delta(\mathcal{O}^1) \leq \sum_{j=1}^{L} \mathrm{diam}(\mathcal{B}_j)^s = L \cdot \mathrm{diam}(\mathcal{O}^L)^s.$$

Let $L \to \infty$, then $\delta \to 0$ and we conclude that $\mathcal{H}^s(\mathcal{O}^1) \leq \mathcal{E}^s(\mathcal{O}^1)$.

Next, we prove that $\mathcal{E}^s(\mathcal{O}^1) \leq C_2 \cdot \mathcal{H}^s(\mathcal{O}^1)$ for some constant $C_2(\widetilde{M}, \gamma, s)$. Let $\delta > 0$. Let $\{B_j\}_{j=1}^J$ be a cover of $\mathcal{O}^1$ such that each B_j is contained in $\mathcal{O}^1$, has diameter $r_j \leq \delta$, and

$$\sum_{j=1}^{J} r_j^s \leq 2 \cdot \mathcal{H}^s_\delta(\mathcal{O}^1).$$

By replacing B_j by its closure, without loss of generality, we can assume that all B_j are closed. For each j, let $B'_j \subseteq \mathcal{O}^1$ be the smallest connected closed subset that contains B_j; in other words, we put

$$\alpha_j = \min\{t | tz \in B_j\}, \quad \beta_j = \max\{t | tz \in B_j\},$$

and

$$B'_j = \{tz | t \in [\alpha_j, \beta_j]\}.$$

Note that by Lemma 5.13,

$$\mathrm{diam}(B'_j) \leq C_1 \cdot d(\alpha_j z, \beta_j z) \leq C_1 r_j.$$

We further modify the cover $\{B'_j\}_{j=1}^J$ to a cover $\{D'_j\}_{j\in I}$ in the following way, where the index set I has cardinality at most J and the adjacent D'_j and D'_{j+1} have exactly one common point. We define the index set

$$I = \{j = 1, ..., J | B'_j \text{ is not contained in } B'_m \text{ for any } m \neq j\}.$$

Clearly, $\{B'_j\}_{j\in I}$ still covers $\mathcal{O}^1$. We rearrange $\{B'_j\}_{j\in I}$ by the order of their left endpoints and then relabel the sets in $\{B'_j\}_{j\in I}$ to $\{D_j\}_{j=1}^{|I|}$. Let

$$\alpha'_j = \min\{t|tz \in D_j\}, \quad \beta'_j = \max\{t|tz \in D_j\};$$

then $\alpha'_j < \alpha'_{j+1}$, $\beta'_j < \beta'_{j+1}$ for all $j = 1, ..., |I|-1$ and $\beta'_{|I|} = 1$. Now we define

$$D'_j = \{tz|t \in [\alpha'_j, \alpha'_{j+1}]\}$$

for each $j = 1, ..., |I|-1$ and define the last one $D'_{|I|} = D_{|I|}$. By construction, $\{D'_j\}_{j=1}^{|I|}$ covers $\mathcal{O}^1$; moreover, each D'_j is not a single point and the adjacent two share exactly one common point.

Let $d_j = \mathrm{diam}(D'_j)$. By construction, each D'_j is contained in D_j, and each D_j is indeed the relabel of some unique B'_m, thus

$$\sum_{j=1}^{|I|} d_j^s \le \sum_{j=1}^{J} \mathrm{diam}(B'_j)^s \le C_1^s \cdot \sum_{j=1}^{J} r_j^s \le 2C_1^s \cdot \mathcal{H}^s_\delta(\mathcal{O}^1).$$

For each $j = 1, ..., |I|$, we put

$$\tau_j = \max\{t|tz \in D'_j\} - \min\{t|tz \in D'_j\}.$$

Because two adjacent D'_j and D'_{j+1} share exactly one common point, we see that $\sum_{j=1}^{|I|} \tau_j = 1$. Let $j_0 \in \{1, ..., |I|\}$ such that

$$\frac{d_{j_0}^s}{\tau_{j_0}} = \min_{j\in I} \frac{d_j^s}{\tau_j} =: \rho.$$

We choose $L \in \mathbb{Z}_+$ with $1 \le L\tau_{j_0} \le 2$. Then

$$\begin{aligned} L \cdot \mathrm{diam}(\mathcal{O}^L)^s &\le L \cdot \mathrm{diam}(D'_{j_0})^s = L \cdot \rho\tau_{j_0} \\ &\le 2\rho = 2\rho \sum_{j=1}^{|I|} \tau_j \le 2\sum_{j=1}^{|I|} d_j^s \le 4C_1^s \cdot \mathcal{H}^s_\delta(\mathcal{O}^1). \end{aligned}$$

By triangle inequality,

$$L \cdot \mathrm{diam}(\mathcal{O}^L) \ge L \cdot d((1/L)z, y) \ge d(z, y) = 1.$$

Thus

$$L \ge \frac{1}{\mathrm{diam}(\mathcal{O}^L)} \ge \frac{1}{\mathrm{diam}(D'_{j_0})} \ge \frac{1}{C_1\delta}.$$

This means that the above chosen $L \to \infty$ as $\delta \to 0$. Let $\delta \to 0$; we conclude that

$$\mathcal{E}^s(\mathcal{O}^1) \le 4C_1^s \cdot \mathcal{H}^s(\mathcal{O}^1).$$

□

Proposition 6.4. $\mathcal{D} := \sup\{\dim_{\mathcal{H}}(Hy)|(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)\}$ *can be obtained.*

Proof. If $\mathcal{D} = 1$, then $\dim_{\mathcal{H}}(Hy) = 1$ for all $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$ and the result holds trivially. Below, we assume that $\mathcal{D} > 1$.

Let $\{(Y_j, y_j, H_j, z_j)\}_j$ be a sequence of spaces in $\Omega(\widetilde{M}, \langle\gamma\rangle, 1)$ such that

$$\lim_{j\to\infty} \dim_{\mathcal{H}}(H_j y_j) \to \mathcal{D},$$

Let $s \in (1, \mathcal{D})$, then $s < \dim_{\mathcal{H}}(H_j y_j)$ for all j large. By Lemma 6.3, this implies that

$$L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y_j,y_j,H_j,z_j)})^s \to \infty$$

as $L \to \infty$. Together with Lemma 6.1(2), for each $j \in \mathbb{N}$, there is an integer L_j such that
(1) $L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y_j,y_j,H_j,z_j)})^s \geq 2^j$ for all $L \geq L_j$, and
(2) $L_j \cdot \mathrm{diam}(\mathcal{O}^{L_j}_{(Y_j,y_j,H_j,z_j)})^s \in [2^j, 2^j + C_1^s]$.
Let $r_j = d((1/L_j)z_j, y_j)$. By Lemma 6.1(1), $r_j \to 0$ as $j \to \infty$. Passing to a subsequence if necessary, we consider

$$(r_j^{-1} Y_j, y_j, H_j, (1/L_j)z_j) \xrightarrow{GH} (Y', y', H', z') \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1).$$

Let $K \in \mathbb{Z}_+$. We estimate that

$$\begin{aligned} r_j^{-1} \cdot \mathrm{diam}(\mathcal{O}^{KL_j}_{(Y_j,y_j,H_j,z_j)}) \geq & \frac{\mathrm{diam}(\mathcal{O}^{KL_j}_{(Y_j,y_j,H_j,z_j)})}{\mathrm{diam}(\mathcal{O}^{L_j}_{(Y_j,y_j,H_j,z_j)})} \\ \geq & \frac{\left(\dfrac{2^j}{KL_j}\right)^{1/s}}{\left(\dfrac{2^j + C_1^s}{L_j}\right)^{1/s}} \to \left(\frac{1}{K}\right)^{1/s} \end{aligned}$$

as $j \to \infty$. Note that for any $K \in \mathbb{Z}_+$, it holds the convergence

$$(r_j^{-1} Y_j, y_j, \mathcal{O}^{KL_j}_{(Y_j,y_j,H_j,z_j)}) \xrightarrow{GH} (Y', y', \mathcal{O}^K_{(Y',y',H',z')}).$$

Thus

$$K \cdot \mathrm{diam}(\mathcal{O}^K_{(Y',y',H',z')})^s \geq K \cdot \frac{1}{K} = 1$$

for any integer $K \geq 1$. This shows that $\dim_{\mathcal{H}}(H'y') \geq s$ by Lemma 6.3. Since s is arbitrarily chosen within $(1, \mathcal{D})$, it follows that $\dim_{\mathcal{H}}(H'y') = \mathcal{D}$. □

6.2. **Proof of Proposition C(2).** We continue to use the notation $\mathcal{D}$ as in Proposition 6.4.

Lemma 6.5. *Let $s > \mathcal{D}$ and $\epsilon > 0$. Then there is a constant $L_0 = L_0(\epsilon, s, \widetilde{M}, \gamma)$ such that for all $(Y, y, H, z) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$, there exists an integer $2 \leq L \leq L_0$ with*

$$L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y,y,H,z)})^s \leq \epsilon.$$

Proof. We argue by contradiction. Suppose that for each integer $L_j = j$, there is some space $(Y_j, y_j, H_j, z_j) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$ such that

$$L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y_j,y_j,H_j,z_j)})^s > \epsilon$$

for all $2 \le L \le L_j$. After passing to a subsequence, we consider

$$(Y_j, y_j, H_j, z_j) \xrightarrow{GH} (Y', y', H', z').$$

For any integer $L \ge 2$, we observe that

$$L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y',y',H',z')})^s = \lim_{j\to\infty} L \cdot \mathrm{diam}(\mathcal{O}^L_{(Y_j,y_j,H_j,z_j)})^s \ge \epsilon.$$

Thus $\dim_{\mathcal{H}}(H'y') \ge s$ by Lemma 6.3, which is a contradiction to $s > \mathcal{D}$. $\square$

Below we write $|\gamma| = d(\gamma\tilde{p}, \tilde{p})$ for convenience. Next, we transfer Lemma 6.5 to a distance estimate of $\langle\gamma\rangle$-action on $\widetilde{M}$.

Lemma 6.6. *Let $s > \mathcal{D}$. Then there are constants $L' = L'(s, \widetilde{M}, \gamma)$ and $R = R(s, \widetilde{M}, \gamma)$ such that the following holds.*

For any $\gamma^b \in \langle\gamma\rangle$ with $|\gamma^b| \ge R$, where $b \in \mathbb{Z}_+$, there exists an integer $2 \le L \le L'$ such that

$$|\gamma^b| \ge L^{1/s} \cdot |\gamma^{\lceil b/L \rceil}|,$$

where $\lceil \cdot \rceil$ means the ceiling function.

Proof. We choose $L'(s, \widetilde{M}, \gamma) = L_0(\frac{1}{2}, s, \widetilde{M}, \gamma)$, the constant in Lemma 6.5. We argue by contradiction to prove the statement. Suppose that there is a sequence $b_i \to \infty$ such that

$$|\gamma^{b_i}| \le L^{1/s} \cdot |\gamma^{\lceil b_i/L \rceil}|$$

for all $L = 2, ..., L'$. Let $r_i = |\gamma^{b_i}| \to \infty$, we consider

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \langle\gamma\rangle, \gamma^{b_i}) \xrightarrow{GH} (Y, y, H, g).$$

It is clear that $g \in H$ satisfies $d(gy, y) = 1$. We put $z = gy$.

We claim that for each integer $L \in \mathbb{Z}_+$, we have convergence

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{\lceil b_i/L \rceil}\tilde{p}) \overset{GH}{\to} (Y, y, (1/L)z).$$

In fact, we can write

$$\lceil b_i/L \rceil \cdot L = b_i + o_i,$$

where $o_i \in \{0, 1, ..., L-1\}$. After passing to a subsequence, we have

$$(r_i^{-1}\widetilde{M}, \tilde{p}, \gamma^{\lceil b_i/L \rceil}, \gamma^{o_i}) \xrightarrow{GH} (Y, y, h, \delta)$$

with $h^L = g\delta$; moreover, $\delta y = y$ because each o_i is at most $L-1$. Thus we have

$$h^L y = g\delta y = gy = z.$$

By Lemma 5.8(3), we have a one-parameter subgroup $\sigma : \mathbb{R} \to H_0$ with $\sigma(1) = g'$ and $g'y = z$. Then by Corollary 5.11 and the fact that $h^L y = \sigma(1)y$, we conclude that

$$hy = \sigma(1/L)y = (1/L)z.$$

This proves the Claim.

From the above Claim and the hypothesis, we have

$$\operatorname{diam}(\mathcal{O}^L_{(Y,y,H,z)}) \geq d((1/L)z, y) = \lim_{i\to\infty} \frac{d(\gamma^{\lceil b_i/L\rceil}\tilde{p},\tilde{p})}{d(\gamma^{b_i}\tilde{p},\tilde{p})} \geq \left(\frac{1}{L}\right)^{1/s}$$

for all $L \in \{2, ..., L'\}$. On the other hand, by the choice $L' = L_0(\frac{1}{2}, s, \widetilde{M}, \gamma)$ and Lemma 6.5,

$$\operatorname{diam}(\mathcal{O}^L_{(Y,y,H,z)}) \leq \left(\frac{1}{2L}\right)^{1/s}$$

for some $L \in \{2, ..., L'\}$. A contradiction. □

Then we use Lemma 6.6 repeatedly to derive a lower bound for $|\gamma^b|$.

Proposition 6.7. *Let $s > \mathcal{D}$. Then there is a constant $C_3 = C_3(s, \widetilde{M}, \gamma)$ such that*

$$|\gamma^b| \geq C_3 \cdot b^{1/s}$$

for all $b \in \mathbb{Z}_+$ large.

Proof of Proposition C(2). Let $s > \mathcal{D}$ and let P_0 be a large integer such that

$$|\gamma^b| \geq R(s, \widetilde{M}, \gamma)$$

holds for all $b \geq P_0$, where $R(s, \widetilde{M}, \gamma)$ is the constant in Lemma 6.6.

Let $b > P_0$. By Lemma 6.6, there is some integer $L_1 \in \{2, ..., L'\}$ such that

$$|\gamma^b| \geq L_1^{1/s} \cdot |\gamma^{\lceil b/L_1\rceil}|.$$

If $\lceil b/L_1\rceil \leq P_0$, then we stop here. Otherwise, we apply Lemma 6.6 again to find some integer $L_2 \in \{2, ..., L'\}$ such that

$$|\gamma^b| \geq L_1^{1/s} \cdot |\gamma^{\lceil b/L_1\rceil}| \geq (L_1L_2)^{1/s} \cdot |\gamma^{\lceil\lceil b/L_1\rceil/L_2\rceil}|.$$

Repeating this process, we eventually obtain

$$|\gamma^b| \geq \left(\prod_j L_j\right)^{1/s} \cdot |\gamma^{\lceil...\lceil b/L_1\rceil/L_2.../L_k\rceil}| \geq \left(\prod_j L_j\right)^{1/s} \cdot r_0,$$

where $\lceil...\lceil b/L_1\rceil/L_2.../L_k\rceil \leq P_0$ and $r_0 = \min_{m\in\mathbb{Z}} |\gamma^m|$. Note that

$$\frac{b}{\prod_j L_j} \leq \lceil...\lceil b/L_1\rceil/L_2.../L_k\rceil \leq P_0.$$

It follows that

$$|\gamma^b| \geq \left(\frac{b}{P_0}\right)^{1/s} \cdot r_0 = C_3 \cdot b^{1/s},$$

where $C_3 = r_0/P_0^{1/s}$. □

As indicated in the introduction, Proposition C(2) follows immediately from Proposition 6.7 and Corollary 2.10.

Proof of Proposition C(2). Let $s > \mathcal{D}$. By Proposition 6.7, we have a lower bound
$$|\gamma^b| \geq C_3 \cdot b^{1/l}$$
for all $b \in \mathbb{Z}_+$ large. On the other hand, by Corollary 2.10, we have an upper bound
$$|\gamma^b| \leq C_4 \cdot b^{1/l}$$
for all $b \in \mathbb{Z}_+$. Combining these two inequalities together, we obtain
$$C_3 \cdot b^{1/s} \leq C_4 \cdot b^{1/l}$$
for all b large. We conclude that $s \geq l$. Recall that $s > \mathcal{D}$ is arbitrary, thus $\mathcal{D} \geq l$. Lastly, by Proposition 6.4, there exists $(Y, y, H) \in \Omega(\widetilde{M}, \langle\gamma\rangle)$ such that
$$\dim_{\mathcal{H}}(Hy) = \mathcal{D} \geq l.$$
This completes the proof. $\square$

6.3. **Relations to previous results on virtual abelianness.** As indicated in the introduction, Proposition C(2) immediately recovers the main result on metric cones and virtual abelianness in [24]. Recall that an open manifold with $\mathrm{Ric} \geq 0$ is *conic at infinity*, if every asymptotic cone (Y, y) is a metric cone with vertex y.

Corollary 6.8. *Let (M, p) be an open n-manifold with $\mathrm{Ric} \geq 0$ and $E(M, p) \neq 1/2$. If its Riemannian universal cover is conic at infinity, then $\pi_1(M)$ is virtually abelian.*

Proof. We shall show that
$$\dim_{\mathcal{H}}(Ly) = 1$$
for any $(Y, y) \in \Omega(\widetilde{M})$ and any closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$. Then by Proposition C(2), we have
$$\mathrm{step}(\pi_1(M)) \leq 1,$$
that is, $\pi_1(M)$ is virtually abelian (see Definition 2.8).

The verification of $\dim_{\mathcal{H}}(Ly) = 1$ is standard given the metric cone structure. We give some details below for readers' convenience. Let (Y, y) be any asymptotic cone of $\widetilde{M}$. By assumption, Y is a metric cone with vertex y. By Cheeger-Colding splitting theorem, (Y, y) splits isometrically as
$$(\mathbb{R}^k \times C(Z), (0, z)),$$
where $k \in \mathbb{N} \cap [0, n]$, $C(Z)$ is a metric cone without lines, and z is the unique vertex of $C(Z)$. Moreover, its isometry group also splits as a product
$$\mathrm{Isom}(Y) = \mathrm{Isom}(\mathbb{R}^k) \times \mathrm{Isom}(C(Z)).$$
Because z is the unique vertex of $C(Z)$, any isometry of $C(Z)$ must fix z. As a consequence,
$$g \cdot y = g \cdot (0, z) \in \mathbb{R}^k \times \{z\}$$
for any isometry g of Y. In particular, we have the orbit
$$Ly \subseteq \mathbb{R}^k \times \{z\}$$

for any closed $\mathbb{R}$-subgroup L of $\mathrm{Isom}(Y)$. Thus we can view Ly as a C^1-curve in the Euclidean factor $\mathbb{R}^k \times \{z\}$. Consequently, Ly must have Hausdorff dimension 1. □

Proposition C(2) also extends the main result on small escape rate and virtual abelianness in [23]. To explain this, we prove a proposition below, which is based on the results in this paper and [23]. We continue to use the notation $\mathcal{D}$ from Proposition 6.4.

Proposition 6.9. *Let M be an open n-manifold with $\mathrm{Ric} \geq 0$ and $E(M,p) \leq \epsilon$, where $\epsilon > 0$ is a small number. Then*

$$\mathcal{D} \leq 1 + \delta(\epsilon|n),$$

where $\delta(\epsilon|n) \to 0$ as $\epsilon \to 0$.

Proof. Let $\delta > 0$. Suppose that $\mathcal{D} = 1 + \delta$. We set $s = 1 + \delta/2$. Following the proof of Proposition 6.4, we can find a space $(Y, y, H, z) \in \Omega(\widetilde{M}, \langle\gamma\rangle, 1)$ such that

$$K \cdot \mathrm{diam}(\mathcal{O}^K)^s \geq 1$$

for any integer $K \geq 1$. On the other hand, using the small escape rate condition, it follows from [23, Theorem 0.1 and Lemma 4.6] that

$$d_{GH}((Y, y, H, z), (\mathbb{R}^k \times X, (0, x), \mathbb{R}, (1, x))) \leq \Psi(\epsilon|n),$$

where $\times$ means a metric product and the group $\mathbb{R}$ acts as translations in $\mathbb{R}^k \times X$. For a fixed $\delta > 0$ and $s = 1 + \delta/2$, we can choose a large integer K such that

$$2^s K^{1-s} \leq 1/2.$$

When ϵ is so small that $\Psi(\epsilon|n) \leq 1/K$, we have

$$1 \leq K \cdot \mathrm{diam}(\mathcal{O}^K_{(Y,y,H,z)})^s \leq K \cdot (\frac{1}{K} + \Psi)^s \leq 2^s K^{1-s} \leq 1/2.$$

This clear contradiction shows that $\delta \to 0$ as $\epsilon \to 0$. □

Combining Proposition 6.9 and Proposition C(2), it is clear that $\mathrm{step}(\pi_1(M)) \geq 2$ implies $E(M,p) > \epsilon(n)$ for some universal constant $\epsilon(n)$, which is the main result in [23].

Appendix A. An application of the slice theorem

In this appendix, we prove Proposition A.1 used in the proof of Lemma 4.6.

Proposition A.1. *Let Y be a Ricci limit space and let G be a closed nilpotent subgroup of $\mathrm{Isom}(Y)$. Suppose that $g \in G$ satisfies $gy \in G_0 y$ for all $y \in Y$, then $g \in G_0$.*

The proof of Proposition A.1 depends on a slice theorem by Palais [19]. For other applications of the slice theorem to Ricci limit spaces, see [26].

For the convenience of readers, we recall some basic notions. Let Y be a completely regular topological space and let G be a Lie group acting properly and effectively on Y by homeomorphisms. Let $y \in Y$ and let $H = \mathrm{Iso}(G, y)$ be the isotropy subgroup at y. We say a subset $S \subseteq Y$ is H-invariant, if $H \cdot S = S$. Given an H-invariant subset S, we define a space $G \times_H S$ as a quotient $(G \times S)/\sim$, where the equivalence relation on $G \times S$ is given by

$$(g, s) \sim (gh, h^{-1}s)$$

for all $g \in G$, $h \in H$ and $s \in S$. We use the $[g, s]$ to write elements in this quotient space $G \times_H S$, where $g \in G$ and $s \in S$. The space $G \times_H S$ has a natural G-action by

$$g' \cdot [g, s] = [g'g, s],$$

where $g' \in G$ and $[g, s] \in G \times_H S$.

Definition A.2. We call $S \subseteq Y$ a slice at a point $y \in Y$, if the followings hold:
(1) S is H-invariant, where $H = \mathrm{Iso}(G, y)$;
(2) $y \in S$ and $G \cdot S$ is an open neighborhood of y;
(3) the map

$$\psi : G \times_H S \to G \cdot S, \quad [g, s] \mapsto g \cdot s$$

is an G-equivariant homeomorphism.

The slice theorem by Palais [19] guarantees the existence of a slice when G is a Lie group. We formulate its statement in our context as follows. Because the isometry group of a Ricci limit space is always Lie [7], the slice theorem applies.

Theorem A.3. [19] *Let Y be a Ricci limit space and let G be a closed subgroup of* $\mathrm{Isom}(Y)$. *Then for every $y \in Y$, there is a slice S at y.*

Note that the isotropy subgroup at y and at gy are related by a conjugation

$$\mathrm{Iso}(G, gy) = g \cdot \mathrm{Iso}(G, y) \cdot g^{-1}.$$

Hence each orbit Gy corresponds to a conjugacy class of isotropy subgroups. Let

$$\mathcal{I}(G) = \{H \leq G | H = \mathrm{Iso}(G, y) \text{ for some } y \in Y\}$$

be the set of all isotropy subgroups. For $H \in \mathcal{I}(G)$, we denote by $[H]$ the conjugacy class of H. We define a partial order on the set of conjugacy classes: $[H] \leq [K]$ if H is conjugate to a subgroup of K by some element $g \in G$.

Definition A.4. Let $H \in \mathcal{I}(G)$. We say that $[H]$ is principal, if $[K] \leq [H]$ implies $[K] = [H]$, where $K \in \mathcal{I}(G)$.

Lemma A.5. *Let Y be a Ricci limit space and let G be a closed subgroup of* $\mathrm{Isom}(Y)$. *Then there is $H \in \mathcal{I}(G)$ such that $[H]$ is principle.*

Proof. It suffices to show that any descending sequence

$$[H_1] \geq [H_2] \geq ... \geq [H_i] \geq ...$$

must stabilize. Since $H_i \in \mathcal{I}(G)$, it is a compact subgroup of G; in particular, each H_i is Lie and has only finitely many components. If H_{i+1} is conjugate to a proper subgroup of H_i, then either $\dim(H_{i+1}) < \dim(H_i)$, or $\dim(H_{i+1}) = \dim(H_i)$ but H_{i+1} has strictly less connected components than H_i. It follows that the sequence stabilizes. $\square$

For a nilpotent isometric G-action on a Ricci limit space Y, we show that a principal $[H]$ must be trivial.

Theorem A.6. *Let Y be a Ricci limit space and let G be a closed nilpotent subgroup of* $\mathrm{Isom}(Y)$*. Suppose $H \in \mathcal{I}(G)$ such that $[H]$ is principal, then $H = \{\mathrm{id}\}$.*

To prove Theorem A.6, beside the slice theorem, we need Lemma 2.13 and the following characterization of the identity map from [25, Lemma 2.1].

Lemma A.7. [25] *Let X be a Ricci limit space and let g be an isometry of X. Suppose that g fixes every point in an open ball in X, then g is the identity map.*

Proof of Theorem A.6. The proof is similar to that of the manifold case.

Let $y \in Y$ such that $\mathrm{Iso}(G, y) = H$. By Theorem A.3, there is a slice S at y. Because the open neighborhood $G \cdot S$ is G-equivariantly homeomorphic to $G \times_H S$, we study this local model $G \times_H S$ to understand $G \cdot S$.

Claim: In $G \times_H S$, $\mathrm{Iso}(G, [g, s])$ is conjugate to H by g. By definition, if $g'[g, s] = [g, s]$, then there is some $h \in H$ such that

$$(g'g, s) = (gh, h^{-1}s)$$

in $G \times S$. Hence h fixes s and $g' = ghg^{-1}$. As a result, $\mathrm{Iso}(G, [g, s])$ must be conjugate to a subgroup of H by g. Because $[H]$ is principal, the claim follows.

Note that the claim in particular implies $\mathrm{Iso}(G, [e, s]) = H$. Let $h \in H$. Then h fixes every point in S. For any $g_0 \in G_0$ and any $s \in S$, by Lemma 2.13 and compactness of H, we have

$$hg_0 s = g_0 h s = g_0 s.$$

Thus h fixes every point in $G_0 \cdot S$, which is an open neighborhood of y. Applying Lemma A.7, we conclude $h = \mathrm{id}$. Hence $H = \{\mathrm{id}\}$. □

Proof of Proposition A.1. By Theorem A.6, there is a point $z \in Y$ such that the isotropy subgroup $\mathrm{Iso}(G, z)$ is trivial. Then $gz \in G_0 z$ implies $g \in G_0$. □

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