Optimal Unconventional Monetary Policy and Trend Inflation

Jinyoung Seo *
UC Davis

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Abstract

I study optimal unconventional monetary policy in a New Keynesian model with trend inflation. A standard New Keynesian model is extended to feature heterogeneous households, financial intermediaries, and unconventional monetary policy. By optimally designing both conventional and unconventional policy, a central bank can completely stabilize both the output gap and inflation, and restore a divine coincidence despite the endogenous cost-push wedges from trend inflation, which is not possible with only one policy tool. Furthermore, optimal unconventional monetary policy at the ZLB highlights the importance of aligning long-run inflation target with policy makers' objectives between stabilizing output gap and inflation because of the policy trade-offs.

*Department of Economics, University of California, Davis. (Email: dkseo@ucdavis.edu).

1 Introduction

Following the global financial crisis of 2008, Quantitative Easing (QE) emerged as an unconventional monetary policy tool, gaining widespread adoption among central banks across the globe. There has also been increasing interest in unconventional monetary policy in theoretical and empirical literature, leading to significant advancements in our understanding of how it operates. Nevertheless, certain challenges persist within the conventional framework, presenting obstacles to effective analysis in a more relevant model environment.

One area that remains relatively under-explored is the interaction between the implementation of unconventional policy and a central bank's inflation target, particularly in a framework where the inflation target holds significant economic consequences. This question has particular importance given the Federal Reserve's adoption of an explicit 2 percent inflation target in 2012. Standard New Keynesian models, however, struggle to adequately capture this interaction, primarily due to the limited real effects of long-run inflation targets in these models (for instance, Smets and Wouters 2007).

This paper seeks to bridge the gap in the existing literature, addressing the disconnect between studies analyzing the transmission of QE policy in the absence of trend inflation and those examining macroeconomic dynamics in the presence of trend inflation but without the implementation of unconventional policy. I build on the tractable four-equation New Keynesian framework developed by Sims, Wu, and Zhang (2021): the model incorporates heterogeneous households, consisting of parents (savers) and children (borrowers), financial intermediaries engaged in short-term deposit issuance and long-term bond holdings, credit shocks affecting leverage constraints within the financial sector, and a central bank employing unconventional monetary policy through long-term bond portfolio management. Furthermore, the model allows for positive trend inflation, leading to the emergence of endogenous cost-push wedges in the Phillips curve.

The model features two distinct sources of endogenous cost-push effects. First, the distortions induced by positive trend inflation introduce endogenous cost-push pressure. In situations where inflation is positive in the long run, it amplifies dispersion in relative price distribution and gives rise to demand misallocation. This misallocation in demand subsequently leads to an overall reduction in aggregate output, signifying the cost-push consequences stemming from trend inflation. These cost-push wedges generate policy trade-offs because expansionary monetary policy often brings about temporary inflation, which creates output loss through the cost-push mechanism. In a relatively standard New Keynesian framework, Alves (2014) shows that the endogenous cost-push force under positive trend inflation breaks the so-called 'divine coincidence.'

Second, financial frictions in the form of leverage constraints in the financial sector also drive cost-push effects in response to either credit shocks or QE shocks. Shocks to financial conditions trigger the reallocation of resources between borrowers and savers, but only savers supply labor. When financial conditions become more accommodating, borrowers can issue debt and consume more, implying a reallocation from saver to borrower. This shift triggers an adverse wealth effect for savers, compelling them to offer more labor, consequently exerting downward pressure on firms' marginal costs. This negative wealth effect partially offsets the inflation stemming from the expansionary policy, although the dominance of one force over the other hinges on specific parameter values.

I use the model to study how the presence of positive trend inflation affects the transmission of both conventional and unconventional monetary policy. To this end, three key results emerge. First, I explore the influence of trend inflation on the transmission of various shocks, especially policy-induced shocks. I find that any inflationary pressure arising from such shocks results in output loss, thereby diminishing the overall effectiveness of monetary policy. An expansionary policy shock drives a temporary increase in inflation, leading to larger inefficient price dispersion. This prompts the endogenous cost-push wedges originating from trend inflation due to demand misallocation. As a result, the demand misallocation induces output loss and counteracts the initial expansion. Hence, as trend inflation increases, the effectiveness of expansionary monetary policy shocks diminishes in their ability to stimulate the economy.

Second, a central bank can fully stabilize the output gap and inflation simultaneously in response to natural rate shocks, despite the endogenous cost-push wedges from trend inflation. When conventional short-term interest rate policy is the only available option for a central bank, it cannot achieve complete stabilization via short-term interest rate policy due to the endogenous cost-push wedges. However, as unconventional policy also generates endogenous cost-push pressures through the wealth effects by savers, it opens up a new policy dimension that a central bank can leverage to deal with cost-push wedges from trend inflation. When both policy tools are available, a central bank can counteract the cost-push wedges from trend inflation by conducting QE policy appropriately and optimally setting short-term interest rates to fully offset natural rate shocks and QE policy shocks in the IS curve. In this sense, unconventional policy serves as a unique policy tool that a central bank can exploit to manage supply-side shocks, not only when the ZLB is binding.

Third, I investigate optimal unconventional monetary policy when the zero lower bound (ZLB) constraint is binding. This is an important application of the model for both policymakers and academic researchers because there is a consensus that unconventional policy tools serve as a useful toolkit for central banks when the short-term nominal policy rate is constrained by the ZLB. Reflecting central banks' dual mandate, I assume that a central bank minimizes a loss function that is a weighted sum of the variance of inflation and the output gap by adjusting its long-term bond portfolio when the short-term nominal interest rate is at zero. Then I evaluate the effectiveness of optimal unconventional monetary policy following a large adverse natural rate shock for different levels of trend inflation and weights on the output gap.

The optimal paths of the output gap and inflation shed light on the interplay between QE efficacy, the central bank's objective function, and the level of trend inflation. To begin with, trend inflation serves as a mitigating factor for negative natural rate shocks, resulting in less severe economic downturns. When faced with a negative natural rate shock and the ZLB constraint, the absence of QE results in a significant recession and a decline in inflation. However, the severity of this impact diminishes with increasing trend inflation. This intriguing phenomenon occurs because higher trend inflation mitigates some of the adverse effects by improving cost conditions and reducing price dispersion, thereby temporarily alleviating output loss. In essence, higher trend inflation operates akin to a negative cost-push shock, exerting downward pressure on marginal costs and inflation, ultimately assisting in resolving inefficiencies in the economy.

More importantly, the optimal unconventional monetary policy at the ZLB underscores the pivotal role played by trend inflation in shaping the trade-offs that a central bank faces. The optimal targeting rule at the ZLB for discretionary policy adopts a "leaning-against-the-wind" strategy, revealing a significant trade-off inherent in stabilization policy. This trade-off primarily arises from the impact of natural rate shocks and the inefficiencies stemming from trend inflation. My model suggests that if a central bank emphasizes price stability over minimizing output gap variability, then it has a better ability to fight against an adverse shock under low levels of trend inflation. This is because the efficacy of QE shocks on inflation is decreasing in trend inflation, relatively faster than its effectiveness on the output gap. Hence, under high trend inflation, the trade-offs faced by an inflation-focused central bank are larger than those faced by an output-focused one. Conversely, central banks prioritizing output stabilization can counteract an adverse shock under moderate to high trend inflation.

Therefore, the interaction between trend inflation and QE has profound policy implications, especially for central banks tasked with a dual mandate of price stability and real economic outcomes within a framework of moderate long-run inflation targets. This intricate relationship highlights the importance of aligning the long-run inflation target with the central bank's broader objective function when implementing unconventional policies.

This holds particular relevance in the contemporary policy landscape, where QE has become an integral component of a central bank's toolkit. The recent shift by the Federal Reserve to Average Inflation Targeting during the Covid crisis, targeting an average inflation rate over a specified period, underscores the evolving nature of monetary policy. Importantly, the policy implication of my research is that a central bank should carefully consider the level of its inflation target in light of its policy objectives between stabilizing the output gap and inflation. If a central bank places a stronger emphasis on maintaining price stability, it may be prudent to maintain a lower inflation target, especially when facing the zero lower bound (ZLB). Conversely, if the central bank prioritizes stabilizing output gaps, a moderate inflation target could be beneficial. This suggests that the choice of an inflation target should be coherent with the central bank's broader policy objectives.

My paper relates to the literature on the analysis of unconventional monetary policy in the New Keynesian framework. My model shares similarities with Gertler and Karadi (2011) and Carlstrom, Fuerst, and Paustian (2017), all of which have some types of frictions in the financial sector that impose limitations on the intermediation between short-term and long-term bonds, in addition to including a central bank's bond portfolio adjustment. In particular, I build on the four-equation model by Sims, Wu, and Zhang (2021), where the authors propose a tractable New Keynesian framework for unconventional policy analysis. Compared to the existing literature, an important contribution of my paper is to incorporate positive trend inflation, which creates non-trivial policy trade-offs that policymakers could face. Positive trend inflation introduces the endogenous cost-push wedges through firms' pricing decisions, and therefore, my model can be used to analyze the interaction between the long-run inflation target and the efficacy of unconventional monetary policy.

This paper also contributes to the extensive literature on optimal monetary policy. My model builds on the existing literature, such as Clarida, Gali, and Gertler (1999), Erceg, Henderson, and Levin (2000) and Blanchard and Galí (2007). Relative to the literature, my results show that a central bank can achieve a "divine coincidence," which refers to the full stabilization of the output gap and inflation. This resolves the question posed by Alves (2014), where the author shows that in models with trend inflation, a divine coincidence does not hold because of the endogenous cost-push wedges. However, in my model, the presence of a new policy instrument opens up the possibility to deal with the wedges by generating wealth effects for certain types of households.

The remainder of the paper proceeds as follows. Section 2 describes the model. Section 3 considers the optimal policy analysis with particular interest in optimal design of unconventional monetary policy. Section 4 presents concluding remarks.

2 Model

2.1 Model Description

In this section, I describe the model environment in detail. The model is based on the four equations New Keynesian model of Sims, Wu, and Zhang (2021) and the Generalized New Keynesian model of Ascari and Ropele (2007). The model departs from a textbook New Keynesian model, for instance Galí (2015), in several aspects. First, the model features two types of households, referred to as ”parent” and ”child”. The parent derives utility from consumption and disutility from labor. It also saves via one period nominal bonds, or deposits, owns firms and financial intermediaries, and receives lump sum transfer (or tax) from the government. Hence, the parent is the household in a standard New Keynesian model. The child derives utility only from its consumption and does not supply labor. Its consumption is financed by the issuance of long-term nominal bonds and a transfer from the parent. As an illustrative example, the consumption dynamics of child in the model exhibit similarities with the investment behavior observed in Sims and Wu (2021), which will be used to calibrate the share of child in the economy.

Second, financial intermediary holds long-term nominal bonds and reserves that pay out interests, and finance via one period deposits and transfers from the parent. Financial intermediaries lives only one period and exits the industry at the end of the period. This is a special case of financial intermediaries in the literature, such as Gertler and Karadi (2011).

Third, a central bank conducts both conventional monetary policy and unconventional monetary policy that operates through the central bank’s long-term bonds portfolio. As extensively studied in literature, the unconventional policy is particularly effective when the ZLB constraint is binding (Carlstrom, Fuerst, and Paustian, 2017).

Fourth, the model considers positive trend inflation and is approximated around positive steady state inflation. Also, I assume no price indexation of firms Price indexation is widely used in small- and medium-scale New Keynesian models, for example Smets and Wouters (2007) and Justiniano, Primiceri, and Tambalotti (2013). Full price indexation means firms that can not change its price in this period do not automatically ’update’ their prices according to a weighted average of previous inflation and steady state inflation. This ad-hoc assumption introduces lagged inflation in New Keynesian Phillips curve and preserves its single equation structure. But as Cogley and Sbordone (2008) show, price indexation is not only ad-hoc but also inconsistent with firms' pricing behavior. They also find that once positive trend inflation is taken into account, the median estimates of the degree of price indexation is zero, meaning no price indexation is empirically relevant case. Based on their empirical finding and theoretical foundations in the literature (Ascari and Ropele, 2007; Alves, 2014), I assume no price indexation for retailers, which implies non-adjusting firms keep their old prices and do not update prices.

The remaining part of the model consist of final good firm and intermediate goods firms. It is important to note that throughout the paper, I adopt a set of simplifying assumptions deliberately to focus exclusively on analytical tractability and the characterization of optimal monetary policy. This model preserves the qualitative feature of an extended model, hence validating the findings in this paper, although it is intentionally simplified to give the core intuition from the analysis. Importantly, despite this simplification, the quantitative implications of the model closely align with those of more complicated models.

Parent. A continuum of households is populated by two types, parent and child. A representative parent derives utility from consumption, C_t and labor hours, L_t. The parent seeks to maximize its objective function that is a discounted utility from consumption and labor with a discount rate $\beta \in (0,1)$, subject to its flow budget constraint:

$$V_t = \max E_t \sum_{j=0}^{\infty} \Big[\frac{C_{t+j}^{1-\sigma}}{1-\sigma} - \psi \frac{L_{t+j}^{1+\chi}}{1+\chi}\Big] \tag{1}$$

$$P_t C_t + S_t \leq W_t L_t + R_{t-1}^S S_{t_1} + P_t D_t + P_t D_t^{FI} + P_t T_t - P_t X_t^b - P_t X_t^{FI}, \tag{2}$$

where $\sigma > 0$ is the constant relative risk aversion (CRRA), or the inverse of the elasticity of intertemporal substitution, $\chi \geq 0$ is the inverse of Frisch elasticity of labor supply, and ψ is a scale parameter to match steady state labor supply.

P_t is the aggregate price index, which will be defined below. Parent can save only through one-period deposits, S_t. Deposits pay gross risk-free nominal interest rate R_t^S in period $t+1$. Parent earns labor income, $W_t L_t$ and receives dividends from firms and financial intermediaries, which are denoted by D_t and D_t^{FI}, respectively. T_t is a lump sum tax or transfer from the government. As described earlier, the parent makes a transfer to child and financial intermediaries each period: X_t^b is a transfer from parent to child, and X_t^{FI} is a transfer to financial intermediaries. Note that these transfers are exogenously determined, hence not choice variables.

The parent's optimal conditions for consumption, labor supply and savings are as follows:

$$\Lambda_{t,t+1} = \beta E_t \Big(\frac{C_t}{C_{t-1}}\Big)^{-\sigma}, \tag{3}$$

$$\psi L_t^{\chi} = C_t^{-\sigma} w_t, \tag{4}$$

$$1 = R_t^S E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} \tag{5}$$

$w_t = W_t/P_t$ denotes the real wage and $\Pi_t = P_t/P_{t-1}$ is gross inflation, $\Lambda_{t,t+1}\Pi_{t+1}^{-1}$ is the parent's nominal stochastic discount factor (SDF).

Child. The child's utility is only determined by consumption $C_{b,t}$ and does not supply labor. The child maximizes present discounted value of utility but is less patient than parent, hence it discounts future utility using discount factor $\beta_b < \beta$:

$$V_{b,t} = E_t \sum_{j=0}^{\infty} \beta_b^j \Big[\frac{C_{b,t+j}^{1-\sigma} - 1}{1-\sigma}\Big] \tag{6}$$

The child finances its consumption through the transfers from parent and the issuance of nominal long-term bonds. Long-term bonds are perpetual bonds with a declining coupon payment $\kappa \in [0, 1]$ (Woodford, 2001). When issuing one unit of long-term bonds in period t, the bond issuer agrees to the payment of coupon schedule: κ dollars in $t+1$, κ^2 in $t+2$, etc. Let Q_t^B denote the price of a newly issued bond at period t, and NB_t denote the amount of new perpetuities issued in period t. Then the total liability in $t+1$ on past issues is given by

$$B_t = NB_t + \kappa NB_{t-1} + \kappa^2 NB_{t-2} + \cdots \tag{7}$$

Rewriting the equation gives the expression for the amount of new issuance:

$$NB_{t+1} = B_{t+1} - \kappa B_t \tag{8}$$

Because of the decaying coupon structure, bonds issued in period $t-j$ will trade at $\kappa^j P_t^B$. Therefore, it is unnecessary to keep track of the prices of perviously issued bonds, because those prices are a function of the current price. Also, the bond with decaying rate of κ has the duration of $\frac{1}{1-\beta\kappa}$, which will be used to calibrate the average duration of the long-term bonds.

The flow budget constraint of the child is given by

$$P_t C_{b,t} + B_{t-1} \leq Q_t(B_t - \kappa B_{t-1}) + P_t X_t^b \tag{9}$$

Note that the sum of the value of consumption and the total coupon liability can not exceed the new bond issuance and the transfers from parents.

The first order conditions for the child is as follows:

$$\Lambda_{b,t-1,b} = \beta_b \left(\frac{C_{b,t}}{C_{b,t-1}} \right)^{-\sigma} \tag{10}$$

$$1 = E_t \Lambda_{b,t,t+1} R_{t+1}^b \Pi_{t+1}^{-1} \tag{11}$$

where $\Lambda_{b,t,t+1}$ denotes the SDF of the child, and R_t^b the holding period gross return on the long-term bonds from $t-1$ to t:

$$R_t^b = \frac{1 + \kappa Q_t}{Q_{t-1}}. \tag{12}$$

Equation (10) exhibits the intertemporal consumption decision of the child and (11) represents the child's saving decision, or pricing rule for long-term bonds.

Financial intermediaries. I assume that a financial intermediary (FI) enters the period with the transfers from the parent, and exits with probability one at the end of the period. Therefore, FI does not accumulate net worth and pays out dividend to its owner, the parent, when exiting. Because every FI receives the same amount of transfers and does not accumulate net worth, one can think that there is a representative financial intermediaries in the economy.

Note that this simplification is a special case of more general structure for financial intermediaries, for instance the one in Gertler and Karadi (2011), where authors consider financial intermediaries that can live indefinitely but stays financial intermediaries next period with probability less than 1. In their environment, FI accumulates net worth over time and it is an important state variable of the model, which also makes the extended model loses its tractability. The assumption of one-period lived FI allows the model to reduce the state variable by one, hence improves its analytical tractability, while preserving the qualitative aspect of the model with financial frictions.

Upon the entering, the representative FI is endowed with the transfers from the parent household, $P_t X_t^{FI}$, which is exogenously determined. The amount of the transfer is composed of two parts: the fixed amount of new equity injection, $\bar{X}^{FI}$, and the value of outstanding long-term bonds inherited from the FI in the previous period, which is calculated at the current market price of bonds, $\kappa Q_t B_{t-1}^{FI}$:

$$P_t X_t^{FI} = P_t \bar{X}^{FI} + \kappa Q_t B_{t-1}^{FI} \tag{13}$$

The liability of the intermediary consists of the equity transfer, $P_t X_t^{FI}$ and short-term deposits, S_t^{FI}, which the FI takes from the parents household. On asset side, FI holds long-term bonds issued by the child, B_t^{FI} and interest bearing reserves, RE_t^{FI}. As a result, the balance sheet condition of the FI is given by

$$Q_t B_t^{FI} + RE_t^{FI} = S_t^{FI} + P_t X_t^{FI} \tag{14}$$

In period $t+1$, the FI pays out interest on deposits, R_t^s, earns interest on reserves, R_t^{re}, and a one-period holding return on long-term bonds, R_{t+1}^b. At the end of the period, the FI exits and pays the dividend to the parent, $P_{t+1} D_{t+1}^{FI}$, which is given by:

$$P_{t+1} D_{t+1}^{FI} = (R_{t+1}^b - R_t^S) Q_t B_t^{FI} + (R_t^{re} - R_t^S) RE_t^{FI} + R_t^S P_t X_t^{FI} \tag{15}$$

Here, $R_{t+1}^b - R_t^s$ is the spread between long-term and short-term bonds, hence the excess return on long-term bonds. $R_t^{re} - R_t^s$ is the spread between reserves and deposits.

Following Sims, Wu, and Zhang (2021), I introduce a risk-weighted leverage constraint in financial sector. Specifically, the balance sheet of the FI is constrained by the following risk-weighted leverage constraint.

$$Q_t B_t^{FI} \leq \Theta_t P_t \bar{X}^{FI} \tag{16}$$

The left-hand side of (16) represents the risk-weighted assets, which puts zero weights on reserves and the weights of unity on long-term bonds. The right-hand side is the amount of equity, $\bar{X}^{FI}$, multiplied by the leverage multiple Θ_t.

The leverage constraint (16) says that the long-term bonds holdings of the FI cannot exceed a exogenously given multiple of the new equity transferred from the parent, $P_t \bar{X}^{FI}$. I assume that the time-varying leverage multiple Θ_t is governed by a exogenous AR(1) process and refer to the innovation to Θ_t as credit shocks.

The intermediary operates to maximize the expected dividend at the end of the period (15) with the nominal stochastic discount factor of the parent household, $\Lambda_{t,t+1} \Pi_{t+1}^{-1}$, subject to the time-varying risk-weighted leverage constraint (16). The FI decides the amount of holdings of long-term bonds and reserves to maximize the objective function.

$$\max_{B_t^{FI}, RE_t^{FI}} P_{t+1} D_{t+1}^{FI} = (R_{t+1}^b - R_t^S) Q_t B_t^{FI} + (R_t^{re} - R_t^S) RE_t^{FI} + R_t^S P_t X_t^{FI} \tag{17}$$

$$\text{subject to} \quad Q_t B_t^{FI} \leq \Theta_t P_t \bar{X}^{FI} \tag{18}$$

Denote the shadow value of one unit of equity to the FI at t by Ω_t, then the first order conditions for the intermediary is given by

$$E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_{t+1}^b - R_t^S) = \Omega_t \tag{19}$$

$$E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^S) = 0 \tag{20}$$

The intermediary's portfolio decision yields the optimal condition (19), which exhibits the condition for long-term bond holdings by the FI. The condition implies that the FI would purchase long-term bonds up to the point at which the expected return on the long-term bonds is equal to its financing cost, i.e. the interest cost on the deposits. When the leverage constraint is binding, $\Omega_t > 0$ and the long-term bonds give positive excess returns.

Equation (20) displays the arbitrage condition between reserves and deposits. Owing to the fact that reserves are given zero weights, the intermediary would hold an indefinite amount of reserves as long as the return on reserves, R_t^{re}, covers the funding cost, R_t^S. In equilibrium, this arbitrage conditions implies the return on reserves and the short-term interest rate governed by the central bank would coincide, hence $R_t^{re} = R_t^S$.

Production. The production side of the economy consists of final output and intermediate goods firms. In each period t, the final consumption good Y_t is produced by perfectly competitive firms using a continuum of each intermediate goods $Y_t(i),\ i \in [0,1]$ as inputs. Final good producer has access to a CES production technology that aggregates the continuum of intermediate goods into the single final good:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon-1}{\epsilon}}. \tag{21}$$

where $\epsilon > 1$ is the elasticity of substitution. Final good producer's profit maximization and the zero profit condition for the competitive market yield the following condition for the final good price P_t:

$$P_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}} \tag{22}$$

and the downward sloping demand function for each intermediate good i:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t \tag{23}$$

A continuum of monopolistically competitive firms produce a different variety i. Each intermediate goods firm i has access to a CRS technology

$$Y_t(i) = A_t L_t(i) \tag{24}$$

where $L_t(i)$ denote the quantity of labor hired by firm i. A_t is a neutral exogenous technology that is common across firms, and follows a stationary AR(1) process.

Every intermediate goods firm is subject to the Calvo pricing friction. In each period t, a fraction ϕ of intermediate goods firms are not allowed to change its price optimally. The remaining fraction of firms, $1-\phi$, can reset their price optimally to maximize the present value of its future expected cash flow. Because the parent household owns intermediate goods firms, realized profits will be distributed to households. Specifically, optimizing firms set their price $P_t(i)$ to solve the following expected profit maximizing problem:

$$\max_{P_t(i)} E_t \sum_{j=0}^{\infty} \phi^j \Lambda_{t,t+j} \Big[\frac{P_t(i)}{P_{t+j}} Y_{t+j}(i) - w_{t+s} L_{t+j}(i)\Big]. \tag{25}$$

$$\text{subject to} \quad Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t \tag{26}$$

$$Y_t(i) = A_t L_t(i) \tag{27}$$

Firm's first order condition can be expressed as follows:

$$p_t^*(i) = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} \tag{28}$$

$$x_{1,t} = mc_t Y_t^{1-\sigma} + \phi\beta E_t \Big[x_{1,t+1} \Pi_{t+1}^{\epsilon}\Big] \tag{29}$$

$$x_{2,t} = Y_t^{1-\sigma} + \phi\beta E_t \Big[x_{2,t+1}, \Pi_{t+1}^{\epsilon-1}\Big] \tag{30}$$

where $p_t^*(i) = P_t^*(i)/P_t$ is the optimal relative reset price that is common across optimizing firms, and $mc_t = w_t/A_t$ denotes an economy-wide real marginal cost. Non-optimizing firms keep their previous price without any indexation, hence

$$P_t(i) = P_{t-1}(i). \tag{31}$$

To explore how positive trend inflation alters firms' pricing decision, it is useful to see how the optimal reset price (p_t^*) in equation (28) is determined. Note that $x_{1,t}$ is discounted costs and $x_{2,t}$ is discounted revenue. These two economic conditions are affected by future inflation, but the degree of exposure is different. That is, cost conditions are more exposed to inflation and it implies that as inflation is expected to last in the longer future, it negatively affects firms' outlook for its future cost conditions, which leads to higher reset price. It is so when the average inflation is high, or *trend inflation* is high. In this regard, as trend inflation increases, firms' pricing decision becomes more forward looking because it puts more weight on future cost conditions than future revenue.

On the other hand, non-optimizing firms keep their old prices. In zero steady state inflation or full price indexation setting, this generates only limited effects on real outcome because even non-optimizing prices are *on average* on the right track and close to the optimal price. But if steady state inflation is positive, then firms with old prices drift away from the optimal price as time goes. These prices are cheaper than new prices, and this drifting effect is compounding over time. This makes firms prices become more backward looking.

Combining these two effect, compared to zero steady state inflation or full indexation case, firms' prices are more forward and backward looking. This is a crucial feature of the Generalized New Keynesian Phillips curve (GNKPC) and will be explicit in the equilibrium relationship below.

Central bank and fiscal authority. The central bank sets the return on reserves, R_t^{re}, and its portfolio of long-term bonds, B_t^{cb}. As discussed, the return on reserves is the same as the short-term interest rate in equilibrium, hence the central bank has control over the short-term interest rate. The return on reserves is either set optimally by minimizing central bank's loss function or bound by the ZLB.

As for the long-term bond portfolio, the central bank finances its long-term bond holdings, B_t^{cb}, by creating reserves, RE_t. Then the balance sheet of the central bank is as follows:

$$Q_t B_t^{cb} = RE_t \tag{32}$$

Let QE_t denote the real value of the long-term bond holdings by the central bank:

$$QE_t = Q_t b_t^{cb}, \tag{33}$$

where $b_t^{cb} = \frac{B_t^{cb}}{P_t}$. Further assume that the central bank can choose b_t^{cb} (equivalently, RE_t)

without any constraint.

As a result of the long-term bond holdings, the central bank earns operating surplus (or deficits) from the portfolio. I assume that this surplus is remitted to the fiscal authority and then transferred to the parent household via lump sum subsidy or tax. The amount of the surplus is given by

$$P_t T_t = R_t^b Q_{t-1} B_{t-1}^{cb} - R_{t-1}^{re} RE_{t-1} \tag{34}$$

Aggregation and equilibrium. Aggregate output is consumed by the parent and the child household, hence the resource constraint is given by

$$Y_t = C_t + C_{b,t} \tag{35}$$

Aggregate production can be obtained by aggregating each intermediate firm's demand for labor from (24) and demand function for each intermediate good (23):

$$L_t = \int_0^1 L_t(i)di = \int_0^1 \frac{Y_t(i)}{A_t}di = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di \frac{Y_t}{A_t} = v_t^p \frac{Y_t}{A_t}, \tag{36}$$

where the second equality is from the labor demand (24), the third equality from the demand for firm i, (23), and the last equality is from the definition of the price dispersion ($v_t^p \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di$). Rewrite the aggregate production as

$$Y_t = (v_t^p)^{-1} A_t L_t. \tag{37}$$

Equation (37) shows the inverse relationship between aggregate output and the price dispersion. When price dispersion increases, demand for intermediate goods reallocates and it causes inefficiency and raises real wage, hence the marginal cost.

This is summarized by v_t^p, which is akin to a negative productivity shock (Ascari, Castelnuovo, and Rossi, 2011). Because of the nominal rigidity, price dispersion is increasing in inflation and as a result, there exists output loss in the long-run under positive trend inflation and no price indexation (Ascari and Ropele, 2009). In addition, in the short-run, fluctuation in inflation drives fluctuation in price dispersion, which generates output loss. The short-run output loss is due to the endogenous cost-push effects in the Generalized New Keynesian model (Alves, 2014; Seo, 2023). In the following section, how the endogenous cost-push effects alters the transmission of shocks will be discussed in more detail.

A representative financial intermediary holds all reserves, hence the market clearing for reserves is $RE_t = RE_t^{FI}$. Also, the FI is the only supplier of the deposits, hence $S_t =$

S_t^{FI}. Long-term bonds issued by the child is held by the FI and the central banks. The market for long-term bonds clear when $B_t = B_t^{FI} + B_t^{cb}$.

Following Sims, Wu, and Zhang (2021), I make the "full bailout" assumption: in each period, the parent household pays off completely the outstanding debt obligation of the child

$$P_t X_t^b = (1 + \kappa Q_t) B_{t-1}. \tag{38}$$

Therefore, the transfer made by the parent to the child, X_t^b, is time-varying and makes the child consumption decision is effectively static:

$$P_t C_{b,t} = Q_t B_t. \tag{39}$$

The full bailout assumption greatly simplifies the equilibrium conditions by reducing the number of a state variable, which gives the four-equations model in Sims, Wu, and Zhang (2021).

I assume that the logs of the technology shocks, A_t, and the credit shocks, θ_t obey AR(1) processes.

$$a_t = \rho_a a_{t-1} + \epsilon_{a,t}, \tag{40}$$

$$\theta_t = \rho_\theta \theta_{t-1} + \epsilon_{\theta,t}, \tag{41}$$

where $\epsilon_{a,t} \sim N(0, \sigma_a^2)$ and $\epsilon_{\theta,t} \sim N(0, \sigma_\theta^2)$. The potential output, Y_t^*, is defined as the equilibrium output where prices are fully flexible (i.e. $\phi = 0$), and there is no credit shock (i.e. $\theta_t = \Theta$). Then the natural rate of interest, R_t^*, is the real short-term interest rate that is consistent with the potential output. The output gap is the deviation of output from the potential output: $X_t = Y_t / Y_t^*$.

2.2 Log-Linearized Model and the Endogenous Cost-Push Effects

Log-linearizing the equilibrium conditions around a positive steady state inflation $\bar{\pi}$ yields the following equations.

$$x_t = E_t x_{t+1} - \frac{1-z}{\sigma}(r_t^s - E_t \pi_{t+1} - r_t^*) - z\left[\bar{b}^{FI}(E_t \theta_{t+1} - \theta_t) + \bar{b}^{cb}(E_t qe_{t+1} - qe_t)\right], \tag{42}$$

$$\pi_t = \gamma mc_t + \beta E_t \pi_{t+1} + \beta \bar{\pi}(1 - \tilde{\phi})\Big(E_t \psi_{t+1} - (1-\sigma)x_t - \zeta_r r_t^*\Big), \tag{43}$$

$$(44) \quad v_t^p = \frac{\epsilon\tilde{\phi}}{1-\tilde{\phi}}\bar{\pi}\pi_t + \tilde{\phi}\pi v_{t-1}^p,$$

$$(45) \quad \psi_t = (1-\tilde{\phi}\beta\bar{\Pi})\Big(mc_t + (1-\sigma)x_t + \zeta_r r_t^*\Big) + \tilde{\phi}\beta\bar{\Pi}\epsilon\pi_t + \tilde{\phi}\beta\bar{\Pi}E_t\psi_{t+1},$$

$$(46) \quad mc_t = \Big(\chi + \frac{\sigma}{1-z}\Big)x_t + \chi v_t^p - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t]$$

where $\gamma \equiv \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\bar{\Pi})}{\tilde{\phi}}$, $\tilde{\phi} \equiv \phi(1+\bar{\pi})^{\epsilon-1}$, $\bar{\Pi} \equiv 1+\bar{\pi}$, $\zeta_r \equiv \frac{(1-\sigma)(1-z)}{\sigma(\rho_f-1)}$, and $\bar{b}^{FI}$ and $\bar{b}^{cb}$ are parameters measuring the steady state long-term bond holdings of financial intermediaries and the central bank, respectively. Lowercase variables denote log deviations from steady state, $x_t = y_t - y_t^*$ denotes the log output gap. The model is closed by specifying policy rules that determine the short-term interest rate, r_t^s, and the central bank's long-term bond portfolio, qe_t.

Equation (42) is an IS curve that summarizes the demand side of the model. It represents the aggregate output consumed by the parent and the child, and the static consumption decision of the child (39). Note that the consumption of the child is completely financed by the issuance of the long-term bonds, hence affected by both the credit shocks and the QE shocks.

Equation (43) shows the Phillips curve relationship in the model. In standard NK model with zero inflation, a New Keynesian Phillips curve (NKPC) is a single equation that relates output gap and inflation. But under the GNK setting of positive inflation and no indexation, a Generalized New Keynesian Phillips curve (GNKPC) includes additional terms that reflects the distortion in supply side. These terms include backward looking price dispersion (44) and forward looking cost conditions (46). Recall the objective function of optimizing firms in (25). When maximizing expected profit, firms take into account future cost conditions which are affected by future paths of inflation. Hence, as trend inflation increases, reset price p_t^* becomes more forward looking, as in (45), and the GNKPC in (43) demonstrates this force.

On the other hand, non-optimizing firms keep their previous prices (see equation 31). But unlike zero trend inflation, those prices drift away from the optimal reset price over time, which is increasing over time at the average rate of $\bar{\pi}$. Because the probability of reoptimization is random, the distribution of the old prices in t is summarized by the price index in $t-1$. This implies that the evolution of the price dispersion in (44) is a weighted average of the past inflation.

Equations (43) through (46) summarize the supply side of the model and exhibit the distinctive feature of the model. That is, the terms in (43) drive cost-push effects that emerge endogenously. Note that both price dispersion and marginal cost are increasing

Table 1: Calibrated parameters

Parameters	Value	Description
β	0.995	Discount rate
σ	1	CRRA
χ	2	Frisch elasticity
z	0.33	Consumption share of child
ϕ	0.8	Price rigidity
ϵ	11	Steady-state price markup
$\bar{b}^{FI}$	0.7	Weight on credit in IS curve
$\bar{b}^{cb}$	0.3	Weight on QE in IS curve
ϕ_π	1.5	Taylor rule inflation
ϕ_y	0	Taylor rule output gap
ρ_f	0.95	AR natural rate
ρ_θ	0.95	AR credit shock
ρ_q	0.95	AR QE shock

in temporary inflationary pressure. Thus, as inflationary shock hits the economy, it brings about supply side effects due to the price distortion from the forward- and backward-looking forces. The larger price dispersion leads to reallocation of demand across intermediate goods firms due to its downward-sloping demand curve. But the aggregate output is a weighted average of each intermediate inputs, hence the price dispersion leads to the fall in aggregate output.

The endogenous cost push effect is the decisive feature of the GNK model. It indicates the output loss from trend inflation in both steady state and in the short-run. In steady state, because of positive trend inflation with no indexation, steady state price dispersion is larger than 1. From the aggregate production function where price dispersion serves as a negative shock in (37), this leads to steady state output loss. In the short-run, for any temporary inflationary pressure, it triggers the deviation of price dispersion and the marginal cost terms from its steady state. In turn, through the GNKPC in (43), the inflationary pressure shifts supply curves as if there were cost-push shocks. This generates temporary output loss in the short-run.

To explore how the endogenous cost-push mechanism alters equilibrium dynamics, I investigate the model-generated impulse responses to each shock. To close the model, I assume that the central bank uses a Taylor rule to set the short-term interest rate. Furthermore, the QE policy is determined by an exogenous AR(1) process:

$$r_t^s = \rho_r r_{t-1}^s + (1-\rho_r)(\phi_\pi \pi_t + \phi_x x_t) + \epsilon_{r,t} \tag{47}$$

$$qe_t = \rho_q qe_{t-1} + \epsilon_{q,t}. \tag{48}$$

The processes of the natural rate of interest, r_t^*, and the credit shock, θ_t, are given by the following AR(1) processes:

$$r_t^* = \rho_f r_{t-1}^* + \epsilon_t, \tag{49}$$

$$\theta_t = \rho_\theta \theta_{t-1} + \epsilon_{t-1}. \tag{50}$$

Parameter values are standard and taken from the literature. Calibration and the description of each parameter are listed in table 1. The discount factor is set to match the real interest rate of 2 percent. CRRA is set to 1, hence the log preferences. Labor preference parameter χ is chosen to Frisch elasticity of 0.5. Calvo parameter is 0.8 and steady state price markup is 10 percent (Justiniano, Primiceri, and Tambalotti, 2010). The share of child in aggregate consumption is 1/3, which is the share of durable consumption and private investment in private nongovernment domestic expenditure. This choice is based on the observation that the dynamics of the child's consumption resembles that of durable consumption (Sims and Wu, 2021). The share of the financial intermediaries in long-term bonds holdings is 0.7, hence the share of central bank is 0.3. Taylor rule parameters are standard values of $\phi_\pi = 1.5$ but I set $\phi_y = 0$. The exogenous shock processes are persistent and its AR coefficients are set to 0.95.

Figure 1 reports the impulse responses to each shock, i.e. natural rate shock (r_t^*), credit or QE shock (θ_t or qe_t), and monetary policy shock ($\epsilon_{r,t}$). In each panel, black solid lines represent the responses under zero trend inflation ($\bar{\pi} = 0$%), blue dashed lines are the responses under 2% trend inflation ($\bar{\pi} = 2$%), and red broken lines show the responses under 4% trend inflation ($\bar{\pi} = 4$%). I display IRFs of four variables: output gap, inflation, short-term interest rate, and excess return on long-term bonds. The size of the shocks are normalized so that the impact effect of output gap is one percent for all shocks.

Panel (a) shows the IRFs to natural rate shocks. With zero trend inflation, both output gap and inflation increase, which induces the central bank to react by increasing the short-term interest rates. This is fairly consistent with standard models. But when trend inflation is higher than zero, IRFs show different dynamics. If $\bar{\pi} = 4$%, output gap is persistently below zero, meaning large downturn following a positive natural shock. Because the responses of potential output does not change for different values of $\bar{\pi}$, the difference

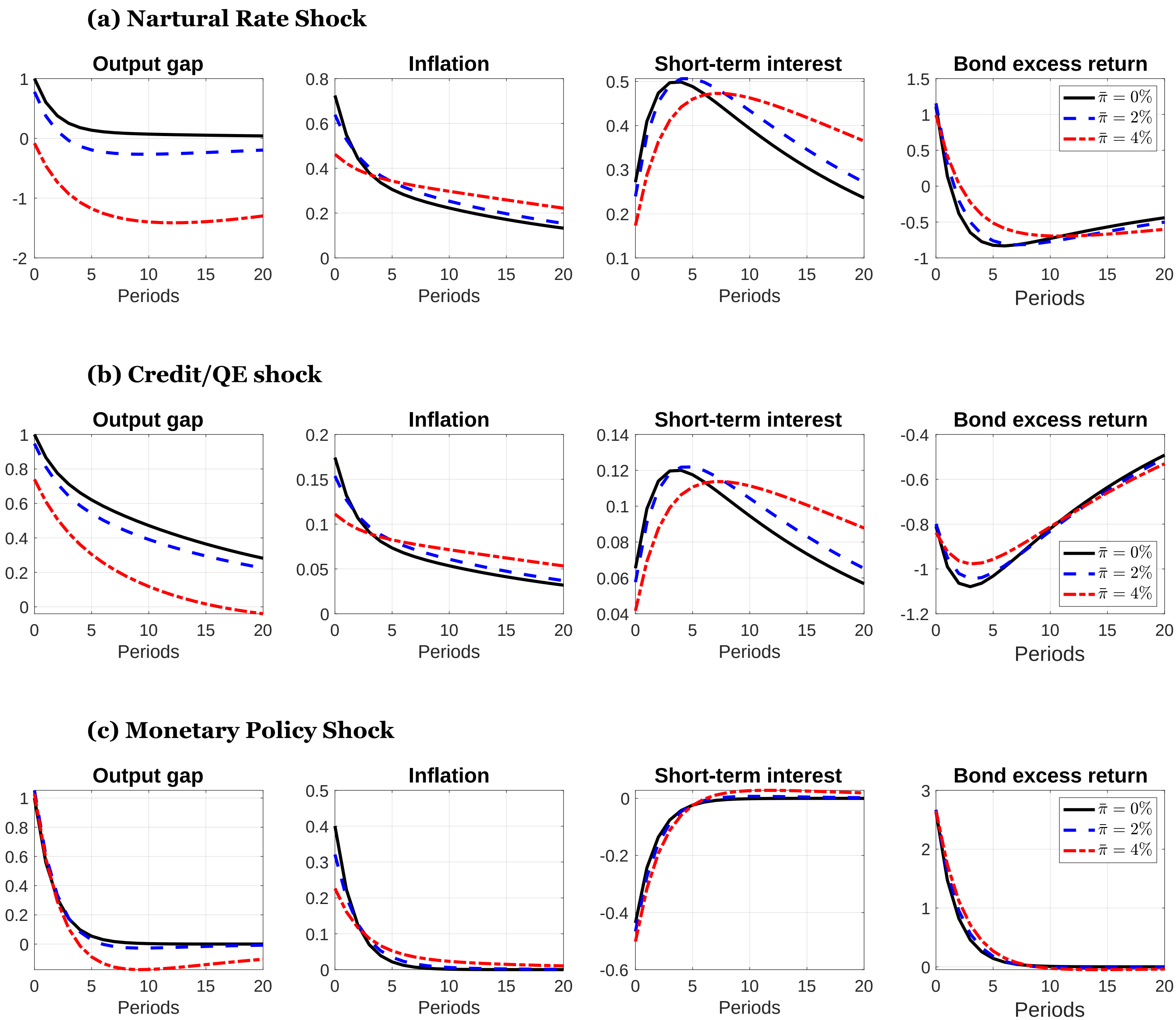


Figure 1: IRFs under Different Trend Inflation

between responses only reflects the endogenous cost push effects from trend inflation. The output loss is significant and due to the fall in demand, inflation response becomes smaller in the first few periods, but more persistent afterwards. This also steepens the yield curve, i.e. increasing excess return.

Panel (b) plots the impulse responses to credit or QE shocks. Because those two shocks are calibrated to have the same AR coefficients and have the same impact effect on output gap, the two shocks generate the identical IRFs. QE shocks loose the leverage constraint of the child and increase the output gap. The expansion in demand also puts inflationary pressure, which leads to increase in inflation and short-term interest rate. The increase in trend inflation has similar effects for QE shocks. That is, the effects on output gap is dampened and inflation becomes more persistent. Therefore, the effectiveness of QE shocks to support output expansion weakens as average inflation rate is increasing. As for the bond excess return, the mechanism through which QE shocks operate is to flatten the yield curve and alleviate the leverage constraint. Thus the excess return on bonds decreases.

Impulse responses to conventional monetary policy shocks are depicted in panel (c). Consistent with standard NK models, monetary loosening raises output gap and inflation but short-term interest rate falls. Because monetary policy shocks operate through the change in short-term interest rate, yield curve steepens. Similar to the impulse responses to QE socks, trend inflation weakens the efficacy of the stabilization effects of monetary policy shocks. For instance, when $\bar{\pi} = 4\%$, the impact effect on output gap is almost identical to the zero inflation case, but the effect is much short-lived, and after few periods, output gap starts to go below the steady state level.

Discussion. The impulse responses show that in the standard calibration, the presence of trend inflation attenuates the output and inflation stabilization effects of the central bank's instruments.

The transmission of conventional monetary policy shocks hinges on the standard intertemporal substitution channel. The parent households face lower short-term real interest rate and increase consumption by substituting with saving. Hence, it is less affected by the household heterogeneity and the financial frictions. But with trend inflation, the stabilization effects diminish, because of the inflationary pressure following an expansionary monetary shock. The GNKPC in (43) indicates that inflation is affected by future cost conditions. When there is an inflationary pressure, it shifts the future cost condition upward, hence firms expect higher marginal cost in the future. This pushes the reset prices higher and it further widens the price dispersion in (44). Therefore, the initial expansionary effect is dampened by the endogenous cost-push effects from trend inflation, hence conventional monetary policy becomes less effective for driving temporary expansion.

The transmission of unconventional policy is through the effects on the leverage constraint. On the one hand, by loosening the leverage constraint, the child can issue larger amount of long-term bonds and consume more, which is a demand shifter for IS curve. On the other hand, favorable credit conditions for the child has a reallocation effect from the saver (the parent) to the borrower (the child). This is because now the borrower can issue more debt to finance their consumption. But the change in the child's demand influences on the marginal cost due to the wealth effects induced by the reallocation effect. Favorable credit conditions generate a negative wealth effect for the parent as this is a reallocation of resources toward the child. In turn, the negative wealth effect generates downward pressure on real wage and real marginal cost. In this sense, the unconventional policy acts as if a positive supply shock that boosts output but drives downward pressure on inflation. This elucidates the presence of qe and θ terms in the marginal costs in (46) and how unconventional policy induces a cost-push effect.

Positive trend inflation, however, puts on another source of cost-push effects through firms' pricing decision. The inflation driven by the unconventional policy triggers more dispersed price distribution and deteriorates cost conditions. This results in significant output loss, which is amplified by the level of trend inflation. Note that this mechanism is stronger for more persistent shocks because those shocks deteriorate even further the future cost conditions. This dampens the output expansion effect and the efficacy of the QE policy weakens.

In both of conventional and unconventional policy, the dominant effect on output is through the increase in demand, which is accompanied by temporary increase in inflation. In other words, both policy effectively serves as demand shifter and induce positive comovement in output and inflation. But trend inflation introduces the endogenous cost-push effects through inefficient price dispersion, and dampens the effects of both policy.

3 Optimal Monetary Policy

In this section, I explore optimal monetary policy when trend inflation is positive and a central bank has two instrument, conventional and unconventional policy. In the model, the central bank faces two sources of cost-push wedges. First, the wealth effect induced by unconventional policy (or credit shocks) aforementioned introduces endogenous cost-push wedges in the Phillips curve through its influence on marginal costs. The reallocation of resources from the saver to the borrower generates a negative wealth effects for the saver, and increases labor supply, which leads to downward pressure on marginal cost and inflation. Second, positive trend inflation also brings about cost-push terms in the Phillips curve. When long-run inflation is positive, price dispersion becomes more dispersed, and has first order effect on equilibrium dynamics. This is because non-degenerate price distribution induces inefficient demand misallocation and leads to aggregate output loss.

These two endogenous cost-push forces complicate the central bank's operation of monetary policy because i) unconventional policy causes the endogenous cost-push effects through financial frictions and ii) any temporary inflationary pressure also triggers the endogenous cost-push effects through trend inflation. In particular, both conventional and unconventional policy contribute to expansion in output through demand shift, which creates temporary inflation. This means in the presence of trend inflation, the central bank's stabilization policy becomes less effective: expansionary conventional and unconventional policy are dampened by the endogenous cost-push effects.

The induced cost-push effect is even more important for the design of optimal policy response. As analyzed in Alves (2014), positive trend inflation breaks the divine coincidence (Blanchard and Galí, 2007) even when only natural rate shocks exist, due to the endogenous cost-push effects. But as Sims, Wu, and Zhang (2021) find, unconventional policy is an effective tool to deal with cost-push shocks exactly because it generates cost-push effects. Then, an important policy question is whether the central bank can recover the divine coincidence by the two policy instrument. Furthermore, what is the optimal conventional and unconventional policy in response to inflationary pressure?

In this section, I answer these questions within the framework in the earlier section. The result indicates that the central bank can exploit the endogenous cost-push wedges induced by unconventional policy to deal with the endogenous cost-push wedges induced by trend inflation, and can recover the divine coincidence. Therefore, in the presence of the two forms of cost-push wedges, it is optimal for the central bank to always use both policy tools to stabilize the economy.

Follow Ascari and Ropele (2007), I assume a loss function defined by a weighted quadratic sum of output gap and inflation:

$$\mathcal{L} = \mu x_t^2 + \pi_t^2 \tag{51}$$

$\mu \geq 0$ is the relative weight attached to the output gap. Note that this loss function reflects policymaker's dual mandate of full employment and price stability. For the rest of this section, I focus on optimal monetary policy under discretion and hence, the instantaneous loss function as in (51) Also, I assume $\chi = 0$, which implies indivisible labor. This greatly simplifies the analysis under non-zero trend inflation because with indivisible labor, non-optimizing firms always exploit labor margin to offset the negative effects from their non-optimizing prices. More importantly, as a consequence of the exploitation of labor margin, price dispersion does not affect economy wide marginal cost anymore (see equation 46).

That being said, price dispersion can be dropped out of the equilibrium conditions and the number of state variable reduces by one. This makes the linearized model more tractable, while does not change the qualitative aspect of the model.

3.1 Optimal Policy under Discretion

In New Keynesian models with trend inflation, the divine coincidence fails (Alves, 2014). This result is coming from the endogenous cost-push wedges from trend inflation in the Generalized New Keynesian Phillips curve in (43). In this case, any inflationary pressure triggers cost-push effect through demand misallocation, which translates into policy trade-offs for a central bank when the only policy instrument is the short-term interest rate policy. Because conventional monetary policy only shifts IS curve, the distortion through the GNKPC cannot be easily handled with the single policy tool.

The result in this section indicates that the divine coincidence can be restored when the central bank has two instruments: both conventional and unconventional monetary policy. This is because unconventional policy affects supply side by altering financial conditions, and can counteract the cost-push wedges from firms' pricing decision. When leverage constraints are loosened, it creates a negative wealth effects for the parents who supply labor. In turn, this puts downward pressure on wages and affects supply side, which is a new policy margin that the central bank can exploit. Hence, the negative wealth effects give the monetary authority a way to offset the endogenous cost-push wedges induced by trend inflation. Consequently, the divine coincidence result still does not hold. The following theorem formally spells out the divine coincidence result.

Theorem 1. *When trend inflation is positive and both policy instruments are available, the central bank can achieves $\pi_t = x_t = 0$, and the equilibrium paths for the policy instruments are*

$$qe_t = \frac{(1-z)\beta\bar{\pi}(1-\tilde{\phi})(\rho_f - 1)}{\gamma\sigma z \bar{b}^{cb}}\zeta_r r_t^* - \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t \tag{52}$$

$$r_t^s = \left(1 + \frac{\beta\bar{\pi}(1-\tilde{\phi})(\rho_f - 1)}{\gamma}\zeta_r\right) r_t^* \tag{53}$$

Proof. See appendix. □

Theorem 1 shows that a central bank can fully offset both natural rate shocks and credit shocks by both policy tools. Note that the optimal policy rules in (52) and (53) become $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$ and $r_t^s = r_t^*$. That is, in the absence of the endogenous cost-push effects from trend inflation, optimal conventional policy is enough to neutralize natural rate shocks

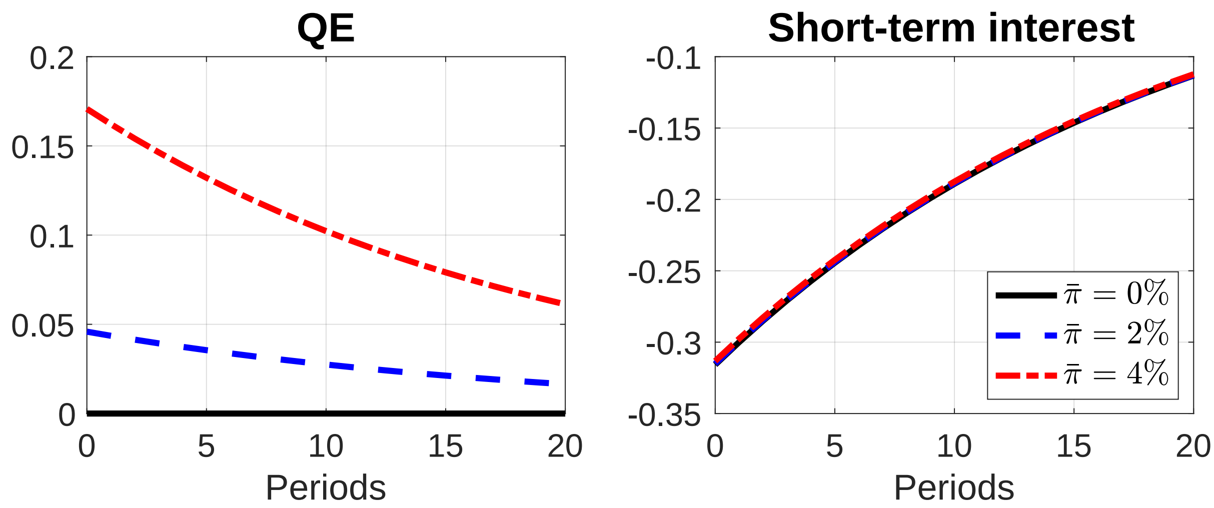


Figure 2: Optimal Policy Responses to r^* Shock

and unconventional policy should only deal with credit shocks. But in more general case, unconventional policy should also react to natural rate shocks because it triggers cost-push effects. Hence, optimal choice of qe_t depends on r_t^*, and optimal r_t^* adjusts in accordance with qe_t.[1] This shows that there is no policy trade-off between stabilizing output gap and inflation even with the endogenous cost-push wedges when a central bank has two instrument. The following corollary confirms that if only one policy instrument is available, the divine coincidence cannot be achieved.

Corollary 1. *When trend inflation is positive and only one policy instrument is available, then there exists a policy trade-off in stabilizing both output gap and inflation, and it is impossible to achieve $x_t = \pi_t = 0$ for all t.*

Proof. See appendix. □

The divine coincidence result implies that it is optimal to always use a mix of the two policy tools to deal with natural rate shocks. But as argued above, higher trend inflation tends to make QE weaker, or putting differently, the larger amount of portfolio adjustment is needed to achieve the same level of expansionary effect.

Figure 2 shows the optimal policy response to natural rate shocks in figure 1, which induces 1 percent increase in output gap under zero steady state inflation. The IRFs show

[1]Note that the cost-push wedge introduced by credit shocks can be neutralized by setting $qe_t = -\bar{b}^{FI}/\bar{b}^{cb}\theta_t$ and conventional monetary policy is not necessary. This is because even though credit shocks distort both demand and supply conditions, the way it affects equilibrium conditions is exactly same as that of the quantitative easing.

distinct effects of trend inflation on the effectiveness of QE. As trend inflation rises, the amount of long-term bonds portfolio adjustment to achieve full stabilization increases, while the optimal short-term interest rate response is essentially identical.

Trend inflation causes the endogenous cost-push effects and generates output loss and because the QE policy also triggers cost-push wedges through wealth effects, the degree of QE response increases in trend inflation, or the degree of output loss. In contrast, conventional monetary policy through short-term interest rate operates mainly through demand side, which makes it invariant to the level of trend inflation.

3.2 Optimal Discretionary QE Policy at the ZLB

QE-type policy should be always used to offset the endogenous cost-push wedge and credit shocks in the model. Another interesting optimal policy design is when an economy is at the ZLB where the short-term nominal rate is zero due to sufficiently negative natural rate. This type of situation has gained attention since the Great Recession but how the level of trend inflation interacts with the optimal QE has not been discovered.

To analyze the optimal QE policy at the ZLB, I make a simplifying assumption following Eggertsson and Woodford (2003). The economy has been in steady state and the central bank has conducted the optimal policy in response to natural rate and credit shocks described in theorem 1 in $t-1$ but then in period t, sufficiently large negative natural rate shock realizes and r_t^* becomes negative. The ZLB constraint binds in period t, i.e. $r_t^s = 0$. Suppose that the economy will continue to stay at the ZLB in each subsequent period with non-zero probability $\alpha \in (0,1)$, which is known public and constant over time. With probability $1-\alpha$, the economy exits the ZLB and the central bank can achieve the full stabilization according to the optimal rule in theorem 1. This means that the path of short-term interest rate is as follows:

$$r_t^s = 0 \tag{54}$$

$$E_t r_{t+1}^s = 0 \text{ with probability } \alpha \tag{55}$$

Considering the inability for the short-term nominal rate policy, the central bank optimally chooses qe_t to minimize its loss function. The optimal targeting rule at the ZLB can be characterized the following lean-against-the-wind rule.

$$x_t = \omega_0 \pi_t \tag{56}$$

$$\text{where} \quad \omega_0 = -\frac{1}{\mu\bar{\Pi}}\Big[\frac{\sigma\gamma}{(1-z)} - \frac{(\pi-1)(1-\tilde{\phi})(1-\sigma)}{\tilde{\phi}}\Big]$$

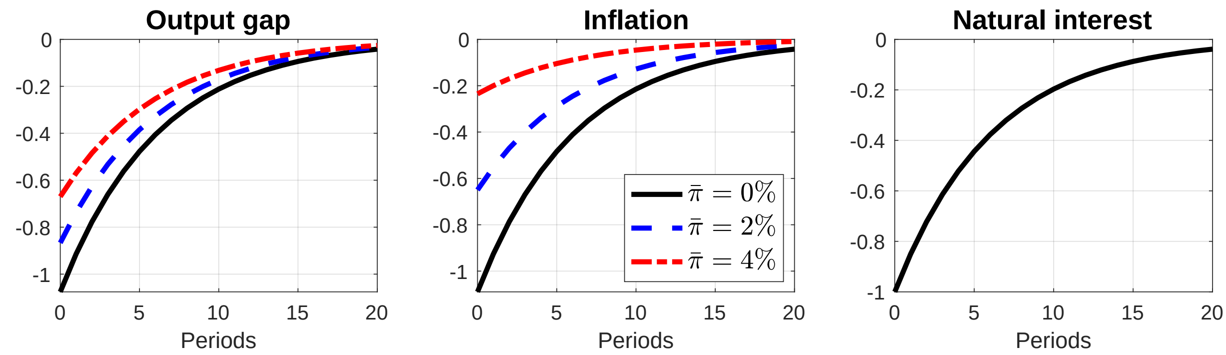


Figure 3: IRFs to r^* Shock at ZLB without QE

Proposition 1 describes the evolution of output gap and inflation, as well as QE policy while the ZLB is binding.

Proposition 1. *When the short rate is constrained by the ZLB, which will continue to bind in the next period with probability α, with QE being the only viable policy instrument, the optimal targeting rule is characterized by equation above. In equilibrium, the paths for inflation, output gap, and QE are*

$$x_t = \omega_1 r_t^* \tag{57}$$

$$\pi_t = \omega_0 \omega_1 r_t^* \tag{58}$$

$$qe_t = \tau r_t^* + \omega_3 \theta_t \tag{59}$$

where

$$\omega_1 = \frac{(1-\rho_f)\Big(\Theta_f \beta(1-\Pi)(1-\tilde{\phi})-(1-\tilde{\phi}\beta\Pi)\Big)\frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}-\Big((1-\tilde{\phi}\beta\bar{\Pi})-\Theta_f\gamma\Big)}{(1-\rho_f)\Big[\Theta_f\Big((1-\beta\rho_f)\omega_0-\beta(1-\Pi)(1-\tilde{\phi})(1-\sigma)\Big)+(1-\tilde{\phi}\beta\Pi)(1-\sigma)\Big]+\Big((1-\tilde{\phi}\beta\bar{\Pi})-\Theta_f\gamma\Big)\omega_0\rho_f} \tag{60}$$

$$\tau = [z\bar{b}^{cb}(1-\alpha\rho_f)]^{-1}\Big((1-\alpha\rho_f-\frac{\omega_0\alpha\rho_f}{\sigma})\omega_1+\frac{1-z}{\sigma}\Big) \tag{61}$$

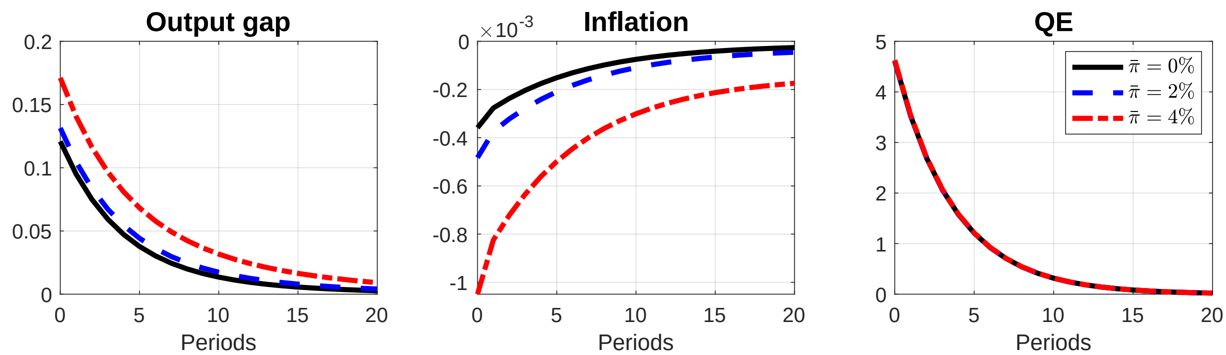
$$\omega_3 = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}} \tag{62}$$

where $\Theta_f \equiv (1-\tilde{\phi}\beta\alpha\rho_f)[\alpha\rho_f\beta(1-\Pi)(1-\tilde{\phi})]^{-1}$.

Proof. See appendix. □

Proposition 1 implies that there exists non-trivial trade-off in stabilization policy. Note that this trade-off is only coming from natural rate shocks. As for credit shocks, cost-push wedges induced by credit shocks can be completely neutralized even when only QE

(a) Inflation-focused loss function ($\mu \to 0$)



(b) Output gap-focused loss function ($\mu = 1$)

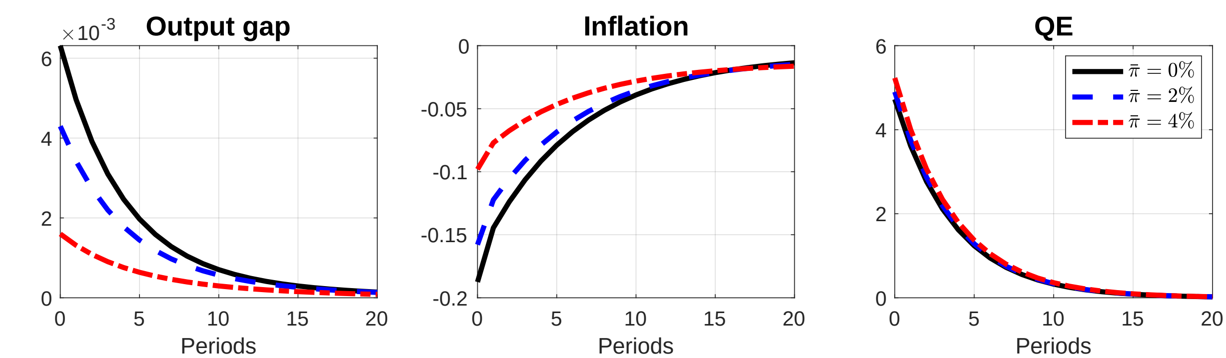


Figure 4: IRFs to r^* Shock at ZLB with Optimal QE

is the option. This is because the workings of QE and credit shocks are similar, hence by appropriate adjustment of long-term bond portfolio, the central bank can fully stabilize output gap and inflation.

In contrast, Natural rate shocks cause inflation and it induces the endogenous cost-push effects, which push further the equilibrium from the optimal. With unconventional policy alone, a central bank can partially offset the adverse shocks but not completely. When trend inflation is high, this additional disturbance is much stronger, so that larger policy response is needed.

Figure 3 shows the paths of output gap and inflation following a negative natural rate shock, when short-term interest rate is bound at zero and QE is absent. Black solid lines plot IRFs under zero steady state inflation, blue dashed lines represent 2% trend inflation and red broken lines exhibit the IRFs under 4% trend inflation. A negative natural rate shock generate significant recession and fall in inflation without any policy response. But it is noteworthy that the negative effect is weaker as trend inflation increases. This is because the fall in inflation due to the natural rate shocks improve cost conditions and reduce price dispersion, which contribute to decrease output loss temporarily. In this sense, the fall in inflation helps to resolve inefficiency in high trend inflation economy, which resemble a negative cost-push shock that put downward pressure on marginal cost and inflation.

In figure 4, I show the paths when the central bank optimally responses by adjusting its QE policy. In panel (a), IRFs are for the case when the central bank places no weight on output gap. In all cases, unconventional policy is fairly effective to counteract negative natural rate shocks. In particular, optimal path of inflation is close to zero, while output gap is slightly positive. For both output gap and inflation, zero trend inflation shows the smallest variability. Although high trend inflation helps to resolve inefficiency and suffer less from the adverse natural rate shock, the efficacy of optimal QE policy is stronger for low trend inflation economy when a central bank is more concerned about inflation.

Panel (b) displays the IRFs when the central bank puts the equal weight on output gap and inflation. Again, QE is an effective tool to stabilize an economy at the ZLB, but high trend inflation economy shows the least variability in output gap and inflation. Trend inflation alters the transmission of QE shock and it drives relatively larger output response for the same amount of inflation change. Therefore, to stabilize output gap, the central bank with high trend inflation faces relatively smaller trade-off than the central bank with low trend inflation.

The IRFs with optimal QE reveal an interesting aspect of the relationship between the efficacy of QE, the objective function of the central bank and the level of trend inflation. If a central bank more focuses on price stability than smaller output gap variability, then its beneficial to keep trend inflation low, so that optimal QE becomes even more effective. On the other hand, if a central bank is more concerned about output stabilization, then having moderate level of trend inflation can be beneficial especially when the ZLB is binding and QE becomes the only policy option, because optimal QE under moderate trend inflation shows smaller output gap and inflation variability.

4 Concluding Remark

This paper addresses the interaction between unconventional monetary policy and a central bank's inflation target, which standard New Keynesian models struggle to capture effectively. The paper employs a tractable four-equation New Keynesian framework, building upon Sims, Wu, and Zhang (2021). This model incorporates heterogenous households, financial intermediaries, credit shocks affecting financial sector leverage constraints, and a central bank employing unconventional monetary policy through long-term bond portfolio management, while allowing for positive trend inflation, leading to the emergence of endogenous cost-push wedges.

The analysis reveals two distinct sources of endogenous cost-push effects, stemming from positive trend inflation and financial sector frictions. These effects create policy trade-offs and challenge the 'divine coincidence' concept. I investigate how positive trend inflation affects the transmission of both conventional and unconventional monetary policy, providing insights into optimal monetary policy at the zero lower bound (ZLB). The findings emphasize the importance of aligning the long-run inflation target with a central bank's broader policy objectives in the context of a dual mandate, holding significant policy implications for central banks in a contemporary policy landscape where QE plays a vital role.

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A The Full Non-Linear Model

Twenty seven variables $\{L_t, C_t, w_t, \Lambda_{t-1,t}, R_t^S, \Pi_t, \Lambda_{b,t-1,t}, R_t^b, Q_t, b_t^{FI}, \Theta_t, re_t, s_t, \Omega_t, R_t^{re}, p_{*,t}, x_{1,t}, x_{2,t},$ $mc_t, Y_t, A_t, C_{b,t}, \nu_t^p, b_{cb,t}, b_t, QE_t, X_t\}$

- The optimality conditions for the parent household:

$$\psi L_t^{\chi} = C_t^{-\sigma} w_t, \tag{63}$$

$$\Lambda_{t-1,t} = \beta\Big(\frac{C_t}{C_{t-1}}\Big)^{-\sigma}, \tag{64}$$

$$1 = R_t^S E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} \tag{65}$$

- Definition of the gross return on the long bond as

$$R_t^b = \frac{1 + \kappa Q_t}{Q_{t-1}} \tag{66}$$

- The optimality conditions for the child

$$\Lambda_{b,t-1,b} = \beta_b \Big(\frac{C_{b,t}}{C_{b,t-1}}\Big)^{-\sigma} \tag{67}$$

$$1 = E_t \Lambda_{b,t,t+1} R_{t+1}^b \Pi_{t+1}^{-1} \tag{68}$$

- The optimality conditions for the FI

$$E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_{t+1}^b - R_t^S) = \Omega_t \tag{69}$$

$$E_t \Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^S) = 0 \tag{70}$$

- The labor demand condition for the wholesale firm

$$w_t = mc_t A_t \tag{71}$$

- The market-clearing condition and aggregate production function

$$Y_t = C_t + C_{b,t} \tag{72}$$

$$Y_t \nu_t^p = A_t L_t \tag{73}$$

- The optimality condition for retail firms

$$p_t^*(i) = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} \tag{74}$$

$$x_{1,t} = mc_t Y_t^{1-\sigma} + \phi\beta E_t \Big[x_{1,t+1} \Pi_{t+1}^{\epsilon}\Big] \tag{75}$$

(76) $$x_{2,t} = Y_t^{1-\sigma} + \phi\beta E_t\Big[x_{2,t+1}, \Pi_{t+1}^{\epsilon-1}\Big]$$

- The aggregate inflation rate evolves according to

(77) $$1 = (1-\phi)p_{*,t}^{1-\epsilon} + \phi\Pi_t^{\epsilon-1}$$

- Price dispersion evolves according to

(78) $$\nu_t^p = (1-\phi)p_{*,t}^{-\epsilon} + \phi\Pi_t^{\epsilon}\nu_{t-1}^p$$

- The balance sheet condition of the FI

(79) $$Q_t b_t^{FI} + re_t = s_t + X_t^{FI}$$

- the leverage constraint of the FI may be written

(80) $$Q_t b_t^{FI} \leq \Theta_t \bar{X}^{FI}$$

- The central bank's balance sheet can be written:

(81) $$Q_t b_t^{cb} = re_t$$

- The market clearing condition for long term bonds in real term is:

(82) $$b_t = b_t^{FI} + b_t^{cb}$$

- The auxiliary QE_t variable is just the real value of the central bank's long bond portfolio

(83) $$QE_t = Q_t b_t^{cb}$$

- The consumption of the child

(84) $$C_{b,t} = Q_t b_t$$

- Output gap X_t

(85) $$\ln X_t = \ln Y_t - \ln Y_t^*$$

- A_t and Θ_t

$$\ln A_t =_A \ln A_{t-1} + s_A \epsilon_{A,t} \tag{86}$$
$$\ln \Theta_t =(1-\rho_\theta)\ln \Theta + \rho_\theta \ln \Theta_{t-1} + s_\theta \epsilon_{\theta,t} \tag{87}$$

- Monetary policy rule and the central bank's bond holdings

$$\ln R_t^S =(1-\rho_r)\ln R^S + (1-\rho_r)[\phi_\pi(\ln \Pi_t - \ln \Pi) + \phi_x(\ln Y_t - \ln Y_t^*)] + s_r \epsilon_{r,t} \tag{88}$$
$$\ln QE_t =(1-\rho_q)\ln QE + \rho_q \ln QE_{t-1} + s_q \epsilon_{q,t} \tag{89}$$

B Details of the Linearized Model

$$\chi l_t =-\sigma c_t + w_t \tag{90}$$
$$\lambda_{t-1,t} =-\sigma(c_t - c_{t-1}) \tag{91}$$
$$0 =E_t\lambda_{t,t+1} + r_t^s \tag{92}$$
$$\lambda_{b,t-1,t} =-\sigma(c_{b,t} - c_{b,t}) \tag{93}$$
$$r_t^b =\frac{\kappa}{R^b} q_t - q_{t-1} \tag{94}$$
$$0 =E_t\lambda_{b,t,t+1} + E_t r_{t+1}^b - E_t\pi_{t+1} \tag{95}$$
$$q_t + b_t^{FI} =0 \tag{96}$$
$$s \cdot s_t =Qb^{FI}(b_t^{FI} - \kappa b_{t-1}^{FI} + (1-\kappa)q_t + \kappa\pi_t) + re \cdot re_t \tag{97}$$
$$w_t = E_t\lambda_{t,t+1} - E_t\pi_{t+1} + \frac{R^b}{sp}(E_t r_{t+1}^b - r_t^s) \tag{98}$$
$$r_t^{re} = r_t^s \tag{99}$$
$$p_{*,t} = x_{1,t} - x_{2,t} \tag{100}$$
$$w_t = mc_t + a_t \tag{101}$$
$$(1-z)c_t + zc_{b,t} = y_t \tag{102}$$
$$x_{1,t} = (1-\phi\beta\pi^\epsilon)(mc_t + (1-\sigma)y_t) + \phi\beta\pi^\epsilon(E_t x_{1,t+1} + \epsilon E_t\pi_{t+1}) \tag{103}$$
$$v_t^p = \frac{\epsilon\phi\pi^{\epsilon-1}}{1-\phi\pi^{\epsilon-1}}(\pi-1)\pi_t + \phi\pi^\epsilon v_{t-1}^p \tag{104}$$
$$\pi = \frac{(1-\phi\pi^{\epsilon-1})(1-\phi\beta\pi^\epsilon)}{\phi\pi^{\epsilon-1}} mc_t + \beta(1+\epsilon(\pi-1)(1-\phi\pi^{\epsilon-1}))E_t\pi_{t+1} \tag{105}$$
$$+ \beta(1-\pi)(1-\phi\pi^{\epsilon-1})((1-\sigma)y_t - E_t x_{1,t}) \tag{106}$$
$$v_t^p = 0 \tag{107}$$
$$\pi_t = \frac{1-\phi}{\phi} p_{*,t} \tag{108}$$

(109) $$q_t + b_t^{cb} = re_t$$

(110) $$b_t = \frac{b^{FI}}{b} b_t^{FI} + \frac{b^{cb}}{b} b_t^{cb}$$

(111) $$c_{b,t} = q_t + b_t$$

(112) $$qe_t = \rho_q qe_{t-1} + s_q \epsilon_{q,t}$$

(113) $$a_t = \rho_A a_{t-1} + s_A \epsilon_{A,t}$$

(114) $$\theta_t = \rho_\theta \theta_{t-1} + s_\theta \epsilon_{\theta,t}$$

(115) $$r_t^{re} = \rho_r r_{t-1}^{re} + (1-\rho_r)[\phi_\pi \pi_t + \phi_x x_t] + s_r \epsilon_{r,t}$$

(116) $$qe_t = re_t$$

(117) $$x_t = y_t - y_t^*$$

Marginal cost

(118) $$mc_t = \Big(\chi + \frac{\sigma}{1-z}\Big) y_t + \chi v_t^p - (1+\chi) a_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t]$$

IS curve

(119) $$y_t = E_t y_{t+1} - \frac{1-z}{\sigma}(r_t^s - E_t \pi_{t+1}) - z[\bar{b}^{FI}(E_t \theta_{t+1} - \theta_t) + \bar{b}^{cb}(E_t qe_{t+1} - qe_t)]$$

Monetary policy

(120) $$r_t^s = \rho_r r_{t-1}^s + (1-\rho_r)[\phi_\pi \pi_t + \phi_x x_t] + s_r \epsilon_{r,t}$$

Long bond portfolio

(121) $$qe_t = \rho_q qe_{t-1} + s_q \epsilon_{q,t}$$

Exogenous shocks

(122) $$a_t = \rho_A a_{t-1} + s_A \epsilon_{A,t}$$

(123) $$\theta_t = \rho_\theta \theta_{t-1} + s_\theta \epsilon_{\theta,t}$$

Output gap

(124) $$x_t = y_t - \frac{(1-\chi)(1-z)}{\chi(1-z)+\sigma} a_t$$

Natural output

(125) $$y_t^* = \frac{(1-\chi)(1-z)}{\chi(1-z)+\sigma} a_t$$

Natural rate ($\rho_f = \rho_A$)

$$r_t^* = \frac{\sigma(\rho_A - 1)(1-\chi)}{\chi(1-z)+\sigma} a_t \tag{126}$$

In terms of output gap

$$x_{1,t} = (1-\phi\beta\pi^\epsilon)(mc_t + (1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)} r_t^*) + \phi\beta\pi^\epsilon(E_t x_{1,t+1} + \epsilon E_t \pi_{t+1}) \tag{127}$$

$$v_t^p = \frac{\epsilon\phi\pi^{\epsilon-1}}{1-\phi\pi^{\epsilon-1}}(\pi - 1)\pi_t + \phi\pi^\epsilon v_{t-1}^p \tag{128}$$

$$\begin{aligned} \pi = {} & \frac{(1-\phi\pi^{\epsilon-1})(1-\phi\beta\pi^\epsilon)}{\phi\pi^{\epsilon-1}} mc_t + \beta(1+\epsilon(\pi-1)(1-\phi\pi^{\epsilon-1}))E_t\pi_{t+1} \\ & + \beta(1-\pi)(1-\phi\pi^{\epsilon-1})((1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)} r_t^* - E_t x_{1,t}) \end{aligned} \tag{129}$$

$$mc_t = \left(\chi + \frac{\sigma}{1-z}\right)x_t + \chi v_t^p - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t] \tag{130}$$

IS curve

$$x_t = E_t x_{t+1} - \frac{1-z}{\sigma}(r_t^s - E_t\pi_{t+1} - r_t^*) - z[\bar{b}^{FI}(E_t\theta_{t+1} - \theta_t) + \bar{b}^{cb}(E_t qe_{t+1} - qe_t)] \tag{131}$$

Monetary policy

$$r_t^s = \rho_r r_{t-1}^s + (1-\rho_r)[\phi_\pi \pi_t + \phi_x x_t] + s_r \epsilon_{r,t} \tag{132}$$

Long bond portfolio

$$qe_t = \rho_q qe_{t-1} + s_q \epsilon_{q,t} \tag{133}$$

Three endogenous cost-push effects. One from credit shocks, one from qe shocks and one from trend inflation.

The qe_t term enters in both the IS and Phillips curve. qe_t term enters with a positive sign in the IS relationship and hence serves as a positive demand shocks, but with a negative sign in the Phillips curve. This is because the reallocation of resources from parent to child induces a negative wealth effect for the parent that puts downward pressure on the wage, and hence real marginal cost. Both of these channels make QE expansionary for output but have competing effects on inflation. An expansionary QE shock in the model is nevertheless inflationary, albeit less so than a conventional monetary policy shock.

C Optimal Policy without QE

Consider an operating framework similar to the one in which the central bank uses the short-term interest rate as its sole policy instrument. The ZLB does not bind. This section studies the optimal adjustment of the short-term rate in this scenario. The optimal choice of the policy rate satisfies

$$\pi_t = -\frac{\mu}{\gamma\zeta}x_t \tag{134}$$

This is the same as the lean-against-the-wind-condition for the policy rate for optimal policy under discretion as in the canonical three-equation model. The equilibrium paths of endogenous variables are given by

Proposition 1. *With the short rate being the only policy instrument, the optimal targeting rule is characterized by the above equation. In equilibrium, the paths for inflation, the output gap, and the policy rate are*

$$\pi_t = \varphi\theta_t \tag{135}$$

$$x_t = -\frac{\gamma\zeta}{\mu}\varphi\theta_t \tag{136}$$

$$r_t^s = r_t^* + \eta\theta_t \tag{137}$$

D Optimal Monetary Policy with Trend Inflation

Simplifying assumption $\chi = 0$ (indivisible labor)

$$x_{1,t} = (1-\phi\beta\pi^{\epsilon})(mc_t + (1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}r_t^*) + \phi\beta\pi^{\epsilon}(E_t x_{1,t+1} + \epsilon E_t\pi_{t+1}) \tag{138}$$

$$\begin{aligned}\pi_t = {} & \frac{(1-\phi\pi^{\epsilon-1})(1-\phi\beta\pi^{\epsilon})}{\phi\pi^{\epsilon-1}}mc_t + \beta(1+\epsilon(\pi-1)(1-\phi\pi^{\epsilon-1}))E_t\pi_{t+1} \\ & + \beta(1-\pi)(1-\phi\pi^{\epsilon-1})((1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}r_t^* - E_t x_{1,t})\end{aligned} \tag{139}$$

$$mc_t = \frac{\sigma}{1-z}x_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t] \tag{140}$$

Now define $\tilde{\phi} \equiv \phi\pi^{\epsilon-1}$,

$$x_{1,t} = (1-\tilde{\phi}\beta\pi)(mc_t + (1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}r_t^*) + \tilde{\phi}\beta\pi(E_t x_{1,t+1} + \epsilon E_t\pi_{t+1}) \tag{141}$$

$$\pi_t = \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}mc_t + \beta(1+\epsilon(\pi-1)(1-\tilde{\phi}))E_t\pi_{t+1} \tag{142}$$

$$+ \beta(1-\pi)(1-\tilde{\phi})\Big[(1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}r_t^* - E_t x_{1,t}\Big] \tag{143}$$

Then, define $\psi_t \equiv x_{1,t} + \epsilon\pi_t$

(144)

$$\psi_t = (1-\tilde{\phi}\beta\pi)\Big[(\frac{\sigma}{1-z} + (1-\sigma))x_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t] + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}r_t^*\Big] + \tilde{\phi}\beta\pi E_t\psi_{t+1}$$

(145)

$$\pi_t = \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}\Big(\frac{\sigma}{1-z}x_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t]\Big) + \beta E_t\pi_{t+1}$$

(146)

$$+ \beta(1-\pi)(1-\tilde{\phi})\Big[(1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}r_t^* - E_t\psi_{t+1}\Big]$$

Then Lagrangian is

$$\begin{aligned}\max_{x_t,\pi_t,\psi_t,r_t^s,qe_t} \mathcal{L} = {} & \mu x_t^2 + \pi_t^2 + \lambda_{1,t}\Big(x_t - E_t x_{t+1} + \frac{1-z}{\sigma}(r_t^s - E_t\pi_{t+1} - r_t^*) \\ & + z\big[\bar{b}^{FI}(E_t\theta_{t+1} - \theta_t) + \bar{b}^{cb}(E_t qe_{t+1} - qe_t)\big]\Big) \\ & + \lambda_{2,t}\Big(\pi_t - \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}\Big(\frac{\sigma}{1-z}x_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t]\Big) - \beta E_t\pi_{t+1} \\ & - \beta(1-\pi)(1-\tilde{\phi})\Big[(1-\sigma)x_t + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}r_t^* - E_t\psi_{t+1}\Big] \\ & + \lambda_{3,t}\Big(\psi_t - (1-\tilde{\phi}\beta\pi)\Big[(\frac{\sigma}{1-z} + (1-\sigma))x_t - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t] + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}r_t^*\Big] \\ & - \tilde{\phi}\beta\pi E_t\psi_{t+1}\Big) - \lambda_{4,t}r_t^s - \lambda_{5,t}qe_t\end{aligned} \tag{147}$$

The first order conditions are

$$\frac{\partial\mathcal{L}}{\partial r_t^s} = \lambda_{1,t}\frac{1-z}{\sigma} - \lambda_{4,t} = 0 \tag{148}$$

$$\frac{\partial\mathcal{L}}{\partial qe_t} = -\lambda_{1,t}z\bar{b}^{cb} + \lambda_{2,t}\frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}(1-z)} + \lambda_{3,t}\frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi}\beta\pi)}{1-z} - \lambda_{5,t} = 0 \tag{149}$$

$$\frac{\partial\mathcal{L}}{\partial\pi_t} = 2\pi_t + \lambda_{2,t} = 0 \tag{150}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_t} = {} & 2\mu x_t + \lambda_{1,t} - \lambda_{2,t}\Big[\frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}\frac{\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big] \\ & - \lambda_{3,t}(1-\tilde{\phi}\beta\pi)(\frac{\sigma}{1-z} + (1-\sigma)) = 0\end{aligned} \tag{151}$$

$$\frac{\partial\mathcal{L}}{\partial E\psi_{t+1}} = \beta(1-\pi)(1-\tilde{\phi})\lambda_{2,t} - \tilde{\phi}\beta\pi\lambda_{3,t} = 0 \tag{152}$$

Theorem 2. *It is not possible to completely stabilize both inflation and output gap with both instruments when both credit and natural rate shocks are present.*

Proof. Suppose $x_t = \pi_t = 0$ for all t. Then $\psi_t = E_t\psi_t = 0$ for all t. This implies that $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t - \frac{(1-\sigma)(1-z)^2}{\sigma^2 z(\rho_A-1)\bar{b}^{cb}}r_t^*$. But this does not cancel out r_t^* in GNKPC, hence contradiction.

D.1 Unconstrained Optimal Policy

Suppose a central bank can adjust both r_t^s and qe_t. Then $\lambda_{4,t} = \lambda_{5,t} = 0$. From (148), $\lambda_{1,t} = 0$. From (149),

(153)
$$\lambda_{2,t} = -\frac{\tilde{\phi}}{1-\tilde{\phi}}\lambda_{3,t}$$

Also, from (151),

$$2\mu x_t = \lambda_{2,t}\Big[\frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}\frac{\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma) - \frac{1-\tilde{\phi}}{\tilde{\phi}}(1-\tilde{\phi}\beta\pi)(\frac{\sigma}{1-z} + (1-\sigma))\Big]$$

(154)
$$= \lambda_{2,t}\Big[\beta(\pi-1)(1-\tilde{\phi})(1-\sigma) - \frac{(1-\tilde{\phi})(1-\sigma)}{\tilde{\phi}}\Big]$$

(155)
$$= \lambda_{2,t}\frac{(1-\tilde{\phi})(1-\sigma)(\tilde{\phi}\beta(\pi-1)-1)}{\tilde{\phi}}$$

(156)
$$= -2\frac{(1-\tilde{\phi})(1-\sigma)(\tilde{\phi}\beta(\pi-1)-1)}{\tilde{\phi}}\pi_t,$$

where the last equality is from (150). Therefore, the optimal path is

(157)
$$x_t = -\frac{(1-\tilde{\phi})(1-\sigma)(\tilde{\phi}\beta(\pi-1)-1)}{\mu\tilde{\phi}}\pi_t.$$

Define $\gamma = \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}$.

(158)
$$\omega_0 = \mu^{-1}\Big[\frac{\sigma\gamma}{\pi(1-z)} - \frac{(\pi-1)(1-\tilde{\phi})(1-\sigma)}{\pi\tilde{\phi}}\Big]$$

D.2 Optimal Policy at the ZLB

Suppose the ZLB binds, so that $\lambda_{4,t} \geq 0$, but QE is available, so $\lambda_{5,t} = 0$. From (150),

(159)
$$\lambda_{2,t} = -2\pi_t$$

From (152),

$$\lambda_{3,t} = \frac{(1-\pi)(1-\tilde{\phi})}{\pi\tilde{\phi}}\lambda_{2,t} \tag{160}$$

$$= -2\frac{(1-\pi)(1-\tilde{\phi})}{\pi\tilde{\phi}}\pi_t \tag{161}$$

Define $\gamma = \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}$. From (149)

$$\lambda_{1,t}z\bar{b}^{cb} = \lambda_{2,t}\frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}(1-z)} + \lambda_{3,t}\frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi}\beta\pi)}{1-z} \tag{162}$$

$$= \frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi}\beta\pi)}{1-z}\Big(\frac{(1-\tilde{\phi})}{\tilde{\phi}}\lambda_{2,t} + \lambda_{3,t}\Big) \tag{163}$$

$$= \frac{\sigma z\bar{b}^{cb}(1-\tilde{\phi}\beta\pi)}{1-z}\frac{(1-\tilde{\phi})}{\pi\tilde{\phi}}\lambda_{2,t} \tag{164}$$

$$= \frac{\gamma\sigma z\bar{b}^{cb}}{\pi(1-z)}\lambda_{2,t} \tag{165}$$

$$\lambda_{1,t} = \frac{\gamma\sigma}{\pi(1-z)}\lambda_{2,t} = -\frac{2\gamma\sigma}{\pi(1-z)}\pi_t. \tag{166}$$

From (151),

$$0 = 2\mu x_t + \frac{\gamma\sigma}{\pi(1-z)}\lambda_{2,t} - \lambda_{2,t}\Big[\frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}\frac{\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)]\Big] \tag{167}$$

$$- \lambda_{2,t}\frac{(1-\pi)(1-\tilde{\phi})}{\pi\tilde{\phi}}(1-\tilde{\phi}\beta\pi)(\frac{\sigma}{1-z} + (1-\sigma)) \tag{168}$$

$$0 = 2\mu x_t + 2\pi_t\Big[-\frac{2\gamma\sigma}{\pi(1-z)} + \frac{\sigma(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\pi\tilde{\phi}(1-z)} - \frac{(1-\pi)(1-\tilde{\phi})(1-\sigma)}{\pi\tilde{\phi}}\Big] \tag{169}$$

$$x_t = \omega_0\pi_t \tag{170}$$

(171)

$$\text{where} \quad \omega_0 = -\mu^{-1}\Big[-\frac{2\gamma\sigma}{\pi(1-z)} + \frac{\sigma(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\pi\tilde{\phi}(1-z)} - \frac{(1-\pi)(1-\tilde{\phi})(1-\sigma)}{\pi\tilde{\phi}}\Big]$$

Define $\gamma = \frac{(1-\tilde{\phi})(1-\tilde{\phi}\beta\pi)}{\tilde{\phi}}$.

$$\omega_0 = \mu^{-1}\Big[\frac{\sigma\gamma}{\pi(1-z)} - \frac{(\pi-1)(1-\tilde{\phi})(1-\sigma)}{\pi\tilde{\phi}}\Big] \tag{172}$$

Guess

$$x_t = \omega_1 r_t^* + \omega_2\theta_t \tag{173}$$

$$\pi_t = \omega_0(\omega_1 r_t^* + \omega_2\theta_t) \tag{174}$$

$$E_t x_{t+1} = \rho_f\omega_1 r_t^* + \rho_\theta\omega_2\theta_t \tag{175}$$

$$E_t \pi_{t+1} = \omega_0(\rho_f\omega_1 r_t^* + \rho_\theta\omega_2\theta_t) \tag{176}$$

Also guess the equilibrium path of QE

$$qe_t = \tau r_t^* + \omega_3\theta_t \tag{177}$$

Then, from GNKPC,

$$\begin{aligned}
\omega_0(\omega_1 r_t^* + \omega_2\theta_t) &= \gamma\Big(\frac{\sigma}{1-z}(\omega_1 r_t^* + \omega_2\theta_t) - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}(\tau r_t^* + \omega_3\theta_t)]\Big) + \beta\omega_0(\rho_f\omega_1 r_t^* + \rho_\theta\omega_2\theta_t) \\
&+ \beta(1-\pi)(1-\tilde{\phi})\Big[(1-\sigma)(\omega_1 r_t^* + \omega_2\theta_t) + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)} r_t^* - E_t\psi_{t+1}\Big] \\
-\beta(1-\pi)(1-\tilde{\phi})E_t\psi_{t+1} &= \Big[(1-\beta\rho_f)\omega_0\omega_1 - \frac{\gamma\sigma}{1-z}(\omega_1 - z\bar{b}^{cb}\tau) - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big(\omega_1 + \frac{1-z}{\sigma(\rho_A-1)}\Big)\Big] r_t^* \\
&+ \Big[(1-\beta\rho_\theta)\omega_0\omega_2 - \frac{\gamma\sigma}{1-z}(\omega_2 - z(\bar{b}^{FI} + \bar{b}^{cb}\omega_3)) - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\omega_2\Big]\theta_t
\end{aligned} \tag{178}$$

Thus

$$E_t\psi_{t+1} = \omega_4\rho_f r_t^* + \omega_5\rho_\theta\theta_t \tag{179}$$

where

(180)
$$\omega_4 = -[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_f)\omega_0\omega_1 - \frac{\gamma\sigma}{1-z}(\omega_1 - z\bar{b}^{cb}\tau) - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big(\omega_1 + \frac{1-z}{\sigma(\rho_A-1)}\Big)\Big]$$

(181)
$$\omega_5 = -[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_\theta)\omega_0\omega_2 - \frac{\gamma\sigma}{1-z}(\omega_2 - z(\bar{b}^{FI} + \bar{b}^{cb}\omega_3)) - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\omega_2\Big]$$

Hence

$$\psi_t = \omega_4 r_t^* + \omega_5\theta_t \tag{182}$$

From (144),

$$\begin{aligned}
\omega_4 r_t^* + \omega_5\theta_t &= (1-\tilde{\phi}\beta\pi)\Big[\Big(\frac{\sigma}{1-z} + (1-\sigma)\Big)(\omega_1 r_t^* + \omega_2\theta_t) - \frac{\sigma z}{1-z}[\bar{b}^{FI}\theta_t + \bar{b}^{cb}(\tau r_t^* + \omega_3\theta_t)] + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)} r_t^*\Big] \\
&+ \tilde{\phi}\beta(\omega_4\rho_f r_t^* + \omega_5\rho_\theta\theta_t)
\end{aligned} \tag{183}$$

From IS curve,

$$\omega_1 r_t^* + \omega_2\theta_t = \rho_f\omega_1 r_t^* + \rho_\theta\omega_2\theta_t - \frac{1-z}{\sigma}(-\omega_0(\rho_f\omega_1 r_t^* + \rho_\theta\omega_2\theta_t) - r_t^*) \tag{184}$$

$$- z[\bar{b}^{FI}(\rho_\theta - 1)\theta_t + \bar{b}^{cb}(\tau(\rho_f - 1)r_t^* + \omega_3(\rho_\theta - 1)\theta_t) \tag{185}$$

Matching coefficients

$$\omega_4 = (1-\tilde{\phi}\beta\pi)\Big[\Big(\frac{\sigma}{1-z} + (1-\sigma)\Big)\omega_1 - \frac{\sigma z}{1-z}\bar{b}^{cb}\tau + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}\Big] + \tilde{\phi}\beta\rho_f\omega_4 \tag{186}$$

$$\omega_1 = \rho_f\omega_1 + \frac{1-z}{\sigma}(\omega_0\rho_f\omega_1 + 1) - z\bar{b}^{cb}(\rho_f - 1)\tau \tag{187}$$

$$\omega_5 = (1-\tilde{\phi}\beta\pi)\Big[\Big(\frac{\sigma}{1-z} + (1-\sigma)\Big)\omega_2 - \frac{\sigma z}{1-z}[\bar{b}^{FI} + \bar{b}^{cb}\omega_3]\Big] + \tilde{\phi}\beta\rho_\theta\omega_5 \tag{188}$$

$$\omega_2 = \rho_\theta\omega_2 + \frac{1-z}{\sigma}\omega_0\rho_\theta\omega_2 - z(\bar{b}^{FI} + \bar{b}^{cb}\omega_3)(\rho_\theta - 1) \tag{189}$$

Rewrite ω_4

$$\begin{aligned}\omega_4 &= -[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_f)\omega_0\omega_1 - \frac{\gamma\sigma}{1-z}(\omega_1 - z\bar{b}^{cb}\tau) - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big(\omega_1 + \frac{1-z}{\sigma(\rho_A-1)}\Big)\Big]\\ &= -[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[\Big((1-\beta\rho_f)\omega_0 - \frac{\gamma\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big)\omega_1\end{aligned}$$

(190)

$$+ \frac{\gamma\sigma z\bar{b}^{cb}}{1-z}\tau - \frac{\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)(1-z)}{\sigma(\rho_A - 1)}\Big]$$

Rewrite (186)

$$\begin{aligned}&-(1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[\Big((1-\beta\rho_f)\omega_0 - \frac{\gamma\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big)\omega_1\\ &+ \frac{\gamma\sigma z\bar{b}^{cb}}{1-z}\tau - \frac{\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)(1-z)}{\sigma(\rho_A-1)}\Big]\\ &= (1-\tilde{\phi}\beta\pi)\Big[\Big(\frac{\sigma}{1-z} + (1-\sigma)\Big)\omega_1 - \frac{\sigma z}{1-z}\bar{b}^{cb}\tau + \frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}\Big]\end{aligned} \tag{191}$$

So

$$\begin{aligned}&\Big[(1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big((1-\beta\rho_f)\omega_0 - \frac{\gamma\sigma}{1-z} - \beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big) + (1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z} + (1-\sigma)\Big)\\ &- \Big((1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\beta(1-\pi)(1-\tilde{\phi}) - (1-\tilde{\phi}\beta\pi)\Big)\frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}\end{aligned}$$

(192)

$$= \frac{\sigma z\bar{b}^{cb}}{1-z}\Big((1-\tilde{\phi}\beta\pi) - (1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\gamma\Big)\tau$$

Rewrite (187)

(193) $$(1-\rho_f-\frac{\omega_0\rho_f(1-z)}{\sigma})\omega_1-\frac{1-z}{\sigma}=z\bar{b}^{cb}(1-\rho_f)\tau$$

Combining these two equations and substituting out τ

(194)
$$(1-\rho_f)\Big[(1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big((1-\beta\rho_f)\omega_0-\frac{\gamma\sigma}{1-z}-\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big)$$
$$+(1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\Big]\omega_1$$
$$-(1-\rho_f)\Big((1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\beta(1-\pi)(1-\tilde{\phi})-(1-\tilde{\phi}\beta\pi)\Big)\frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}$$

(195)
$$=\frac{\sigma}{1-z}\Big((1-\tilde{\phi}\beta\pi)-(1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}\gamma\Big)\Big((1-\rho_f-\frac{\omega_0\rho_f(1-z)}{\sigma})\omega_1-\frac{1-z}{\sigma}\Big)$$

Define $\Theta_f\equiv(1-\tilde{\phi}\beta\rho_f)[\rho_f\beta(1-\pi)(1-\tilde{\phi})]^{-1}$

$$\omega_1=\Bigg\{(1-\rho_f)\Big[\Theta_f\Big((1-\beta\rho_f)\omega_0-\frac{\gamma\sigma}{1-z}-\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big)+(1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\Big]$$
$$-\frac{\sigma}{1-z}\Big((1-\tilde{\phi}\beta\pi)-\Theta_f\gamma\Big)(1-\rho_f-\frac{\omega_0\rho_f(1-z)}{\sigma})\Bigg\}^{-1}\times$$

(196)
$$\Bigg\{(1-\rho_f)\Big(\Theta_f\beta(1-\pi)(1-\tilde{\phi})-(1-\tilde{\phi}\beta\pi)\Big)\frac{(1-\sigma)(1-z)}{\sigma(\rho_A-1)}-\Big((1-\tilde{\phi}\beta\pi)-\Theta_f\gamma\Big)\Bigg\}$$

Thus,

(197) $$\tau=[z\bar{b}^{cb}(1-\rho_f)]^{-1}\Big((1-\rho_f-\frac{\omega_0\rho_f}{\sigma})\omega_1+\frac{1-z}{\sigma}\Big)$$

Next, rewrite ω_5

$$\omega_5=-[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_\theta)\omega_0\omega_2-\frac{\gamma\sigma}{1-z}(\omega_2-z(\bar{b}^{FI}+\bar{b}^{cb}\omega_3))-\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\omega_2\Big]$$
$$=-[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_\theta)\omega_0-\frac{\gamma\sigma}{1-z}-\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big]\omega_2$$

(198)
$$-[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\frac{\gamma\sigma z\bar{b}^{cb}}{1-z}\omega_3-[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\frac{\gamma\sigma z\bar{b}^{FI}}{1-z}$$

Rewrite (188)

$$-(1-\tilde{\phi}\beta\rho_\theta)\Bigg([\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\Big[(1-\beta\rho_\theta)\omega_0-\frac{\gamma\sigma}{1-z}+\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big]\omega_2 \tag{199}$$

$$-[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\frac{\gamma\sigma z\bar{b}^{cb}}{1-z}\omega_3+[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}\frac{\gamma\sigma z\bar{b}^{FI}}{1-z}\Bigg) \tag{200}$$

$$=(1-\tilde{\phi}\beta\pi)\Big[\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\omega_2-\frac{\sigma z}{1-z}[\bar{b}^{FI}+\bar{b}^{cb}\omega_3]\Big] \tag{201}$$

Define $\Theta_\theta\equiv(1-\tilde{\phi}\beta\rho_\theta)[\rho_\theta\beta(1-\pi)(1-\tilde{\phi})]^{-1}$

$$\Big[\Theta_\theta\Big[(1-\beta\rho_\theta)-\frac{\gamma\sigma}{1-z}+\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big]+(1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\Big]\omega_2-\frac{\sigma z\bar{b}^{FI}}{1-z}\Big(\Theta_\theta-(1-\tilde{\phi}\beta\pi)\Big)$$
$$=\frac{\sigma z\bar{b}^{cb}}{1-z}\Big((1-\tilde{\phi}\beta\pi)-\Theta_\theta\gamma\Big)\omega_3 \tag{202}$$

Rewrite (189)

$$\Big(1-\rho_\theta-\frac{1-z}{\sigma}\omega_0\rho_\theta\Big)\omega_2+z\bar{b}^{FI}(\rho_\theta-1)=z\bar{b}^{cb}(1-\rho_\theta)\omega_3 \tag{203}$$

Combining these two and substituting out ω_3

$$(1-\rho_\theta)\Big[\Theta_\theta\Big[(1-\beta\rho_\theta)-\frac{\gamma\sigma}{1-z}+\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big]+(1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\Big]\omega_2$$
$$-(1-\rho_\theta)\frac{\sigma z\bar{b}^{FI}}{1-z}\Big(\Theta_\theta-(1-\tilde{\phi}\beta\pi)\Big)$$
$$=\frac{\sigma}{1-z}\Big((1-\tilde{\phi}\beta\pi)-\Theta_\theta\gamma\Big)\Big[\Big(1-\rho_\theta-\frac{1-z}{\sigma}\omega_0\rho_\theta\Big)\omega_2+z\bar{b}^{FI}(\rho_\theta-1)\Big] \tag{204}$$

Then

$$\omega_2=\Bigg\{(1-\rho_\theta)\Big[\Theta_\theta\Big[(1-\beta\rho_\theta)-\frac{\gamma\sigma}{1-z}+\beta(1-\pi)(1-\tilde{\phi})(1-\sigma)\Big]+(1-\tilde{\phi}\beta\pi)\Big(\frac{\sigma}{1-z}+(1-\sigma)\Big)\Big]$$
$$-\frac{\sigma}{1-z}\Big((1-\tilde{\phi}\beta\pi)-\Theta_\theta\gamma\Big)\Big(1-\rho_\theta-\frac{1-z}{\sigma}\omega_0\rho_\theta\Big)\Bigg\}^{-1}\times$$
$$\Bigg\{(1-\rho_\theta)\frac{\sigma z\bar{b}^{FI}}{1-z}\Big(\Theta_\theta-(1-\tilde{\phi}\beta\pi)\Big)+\frac{\sigma z\bar{b}^{FI}}{1-z}(\rho_\theta-1)\Big((1-\tilde{\phi}\beta\pi)-\Theta_\theta\gamma\Big)\Bigg\} \tag{205}$$

Thus

$$\omega_3 = [z\bar{b}^{cb}(1-\rho_\theta)]^{-1}\Big[\Big(1-\rho_\theta-\frac{1-z}{\sigma}\omega_0\rho_\theta\Big)\omega_2 + z\bar{b}^{FI}(\rho_\theta-1)\Big] \tag{206}$$