Derivatives of Exponential and Logarithmic Functions Unit Assignment

MCV4U

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1. Find the derivatives of each of these functions.

(a) $y = ln(4x^2 + 4)$

- To apply the Chain Rule, set u as $4x^2 + 4$
  - $\frac{d}{du}(ln(u))\frac{d}{dx}(4x^2 + 4)$
- The derivative of $ln(u)$ with respect to u is the following.
  - $\frac{1}{u}\frac{d}{dx}(4x^2 + 4)$
  - $\frac{1}{4x^2+4}\frac{d}{dx}(4x^2 + 4)$
  - $\frac{1}{4x^2+4}(8x)$
  - $\frac{2x}{x^2+1}$
- The answer is $\frac{2x}{x^2+1}$.

(b) $y = \frac{2x^4}{e^{5x}}$

- $y = 2x^4e^{-5x}$
- $y' = 2\frac{d}{dx}(x^4e^{-5x})$

  $= 2(x^4(-5e^{-5x}) + 4x^3e^{-5x})$

  $= -10x^4e^{-5x} + 8x^3e^{-5x}$
- $y' = \frac{-10x^4+8x^3}{e^{5x}}$

(c) $y = 2^{5x+7}(ln(5x + 1))$

- $y' = 2^{5x+7}\frac{d}{dx}(ln(5x + 1)) + ln(5x + 1)\frac{d}{dx}(2^{5x+7})$

  $= \frac{5\bullet 2^{5x+7}}{5x+1} + ln(5x + 1) \bullet 2^{5x+7}ln(2) \bullet 5$
- $y' = \frac{5}{5x+1}2^{5x+7} + 5ln2 \bullet ln(5x + 1) \bullet 2^{5x+7}$

(d) $y' = \frac{4x^3}{e^{5x}+x^4}$

- $y' = 4 \bullet \frac{d}{dx}(\frac{x^3}{e^{5x}+x^4})$

  $= 4(\frac{(e^{5x}+x^4)\frac{d}{dx}(x^3)-x^3\frac{d}{dx}(e^{5x}+x^4)}{(e^{5x}+x^4)^2})$

  $= 4\frac{[3x^2(e^{5x}+x^4)-x^3(5e^{5x}+4x^3)]}{(e^{5x}+x^4)^2}$

  $= \frac{12x^6+12x^2e^{5x}-20x^3e^{5x}-16x^6}{(e^{5x}+x^4)^2}$

  $= \frac{-4x^6+12x^2e^{5x}-20x^3e^{5x}}{(e^{5x}+x^4)^2}$

2. Use the process of implicit differentiation to find $\frac{dy}{dx}$ given that $x^3e^y - ye^x = 0$

- $e^yx^3\frac{d}{dx}[y] + 3e^yx^2 - e^xy - e^x\frac{d}{dx}[y] = 0$

- $e^y x^3 y' + 3e^y x^2 - e^x y - e^x y' = 0$
- $y'(e^y x^3 - e^x) + 3e^y x^2 - e^x y = 0$
- $y'(e^y x^3 - e^x) =- 3e^y x^2 + e^x y$
- $y' =- \frac{3x^2e^y - ye^x}{x^3e^y - e^x}$
- $\frac{dy}{dx} =- \frac{3x^2e^y - ye^x}{x^3e^y - e^x}$

3. Use curve sketching methods, sketch the graph of the function given by the equation $y = \frac{x^2}{e^x}$. Make sure that you include all steps, charts, and derivations details. (10 marks)

- To begin with, find the first derivative.

  $y = \frac{x^2}{e^x}$

  $y' = \frac{d}{dx}(\frac{x^2}{e^x}) = \frac{e^x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{(e^x)^2} = \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{e^x(2x - x^2)}{e^{2x}} = \frac{2x - x^2}{e^x}$
- Find the second derivative.

  $y'' = \frac{dy'}{dx} = \frac{d}{dx}(\frac{2x - x^2}{e^x}) = \frac{e^x \times \frac{d}{dx}(2x - x^2) - (2x - x^2)\frac{d}{dx}(e^x)}{(e^x)^2} = \frac{e^x(2 - 2x) - (2x - x^2)e^x}{e^{2x}} = \frac{e^x(2 - 2x - 2x + x^2)}{e^{2x}}$

  $= \frac{2 - 4x + x^2}{e^x}$

- To find the y-intercept of a given y function.

  $y|_{x=0} = \frac{x^2}{e^x}$

  $y = 0$

  (0,0) is both the x and y intercepts

- To find critical points.

  $y' = 0$

  $\frac{2x - x^2}{e^x} = 0$

  $2x - x^2 = 0$

  $x(2 - x) = 0$

  $x = 0 \; or \; x = 2$

- Slope chart of the first derivative

	x<0	x=0	0<x<2	x=2	x>2
$2x - x^2$	-	0	+	0	-
e^x	+	+	+	+	+

$\frac{2x-x^2}{e^x}$	-	0	+	0	-

There must be a minimum when x=0, and a maximum when x=2.

- To find inflection point

  $y'' = 0$

  $\frac{2-4x+x^2}{e^x} = 0$

  Since $e^x > 0$ is always true, $2 - 4x + x^2$ should be zero.

  $2 - 4x + x^2 = 0$

  $x = 2 + \sqrt{2}\ or\ x = 2 - \sqrt{2}$

- To find concavity chart

	$x < 2 - \sqrt{2}$	$x = 2 - \sqrt{2}$	$2 - \sqrt{2} < x < 2 + \sqrt{2}$	$x = 2 + \sqrt{2}$	$x > 2 + \sqrt{2}$
$2 - 4x + x$	+	0	-	0	+
e^x	+	+	+	+	+
$\frac{2-4x+x^2}{e^x}$	+	0	-	0	+

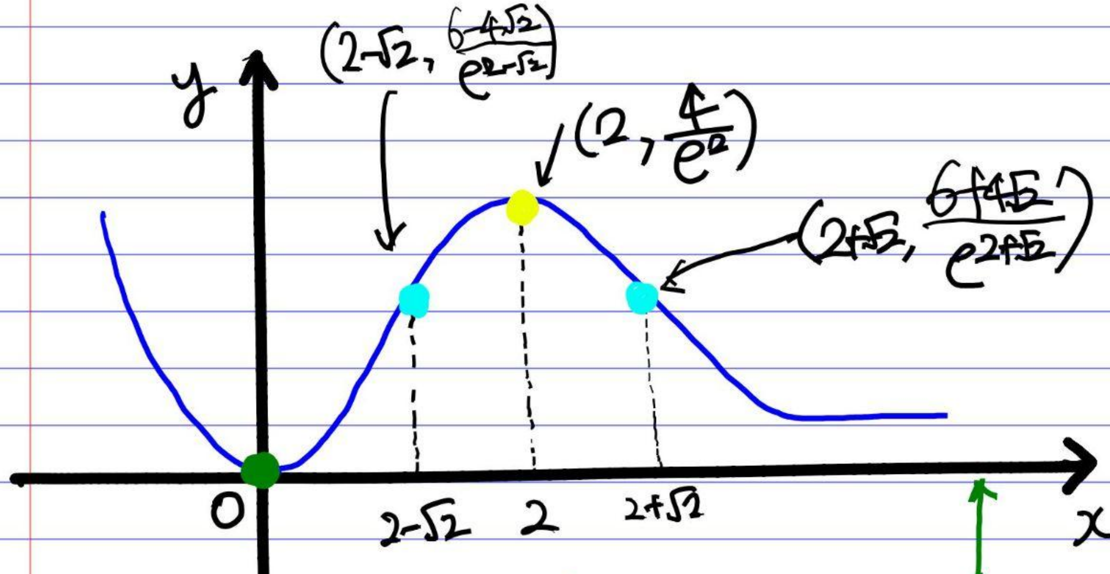
The inflection points must be at $x = 2 + \sqrt{2}\ or\ x = 2 - \sqrt{2}$

- To find end behaviour, send x to infinity or minus infinity.
  - send x to infinity

    Use L'Hospital's rule until the function does not show $\frac{\infty}{\infty}$ form.

    $y = lim_{x\to\infty}(\frac{x^2}{e^x}) = lim_{x\to\infty}\frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^x)} = lim_{x\to\infty}\frac{2x}{e^x} = lim_{x\to\infty}\frac{2}{e^x} = 0$
  - send x to minus infinity

    $y = lim_{x\to-\infty}(\frac{x^2}{e^x}) = \frac{(-\infty)^2}{e^{(-\infty)}} = \infty \times \infty = \infty$



① horizontal asymptote: $y = 0$

② x/y intercept / local min : $x = 0$

③ critical points: $(0,0)$, $\left(2, \frac{4}{e^2}\right)$

④ points of inflection:

$$\left(2-\sqrt{2}, \frac{6-4\sqrt{2}}{e^{2-\sqrt{2}}}\right)$$

$$\left(2+\sqrt{2}, \frac{6+4\sqrt{2}}{e^{2+\sqrt{2}}}\right)$$

4. Use logarithmic differentiation to find the derivative of the function $y = \frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}}$.

$ln(y) = ln(\frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}})$

$ln(y) = ln(2x + 5)^3 + ln(5x^2 - 2x)^4 - ln(\sqrt{2x + 5})$

$ln(y) = 3ln(2x + 5) + 4ln(5x^2 - 2x) - \frac{1}{2}ln(2x + 5)$

Differentiating both sides as the following.

$\frac{dy}{dx}\frac{1}{y} = \frac{3}{(2x+5)}\frac{d}{dx}(2x + 5) + \frac{4}{(5x^2-2x)}\frac{d}{dx}(5x^2 - 2x) - \frac{1}{2(2x+5)}\frac{d}{dx}(2x + 5)$

$= \frac{3}{(2x+5)}(2) + \frac{4}{(5x^2-2x)}(10x - 2) - \frac{1}{2(2x+5)}(2)$

$= \frac{6}{(2x+5)} + \frac{40x-8}{(5x^2-2x)} - \frac{1}{(2x+5)}$

$= \frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)}$

$= \frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)}$

$\frac{dy}{dx} = y(\frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)})$

Thus, the answer would be

$\frac{dy}{dx} = \frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}}(\frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)})$

5. The population (P) of an island y years after colonisation is given by the function $P = \frac{250}{1+4e^{-0.01y}}$.

(a) What was the initial population of the island?

We can get the initial population by substituting y = 0.

$P(0) = \frac{250}{1+4e^{-0.01(0)}} = \frac{250}{1+4} = 50$

Thus, the initial population is 50 people.

(b) How long did it take before the island had a population of 150?

We can get the answer by substituting P = 150.

$150 = \frac{250}{1+4e^{-0.01y}}$

$150(1 + 4e^{-0.01y}) = 250$

$150 + 600e^{-0.01y} = 250$

$600e^{-0.01y} = 100$

$e^{-0.01y} = \frac{1}{6}$

$- 0.01y = ln(\frac{1}{6})$

$- 0.01y =- ln(6)$

$y = \frac{ln(6)}{0.01}$

$y \approx 179.176\ years$

Thus, it took 179.176 years before the island had a population of 150.

(c) After how many years was the population growing the fastest?

Since the population growth is given by $P'(y)$, we can conclude that the population growth is the fastest when $P''(y) = 0$.

$P'(y) = \frac{d}{dy}(\frac{250}{1+4e^{-0.01y}}) = 250\frac{d}{dy}((1 + 4e^{-0.01y})^{-1}) = 250(-\frac{1}{(1+4e^{-0.01y})^2})(-0.04e^{-0.01y})$

$= \frac{10e^{-0.01y}}{(1+4e^{-0.01y})^2}$

$P''(y) = \frac{d}{dy}(\frac{10e^{-0.01y}}{(1+4e^{-0.01y})^2}) = 10 \bullet \frac{\frac{d}{dy}(e^{-0.01y})(1+4e^{-0.01y})^2 - \frac{d}{dy}((1+4e^{-0.01y})^2)e^{-0.01y}}{((1+4e^{-0.01y})^2)^2}$

$= 10 \bullet \frac{(-0.01e^{-0.01y})(1+4e^{-0.01y})^2-(-0.08e^{-0.01y}(1+4e^{-0.01y}))e^{-0.01y}}{((1+4e^{-0.01y})^2)^2}$

$= \frac{10e^{-0.02y}(-0.01e^{0.01y}+0.04)}{(1+4e^{-0.01y})^3}$

$P''(y) = 0 \rightarrow \frac{10e^{-0.02y}(-0.01e^{0.01y}+0.04)}{(1+4e^{-0.01y})^3} = 0$

$=-0.01e^{0.01y} + 0.04 = 0 \rightarrow e^{0.01y} = 4 \rightarrow 0.01y = ln(4) \rightarrow y = 138.63\ years$

Thus, after 138 years, the population is growing the fastest.

(d) Using curve sketching methods, sketch the graph of the function. Make sure that you include all steps, charts, and derivations details.

By using Chain Rules, we can render the first derivatives as the following.

$P'(y) = 250(-1)(1 + 4e^{-0.01y})^{-2}(4e^{-0.01y})(-0.01)$

$P'(y) = \frac{10e^{-0.01y}}{(1+4e^{-0.01y})^2}$

$P'(y) = \frac{10e^{0.01y}}{(4+e^{0.01y})^2} > 0$

Thus, the population or graph is always increasing.

Since the numerator cannot be 0, there is no critical point.

We can render the second derivatives as the following.

$P''(y) = \frac{d}{dy}(\frac{10e^{0.01y}}{(4+e^{0.01y})^2})$

$= 10 \times (\frac{-0.01e^{-0.01y}(1+4e^{-0.01y})^2+2e^{-0.01y}(1+4e^{-0.01y})\times 0.04e^{-0.01y}}{(1+4e^{-0.01y})^4})$

$= 10 \times \left(\frac{-0.01e^{-0.01y}(1+4e^{-0.01y}) + 0.08(e^{-0.01y})^2}{(1+4e^{-0.01y})^3}\right)$

$= \frac{10e^{-0.01y}}{(1+4e^{-0.01y})^3}\left(-0.01(1+4e^{-0.01y}) + 0.08(e^{-0.01y})\right)$

$\frac{d^2p}{dy^2} = 0$

$\Rightarrow \; -0.01 - 0.04e^{-0.01y} + 0.08e^{-0.01y} = 0$

$\Rightarrow \; 0.04e^{-0.01y} = 0.01$

$\Rightarrow \; e^{-0.01y} = \frac{1}{4}$

$\Rightarrow \; 0.01y = ln(4),\ y = 100ln(4)$

Get the point of inflection at $y = 100ln(4)$

$P(100ln(4)) = \frac{250}{1+4e^{-0.01\times 100ln(4)}} = \frac{250}{1+4e^{-ln4}} = \frac{250}{1+4\times\frac{1}{4}}$

$= \frac{250}{2} = 125$

Let us draw the concavity chart of $y = 100ln(4)$.

	$-\infty < x < -2\ln(2)$	$x = \frac{\ln(0.25)}{0.01}$	$-2\ln(2) < x < \infty$
Sign	$f''(x) < 0$	$f''(x) = 0$	$f''(x) > 0$
Behavior	Concave Downward	Inflection	Concave Upward

P is concave up when the second derivative is larger than zero.

$0.04e^{-0.01y} > 0.01$

$e^{-0.01y} > \frac{1}{4}$

$-0.01y > ln(\frac{1}{4})$

$y < 100ln(4)$

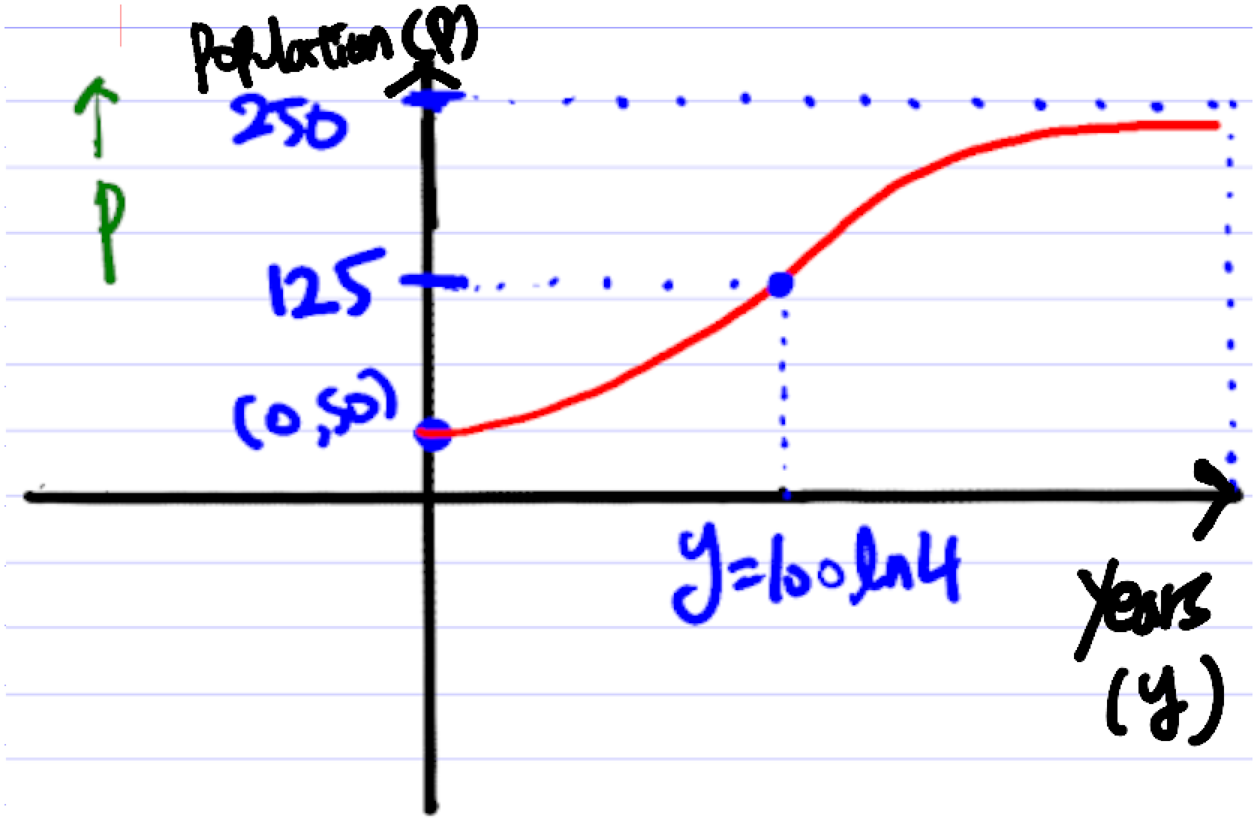
Thus, if y is larger than 100ln(4), the graph is always concave down.

Calculate the end behaviour when x goes infinity or minus infinity.

$\lim_{y\to\infty} P = \lim_{y\to\infty}\left(\frac{250}{1+4e^{-0.01y}}\right) = 250$

$\lim_{y\to-\infty} P = \lim_{y\to-\infty}\left(\frac{250}{1+4e^{-0.01y}}\right) = 0$

Therefore, there are two horizontal asymptotes, y=250 and y=0.



e. Give a possible explanation for the shape of the curve.

Since it is an exponentially increasing function, the maximum y value on the graph is 250 which is the carrying capacity of the population. As the population P(y) in the island increases, the number of years increases. However, the amount of population increment declines after y = 100ln4.

Thus, we can think of the possibility that the population increment is limited according to the water or food resources that island has.