MCV4U Unit 2 Assignment
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1. **Write an example of each of the following (assuming it is in 3-dimensional space).**

- A point lying on the x-axis.
  Example: $(2, 0, 0)$
- A point lying on the yz plane.
  Example: $(0, 2, 3)$
- A point lying on both the xy and xz planes.
  A point can lie on both xy planes and yz planes if and only if the point lies on the x-axis.
  Example: $(1, 0, 0)$
- A point lying on all three planes.
  Example: $(2, 3, 4)$
- A point lying on none of the three planes, but equidistant from the xz and yz planes.
  Example: $(0, 0, 0)$

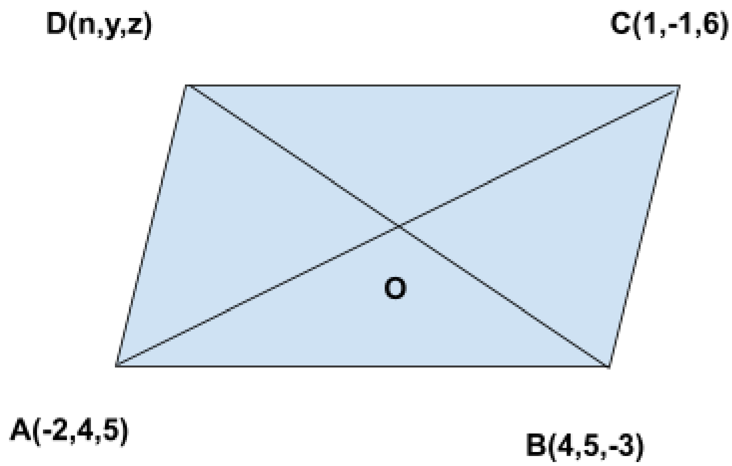
2. **Triangle ABC has vertices A(4, 7, 7), B(1, 6, 5), and C(–2, 9, 8). What kind of triangle is ΔABC?**

- To begin with, getting the distance between two points having coordinates $(x1, y1, z1),\ (x2, y2, z2)$ are $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- We could calculate the edge lengths of the given triangle as the following.
  - $AB = \sqrt{(1-4)^2 + (6-7)^2 + (5-7)^2} = \sqrt{9+1+4} = \sqrt{14}$
  - $AC = \sqrt{(-2-4)^2 + (9-7)^2 + (8-7)^2} = \sqrt{36+4+1} = \sqrt{41}$
  - $BC = \sqrt{(-2-1)^2 + (9-6)^2 + (8-5)^2} = \sqrt{9+9+9} = \sqrt{27}$
- We can get whether the given triangle ABC is a right angle triangle or not using the Pythagorean theorem.
  - $AC^2 = AB^2 + BC^2$
  - $\sqrt{41}^2 = \sqrt{14}^2 + \sqrt{27}^2$
  - $41 = 14 + 27$
  - Hence, the given triangle ABC is a right-ended triangle.

3. **The points (–2, 4, 5), (4, 5, –3), and (1, –1, 6) are three of four vertices of parallelogram ABCD. Explain why there are three possibilities for the location of the fourth vertex, and find the three points.**

There are three possibilities for the 4th vertex because either of the three vertices (A, B, or C) can be the 4th vertex.

First Case.

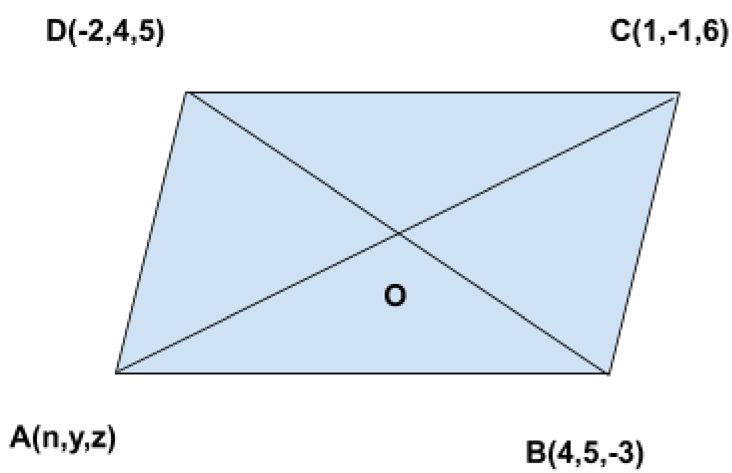


We will find the case where the 4th vertex is at the first vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the mid-point of the two points, which is the value of the summing each of unknown value of 4th vertex to that of opposite point.

With Point D and Point B, we can get the midpoint as follows. $O \equiv (\frac{n+4}{2}, \frac{y+5}{2}, \frac{z-3}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{-2+1}{2}, \frac{4-1}{2}, \frac{5+6}{2})$
We can calculate the values of undetermined midpoint like the following.
$\frac{n+4}{2} = -\frac{1}{2}, \ \frac{y+5}{2} = \frac{3}{2}, \ \frac{z-3}{2} = \frac{11}{2} \ \Rightarrow \ n = -5, \ y = -2, \ z = 14$

Thus, Point D is $(-5, \ 2, \ 14)$.

Second Case.

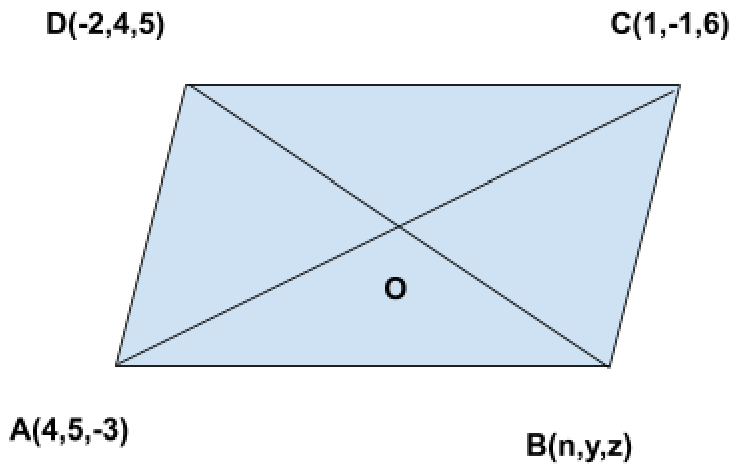


We will find the case where the 4th vertex is on the left lower vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the mid-point of the two points, which is the value of the summing each of unknown value of 4th vertex to that of opposite point.

With Point A and Point C, we can get the midpoint as follows. $O \equiv (\frac{n+1}{2}, \frac{y-1}{2}, \frac{z+6}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{4+2}{2}, \frac{5+4}{2}, \frac{-3+5}{2})$
We can calculate the values of undetermined midpoint like the following.
$\frac{n+1}{2} = 1, \ \frac{y-1}{2} = \frac{9}{2}, \ \frac{z+6}{2} = 1 \ => \ n = 1, \ y = 10, \ z =- 4$

Thus, Point D is $(1, \ 10, \ -4)$.

Third Case.

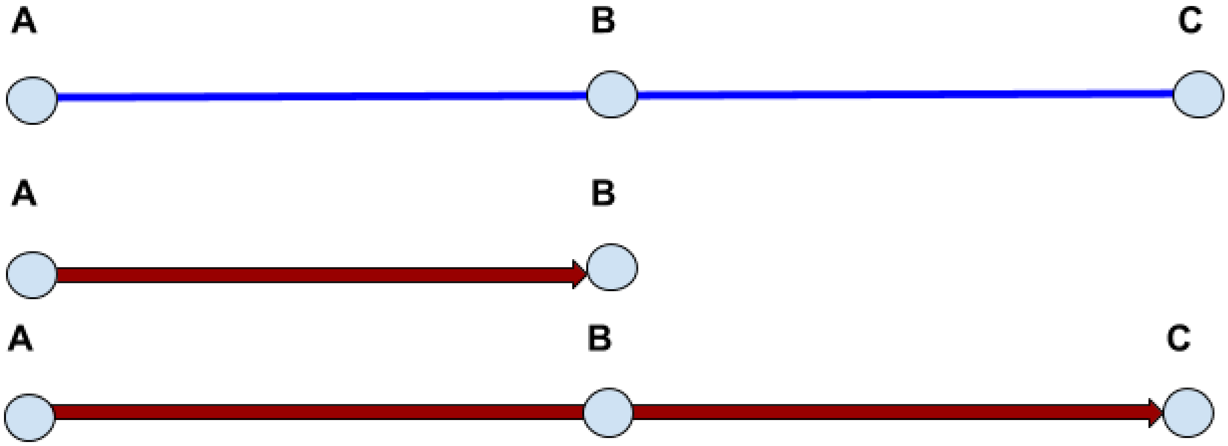


We will find the case where the 4th vertex is on the right lower vertex of the parallelogram. And then, I would like to calculate the known values of the points and use them to determine the value of Point O, the mid-point of the two points, which is the value of the summing each of unknown value of 4th vertex to that of opposite point.

With Point A and Point C, we can get the midpoint as follows. $O \equiv (\frac{n-2}{2}, \frac{y+4}{2}, \frac{z+5}{2})$
With Point A and Point C, we can get the determined value of midpoint as follows.
$O \equiv (\frac{5}{2}, 2, \frac{3}{2})$
We can calculate the values of undetermined midpoint like the following.
$\frac{n-2}{2} = \frac{5}{2}, \ \frac{y+4}{2} = 2, \ \frac{z+5}{2} = \frac{3}{2} \ => \ n = 7, \ y = 0, \ z =- 2$

Thus, Point D is $(7, \ 0, \ -2)$.

4. **The points A(–2, –1, z), B(2, 4, 3), and C(10, y, –1) are collinear. Find the values of y and z.**
   To understand the concept of collinear values, let us draw two possible vectors, $\overline{AB} \ and \ \overline{AC}$.

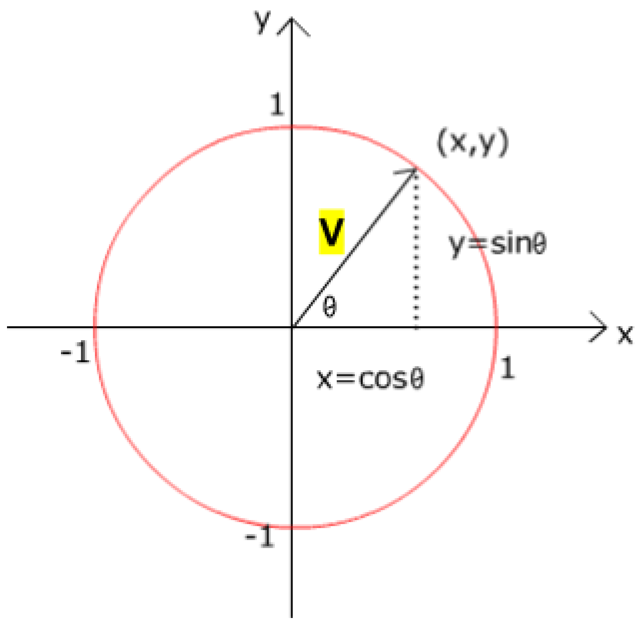


It is clearly appreciatable that the aforementioned vectors can be scaled up and down in length to shrink to another vector. In addition, these vectors are only inclined to change according to the sign of the scaling factor. For example, $2 \rightarrow_i + 4 \rightarrow_j$ is a multiple of $1 \rightarrow_i + 2 \rightarrow_j$ by a factor of 2, while $-2 \rightarrow_i - 4 \rightarrow_j$ is only available for scaled down by a factor of 2.

$\overline{AB} = (2, 4, 3) - (-2, -1, z) = (4, 5, 3 - z)$
$\overline{AC} = (10, y, -1) - (-2, -1, z) = (12, y + 1, -1 - z)$
Using scaling logic, let the following.
$\overline{AB} = k\overline{AC} \Rightarrow (4, 5, 3 - z) = k(12, y + 1, -1 - z) = (12k, ky + k, -k - kz)$
Comparing the respective given elements.
$12k = 4,\ ky + k = 5,\ \ -k - zk = 3 - z$
$element\ X\ = k = \frac{1}{3}$
$element\ Y\ = \frac{1}{3}(y + 1) = 5 \Rightarrow y = 14$
$element\ Z = \frac{1}{3}(-1 - z) = 3 - z \Rightarrow z = 5$

Thus, the answer would be $y = 14,\ z = 5$

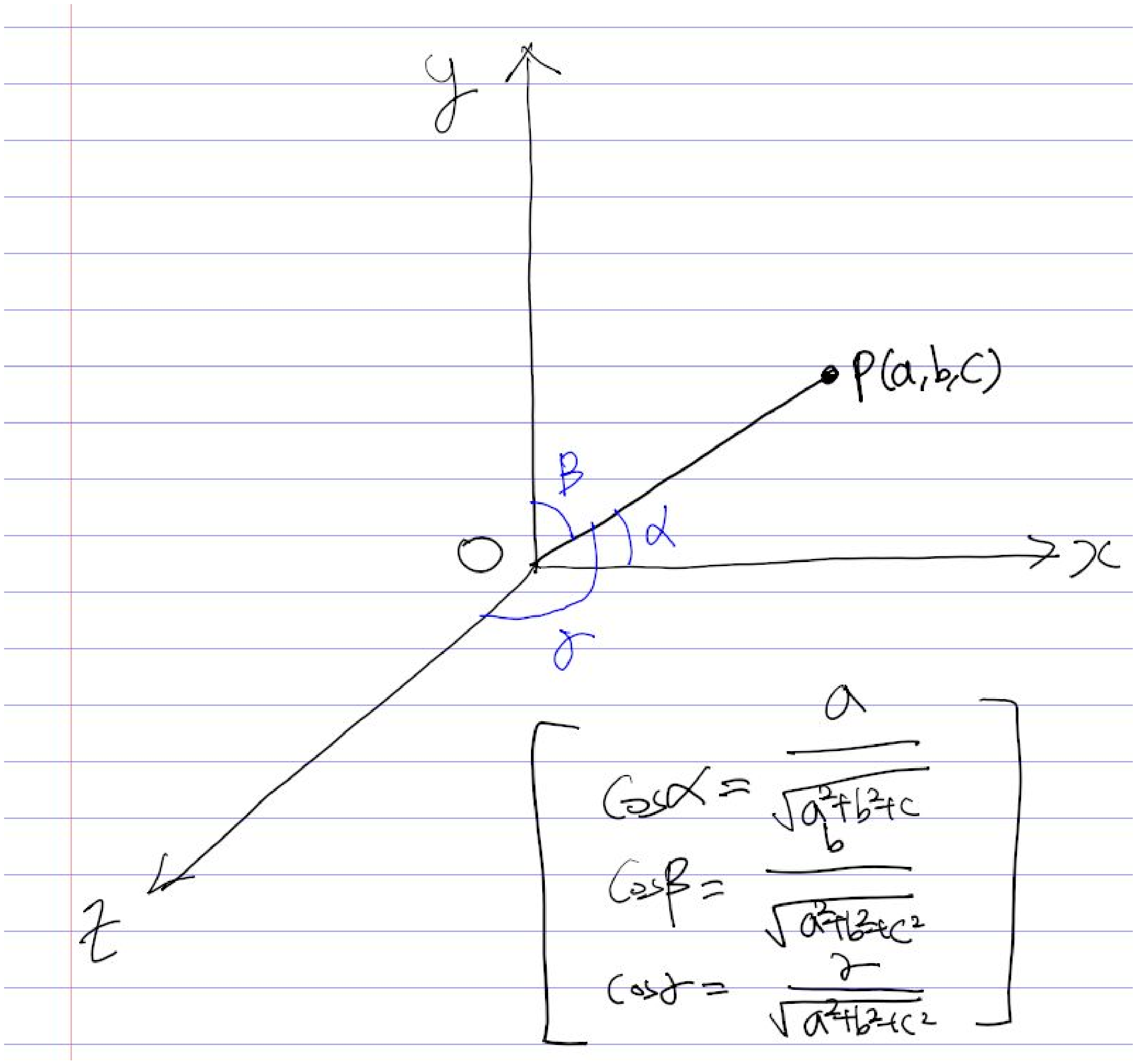
5. **Explain the meaning of direction angles and their relation to direction vectors.**
   Direction angle means the angle between a vector and the positive x-axis.



In the picture shown above, vector $\vec{v}$ has a direction angle θ while $\vec{v} = |\vec{v}|\,cos\theta i + |\vec{v}|\,sin\theta j$ where $|\vec{v}|$ is the magnitude of vector $\vec{v}$.

Let the vectors be $\vec{a}, \vec{b}$, then directional cosine is $cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$, where θ is directional angle between a, b and $\vec{a} \cdot \vec{b}$ is a dot product between two vectors while $|\vec{a}|, |\vec{b}|$ is a magnitude of each vector respectively.

1. What are the direction angles of the vector [–5, 1, 8]?



P is a general point with (a, b, c) while they are (x, y, z)'s coordinate respectively.
α, β, γ are the direction angle of the vector $\overline{OP}$ with x, y, z-axis respectively.

Given the vector, $u = -5\widehat{i} + \widehat{j} + 8\widehat{k}$. $a = -5, b = 1, c = 8$

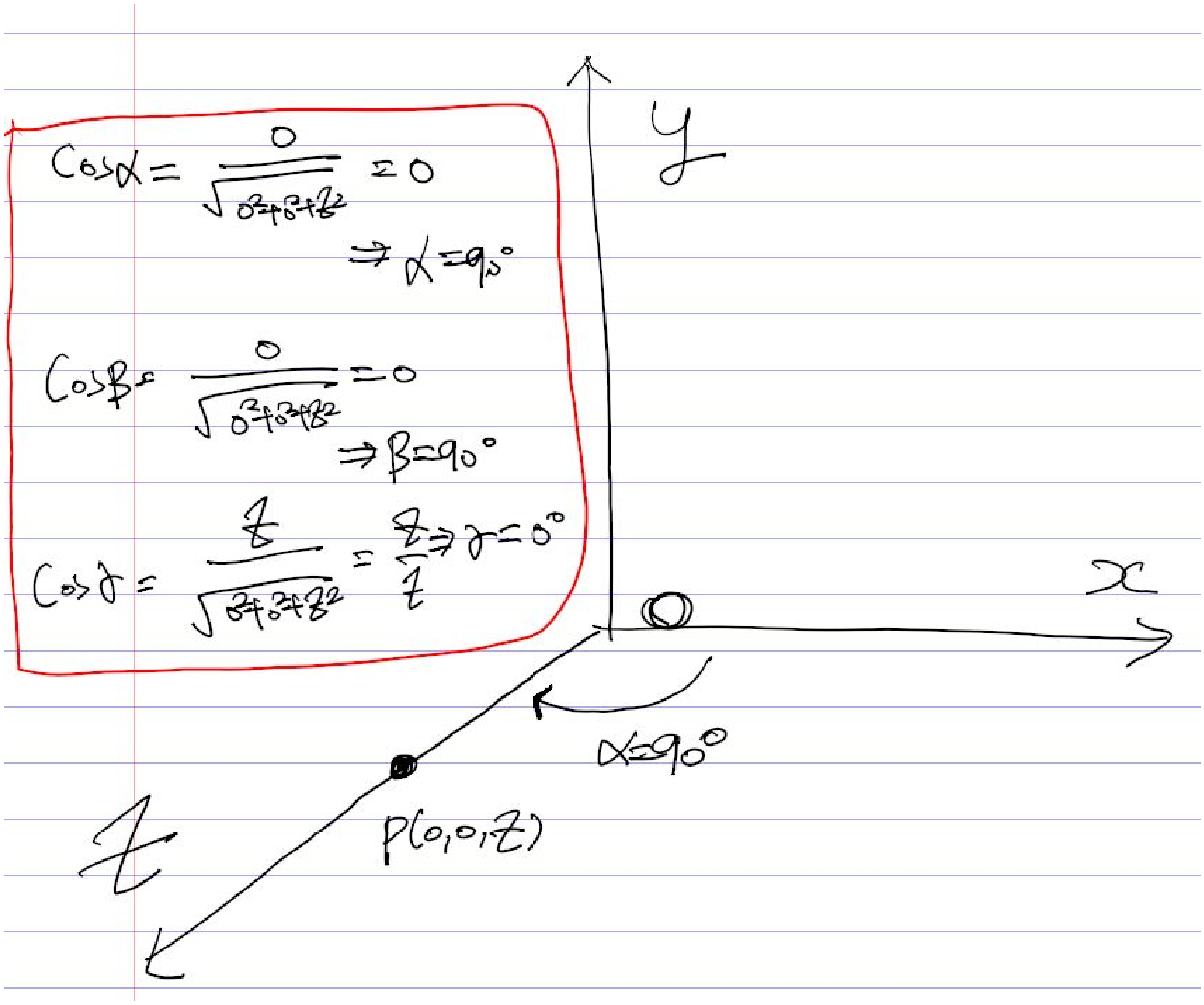
$cos\ \alpha = \frac{-5}{\sqrt{(-5)^2+1^2+8^2}} = -\frac{\sqrt{10}}{6},\ \alpha = cos^{-1}\left[\frac{-\sqrt{10}}{6}\right] = 121.81^o$

$cos\ \beta = \frac{1}{\sqrt{(-5)^2+1^2+8^2}} = \frac{\sqrt{10}}{30},\ \beta = cos^{-1}\left[\frac{\sqrt{10}}{30}\right] = 83.95^o$

$cos\ \gamma = \frac{8}{\sqrt{(-5)^2+1^2+8^2}} = \frac{4\sqrt{10}}{15},\ \gamma = cos^{-1}\left[\frac{4\sqrt{10}}{15}\right] = 32.51^o$

Thus, directional angles are $\alpha = 121.81^o,\ \beta = 83.95^o,\ \gamma = 32.51^o$.

2. If a point P lies on the z-axis, what are the direction angles of the position vector $\overline{OP}$?



If Point P lies on the z-axis, then the x, y coordinates of a point are zero. (ex. $P(0,0,z)$)

$cos\ \alpha = \frac{0}{\sqrt{0^2+0^2+z^2}} = 0,\ \alpha = 90^o$

$cos\ \beta = \frac{0}{\sqrt{0^2+0^2+z^2}} = 0,\ \beta = 90^o$

$cos\ \gamma = \frac{z}{\sqrt{0^2+0^2+z^2}} = \frac{z}{z} = 1,\ \gamma = 0^o$

Thus, the directional angles are $\alpha = 90^o,\ \beta = 90^o,\ \gamma = 0^o$.

c. Prove that $cos^2(\alpha) + cos^2(\beta) + cos^2(\gamma) = 1$.

o

To prove that $cos^2(\alpha) + cos^2(\beta) + cos^2(\gamma) = 1$, where α, β, γ are direction angles of a vector with x, y, z axis respectively.

Let the vector $\overline{OP}$ with Point P (a, b, c) , from we can get the following.

$cos\ \alpha = \frac{a}{\sqrt{a^2+b^2+c^2}},\ cos^2\alpha = \frac{a^2}{a^2+b^2+c^2}$

$cos\ \beta = \frac{b}{\sqrt{a^2+b^2+c^2}},\ cos^2\beta = \frac{b^2}{a^2+b^2+c^2}$

$cos\ \gamma = \frac{c}{\sqrt{a^2+b^2+c^2}},\ cos^2\gamma = \frac{c^2}{a^2+b^2+c^2}$

Thus, $cos^2\alpha + cos^2\beta + cos^2\gamma = \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2} = 1$

d. A vector has direction angles $\alpha = 85^o,\ \beta = 65^o$.

(a) Find the value γ .

We know that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$.

Substitute α, β into $85^o,\ 65^o$.

$cos^2(85) + cos^2(65) + cos^2(\gamma) = 1.$

$7.596 * 10^{-3} + 0.1786 + cos^2\gamma = 1.$

$cos^2\gamma = 0.8138 => cos\gamma = 0.9021$

Thus, $cos^{-1}(0.9021) = 25.56^o$.

(b) Find a vector that has those direction angles. $\alpha = 85^o,\ \beta = 65^o,\ \gamma = 25.65^o$.

$cos\ \alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}\ =>\ \alpha = \sqrt{a^2 + b^2 + c^2} * cos(\alpha) = 1 * cos85^o => a = 0.0871$

$cos\ \beta = \frac{b}{\sqrt{a^2+b^2+c^2}}\ =>\ \beta = \sqrt{a^2 + b^2 + c^2} * cos(\beta) = 1 * cos65^o => b = 0.4226$

$cos\ \gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}\ =>\ \alpha = \sqrt{a^2 + b^2 + c^2} * cos(\gamma) = 1 * cos25.65^o => c = 0.9021$

The vector is $[0.0871,\ 0.4226,\ 0.9021]$ which is having directional angles as α, β, γ are specified.

(c) Explain why it is not possible for two of a vector's direction angles to be less than 45°.

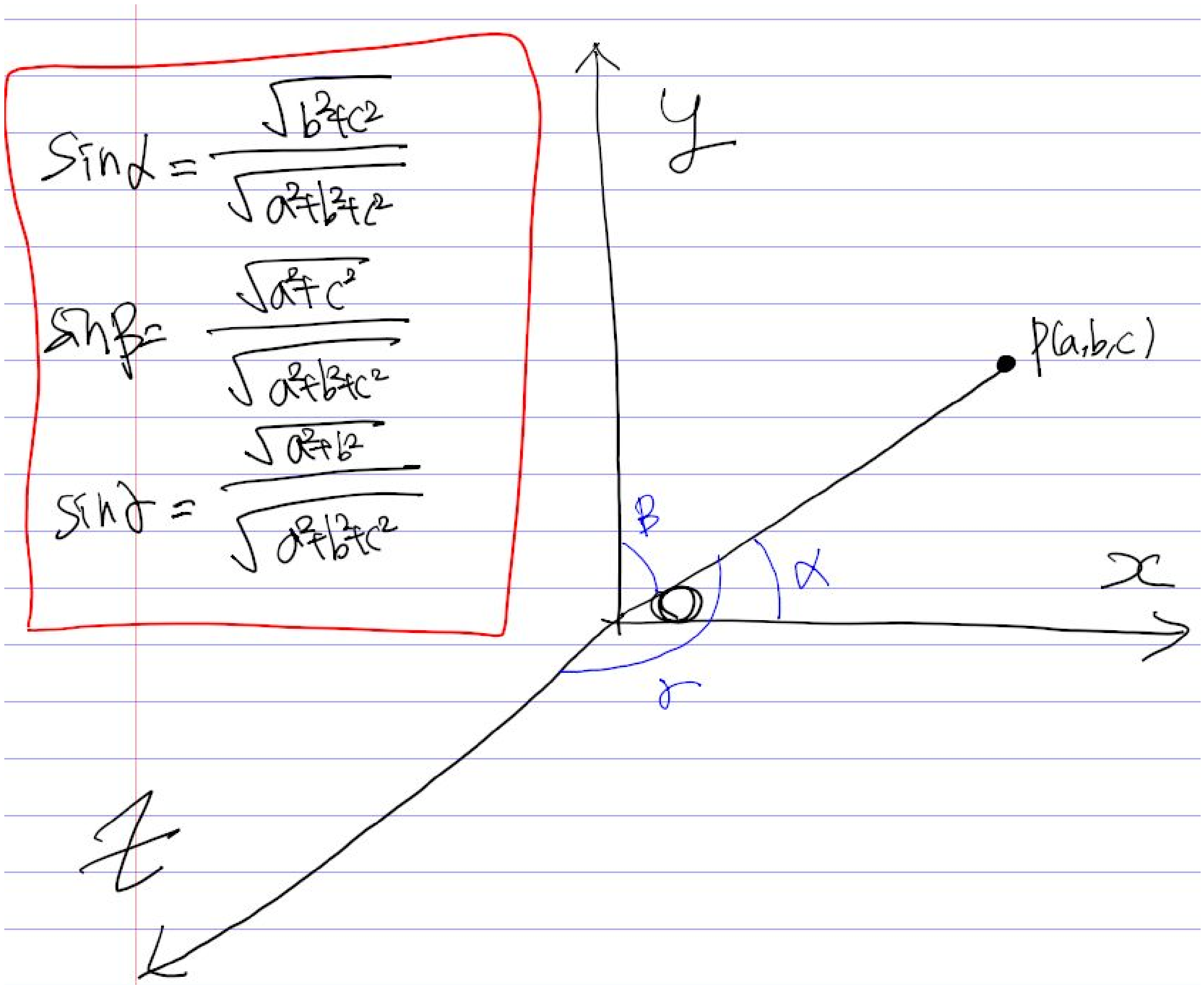
It is not possible for two of the vector’s direction angles to less than 45 degrees because it might give the following. $cos^2\alpha + cos^2\beta + cos^2\gamma \neq 1$

Let vector be $r(cos\alpha, cos\beta, cos\gamma)$ where $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$ and $0 \leq \alpha, \beta, \gamma \leq 180$.

If both $\alpha < 45\ and\ \beta < 45$ then $cos\alpha > 1/\sqrt{2}\ and\ cos\beta > 1/\sqrt{2}$ and so $cos^2\alpha + cos^2\beta > 1$.

This implies that $cos^2\gamma < 0$ which is impossible. Hence both α, β cannot be <45.

(d) What is the value of $sin^2\alpha + sin^2\beta + sin^2\gamma$? Why?



$sin\alpha = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$, $sin\beta = \frac{\sqrt{a^2+c^2}}{\sqrt{a^2+b^2+c^2}}$, $sin\gamma = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+c^2}}$

$sin\alpha^2 + sin\beta^2 + sin\gamma^2 = sin\alpha = \frac{b^2+c^2+a^2+c^2+a^2+b^2}{a^2+b^2+c^2} = \frac{2(a^2+b^2+c^2)}{a^2+b^2+c^2} = 2$

Thus, $sin\alpha^2 + sin\beta^2 + sin\gamma^2 = 2$

6. Explain the meanings of the terms linearly dependent and coplanar. Make sure you demonstrate that you understand the difference between the terms, and the situation in which linear dependency implies coplanarity.

Linear dependencies: The theory of vector space, a set of vectors, is said to be linearly dependent if one vector in a set can be defined as a linear combination of another vector. If there are no vectors in the set that cannot be written in this way, the vectors are called linearly independent.

Coplanar: A set of points, line segments, rays, or other geometric shapes in the same plane is called the coplanar. If there are three linearly dependent vectors, they are equilibrium. This simply means that the third vector can be represented in two different linear combinations.

If there are more than two coordinates, the above coordinates remain the same. The three linearly dependent vectors are always in the same plane. This statement is true, and only the above is true.

If there are four-vectors, and three vectors are linearly independent in three-dimensional space, when the vectors combine all three vectors linearly, the vectors are not the same as the underlying vectors that are normally required to represent all three vectors.

7. Determine if the vectors [2, 4, –1], [8, –10, 5], and [5, –3, 2] are coplanar.

Let $\vec{u} = s\vec{v} + t\vec{w}$ for some scalars s and t but they are not both zeros.
$\vec{u} = s\vec{v} + t\vec{w}$, $[2, 4, -1] = s[8, -10, 5] + t[5, -3, 2]$
$[2, 4, -1] = [8s, -10s, 5s] + [5t, -3t, 2t]$

$2 = 8s + 5t$ -> (1)
$4 = -10s - 3t$ -> (2)
$-1 = 5s + 2t$ -> (3)

From equations (1) and (2): We can solve for s and t by the elimination method. Multiply equation (1) by 3, and multiply equation (2) by 5 then copy equation (1) and subtract equation (2) from (1) and solve for s.

$$\begin{array}{rl} & 24s + 15t = 6 \\ + & -50s - 15t = 20 \\ \hline & -26s = 26 \\ & s = -1 \end{array}$$

Substitute $s = -1$ into equation (1) and solve for t.
$24(-1) + 15t = 6$, $-24 + 15t = 6$, $15t = 30$, $t = 2$

Check $s = -1$, $t = 2$ satisfy the equation.
$-1 = -5 + 4$, $SATISFIED$.
Therefore, the vectors $[2, 4, -1], [8, -10, 5], [5, -3, 2]$ are coplanar.
8. Give examples of sets of three vectors that are.

(a) Collinear

If three vectors lie on the same line, we can call that they are collinear vectors. Let us have the following example.

$A(3\widehat{i} + 4j - 2k),\ B(7\widehat{i} + 8j - 8k),\ C(13\widehat{i} + 14\widehat{j} - 17k)$

$\rightarrow_{AB} = \rightarrow_{B} - \rightarrow_{A} = [4, 4, -6] = 2[2, 2, -3]$

$\rightarrow_{BC} = \rightarrow_{B} - \rightarrow_{C} = [6, 6, -9] = 3[2, 2, -3]$

We can see that the direction of $\rightarrow_{AB}\ and \rightarrow_{BC}$ are in the same direction. Thus, they are collinear vectors.

(b) Coplanar

We can say that three vectors $\rightarrow_{a}, \rightarrow_{b}, \rightarrow_{c}$ are coplanar vectors if they lie on the same plane and triple product value $(\rightarrow_{a} * (\rightarrow_{b}\ x \rightarrow_{c}) = 0)$ is zero.

Example abounds in the following.

$Let\ vector\ a\ =\ < 1, 1, 1 >,\ vector\ b\ =\ < 1, 3, 1 >,\ vector\ c =\ < 2, 2, 2 >$

Let us calculate the three product value.

$$\vec{b} \times \vec{c} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$= \hat{i}[6-2] + \hat{j}[2-2] + \hat{k}[2-1]$$

$$= 4\hat{i} - 4\hat{k}$$

Let us multiply vector a to the dot product of vector b and vector c.

$\rightarrow_{a}(\rightarrow_{b} * \rightarrow_{c}) = (\widehat{i} + \widehat{j} + \widehat{k}) * (4\widehat{i} - 4\widehat{k}) = (1)(4) + 1(0) + 1(-4) = 0$

Thus, the aforementioned vectors as an example are coplanar vectors.

(c) Not coplanar

We can say that three vectors $\rightarrow_{a}, \rightarrow_{b}, \rightarrow_{c}$ are NOT coplanar vectors if they don't lie on the same plane and triple product value $(\rightarrow_{a} * (\rightarrow_{b}\ x \rightarrow_{c}) = 0)$ is not zero.

Example abounds in the following.

$Let\ vector\ a\ =\ < 1, 2, 3 >,\ vector\ b\ =\ < 1, 1, 1 >,\ vector\ c =\ < 1, 2, 1 >$

Let us calculate the three product value.

$$\vec{b} \times \vec{c} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \hat{i}[1-2] + \hat{j}[1-1] + \hat{k}[2-1]$$

$$= -\hat{i} + \hat{k}$$

Let us multiply vector a to the dot product of vector b and vector c.

$\rightarrow_a(\rightarrow_b * \rightarrow_c) = (\hat{i} + 2\hat{j} + 3\hat{k}) * (-\hat{i} + \hat{k}) = (1)(-1) + 2(0) + 3(1) = 2 \neq 0$

Thus, the aforementioned vectors as an example are NOT coplanar vectors.

9. Explain how you would prove if four given points are coplanar. Use your method to determine if A(3, 4, –2), B(8, 5, 0), C(1, 10, –6), and D(9, 2, 2) are coplanar.

For $vector\ AB = [(8-3)\hat{i}, (5-4)\hat{j}, (0-(-2)\hat{k}] = (5, 1, 2)$

For $vector\ BC = [(1-8)\hat{i}, (10-5)\hat{j}, (-6-(0)\hat{k}] = (-7, 5, -6)$

For $vector\ AC = [(9-1)\hat{i}, (2-10)\hat{j}, (2-(-6)\hat{k}] = (8, -8, 8)$

To prove that they are coplanar vectors, show that $[vector\ AB * vector\ BC * vector\ CD] = 0$

$$\begin{bmatrix} 5 & 1 & 2 \\ -7 & 5 & 6 \\ 8 & -8 & 8 \end{bmatrix}$$

$$= 5(40-48) - 1(-56+48) + 2(56-40)$$

$$= 5(-8) - (-8) + 2(16)$$

$$= -40 + 8 + 32$$

$$= 0$$

Thus, since the three scalar product value is zero, the aforementioned vectors are coplanar.

10. Determine if the following vectors are coplanar. Assume that vector v1, vector v2, and vector v3 are not coplanar.

The scalar triple product is $(vector\ w1\ *\ (vector\ w2\ *\ vector\ w3))$ while the value should be zero if the following vectors are coplanar.

Scalar Triple product

$$=(\vec{W_1}\times\vec{W_2})\cdot\vec{W_3}$$

$$\vec{W_1}\times\vec{W_2}=(2\vec{V_1}+7\vec{V_2})\times(\vec{V_2}+2\vec{V_3}).$$

$$=2(\vec{V_1}\times\vec{V_2})+4(\vec{V_1}\times\vec{V_3})+7|V_2|^2+14(\vec{V_2}\times\vec{V_3})$$

$$\vec{W_3}\cdot(\vec{W_1}\times\vec{W_2})$$

$$\Rightarrow(-\vec{V_1}-7\vec{V_3})\cdot\left(2(\vec{V_1}\times\vec{V_2})+4(\vec{V_1}\times\vec{V_3})+7|V_2|^2+14(\vec{V_2}\times\vec{V_3})\right)$$

$$\Rightarrow\ \cancelto{0}{-2\vec{V_1}\cdot(\vec{V_1}\times\vec{V_2})}\cancelto{0}{-4\vec{V_1}(\vec{V_1}\times\vec{V_3})}-7\vec{V_1}|V_2|^2-14(\vec{V_1}\cdot(\vec{V_2}\times\vec{V_3})$$

$$-14\vec{V_3}(\vec{V_1}\times\vec{V_2})-28\cancelto{0}{(\vec{V_1}\times\vec{V_3})\cdot\vec{V_3})}-49\vec{V_3}|V_2|^2\cancelto{0}{-98\vec{V_3}\cdot(\vec{V_2}\times\vec{V_3})}$$

Use property $\vec{x}\cdot(\vec{x}\times\vec{y})=0$

$$\rightarrow\ -7\vec{V_1}|\vec{V_2}|^2-14\vec{V_1}\cdot(\vec{V_2}\times\vec{V_3})+14\vec{V_1}(\vec{V_2}\times\vec{V_3})-49\vec{V_3}|V_2|^2$$

$$=\ -7\vec{V_1}|\vec{V_2}|^2-49\vec{V_3}|\vec{V_2}|^2$$

Scalar triple product is Not zero So

We can say these vectors are Non-coplanar