Vector Applications Unit Assignment
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1. Given $vector\ a\ = [2, 5, -7]$ and $vector\ b = [3, -6, -2]$, find

a) $vector\ a\ dot\ vector\ b$

$= [2, 5, -7] \bullet [3, -6, -2] = (2 * 3) + (5 * (-6)) + (-7) * (-2) = 6 - 30 + 14 = -24 + 14 = -10$

Thus, the answer is -10

b) A unit vector in the direction $vector\ b$

= $\widehat{h} = \frac{vector\ b}{|b|}$

= $\frac{[3, -6, -2]}{\sqrt{3^2+6^2+2^2}} = \frac{[3, -6, -2]}{\sqrt{9+36+4}} = \frac{[3, -6, -2]}{7}$

= $\widehat{h} = [\frac{3}{7}, \frac{-6}{7}, \frac{-2}{7}]$

c) the angle between vector a and vector b

$\theta = cos^{-1}(\frac{vector\ a \bullet vector\ b}{|vector\ a||vector\ b|}) = cos^{-1}(\frac{-10}{\sqrt{2^2+5^2+7^2}\sqrt{3^2+6^2+2^2}}) = cos^{-1}(\frac{-10}{7\sqrt{78}})$

$\theta = cos^{-1}(\frac{-10}{7\sqrt{78}}),\ \theta \approx 99.31^o$

d) A vector perpendicular to vector a

Let us say that $[vector\ a,\ b,\ c]\ is\ perpendicular\ to\ vector\ a$

$[a,\ b,\ c]\ \cdot\ [2,\ 5,\ -7]\ = 0$

$2a + 5b - 7c = 0$

Substitute $a = 1, b = 1, c = 1$ to get the formula satisfied

$[1, 1, 1]\ and\ [2, 5, -7]\ are\ perpendicular\ each\ other$

2. A force $vector\ F\ = [-2, 1, 5]$ in Newtons, pulls a sled through a displacement $vector\ s = [-3, 5, 4]$ in meters. The link between the dot product and geometric vectors and the calculation of work is $Work = |vector\ F||vector\ s|cos\theta$

a) How much work is done on the sled by the force?

$Work = |vector\ F||vectorS|cos\theta$

As we learned in this lesson, If θ is the angle between the vectors a and b, then $a \bullet b = |a||b|cos\theta$.

Thus, we can replace the work formula with $Work = |vector\ F| \bullet |vector\ S|$.

As we can write the vectors as following, $vector\ F\ = -2i + j + 5k,\ vector\ s\ = -3i + 5j + 4k$

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the following dot product calculation would be valid.

$(-2 * -3) + (1 * 5) + (5 * 4) = 6 + 5 + 20 = 31$

Thus, the answer is $31J$.

b) What is the minimum magnitude of force that could have been applied to the sled to obtain the same displacement? Explain your answer.

As stated earlier, $vector\ F\ \bullet vector\ S = |vector\ F||vectorS|\ cos\ \theta = 31$.

Thus, we can transpose an equation as the following. $vector\ F\ = \frac{31}{|vector\ S|cos\theta}$.

As we know that $|vector\ F|$ is smallest when the value of the denominator is largest.
To make the denominator the largest, we can render the case where $\theta = 0,\ cos\theta = 1$.
$|vector\ F\ min| = \frac{31}{|vector\ s|} = \frac{31}{|(-3i+5j+4k)|} = \frac{31}{\sqrt{9+25+16}} = \frac{31}{\sqrt{50}} = \frac{31}{5\sqrt{2}}$

Since $1\ joule\ (J)\ =\ 1.00\ newton\ meters\ (N-m)$, $\frac{31J}{5\sqrt{2}}$ is equal to $5.09N$.

3. Given that $vector\ a = [1,-3,6]\ and\ vector\ b = [4,-5,-2]$, find

a) $vector\ a\ \times\ vector\ b$ and verify that it is perpendicular to both $vector\ a\ and\ vector\ b$.

As stated above, we can get $vector\ a\ \times vector\ b\ = 36\widehat{i} + 26\widehat{j} + 7\widehat{k}$.
We know that the condition of being perpendicular between vector p and q is $\rightarrow_P \times \rightarrow_Q = 0$.
Thus, we need to show that $\rightarrow_a \times (\rightarrow_a \times \rightarrow_b) = 0$ and $\rightarrow_b \times (\rightarrow_a \times \rightarrow_b) = 0$.

To begin with, let me show that $\rightarrow_a \times (\rightarrow_a \times \rightarrow_b) = 0$ is a valid condition.
$\rightarrow_a \times (\rightarrow_a \times \rightarrow_b) = (\widehat{i} - 3\widehat{j} + 6\widehat{k}) \times (36\widehat{i} + 26\widehat{j} + 7\widehat{k}) = 0$
$= 36 - 78 + 42$
$=- 42 + 42$
$= 0$

Besides, It is to be proved that $\rightarrow_b \times (\rightarrow_a \times \rightarrow_b) = 0$ is a valid condition.
$\rightarrow_b \times (\rightarrow_a \times \rightarrow_b) = (4\widehat{i} - 5\widehat{j} - 2\widehat{k}) \times (36\widehat{i} + 26\widehat{j} + 7\widehat{k}) = 0$
$= 144 - 130 - 14$
$= 14 - 14$
$= 0$

Thus, the given two vectors are perpendicular.

b) A $vector\ c$ such that $vector\ a \bullet (vector\ b\ \times vector\ c) = 0$. What is the relationship between the vectors $vector\ a$, $vector\ b$, and $vector\ c$ in this case, and why? Verify this.

Let such $\rightarrow_c = s\rightarrow_a + t\rightarrow_b$ and let $s = 1,\ t = 3$.
$\rightarrow_c = \rightarrow_a + 3\rightarrow_b = [1 + 3 \times 4,\ (-3) + 3 \times (-5),\ 6 + 3 \times (-2)]$
$\rightarrow_c = [13,-18,0]$

To find the cross product, we form a determinant the first row of which is a unit vector, the second row is our first vector, and the third row is our second vector.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & -2 \\ 13 & -18 & 0 \end{vmatrix}.$$

Then, expand along the first row as the following.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & -2 \\ 13 & -18 & 0 \end{vmatrix} = \begin{vmatrix} -5 & -2 \\ -18 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -2 \\ 13 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & -5 \\ 13 & -18 \end{vmatrix}\mathbf{k} =$$

$$= (-5 \cdot (0) - (-18) \cdot (-2))\mathbf{i} - (4 \cdot (0) - (13) \cdot (-2))\mathbf{j} + (4 \cdot (-18) - (13) \cdot (-5))\mathbf{k} =$$

$$= -36\mathbf{i} - 26\mathbf{j} - 7\mathbf{k}$$

So, $(4, -5, -2) \times (13, -18, 0) = (-36, -26, -7)$.

Answer: $(4, -5, -2) \times (13, -18, 0) = (-36, -26, -7)$.

- $\vec{b} \times \vec{c} = (-36, -26, -7)$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (1 \bullet (-36)) + (-3 \bullet (-26)) + (6 \bullet (-7)) = 0$

Thus, we can say that $vector\ a,\ vector\ b\ and\ vector\ c$ are coplanar.

4. Given $vector\ v = [3, 5, -4]\ and\ vector\ w = [4, -3, -2]$, find

a) $vector\ v \downarrow vector\ w$

To solve the projection v on w, $proj_w(V) = (\frac{v \bullet w}{|w|^2}) = \frac{12-15+8}{(4)^2+(-3)^2+(-2)^2}(4, -3, -2)$

$= \frac{5}{16+9+4}(4, -3, -2)$

$= (\frac{20}{29}, -\frac{15}{29}, -\frac{10}{29})$

b) $vector\ w \downarrow vector\ v$

To solve the projection w on v, $proj_v(W) = (\frac{v \bullet w}{|w|^2})v = \frac{12-15+8}{(3)^2+(5)^2+(-4)^2}(3, 5, -4)$

$= \frac{5}{9+25+16}(3, 5, -4)$

$= \frac{1}{10}(3, 5, -4)$

$= (\frac{3}{10}, \frac{5}{10}, -\frac{4}{10})$

$= (\frac{3}{10}, \frac{1}{2}, -\frac{2}{5})$

c) What does the magnitude of $vector\ w \downarrow vector\ v$ depend on?

$vector\ w \downarrow vector\ v\ = (\frac{3}{10}, \frac{1}{2}, -\frac{2}{5})$

Magnitude is equal to $\sqrt{\frac{9}{100} + \frac{1}{4} + \frac{4}{25}} = \sqrt{\frac{900+2500+1600}{10000}} = \sqrt{\frac{1}{2}}$

It depends on both $vector\ w\ and\ vector\ v$, as the formula is $\frac{v \bullet w}{|w|^2}$.

d) What does the direction of $vector\ w \downarrow vector\ v$ depend on?

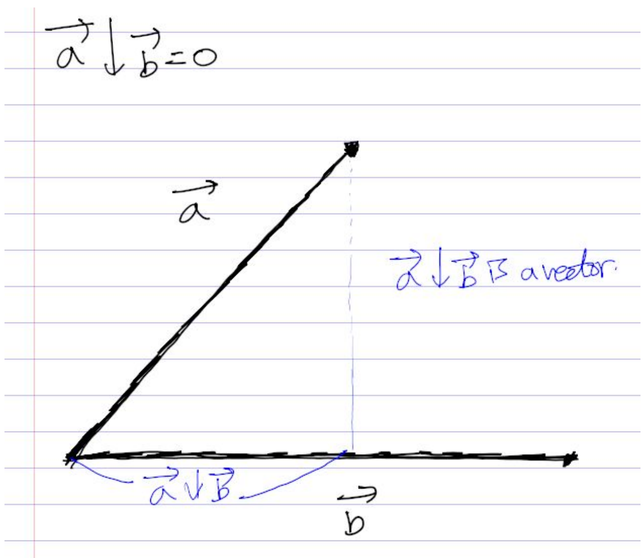
$\vec{w} \downarrow \vec{u} = \frac{\vec{w} \cdot \vec{u}}{|\vec{u}|^2}\vec{u} \quad = \frac{\vec{w} \cdot \vec{u}}{|\vec{u}|} \cdot \frac{\vec{u}}{|\vec{u}|} \ = \frac{\vec{w} \cdot \vec{u}}{|\vec{u}|} \cdot \hat{u}$

$\frac{\vec{w} \cdot \vec{u}}{|\vec{u}|}$ determines the magnitude of the projection which is a scalar projection.

The direction of projection is the same as the direction of vector v.

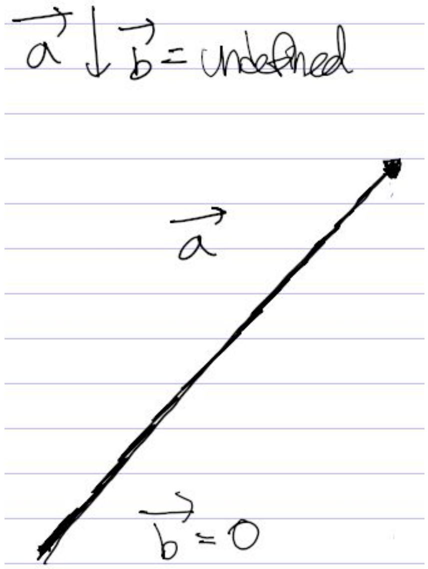
5. Draw diagrams to explain the answers to the following questions.

a) Is it possible to have $vector\ a\ \downarrow vector\ b = 0$?



Since $vector\ a\ \downarrow vector\ b$ is a vector, $vector\ a\ \downarrow vector\ b$ cannot be zero.

b) Is it possible to have $vector\ a\ \downarrow vector\ b\ undefined$?



$(vector\ a\ \downarrow vector\ b) = \frac{(vector\ a)\bullet(vector\ b)}{|vector\ b|^2}(vector\ b)$

The aforementioned image shows that $vector\ b = 0$

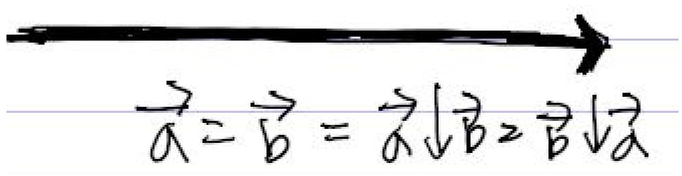
Thus, $(vector\ a\ \downarrow vector\ b) = \frac{(vector\ a)\bullet(vector\ b)}{|0|^2}(0)$

Since a denominator cannot be zero, if it should be zero, we can say that the result is undefined.

c) Is it possible to have $vector\ a\ \downarrow vector\ b = vector\ b\ \downarrow vector\ a$

Let us explore two possible cases.

Case 1) $(vector\ a\ \downarrow vector\ b)\ is\ not\ zero$



$(vector\ a \downarrow vector\ b) = \frac{(vector\ a)\bullet(vector\ b)}{(vector\ b)^2}(vector\ b)$

According to the picture above, $vector\ a$ is equal to $vector\ b$ thus we can substitute it.

$(vector\ a \downarrow vector\ b) = \frac{(vector\ a)\bullet(vector\ a)}{(vector\ a)^2}(vector\ a)$

$= \frac{(vector\ a)^2}{(vector\ a)^2}(vector\ a) = (vector\ a)$

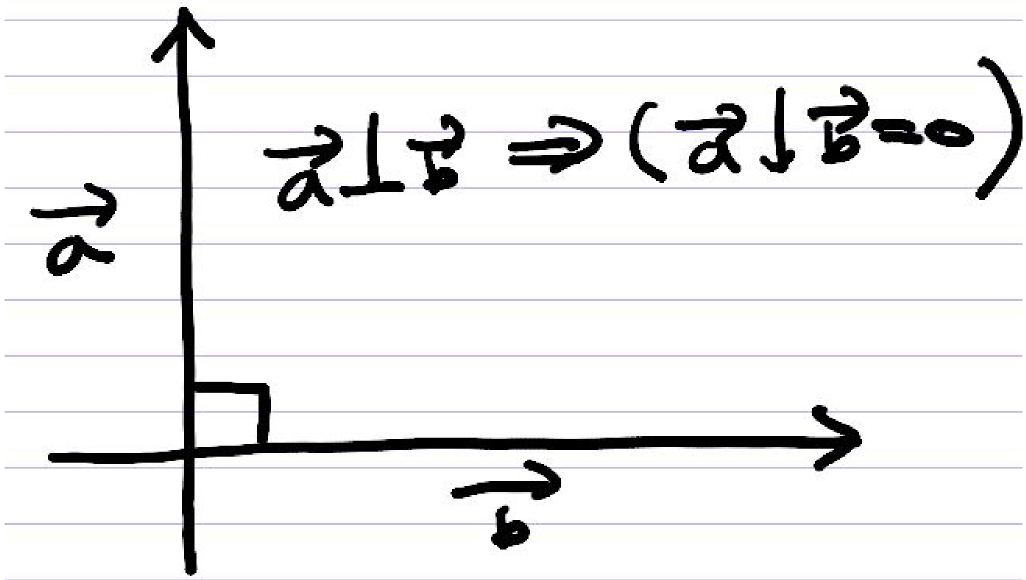
$(vector\ b \downarrow vector\ a) = \frac{(vector\ b)\bullet(vector\ a)}{(vector\ a)^2}(vector\ a)$

According to the picture above, $vector\ b$ is equal to $vector\ a$ thus we can substitute it.

$(vector\ b \downarrow vector\ a) = \frac{(vector\ a)\bullet(vector\ a)}{(vector\ a)^2}(vector\ a)$

In this case, $vector\ a\ \downarrow vector\ b = vector\ b\ \downarrow vector\ a$ is satisfied.

Case 2) $(vector\ a \downarrow vector\ b) = 0$ is zero



$(vector\ a \downarrow vector\ b) = \frac{(vector\ a)\bullet(vector\ b)}{(vector\ b)^2}(vector\ b)$

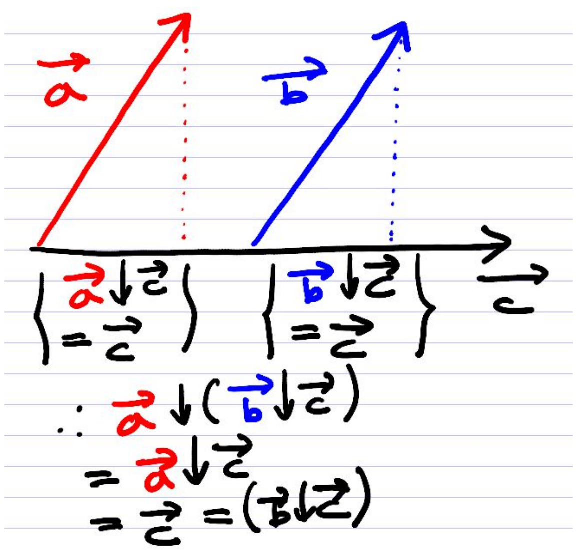
$= 0 \bullet vector\ b = 0$

$(vector\ b \downarrow vector\ a) = \frac{(vector\ b)\bullet(vector\ a)}{(vector\ a)^2}(vector\ a)$

$= 0 \bullet vector\ a = 0$

In this case, since two vectors are perpendicular, the result is zero.

d) Explain why $vector\ a \downarrow vector\ c = vector\ a \downarrow (vector\ b \downarrow vector\ c)$.



Therefore, the given formula is satisfied.

6. Answer the following with either an explanation, a diagram or a proof.

a) If $vector\ a \bullet vector\ b = vector\ a \bullet vector\ c$, what is the relationship between $vector\ b \bullet vector\ c$?

$(vector\ a \bullet vector\ b) - (vector\ a \bullet vector\ c) = 0$

$vector\ a \bullet (vector\ b - vector\ c) = 0$

Thus, the answer would be the following. It shows that we have three cases to satisfy the equation.

$vector\ a = \vec{0}\ or\ (vector\ b - vector\ c) = \vec{0}\ or\ (vector\ a \perp (vector\ b - vector\ c))$

Therefore, vector b does not **always** need to be equal to vector c, but it could be.

b) If $vector\ a \times vector\ b = vector\ a \times vector\ c$, what is the relationship between $vector\ b \times vector\ c$?

$vector\ a \times vector\ b = vector\ a \times vector\ c$

$vector\ a \times (vector\ b - vector\ c) = 0$

Thus, the answer would be the following. It shows that we have three cases to satisfy the equation.

$vector\ a = \vec{0}\ or\ (vector\ b - vector\ c) = \vec{0}\ or\ (vector\ a\ is\ parallel\ to\ (vector\ b - vector\ c))$

Therefore, vector b does not **always** need to be equal to vector c, but it could be.

7. Prove that

$vector\ a \bullet (vector\ b + vector\ c) = (vector\ a \bullet vector\ b) + (vector\ a \bullet vector\ c)\ for\ all\ vector\ a,\ b,\ c \in R^3$

Let $\vec{a} =< a_1, b_1, c_1 >,\ \vec{b} =< a_2, b_2, c_2 >, \vec{c} =< a_3, b_3, c_3 >$

LHS is the following.

$\vec{a} \bullet (\vec{b} + \vec{c}) =< a_1, b_1, c_1 > \bullet (< a_2, b_2, c_2 > + < a_3, b_3, c_3 >)$

$=< a_1, b_1, c_1 > \bullet (< a_2 + a_3, b_2 + b_3, c_2 + c_3 >)$

$= a_1(a_2 + a_3) + b_1(b_2 + b_3) + c_1(c_2 + c_3)$

$= (a_1a_2 + a_1a_3) + (b_1b_2 + b_1b_3) + (c_1c_2 + c_1c_3)$

$= (a_1a_2 + b_1b_2 + c_1c_2) + (a_1a_3 + b_1b_3 + c_1c_3)$

$= \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

RHS is the following.

$= \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

Thus, LHS and RHS is the same and it means that dot product among vectors is distributive.

8. Given vectors $vector\ a$, $vector\ b$, $vector\ c$, and $vector\ d$, state whether each of the following results in a scalar, a vector, or is not possible. Justify each response.

a. To prove that $vector\ a \bullet (vector\ b \times vector\ c)$ is a scalar.

Since $(vector\ b \times vector\ c)$ is a vector while $vector \bullet (vector)$ is scalar product of two vectors, the answer would be scalar.

b. To prove that $(vector\ a\ \bullet\ vector\ b) \times vector\ c$ is not possible

It is to be noted that dot and cross product is only available for vectors. Since the result of $(vector\ a\ \bullet\ vector\ b)$ is a scalar, we cannot do cross product in the formula.

c. To prove that $(vector\ a \times vector\ b) + (vector\ c \bullet vector\ d)$ is not possible

Since the addition between vector and scalar is not available, we can say that the formula cannot be done.

d. To prove that $(vector\ a\ \bullet\ vector\ b) + (vector\ c\ \bullet\ vector\ d)$ is a scalar.

Since the addition between two scalar values rendered by the dot product calculation is a scalar value, we can say that the formula is possible.

e. To prove that $(vector\ a\ \times vector\ b) \bullet (vector\ c \times vector\ d)$ is a scalar.

Since the result of the cross product is a vector, we can say that the formula is trying to calculate the dot product of two vectors, which results in a scalar. Thus, the formula is possible.

f. To prove that $(vector\ a \bullet vector\ b) \times (vector\ c \bullet vector\ d)$ is not possible

Since the cross product between scalars is not supported, we can say the formula is not possible.

9. Charlie is trying to hold on to his toy fire truck. His brother Noah is pulling with a force of 8 N on a bearing of 023° and his brother Jude with a force of 5 N on a bearing of 155°. What force does Charlie need to exert to keep the toy in equilibrium?

When it comes to Noah, the component of force is in the North direction is $8cos23^o$ while the East direction is $8sin23^o$. Since the bearing of 155^o is southeast, we can easily render the result of the angle south of east as the following: $155^o - 90^o = 65^o$.

Thus, for Jude, the component of force is in the East direction is $5cos65^o$ while that in the South direction is $5sin65^o$.

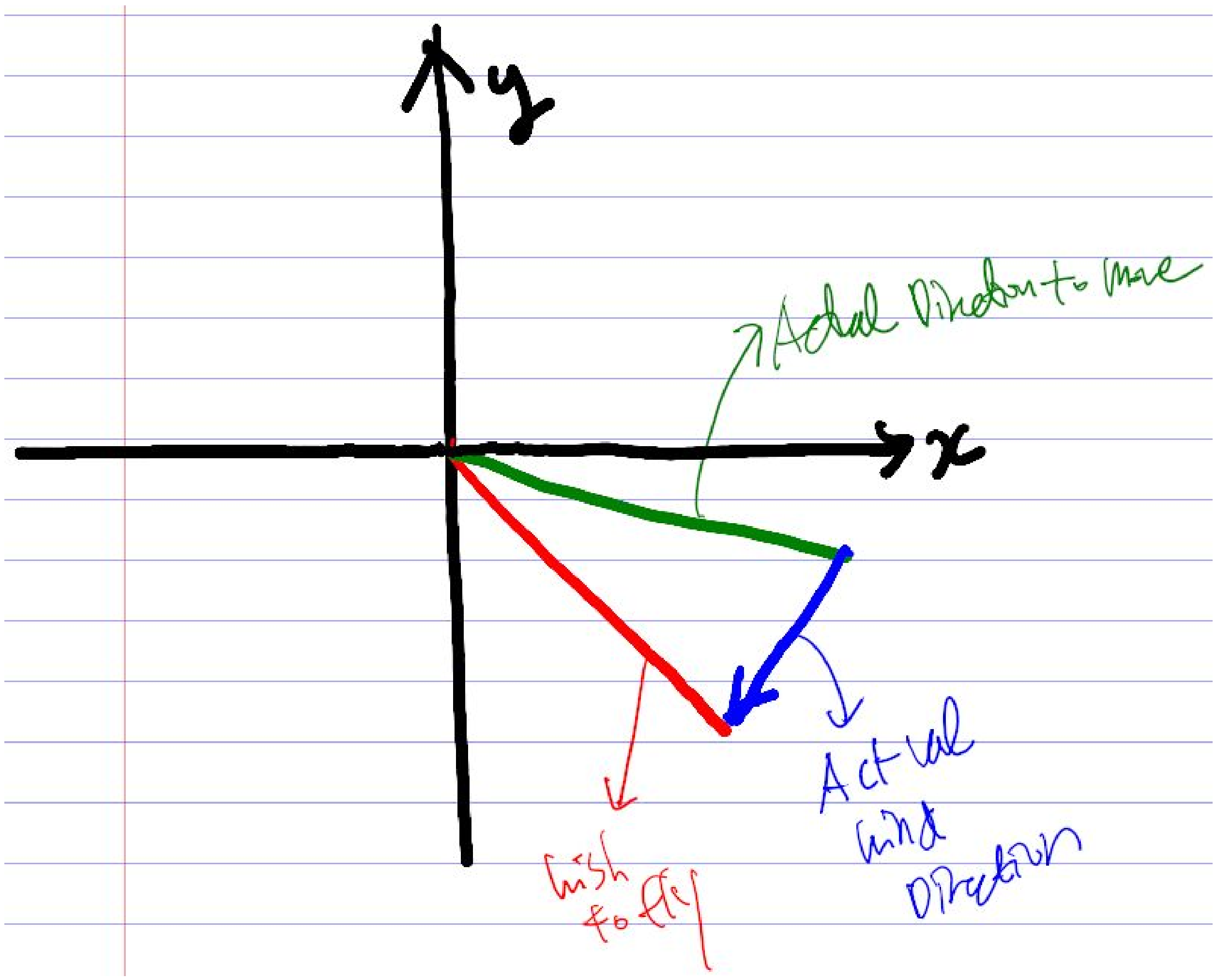
Thus, the total force required for North Direction is $8cos23^o - 5cos65^o \approx 2.83N$ while due East is $8sin23^o + 5cos65^o \approx 5.23N$

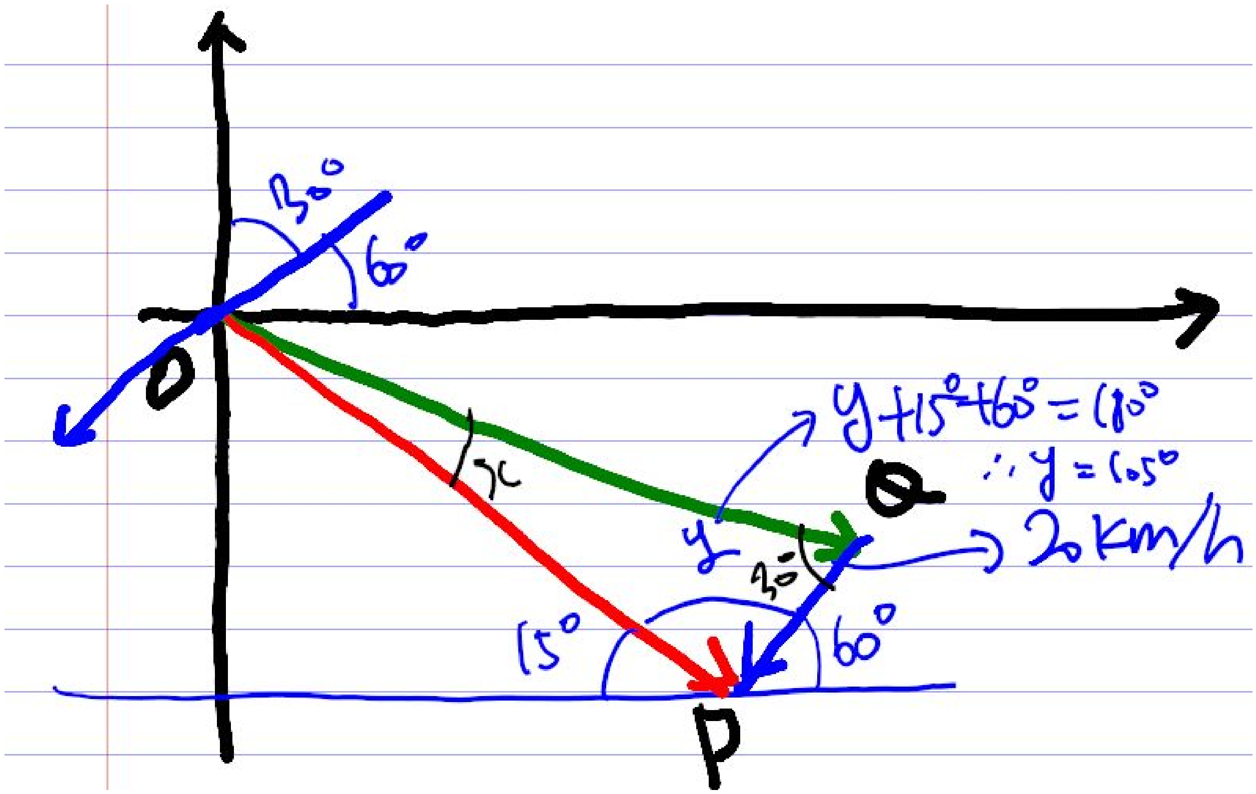
To make the net force equal to 0 to keep the toy in equilibrium, Charlie must exert the same amount of forces in the opposite direction with a bearing of 65 degree.

Thus, Charlie should apply $\sqrt{(2.83)^2 + (5.23)^2} \approx 5.95N$

10. A pilot wishes to fly from Bayfield to Kitchener, a distance of 100 km on a bearing of 105°. The speed of the plane in still air is 240 km/h. A 20 km/h wind is blowing on a bearing of 210°. Remembering that she must fly on a bearing of 105° relative to the ground (i.e. the resultant must be on that bearing), find

- the heading she should take to reach her destination.
- how long the trip will take.





In $\triangle OPQ$, using Lemi's theorem, $\frac{\sin y}{240} = \frac{\sin x}{20}$

$\sin x = \frac{20}{240}(\sin 105^o) = 0.08049$

$x = 4.62$

$\angle Q = 180 - 4.62 - 105 = 70.38$

As the same Lemi's theorem is applied, $\frac{\sin Q}{OP} = \frac{\sin Y}{OQ=240}$

$\overline{OP} = 240 * \frac{\sin Q}{\sin Y} = 240 * \frac{\sin 70.38}{\sin 105} = 234 km/h$

Thus, the speed of the wind is 20km/h and the actual speed of aircraft is 240km/h while relative velocity is 234km/h.

The heading she should take to reach her destination is $\theta = 105^o - x = 100.38$

The time that she needs to take for the trip is $t = \frac{distance}{velocity} = \frac{distance\ OP}{234km/h} = \frac{100}{234} \approx 25.64min$