Derivatives of Exponential and Logarithmic Functions Unit Assignment

MCV4U

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1. Find the derivatives of each of these functions.

(a) $y = ln(4x^2 + 4)$

- To apply the Chain Rule, set u as $4x^2 + 4$
  - $\frac{d}{du}(ln(u))\frac{d}{dx}(4x^2 + 4)$
- The derivative of $ln(u)$ with respect to u is the following.
  - $\frac{1}{u}\frac{d}{dx}(4x^2 + 4)$
  - $\frac{1}{4x^2+4}\frac{d}{dx}(4x^2 + 4)$
  - $\frac{1}{4x^2+4}(8x)$
- The answer is $\frac{8x}{4x^2+4}$.

(b) $y = \frac{2x^4}{e^{5x}}$

- $y = 2x^4e^{-5x}$
- $y' = 2\frac{d}{dx}(x^4e^{-5x})$
  $= 2(x^4(-5e^{-5x}) + 4x^3e^{-5x})$
  $=- 10x^4e^{-5x} + 8x^3e^{-5x}$
- $y' = \frac{-10x^4+8x^3}{e^{5x}}$

(c) $y = 2^{5x+7}(ln(5x + 1))$

- $y' = 2^{5x+7}\frac{d}{dx}(ln(5x + 1)) + ln(5x + 1)\frac{d}{dx}(2^{5x+7})$
  $= \frac{5 \cdot 2^{5x+7}}{5x+1} + ln(5x + 1) \cdot 2^{5x+7}ln(2) \cdot 5$
- $y' = \frac{5}{5x+1}2^{5x+7} + 5ln2 \cdot ln(5x + 1) \cdot 2^{5x+7}$

(d) $y' = \frac{4x^3}{e^{5x}+x^4}$

- $y' = 4 \cdot \frac{d}{dx}(\frac{x^3}{e^{5x}+x^4})$
  $= 4(\frac{(e^{5x}+x^4)\frac{d}{dx}(x^3)-x^3\frac{d}{dx}(e^{5x}+x^4)}{(e^{5x}+x^4)^2})$
  $= 4\frac{[3x^2(e^{5x}+x^4)-x^3(5e^{5x}+4x^3)]}{(e^{5x}+x^4)^2}$
  $= \frac{12x^6+12x^2e^{5x}-20x^3e^{5x}-16x^6}{(e^{5x}+x^4)^2}$

2. Use the process of implicit differentiation to find $\frac{dy}{dx}$ given that $x^3e^y - ye^x = 0$

- $e^yx^3\frac{d}{dx}[y] + 3e^yx^2 - e^xy - e^x\frac{d}{dx}[y] = 0$
- $e^yx^3y' + 3e^yx^2 - e^xy - e^xy' = 0$
- $y'(e^yx^3 - e^x) + 3e^yx^2 - e^xy = 0$
- $y' =- \frac{3x^2e^y-ye^x}{x^3e^y-e^x}$
- $\frac{dy}{dx} =- \frac{3x^2e^y-ye^x}{x^3e^y-e^x}$

3. Use curve sketching methods, sketch the graph of the function given by the equation $y = \frac{x^2}{e^x}$. Make sure that you include all steps, charts, and derivations details. (10 marks)

- To begin with, find the first derivative.

  $y = \frac{x^2}{e^x}$

  $y' = \frac{d}{dx}(\frac{x^2}{e^x}) = \frac{e^x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(e^x)}{(e^x)^2} = \frac{e^x(2x)-x^2e^x}{e^{2x}} = \frac{e^x(2x-x^2)}{e^{2x}} = \frac{2x-x^2}{e^x}$
- Find the second derivative.

  $y'' = \frac{dy'}{dx} = \frac{d}{dx}(\frac{2x-x^2}{e^x}) = \frac{e^x\times\frac{d}{dx}(2x-x^2)-(2x-x^2)\frac{d}{dx}(e^x)}{(e^x)^2} = \frac{e^x(2-2x)-(2x-x^2)e^x}{e^{2x}} = \frac{e^x(2-2x-2x+x^2)}{e^{2x}}$

$= \frac{2-4x+x^2}{e^x}$

- To find the x-intercept of a given y function.
  $y|_{x=0} = \frac{x^2}{e^x}|_{x=0}$
  $x^2 = 0,\ x = 0$
- To find critical points.
  $y' = 0$
  $\frac{2x-x^2}{e^x} = 0$
  $2x - x^2 = 0$
  $x(2-x) = 0$
  $x = 0\ or\ x = 2$
- Slope chart of the first derivative

	x<0	0<x<2	x>2
$2x - x^2$	-	+	-
e^x	+	+	+
$\frac{2x-x^2}{e^x}$	-	+	-

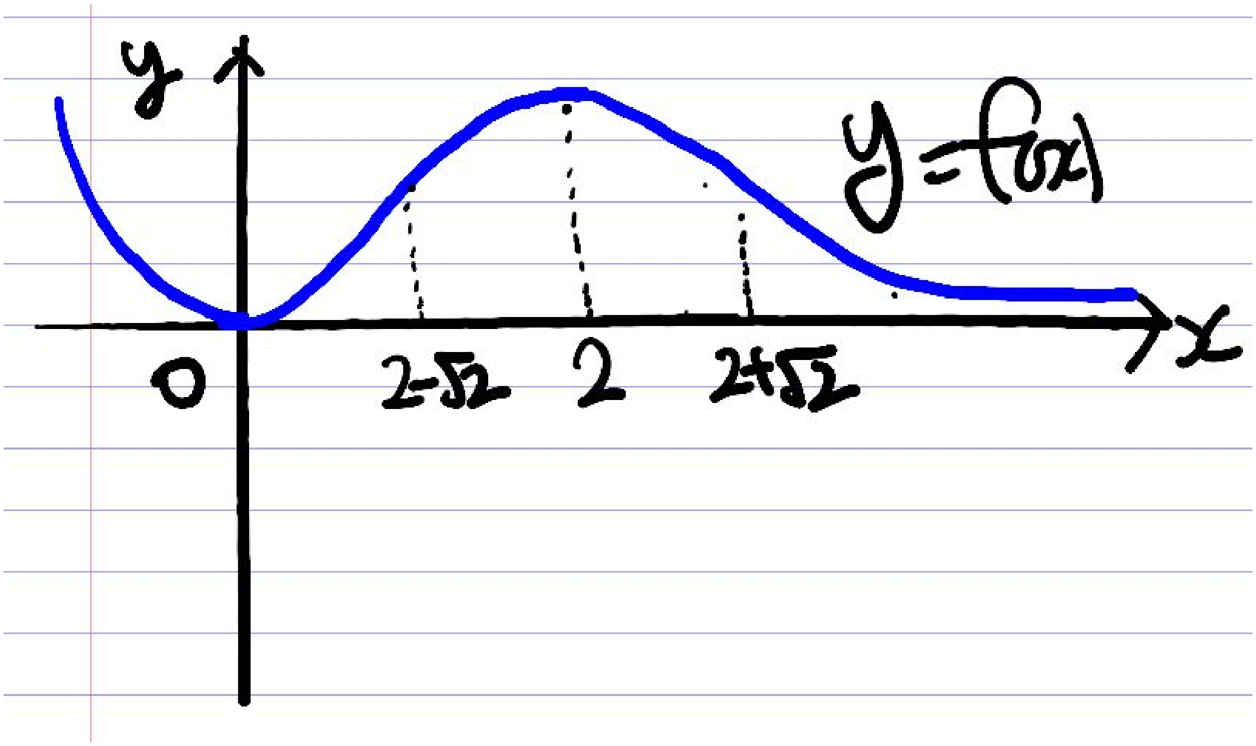
- To find inflection point
  $y'' = 0$
  $\frac{2-4x+x^2}{e^x} = 0$
  Since $e^x > 0$ is always true, $2-4x+x^2$ should be zero.
  $2-4x+x^2 = 0$
  $x = 2+\sqrt{2}\ or\ x = 2-\sqrt{2}$

- To find concavity chart

	$x < 2-\sqrt{2}$	$2-\sqrt{2} < x < 2+\sqrt{2}$	$x > 2+\sqrt{2}$
$2-4x+x^2$	+	-	+
e^x	+	+	+
$\frac{2-4x+x^2}{e^x}$	+	-	+

- To find end behaviour, send x to infinity or minus infinity.
  - send x to infinity
    Use L'Hospital's rule until the function does not show $\frac{\infty}{\infty}$ form.
    $y = lim_{x->\infty}(\frac{x^2}{e^x}) = lim_{x->\infty}\frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^x)} = lim_{x->\infty}\frac{2x}{e^x} = lim_{x->\infty}\frac{2}{e^x} = 0$

- send x to minus infinity
  $y = lim_{x->-\infty}(\frac{x^2}{e^x}) = \frac{(-\infty)^2}{e^{(-\infty)}} = \infty \times \infty = \infty$



4. Use logarithmic differentiation to find the derivative of the function $y = \frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}}$ **.**

$ln(y) = ln(\frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}})$
$ln(y) = ln(2x+5)^3 + ln(5x^2-2x)^4 - ln(\sqrt{2x+5})$
$ln(y) = 3ln(2x+5) + 4ln(5x^2-2x) - \frac{1}{2}ln(2x+5)$

Differentiating both sides as the following.

$\frac{dy}{dx}\frac{1}{y} = \frac{3}{(2x+5)}\frac{d}{dx}(2x+5) + \frac{4}{(5x^2-2x)}\frac{d}{dx}(5x^2-2x) - \frac{1}{2(2x+5)}\frac{d}{dx}(2x+5)$
$= \frac{3}{(2x+5)}(2) + \frac{4}{(5x^2-2x)}(10x-2) - \frac{1}{2(2x+5)}(2)$
$= \frac{6}{(2x+5)} + \frac{40x-8}{(5x^2-2x)} - \frac{1}{(2x+5)}$
$= \frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)}$

Thus, the answer would be
$\frac{dy}{dx} = \frac{(2x+5)^3(5x^2-2x)^4}{\sqrt{2x+5}}(\frac{5}{(2x+5)} + \frac{40x-8}{(5x^2-2x)})$

5. The population (P) of an island y years after colonisation is given by the function $P = \frac{250}{1+4e^{-0.01y}}$ **.**

(a) What was the initial population of the island?
We can get the initial population by substituting y = 0.

$P(0) = \frac{250}{1+4e^{-0.01(0)}} = \frac{250}{1+4} = 50$

Thus, the initial population is 50 people.

(b) How long did it take before the island had a population of 150?

We can get the answer by substituting P = 150.

$150 = \frac{250}{1+4e^{-0.01y}}$

$150(1 + 4e^{-0.01y}) = 250$

$150 + 600e^{-0.01y} = 250$

$600e^{-0.01y} = 100$

$e^{-0.01y} = \frac{1}{6}$

$-0.01y = ln(\frac{1}{6})$

$-0.01y =- ln(6)$

$y = \frac{ln(6)}{0.01}$

$y \approx 179.176\ years$

Thus, the answer would be 179.176 years.

(c) Using curve sketching methods, sketch the graph of the function. Make sure that you include all steps, charts, and derivations details.

By using Chain Rules, we can render the first derivatives as the following.

$P'(y) = 250(-1)(1 + 4e^{-0.01y})^{-2}(4e^{-0.01y})(-0.01)$

$P'(y) = \frac{10e^{-0.01y}}{(1+4e^{-0.01y})^2}$

$P'(y) = \frac{10e^{0.01y}}{(4+e^{0.01y})^2} > 0$

Thus, the population or graph is always increasing.

We can render the second derivatives as the following.

$P''(y) = \frac{d}{dy}(\frac{10e^{0.01y}}{(4+e^{0.01y})^2})$

$= 10 \times (\frac{-0.01e^{-0.01y}(1+4e^{-0.01y})^2+2e^{-0.01y}(1+4e^{-0.01y})\times 0.04e^{-0.01y}}{(1+4e^{-0.01y})^4})$

$= 10 \times (\frac{-0.01e^{-0.01y}(1+4e^{-0.01y}) +0.08(e^{-0.01y})^2}{(1+4e^{-0.01y})^3})$

$= \frac{10e^{-0.01y}}{(1+4e^{-0.01y})^3}(-0.01(1 + 4e^{-0.01y}) + 0.08(e^{-0.01y}))$

$\frac{d^2p}{dy^2} = 0$

$=> \ -0.01 - 0.04e^{-0.01y} + 0.08e^{-0.01y} = 0$

$=> \ 0.04e^{-0.01y} = 0.01$

$=> \ e^{-0.01y} = \frac{1}{4}$

$=> \ 0.01y = ln(4),\ y = 100ln(4)$

Get the point of inflection at $y = 100ln(4)$

$P(100ln(4)) = \frac{250}{1+4e^{-0.01\times 100ln(4)}} = \frac{250}{1+4e^{-ln4}} = \frac{250}{1+4\times\frac{1}{4}}$

$= \frac{250}{2} = 125$

P is concave up when the second derivative is larger than zero.

$0.04e^{-0.01y} > 0.01$

$e^{-0.01y} > \frac{1}{4}$

$-0.01y > ln(\frac{1}{4})$

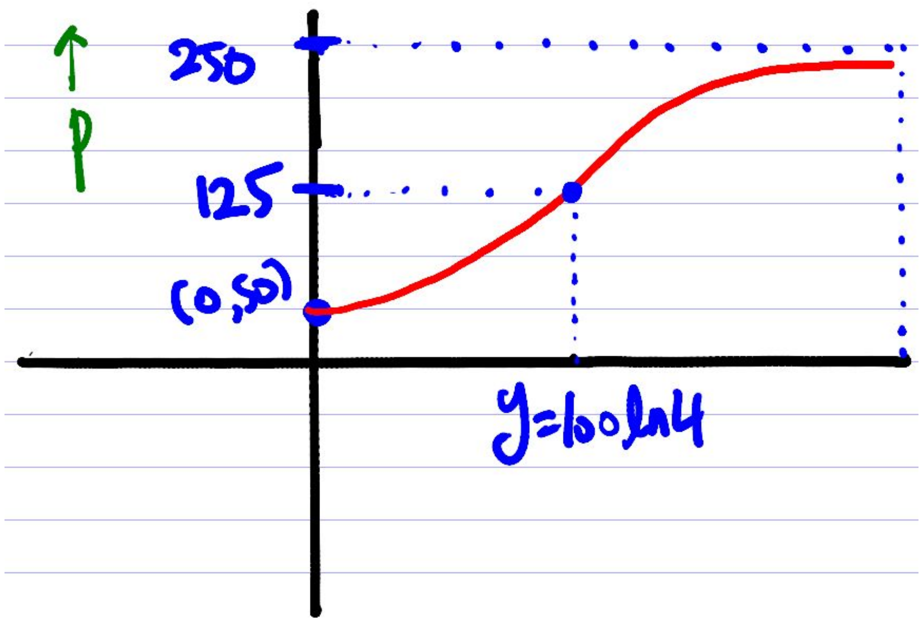
$y < 100ln(4)$

Thus, if y is larger than 100ln(4), the graph is always concave down.

Calculate the end behaviour when x goes infinity or minus infinity.

$lim_{y->\infty}P = lim_{y->\infty}(\frac{250}{1+4e^{-0.01y}}) = 250$

$lim_{y->-\infty}P = lim_{y->-\infty}(\frac{250}{1+4e^{-0.01y}}) = 0$



e. Give a possible explanation for the shape of the curve.

Since it is an exponential increasing function, the carrying capacity is 250 which is the limit of population. As the population P(y) in the island increases, the number of years increases but the amount of increase declines after y=100ln4.

Thus, we can think of the possibility that the population increment is limited according to the water or food resources that island has.