Linear Dependence and Coplanarity Unit Assignment
MCV4U

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1. **Write an example of each of the following (assuming it is in 3-dimensional space).**

- A point lying on the x-axis.
  Example: $(2, 0, 0)$
- A point lying on the yz plane.
  Example: $(0, 2, 3)$
- A point lying on both the xy and xz planes.
  A point can lie on both xy planes and yz planes if and only if the point lies on the x-axis.
  Example: $(1, 0, 0)$
- A point lying on all three planes.
  Example: $origin\ (0, 0, 0)$
- A point lying on none of the three planes, but equidistant from the xz and yz planes.
  Example: $(2, 4, 4)$

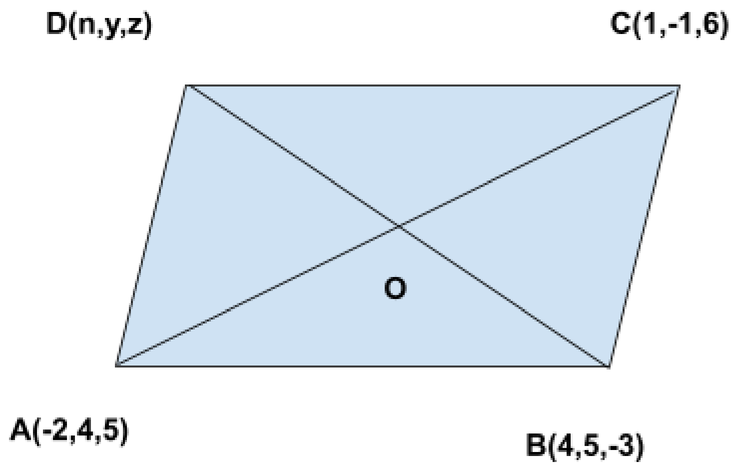
2. **Triangle ABC has vertices A(4, 7, 7), B(1, 6, 5), and C(–2, 9, 8). What kind of triangle is ΔABC?**

- To begin with, getting the distance between two points having coordinates $(x1, y1, z1),\ (x2, y2, z2)$ are $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- We could calculate the edge lengths of the given triangle as the following.
  - $AB = \sqrt{(1-4)^2 + (6-7)^2 + (5-7)^2} = \sqrt{9+1+4} = \sqrt{14}$
  - $AC = \sqrt{(-2-4)^2 + (9-7)^2 + (8-7)^2} = \sqrt{36+4+1} = \sqrt{41}$
  - $BC = \sqrt{(-2-1)^2 + (9-6)^2 + (8-5)^2} = \sqrt{9+9+9} = \sqrt{27}$
- We can get whether the given triangle ABC is a right angle triangle or not using the Pythagorean theorem.
  - $AC^2 = AB^2 + BC^2$
  - $\sqrt{41}^2 = \sqrt{14}^2 + \sqrt{27}^2$
  - $41 = 14 + 27$
  - Hence, the given triangle ABC is a right-angled triangle.

3. **The points (–2, 4, 5), (4, 5, –3), and (1, –1, 6) are three of four vertices of parallelogram ABCD. Explain why there are three possibilities for the location of the fourth vertex, and find the three points.**

There are three possibilities for the 4th vertex because either of the three vertices (A, B, or C) can be the 4th vertex.

First Case.

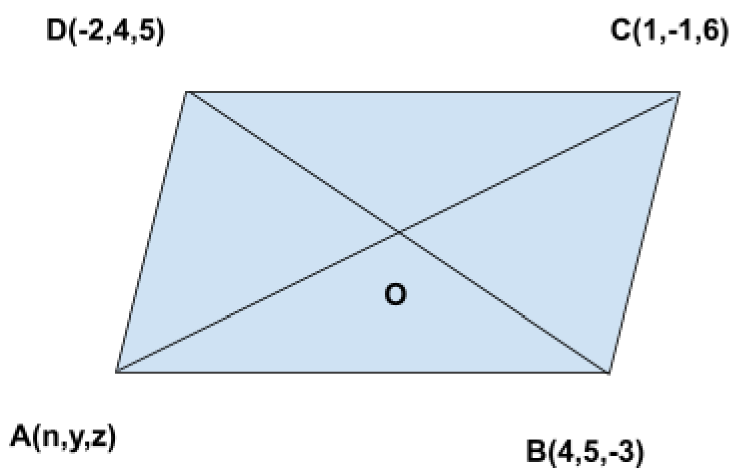


We will find the case where the 4th vertex is at the first vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the midpoint of the two points, which is the value of the summing each unknown value of the 4th vertex to that of the opposite point.

With Point D and Point B, we can get the midpoint as follows. $O \equiv (\frac{n+4}{2}, \frac{y+5}{2}, \frac{z-3}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{-2+1}{2}, \frac{4-1}{2}, \frac{5+6}{2})$
We can calculate the values of undetermined midpoints like the following.
$\frac{n+4}{2} = -\frac{1}{2}, \frac{y+5}{2} = \frac{3}{2}, \frac{z-3}{2} = \frac{11}{2} \Rightarrow n = -5, y = -2, z = 14$

Thus, Point D is $(-5, 2, 14)$.

Second Case.

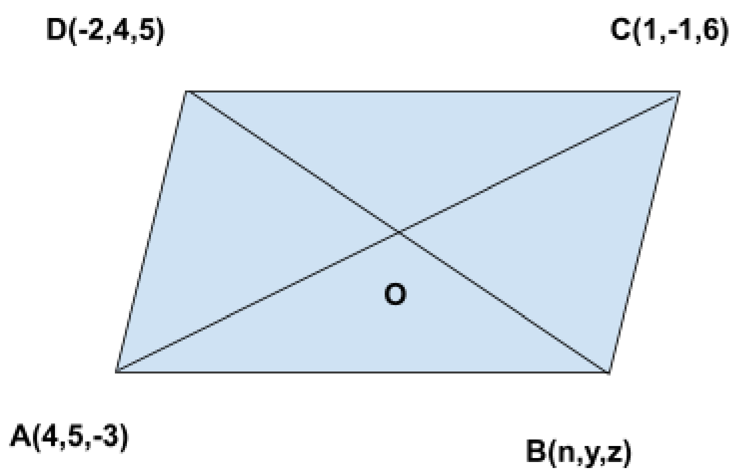


We will find the case where the 4th vertex is on the left lower vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the midpoint of the two points, which is the value of the summing each unknown value of the 4th vertex to that of the opposite point.

With Point A and Point C, we can get the midpoint as follows. $O \equiv (\frac{n+1}{2}, \frac{y-1}{2}, \frac{z+6}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{4+2}{2}, \frac{5+4}{2}, \frac{-3+5}{2})$
We can calculate the values of undetermined midpoints like the following.
$\frac{n+1}{2} = 3,\ \frac{y-1}{2} = \frac{9}{2},\ \frac{z+6}{2} = 1 \ \Rightarrow\ n = 5,\ y = 10,\ z = -4$

Thus, Point D is $(5,\ 10,\ -4)$.

<u>Third Case.</u>



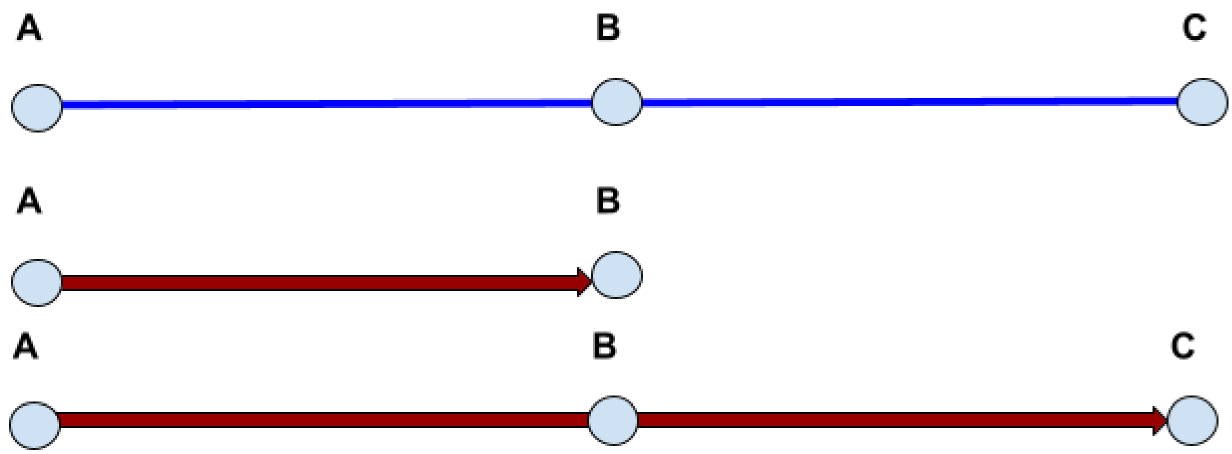
We will find the case where the 4th vertex is on the right lower vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the midpoint of the two points, which is the value of the summing each unknown value of the 4th vertex to that of the opposite point.

With Point A and Point C, we can get the midpoint as follows. $O \equiv (\frac{n-2}{2}, \frac{y+4}{2}, \frac{z+5}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{5}{2}, 2, \frac{3}{2})$
We can calculate the values of undetermined midpoints like the following.
$\frac{n-2}{2} = \frac{5}{2},\ \frac{y+4}{2} = 2,\ \frac{z+5}{2} = \frac{3}{2} \ \Rightarrow\ n = 7,\ y = 0,\ z = -2$

Thus, Point D is $(7,\ 0,\ -2)$.

4. **The points A(–2, –1, z), B(2, 4, 3), and C(10, y, –1) are collinear. Find the values of y and z.**

To understand the concept of collinear values, let us draw two possible vectors, $\overline{AB}$ *and* $\overline{AC}$.

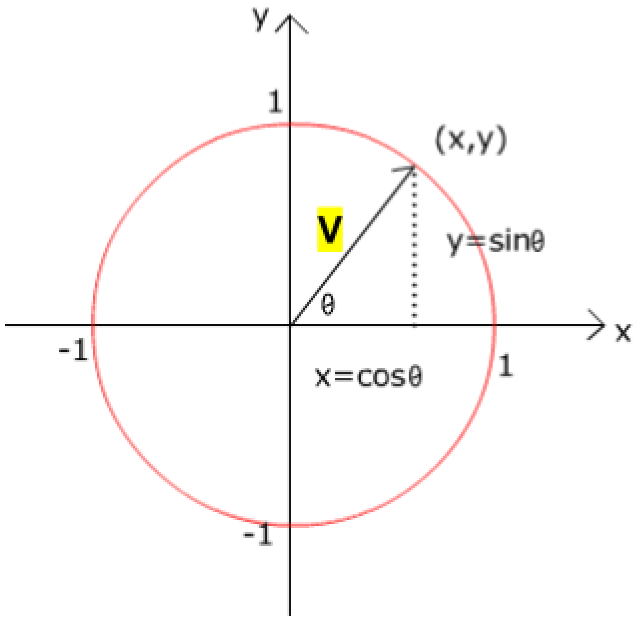


It is clearly appreciable that the aforementioned vectors can be scaled up and down in length to shrink to another vector. In addition, these vectors are only inclined to change according to the sign of the scaling factor. For example, $2\rightarrow_i + 4\rightarrow_j$ is a multiple of $1\rightarrow_i + 2\rightarrow_j$ by a factor of 2, while $-2\rightarrow_i - 4\rightarrow_j$ is only available for scaled down by a factor of 2.

$\overline{AB} = (2,4,3) - (-2,-1,z) = (4,5,3-z)$
$\overline{AC} = (10,y,-1) - (-2,-1,z) = (12,y+1,-1-z)$
Using scaling logic, let the following.
$\overline{AB} = k\overline{AC} \Rightarrow (4,5,3-z) = k(12,y+1,-1-z) = (12k,ky+k,-k-kz)$
Comparing the respective given elements.
$12k = 4,\ ky + k = 5,\ -k - zk = 3 - z$
$element\ X\ = k = \frac{1}{3}$
$element\ Y\ = \frac{1}{3}(y+1) = 5 \Rightarrow y = 14$
$element\ Z = \frac{1}{3}(-1-z) = 3 - z \Rightarrow z = 5$

Thus, the answer would be $y = 14,\ z = 5$

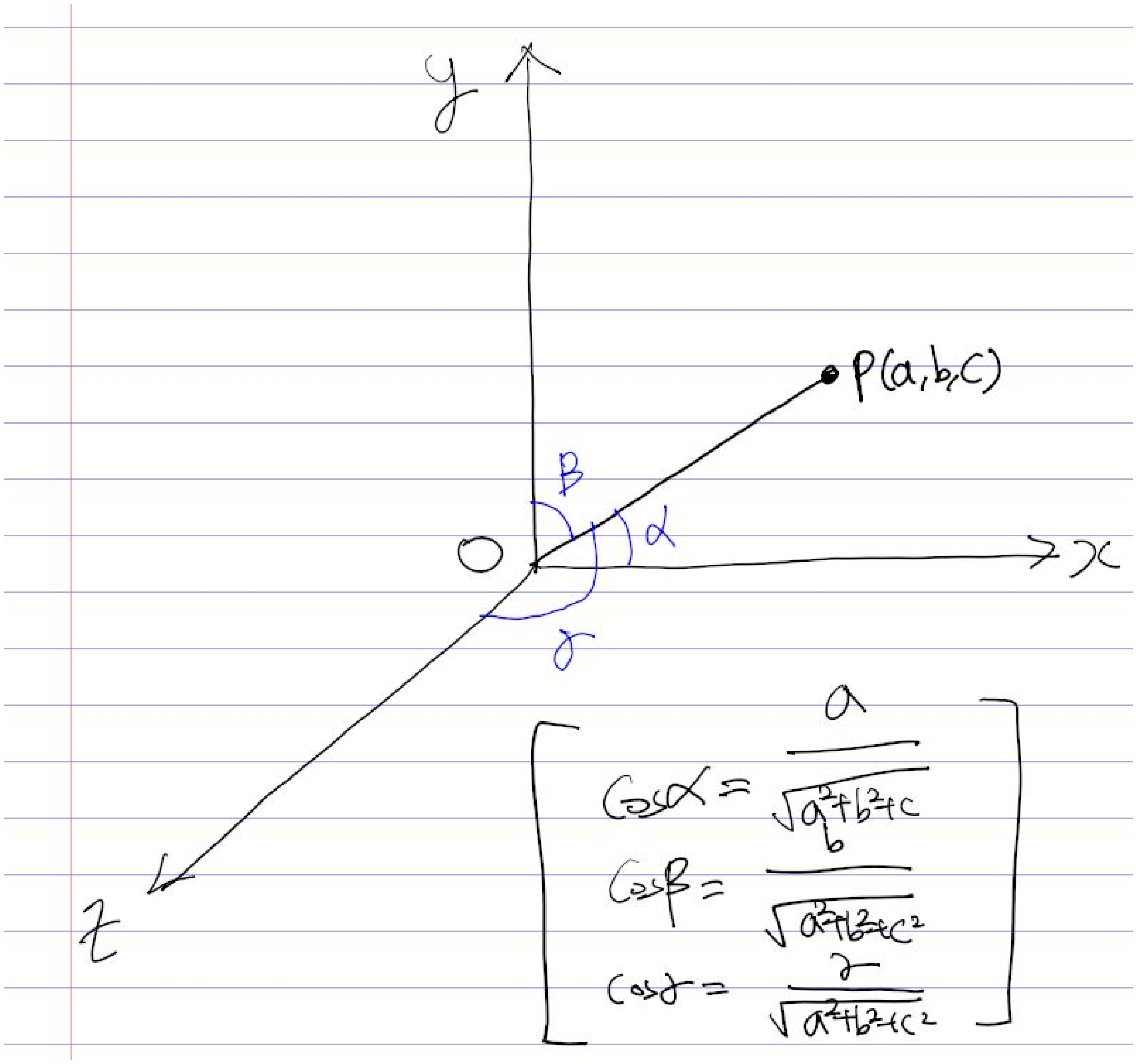
5. **Explain the meaning of direction angles and their relation to direction vectors. Direction angle means the angle between a vector and the positive x-axis.**



In the picture shown above, vector $\vec{v}$ has a direction angle θ while $\vec{v} = |\vec{v}|\,cos\theta i + |\vec{v}|\,sin\theta j$ where $|\vec{v}|$ is the magnitude of vector $\vec{v}$.

Let the vectors be $\vec{a}, \vec{b}$, then directional cosine is $cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$, where θ is directional angle between a, b and $\vec{a} \bullet \vec{b}$ is a dot product between two vectors while $|\vec{a}|, |\vec{b}|$ is the magnitude of each vector respectively.

a. What are the direction angles of the vector [–5, 1, 8]?



P is a general point with (a, b, c) while they are (x, y, z)'s coordinate respectively.

α, β, γ are the direction angle of the vector $\overline{OP}$ with x, y, z-axis respectively.

Given the vector, $u = -5\hat{i} + \hat{j} + 8\hat{k}$. $a = -5, b = 1, c = 8$

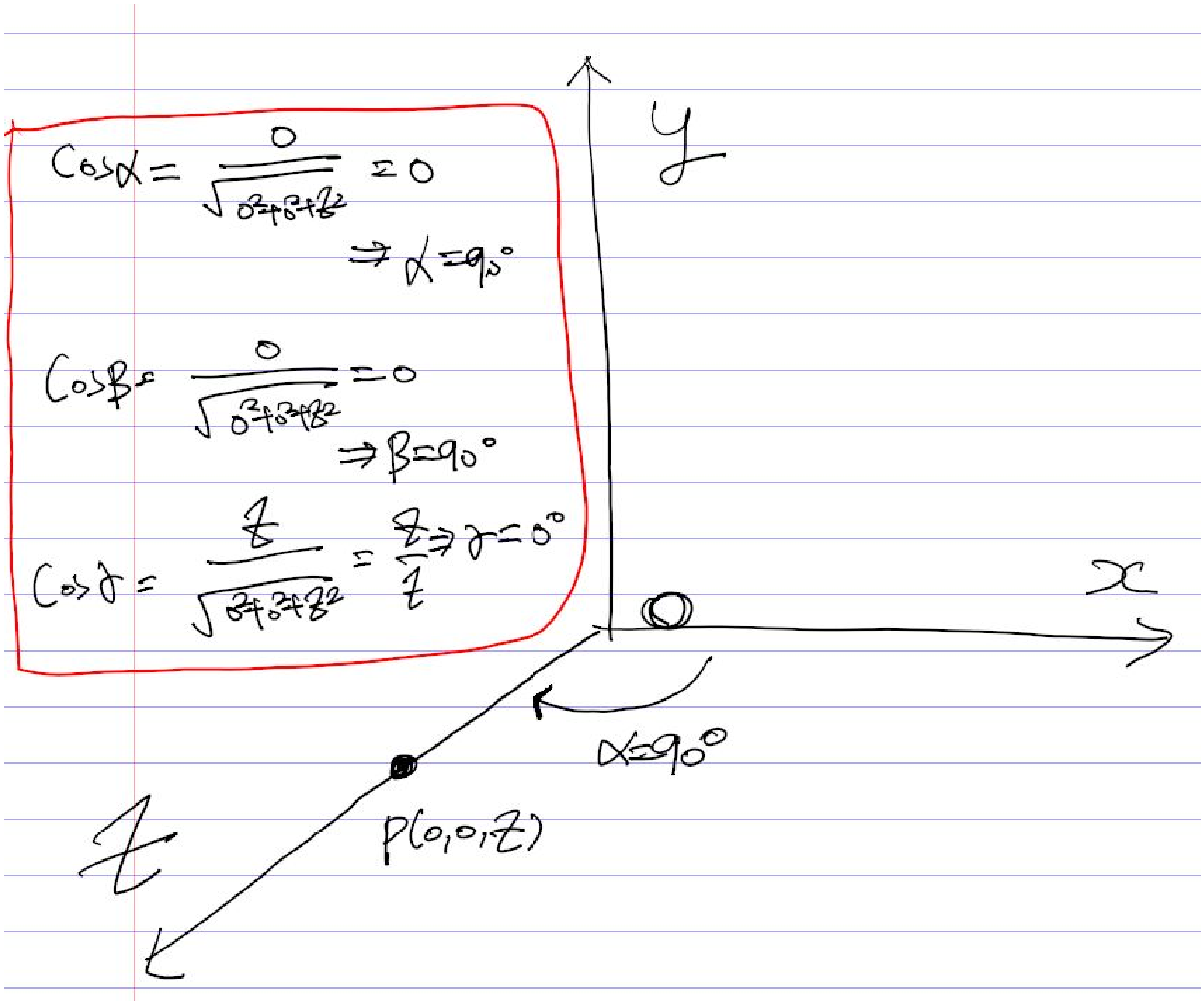
$cos\ \alpha = \frac{-5}{\sqrt{(-5)^2+1^2+8^2}} = -\frac{\sqrt{10}}{6},\ \alpha = cos^{-1}\left[\frac{-\sqrt{10}}{6}\right] = 121.81^o$

$cos\ \beta = \frac{1}{\sqrt{(-5)^2+1^2+8^2}} = \frac{\sqrt{10}}{30},\ \beta = cos^{-1}\left[\frac{\sqrt{10}}{30}\right] = 83.95^o$

$cos\ \gamma = \frac{8}{\sqrt{(-5)^2+1^2+8^2}} = \frac{4\sqrt{10}}{15},\ \gamma = cos^{-1}\left[\frac{4\sqrt{10}}{15}\right] = 32.51^o$

Thus, directional angles are $\alpha = 121.81^o,\ \beta = 83.95^o,\ \gamma = 32.51^o$.

b. If a point P lies on the z-axis, what are the direction angles of the position vector $\overline{OP}$?



If Point P lies on the z-axis, then the x, y coordinates of a point are zero. (ex. $P(0,0,z)$)

$cos\ \alpha = \frac{0}{\sqrt{0^2+0^2+z^2}} = 0,\ \alpha = 90^o$

$cos\ \beta = \frac{0}{\sqrt{0^2+0^2+z^2}} = 0,\ \beta = 90^o$

$cos\ \gamma = \frac{z}{\sqrt{0^2+0^2+z^2}} = \frac{z}{z} = 1,\ \gamma = 0^o$

Thus, the directional angles are $\alpha = 90^o,\ \beta = 90^o,\ \gamma = 0^o$.

c. Prove that $cos^2(\alpha) + cos^2(\beta) + cos^2(\gamma) = 1$.

To prove that $cos^2(\alpha) + cos^2(\beta) + cos^2(\gamma) = 1$, where α, β, γ are direction angles of a vector with x, y, z axis respectively.

Let the vector $\overline{OP}$ with Point P (a, b, c) , from we can get the following.

$cos\ \alpha = \frac{a}{\sqrt{a^2+b^2+c^2}},\ cos^2\alpha = \frac{a^2}{a^2+b^2+c^2}$

$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $\cos^2\beta = \frac{b^2}{a^2+b^2+c^2}$

$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$, $\cos^2\gamma = \frac{c^2}{a^2+b^2+c^2}$

Thus, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2} = 1$

d. A vector has direction angles $\alpha = 85^o,\ \beta = 65^o$.

(a) Find the value γ.

We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Substitute α, β into $85^o,\ 65^o$.

$\cos^2(85) + \cos^2(65) + \cos^2(\gamma) = 1.$

$7.596 * 10^{-3} + 0.1786 + \cos^2\gamma = 1.$

$\cos^2\gamma = 0.8138$

$\cos(\gamma) = \pm\sqrt{0.8138}$

$\cos(\gamma) = \pm 0.9021$

$\gamma = \cos^{-1}(0.9021) = 25.56^o,\ \cos^{-1}(-0.9021) = 154.435^o$.

(b) Find a vector that has those direction angles. $\alpha = 85^o,\ \beta = 65^o,\ \gamma = 25.65^o (or\ 154.435^o)$.

$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}} \Rightarrow \alpha = \sqrt{a^2+b^2+c^2} \times \cos(\alpha) = 1 \times \cos 85^o \Rightarrow a = 0.0871$

$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}} \Rightarrow \beta = \sqrt{a^2+b^2+c^2} \times \cos(\beta) = 1 \times \cos 65^o \Rightarrow b = 0.4226$

$\cos\gamma(when\ 25.65^o) = \frac{c}{\sqrt{a^2+b^2+c^2}} \Rightarrow \alpha = \sqrt{a^2+b^2+c^2} \times \cos(\gamma) = 1 \times \cos 25.65^o \Rightarrow c = 0.9021$

The vector is $[0.0871,\ 0.4226,\ 0.9021]$ which is having directional angles as α, β, γ are specified.

$\cos\gamma(when\ 154.435^o) = \frac{c}{\sqrt{a^2+b^2+c^2}} \Rightarrow \alpha = \sqrt{a^2+b^2+c^2} \times \cos(\gamma) = 1 \times \cos 154.435^o \Rightarrow c = -0.9021$

The vector is $[0.0871,\ 0.4226,\ -0.9021]$ which is having directional angles as α, β, γ are specified.

e. Explain why it is not possible for two of a vector's direction angles to be less than 45°.

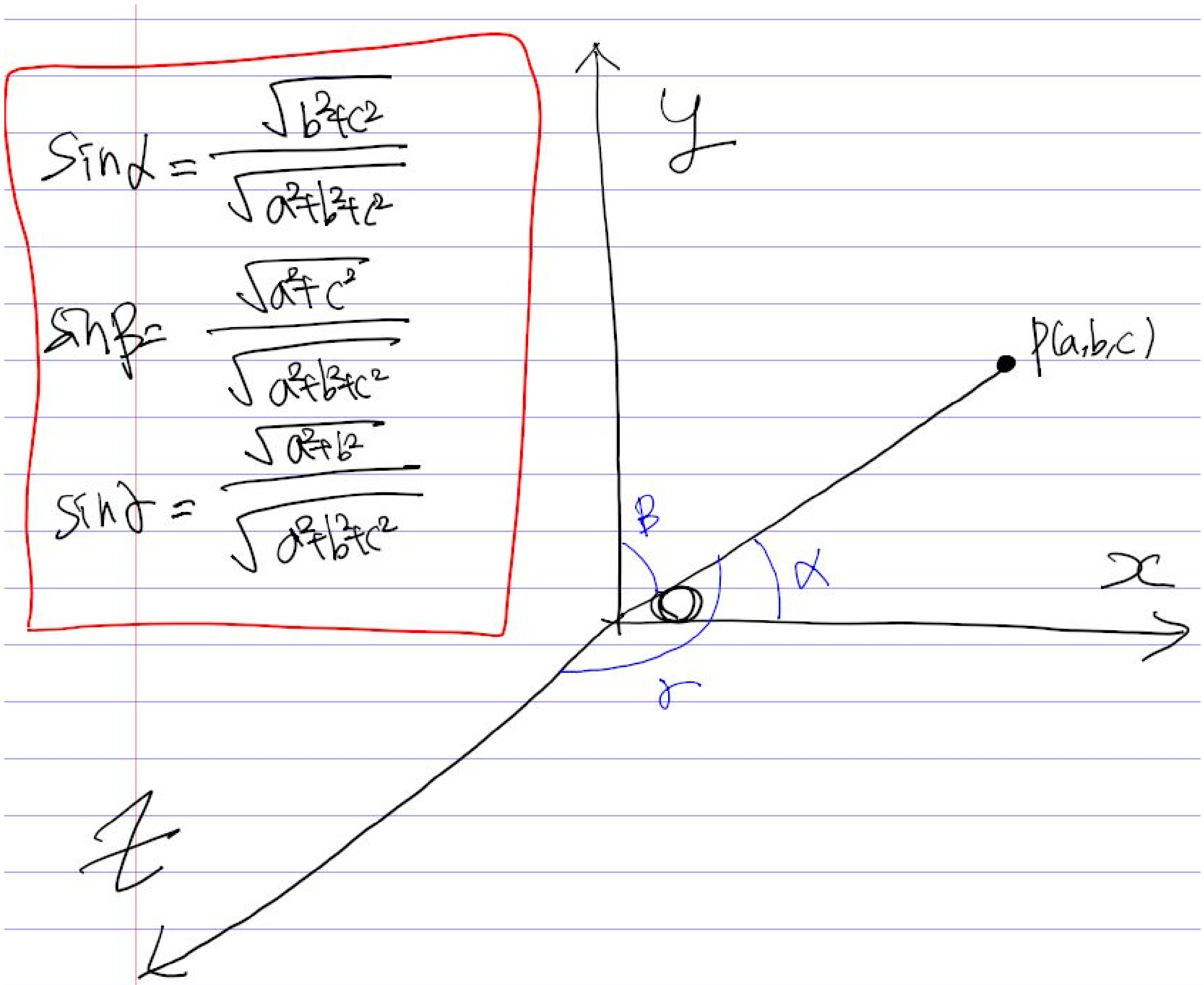
It is not possible for two of the vector's direction angles to less than 45 degrees because it might give the following. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \neq 1$

Let such vector be $r(\cos\alpha, \cos\beta, \cos\gamma)$ where $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and $0 \le \alpha, \beta, \gamma \le 180$.

If both $\alpha < 45\ and\ \beta < 45$ then $\cos\alpha > 1/\sqrt{2}\ and\ \cos\beta > 1/\sqrt{2}$ and so $\cos^2\alpha + \cos^2\beta > 1$.

This implies that $\cos^2\gamma < 0$ which is impossible. Hence both α, β cannot be <45.

f. What is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$? Why?



$sin\alpha = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$, $sin\beta = \frac{\sqrt{a^2+c^2}}{\sqrt{a^2+b^2+c^2}}$, $sin\gamma = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+c^2}}$

$sin\alpha^2 + sin\beta^2 + sin\gamma^2 = sin\alpha = \frac{b^2+c^2+a^2+c^2+a^2+b^2}{a^2+b^2+c^2} = \frac{2(a^2+b^2+c^2)}{a^2+b^2+c^2} = 2$

Thus, $sin\alpha^2 + sin\beta^2 + sin\gamma^2 = 2$

6. Explain the meanings of the terms linearly dependent and coplanar. Make sure you demonstrate that you understand the difference between the terms, and the situation in which linear dependency implies coplanarity.

Linear dependence means the following. If one of the vectors can be defined as a linear combination of other vectors; if no vector in the set can be written this way, the vector is linearly independent. On the other hand, coplanar means the case when a set of vectors is in the same plane. There is a case when two vectors are always coplanar but they can be linearly dependent or linearly independent. However, if three vectors are linearly dependent, we can say that they are coplanar.

7. Determine if the vectors [2, 4, –1], [8, –10, 5], and [5, –3, 2] are coplanar.

Let $\rightarrow_u = s\rightarrow_v + t\rightarrow_w$ for some scalars s and t but they are not both zeros.

$\rightarrow_u = s\rightarrow_v + t\rightarrow_w$, $[2, 4, -1] = s[8, -10, 5] + t[5, -3, 2]$

$[2, 4, -1] = [8s, -10s, 5s] + [5t, -3t, 2t]$

$2 = 8s + 5t$ –> (1)

$4 = -10s - 3t$ –> (2)

$-1 = 5s + 2t$ –> (3)

From equations (1) and (2): We can solve for s and t by the elimination method. Multiply equation (1) by 3, and multiply equation (2) by 5 then copy equation (1) and subtract equation (2) from (1) and solve for s.

$$\begin{array}{rl} & 24s + 15t = 6 \\ + & -50s - 15t = 20 \\ \hline & -26s = 26 \\ & s = -1 \end{array}$$

Substitute $s = -1$ into equation (1) and solve for t.

$24(-1) + 15t = 6,\ \ -24 + 15t = 6,\ \ 15t = 30,\ \ t = 2$

Check $s = -1,\ t = 2$ satisfy the equation.

$-1 = -5 + 4,\ SATISFIED.$

Therefore, the vectors $[2, 4, -1], [8, -10, 5], [5, -3, 2]$ are coplanar.

8. Give examples of sets of three vectors that are.

(a) Collinear

If three vectors lie on the same line, we can call them collinear vectors. Let us have the following example.

$\vec{u} = m\vec{v},\ \vec{v} = n\vec{u}\ when\ m \neq 0\ and\ n \neq 0$

$\vec{u} = [1, 5, 7]$

$\vec{v} = [2, 10, 14]$

$\vec{w} = [6, 30, 42]$

We can express vector $\vec{v}$ and $\vec{w}$ in vector $\vec{u}$.

$\vec{v} = 2[1, 5, 7] = 2\vec{u}$

$\vec{w} = 6[1, 5, 7] = 6\vec{u}$

(b) Coplanar

We can say that three vectors $\rightarrow_a, \rightarrow_b, \rightarrow_c$ are coplanar vectors if they lie on the same plane and triple product value $(\rightarrow_a \cdot (\rightarrow_b \times \rightarrow_c) = 0)$ is zero.

Example abounds in the following.

$Let\ vector\ a\ =\ <1,1,1>,\ vector\ b\ =\ <1,3,1>,\ vector\ c =\ <2,2,2>$

Let us calculate the three product value.

$$\vec{b} \times \vec{c} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$= \hat{i}[6-2] + \hat{j}[2-2] + \hat{k}[2-1]$$

$$= 4\hat{i} - 4\hat{k}$$

Let us multiply vector a to the dot product of vector b and vector c.

$\rightarrow_a \cdot (\rightarrow_b \times \rightarrow_c) = (\widehat{i} + \widehat{j} + \widehat{k}) \cdot (4\widehat{i} - 4\widehat{k}) = (1)(4) + 1(0) + 1(-4) = 0$

Thus, the aforementioned vectors as an example are coplanar vectors.

(c) Not coplanar

We can say that three vectors $\rightarrow_a, \rightarrow_b, \rightarrow_c$ are NOT coplanar vectors if they don't lie on the same plane and triple product value $(\rightarrow_a \cdot (\rightarrow_b \times \rightarrow_c) = 0)$ is not zero.

Example abounds in the following.

$Let\ vector\ a\ =\ <1,2,3>,\ vector\ b\ =\ <1,1,1>,\ vector\ c =\ <1,2,1>$

Let us calculate the three product value.

$$\vec{b} \times \vec{c} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \hat{i}\,[1-2] + \hat{j}\,[1-1] + \hat{k}\,[2-1]$$

$$= -\hat{i} + \hat{k}$$

Let us multiply vector a to the dot product of vector b and vector c.

$\rightarrow_a \cdot (\rightarrow_b \times \rightarrow_c) = (\widehat{i} + 2\widehat{j} + 3\widehat{k}) \cdot (-\widehat{i} + \widehat{k}) = (1)(-1) + 2(0) + 3(1) = 2 \neq 0$

Thus, the aforementioned vectors as an example are NOT coplanar vectors.

9. Explain how you would prove if four given points are coplanar. Use your method to determine if A(3, 4, –2), B(8, 5, 0), C(1, 10, –6), and D(9, 2, 2) are coplanar.

For $vector\ AB = [(8-3)\widehat{i}, (5-4)\widehat{j}, (0-(-2)\widehat{k}] = (5, 1, 2)$

For $vector\ BC = [(1-8)\widehat{i}, (10-5)\widehat{j}, (-6-(0)\widehat{k}] = (-7, 5, -6)$

For $vector\ AC = [(9-1)\widehat{i}, (2-10)\widehat{j}, (2-(-6)\widehat{k}] = (8, -8, 8)$

To prove that they are coplanar vectors, show that $[vector\ AB * vector\ BC * vector\ CD] = 0$

$$\begin{bmatrix} 5 & 1 & 2 \\ -7 & 5 & 6 \\ 8 & -8 & 8 \end{bmatrix}$$

$$= 5(40-48) - 1(-56+48) + 2(56-40)$$
$$= 5(-8) - (-8) + 2(16)$$
$$= -40 + 8 + 32$$
$$= 0$$

Thus, since the three scalar product values are zero, the aforementioned vectors are coplanar.

10. Determine if the following vectors are coplanar. Assume that vector v1, vector v2, and vector v3 are not coplanar.

$\vec{w1} = 2\vec{v1} + 7\vec{v2}$

$\vec{w2} = \vec{v2} + 2\vec{v3}$

$\vec{w3} =- \vec{v1} - 7\vec{v3}$

Vectors are not coplanar as their scalar triple product is not zero.
We need to get a Scalar Triple Product, which is the following. $(\vec{W}_1 \times \vec{W}_2) \bullet \vec{W}_3$

$(\vec{W}_1 \times \vec{W}_2) = (2\vec{v1} + 7\vec{v2}) \times (\vec{v2} + 2\vec{v3})$
$= 2(\vec{v1} + \vec{v2}) + (2\vec{v1} \times 2\vec{v3}) + (7\vec{v2} \times \vec{v2}) + (7\vec{v2} \times 2\vec{v3})$
$= 2(\vec{v1} + \vec{v2}) + 4(\vec{v1} \times \vec{v3}) + 7|v2|^2 + 14(\vec{v2} \times \vec{v3})$

$(\vec{W}_1 \times \vec{W}_2) \bullet \vec{W}_3 = (-\vec{v1} - 7\vec{v3}) \bullet (2(\vec{v1} \times \vec{v2}) + 4(\vec{v1} \times \vec{v3}) + 7|v2|^2 + 14(\vec{v2} \times \vec{v3}))$
$=- 2\vec{v1} \bullet (\vec{v1} \times \vec{v2}) - 4\vec{v1}(\vec{v1} \times \vec{v3}) - 7\vec{v1}|\vec{v2}|^2 - 14(\vec{v1} \bullet (\vec{v2} \times \vec{v3})) - 14\vec{v3}(\vec{v1} \times \vec{v2})$
$- 28(\vec{v1} \times \vec{v3}) \bullet \vec{v3} - 49\vec{v3}|v2|^2 - 98\vec{v3} \bullet (\vec{v2} \times \vec{v3})$

To use the property $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0$, we can say the following.

$$-7\vec{v1}|\vec{v2}|^2 - 14\vec{v1} \cdot (\vec{v2} \times \vec{v3}) + 14\vec{v1}(\vec{v2} \times \vec{v3}) - 49\vec{v3}|v2|^2$$

$$= -7\vec{v1}|\vec{v2}|^2 - 49\vec{v3}|\vec{v2}|^2$$

Scalar triple product is not zero.

Thus, we can conclude that these vectors are non-coplanar.