Derivative Applications Unit Assignment
MCV4U

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1. **The Louvre museum in Paris features an inverted square pyramid with a height of 10 m. The side of the square is 13 m. Super-villains decide to fill the pyramid with water and do so at a rate of 10 m3 per second. How quickly is the water level rising when it reaches the top?**

We should consider that the volume of the pyramid would be the following.

$V = \frac{1}{3}bh$, where b is a base, which is the square.

Let s be the measure of sides of the square. Thus, substitute b with s.

$V = \frac{1}{3}s^2h$

The volume of water would be a square pyramid at any point in time which is similar to the container.

Thus, $\frac{s}{h} = \frac{13}{10}$, $s = \frac{13}{10}h$

We can express the volume in one variable as follows.

$V = \frac{1}{3}(\frac{13}{10}h^2)h = \frac{169}{300}h^3$

Now, differentiate the equation above.

$\frac{dv}{dt} = \frac{169}{300}(3h^2)\frac{dh}{dt}$

$\frac{dv}{dt} = \frac{169}{100}(h^2)\frac{dh}{dt} -> (1)$

We should acknowledge the following given by the question.

$\frac{dv}{dt} = \frac{10m^3}{s}$

Thus, substitute $\frac{dy}{dt} = 10$ with the equation (1) above.

$10 = \frac{169}{100}(h^2)\frac{dh}{dt}$

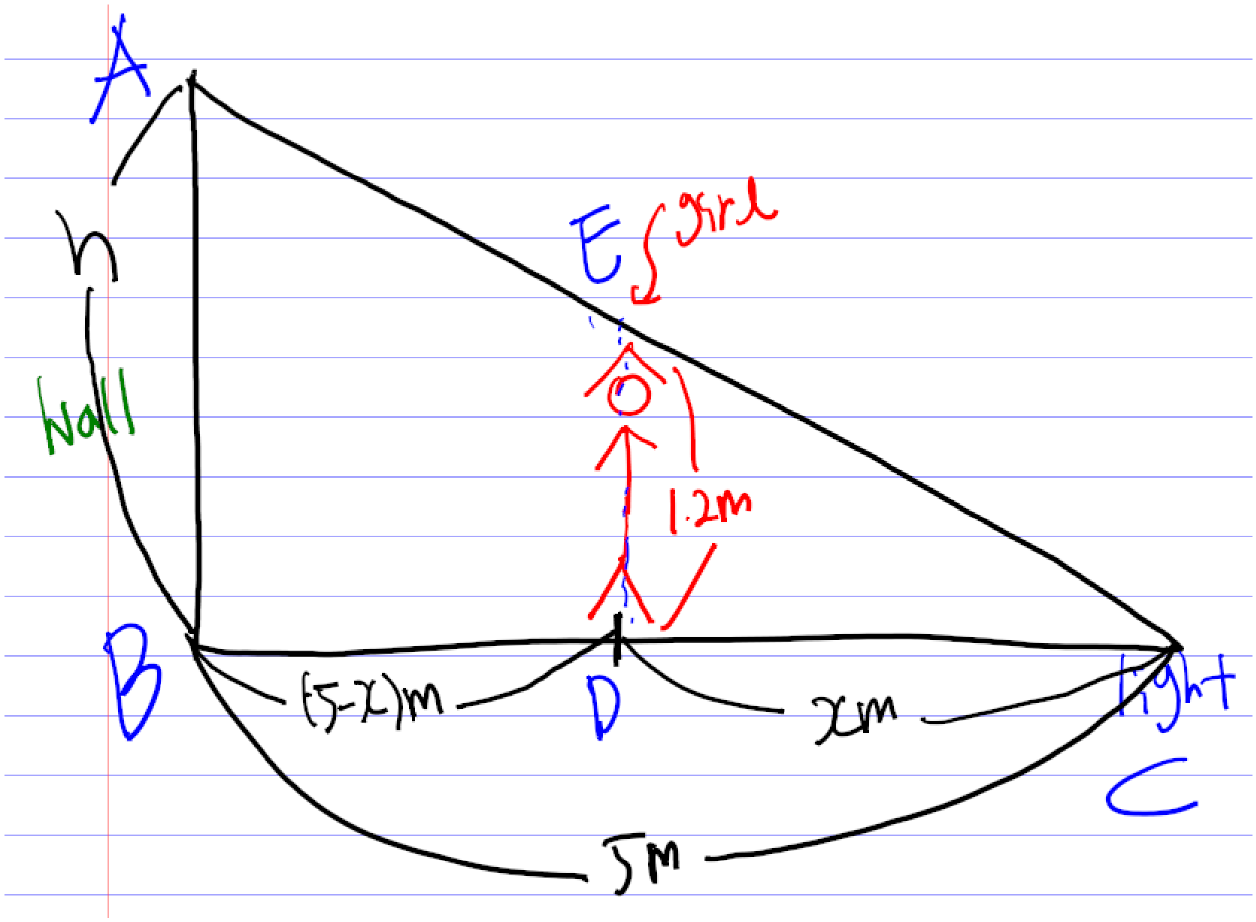
$\frac{dh}{dt} = \frac{1000}{169h^2}$

Thus, for example, when h is 10, we can conclude when the pyramid is full as follows.

$\frac{dh}{dt}|_{h=10} = \frac{1000}{169(100)} = \frac{1000}{16900} = \frac{10}{169}m/s$

The answer would be $\frac{10}{169}m/s$.

2. **A girl of height 120 cm is walking toward a light on the ground at a speed of 0.6 m/s. Her shadow is being cast on a wall behind her that is 5 m from the light. How is the size of her shadow changing when she is 1.5 m from the light?**



As the question gives, the height of the girl is 1.2m. Since we do not know the distance between the girl and light, we can let the distance as x. In addition, let the height of shadow as h.

$\frac{dx}{dt} = -\ 0.6m/s$, as x is decreasing as she moves toward light.

Besides, $\Delta ABC\ and\ \Delta EDC$ are similar to each other, we can write the following.

$\frac{AB}{DE} = \frac{BC}{DC}$

$\frac{h}{1.2} = \frac{5}{x}$

$h = \frac{6}{x}$

Let us get the rate of change of her shadow size according to the time.

$\frac{dh}{dt} = 6(-\frac{1}{x^2})\frac{dx}{dt}$

$= \frac{-6}{x^2}(-\ 0.6) = \frac{3.6}{x^2}$

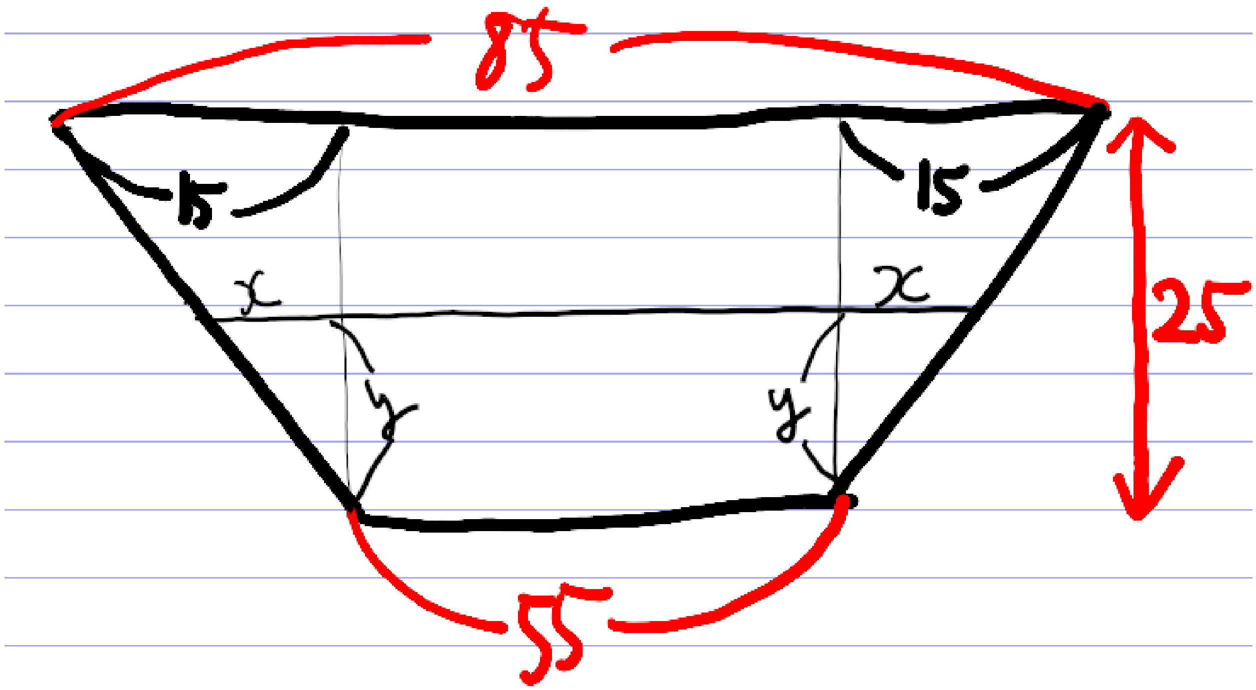
To get her shadow size when she is 1.5m from the light, solve it by substitute x with 1.5.

$\frac{dh}{dt}|_{x=1.5} = \frac{3.6}{(1.5)^2} = 1.6m/s$

Thus, the answer would be $1.6m/s$.

3. **Water is leaking from a trough at the rate of 0.8 L/s. The trough has a trapezoidal cross section, where the width at the bottom is 55 cm, at the top is 85 cm, and the height is 25 cm. The length of the trough is 3 m. Find the rate at which the height is changing when the depth of water is 11 cm.**

The illustration states the situation as given above.



We can use the proportional expression like the following.

$\frac{x}{y} = \frac{15}{25}$

$x = \frac{3}{5}y$

The volume of water which is in the trough from the bottom to the depth y is the multiplication of base area and height.

$Volume(V) = \frac{1}{2}(110 + 2x)y(300) = \frac{300}{2}y(110 + 2(\frac{3}{5}y))$

$= \frac{300}{2}y(110 + \frac{6}{5}y) = \frac{300}{2}y(\frac{550+6y}{5})$

$= \frac{300}{10}(550y + 6y^2)$

Differentiate the volume with respect to t as the following.

$\frac{dV}{dt} = \frac{300}{10}\frac{d}{dt}(550y + 6y^2)$

$-800(mL/s) = \frac{300}{10}(550\frac{dy}{dt} + 12y\frac{dy}{dt})$

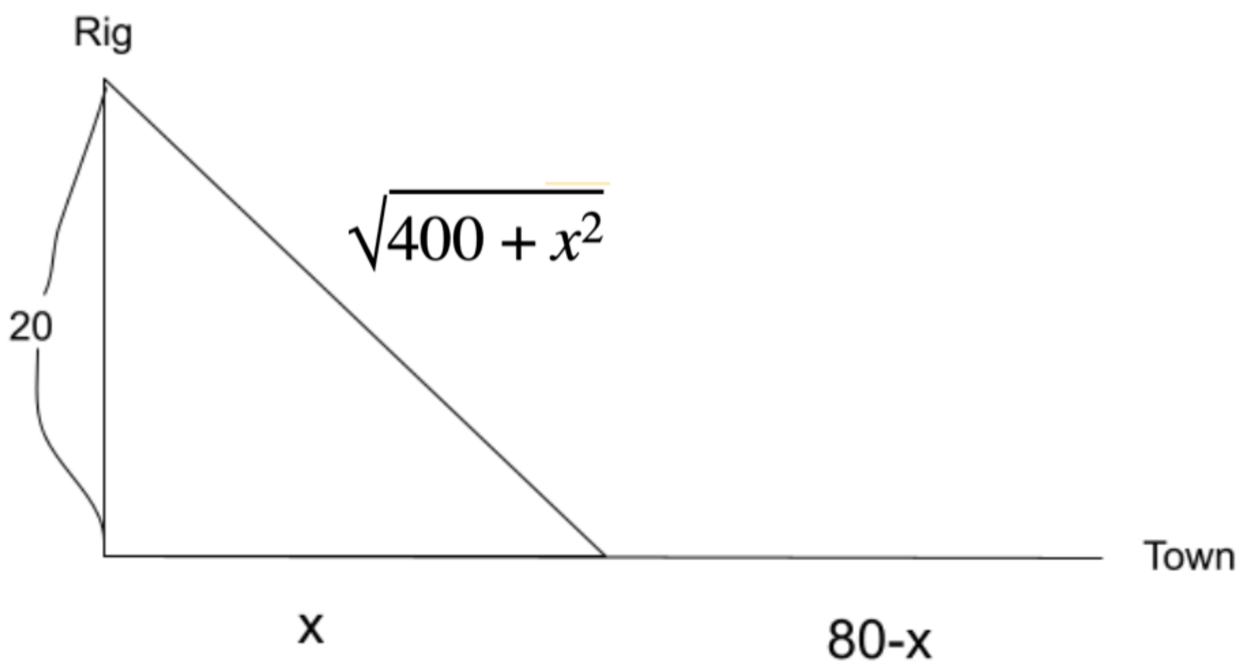
$-\frac{8000}{300} = (550 + 12y)\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-8000}{300(550+12y)}$

$= \frac{-8000}{300(550+12\times11)}$
$= \frac{-8000}{300(682)}$
$= \frac{-8000}{204600} \approx -\ 0.04 cm/sec$

Thus, the answer would be -0.04cm/sec.

4. **An oil rig lies 20 km off the coast of Newfoundland. A town lies 80 km along the coast from the nearest point on land to the rig. A pipe is to be drilled from the rig to the town. The cost per km of the pipe under water is $2.5 million, but on land is $1.5 million. Find the route that results in the cheapest pipeline, and determine that lowest cost.**



$C\ =\ total\ cost\ of\ millions\ of\ dollars$
$x\ =\ the\ distance\ in\ kilometers\ between\ the\ nearest\ point\ to\ the\ rig\ and\ the\ start\ of\ the\ land\ pipeline.$

- On the passage, costs for land pipeline is $1.5 million/km and $2.5 million/km for underwater pipeline.

$C\ =\ 2.5(\sqrt{400\ +\ x^2})\ +\ 1.5(80\ -\ x) = \ 2.5(400\ +\ x^2)^{\frac{1}{2}}\ -\ 1.5x\ +\ 100$

Get the first derivative.

$C'\ =\ 1.25(400\ +\ x^2)^{\frac{-1}{2}}(2x)\ -\ 1.5$

Substitute $C'$with 0.

$0 = \frac{2.5x}{(400+x^2)^{\frac{1}{2}}} - 1.5$

$1.5 = \frac{2.5x}{(400+x^2)^{\frac{1}{2}}}$

$2.5x = 1.5(400 + x^2)^{\frac{1}{2}}$

$6.25x^2 = 2.25(400 + x^2)$

$6.25x^2 = 900 + 2.25x^2$

$4x^2 = 900$

$x^2 = \sqrt{225}$

$x = \pm 15$

Since the distance can not be a negative value, we need to determine whether the critical point is max or min.

	$x < 15$	$x = 15$	$x > 15$
C'	$-$	0	$+$
	\	_	/
		Min	

The two interval endpoints must be compared to see whether the critical point is an absolute minimum between the intervals.

- $x = 0$
  $C = 2.5(\sqrt{400 + 0}) + 1.5(80 - 0)$
  $C = 170$
- $x = 80$
  $C = 2.5(\sqrt{400 + 80^2}) + 1.5(80 - 80)$
  $C \simeq 206.15...$
- $x = 15$
  $C = 2.5(\sqrt{400 + 15^2}) + 1.5(80 - 15)$
  $C = 160$

Thus, the absolute minimum point is at $x = 15, when\ 0 \leq x \leq 80$.

The length of the underwater pipeline is the following.

- $\sqrt{(15)^2 + 400} = 25$

The length of the land pipeline is the following.

$(80 - 15) = 65$

As a result, the pipeline's optimal path will have 25 kilometers of underwater pipeline and 65 kilometers of land pipeline. $160 million will be the optimal cost.