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Machine Learning  
CS 600.475

## Support Vector Machines

## Algorithm: Perceptron

- Initialize  $w$  and  $\eta$
- On each round
  - Receive example  $x$
  - Predict  $\hat{y} = \text{sign}(w \cdot x)$
  - Receive correct label  $y \in \{+1, -1\}$
  - Suffer loss  $\ell_{0/1}(y, \hat{y})$
  - Update  $w$ :  $w^{i+1} = w^i + \eta y_i x_i$

## Perceptron

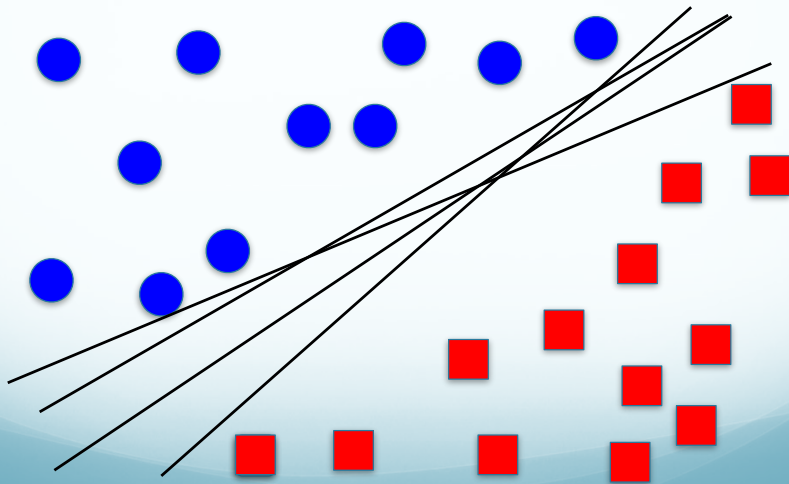
### Fitting a function to data

- Fitting: Stochastic gradient descent
- Function: 0/1 loss with linear function
- Data: Update using a single example at a time

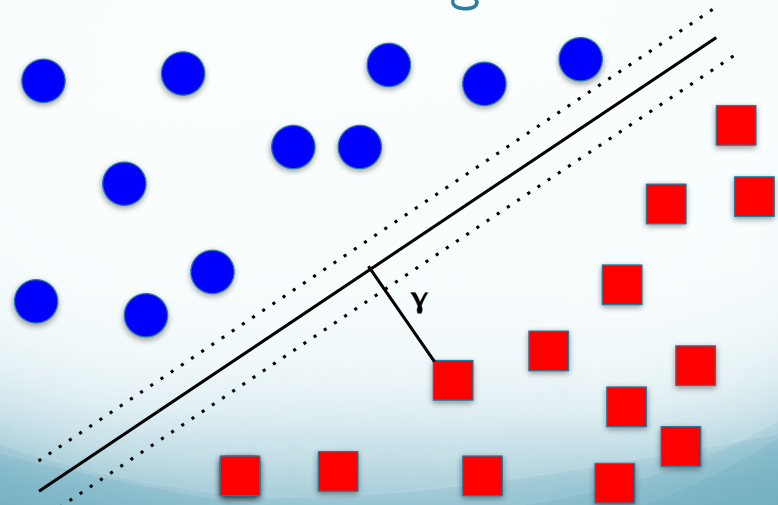
## Questions

- Perceptron picks one separating hyperplane (of many)
  - What would we do if we saw all of the data (batch)?
  - We'd pick the best separating hyperplane!
- Which separating hyperplane is the best?
  - Let's look at the geometric model
- Better solutions for non-linear data?

## Geometric Representation



## The Margin



## Functional Margin

- Prediction and  $y$  should agree to get large margin

$$\hat{\gamma}^i = y_i(w^T x + b)$$

- What if we double  $w$ ?

$$\hat{\gamma}^i = y_i(2w^T x + 2b)$$

- Doubles margin, but no practical change
  - We will address this in a moment

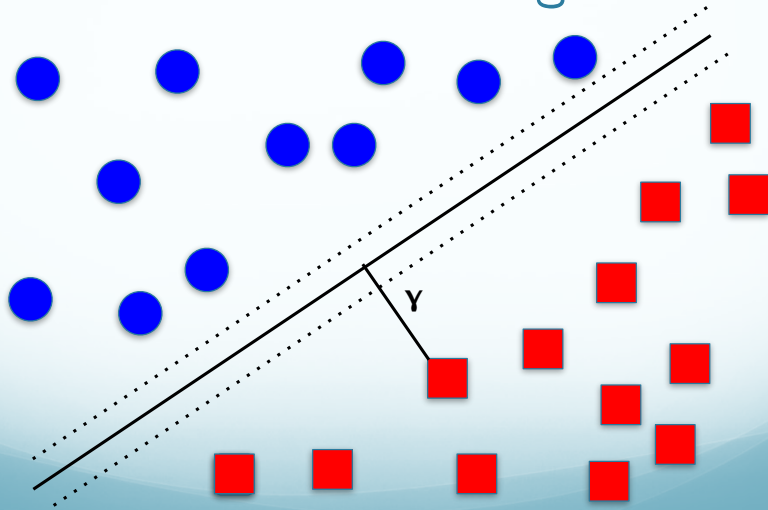
## Functional Margin of Data

- Given a training set of size  $N$ :

- Smallest margin

$$\hat{\gamma} = \min_{i=1, \dots, N} \hat{\gamma}^i$$

## Geometric Margin



## Geometric Margin

- Size of  $\gamma$ ?
- $\frac{w}{\|w\|}$  is a unit length vector pointing in the direction of  $w$

- $\gamma$  intersects with the decision boundary at

$$x_i - \gamma^i \cdot \frac{w}{\|w\|}$$

and points on the boundary must give a prediction of 0

$$\gamma^i = y_i \left( \left( \frac{w}{\|w\|} \right)^T x_i + \frac{b}{\|w\|} \right)$$

if  $\|w\| = 1$  then functional = geometric margin

## Max-Margin Principle

- Assuming the observed data is linearly separable
- Select the hyperplane that separates the data with the maximal margin
- Why?
  - New examples are likely to be close to old examples
  - Gives the best generalization error on new data

## Maximum Geometric Margin

$$\max_{\gamma, w, b} \gamma$$

$$s.t. \quad y_i(w^T x_i + b) \geq \gamma, i = 1, \dots, N$$

$$\|w\| = 1$$

- Every training instance has margin at least  $\gamma$
- $\|w\|$  constraint means geometric = functional margin
- Problem:  $\|w\|$  constraint is non-convex!

## Maximum Geometric Margin

- Functional and geometric related by  $\gamma = \frac{\hat{\gamma}}{\|w\|}$

$$\max_{\hat{\gamma}, w, b} \frac{\hat{\gamma}}{\|w\|}$$

$$\text{s.t. } y_i(w^T x_i + b) \geq \hat{\gamma}, i = 1, \dots, N$$

## Maximum Geometric Margin

- Recall: we can arbitrarily scale w!
- Arbitrarily set  $\gamma = 1$

$$\min_{w, b} \frac{1}{2} \|w\|^2$$

$$\text{s.t. } y_i(w^T x_i + b) \geq 1, i = 1, \dots, N$$

- $\min \|w\|^2$  same as  $\max 1/\|w\|$
- Quadratic program (QP): quadratic objective with linear constraints

## Support Vector Machines

### Fitting a function to data

- Fitting: Batch optimization method: QP solver
- Function: hyperplane with functional margin  $\geq 1$ 
  - New loss function?
- Data: Train in batch mode

## SVM vs. Logistic Regression

- Both minimize the empirical loss with some regularization

- SVM: 
$$\frac{1}{n} \sum_{i=1}^n (1 - y_i [w \cdot x_i])^+ + \lambda \frac{1}{2} \|w\|^2$$

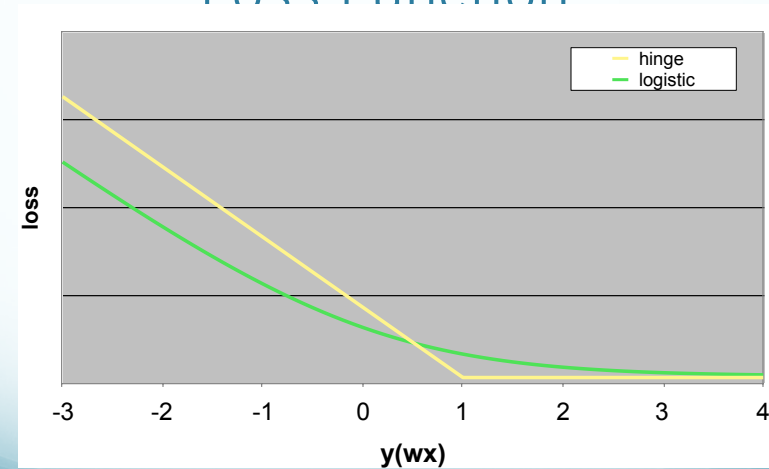
- Logistic: 
$$\frac{1}{n} \sum_{i=1}^n \underbrace{-\log \mathcal{G}(y_i [w \cdot x_i])}_{-P(y_i | x_i, w)} + \lambda \frac{1}{2} \|w\|^2$$

- $(z)^+$  indicates only positive values
- $\mathcal{G}(z) = (1 + \exp(-z))^{-1}$  is the logistic function

## Loss Function

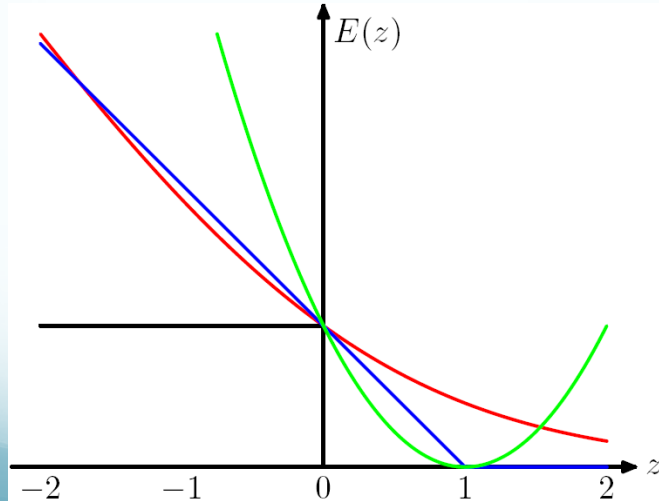
- Both minimize  $\frac{1}{n} \sum_{i=1}^n \ell(y_i [w \cdot x_i]) + \lambda \frac{1}{2} \|w\|^2$
- Different loss functions
- SVM: Hinge Loss  
 $\ell(w, x, y) = \max(0, 1 - y[w \cdot x])$
- Logistic regression: Logistic loss  
 $\ell(w, x, y) = \log(1 + \exp\{-y[w \cdot x]\})$

## Loss Function



Thanks to Jason Eisner for figure

## Rethinking Loss Functions



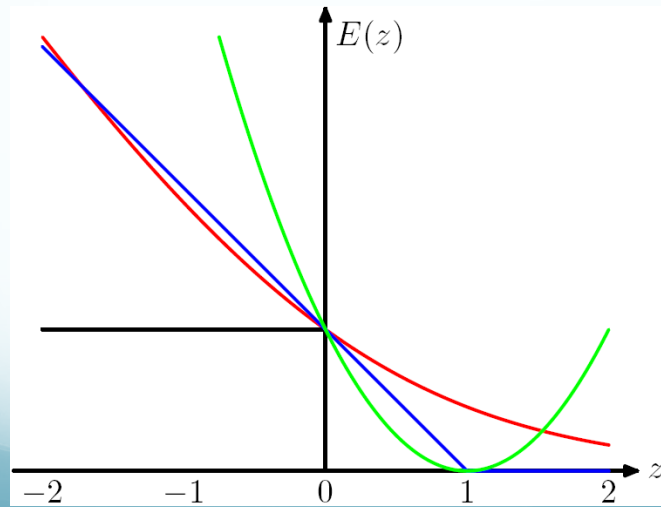
## Perceptron: How to Update?

- How do we update  $w$  to improve our loss?
- Define an error function based on 0/1 loss

$$L_w(y) = \sum_i^N \max(0, -y_i w \cdot x_i)$$

- What is the difference?

## Rethinking Loss Functions



## The Perceptron Connection

- SVM minimizes the Perceptron but goes further
- Perceptron gives local updates, SVM gives global updates
- SVM is more aggressive: max-margin principle
- Could we apply max-margin to online learning?
  - Yes! Perceptron with margin
  - Other methods as well

## Support Vector Machines

### Fitting a function to data

- Fitting: Batch optimization method
- Function: select hyperplane that ensures a fixed margin, L2 regularization
  - Loss: hinge loss
- Data: Train in batch mode

## Another Formulation



## Dual Formulation

- The primal and dual formulations are complimentary
  - Solving one will give the solution for the other
- Primal problem: objective function is a combination of the m variables
  - Minimize the objective function
  - Solution is a vector of m values that minimize function
- Dual problem: objective function is a combination of n variables
  - Maximize the objective function
  - Solution is a vector of n values called the dual variables

## SVM Solution

- Select  $\alpha$ s that maximize
 
$$\sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n y_i y_j \alpha_i \alpha_j (x_i x_j^T)$$
 such that  $\alpha_i \geq 0$  and  $\sum_{i=1}^n \alpha_i y_i = 0$
- Predictions for new examples
 
$$x^T \cdot w = x^T \cdot \sum_{i=1}^n [\alpha_i y_i x_i] = \sum_{i=1}^n \alpha_i y_i (x^T \cdot x_i)$$

## New Approach

### Fitting a function to data

- Fitting: Maximize objective in the dual using a QP solver
- Function: max margin linear classifier
 
$$\hat{y} = \text{sign}(x^T \cdot w) = \text{sign}\left(\sum_{i=1}^n \alpha_i y_i (x^T \cdot x_i)\right)$$
- Data: Train in batch mode

## Dual vs. Primal Formulation

- In the primal we have M variables to solve
  - Solve for the vector  $w$  (length of features)
- In the dual we have N variables to solve
  - Solve for the vector  $\alpha$  (length of examples)
- When to use the primal?
  - Lots of examples without many features
- When to use the dual?
  - Lots of features without many examples
  - Some other reasons (we'll talk about later)

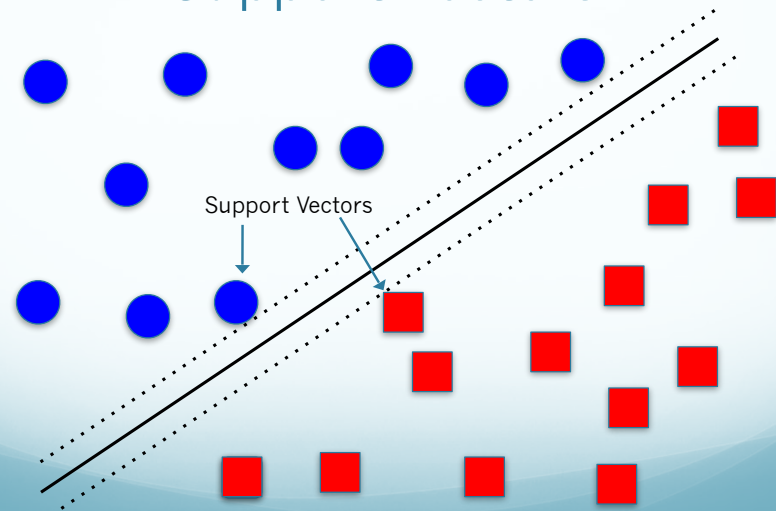
## Support Vectors

- Why is it called support vector machine?
- Only some of the  $\alpha$ s will be non-zero
  - All misclassified examples will be support vectors

$$\sum_{i=1}^n \alpha_i y_i (x^T \cdot x_i)$$

- Only these vector support the hyperplane
- These are the vectors closest to the hyperplane
- These are called “support vectors”

## Support Vectors



## By the Way

- We represented  $w$  in terms of the input  $X$
- $w$  is a *linear combination* of the inputs
  - Before: prediction was linear combination of  $w$  and  $x$

$$w = \sum_{i=1}^n [\alpha_i y_i x_i]$$

- The same is true of Perceptron
  - If we store the support examples

## Dual Perceptron



## Non-Separable Data

- But not all data is linearly separable
  - Previous solution: add a unique feature to every example to make it separable
- What will SVMs do?
  - The regularization forces the weights to be small
  - But it must still find a max margin solution
  - Result: even with significant regularization, still leads to over-fitting

## Slack Variables

$$\min_w \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i$$

such that  $(w x_i) y_i + \xi_i \geq 1, \quad \forall i$   
 $\xi_i \geq 0, \quad \forall i$

- We can always satisfy the margin using  $\xi$ 
  - We want these  $\xi$ s to be small
  - Trade off parameter C (similar to  $\lambda$  before)
- $\xi$ s are called slack variables
  - They cut the margin some “slack”

## Non-Separable Solution

- Similar form to the separable solution
- Extra term added to objective

## Bias vs. Variance

- Smaller C means more slack (larger  $\xi$ )
  - More training examples are wrong
  - More bias (less variance) in the output
- Larger C means less slack (smaller  $\xi$ )
  - Better fit to the data
  - Less bias (more variance) in the output
- For non-separable data we can't learn a perfect separator so we don't want to try too hard
  - Finding the right balance is a tradeoff

## Lingering Questions

- What would we do if we saw all of the data (batch)?
  - We'd pick the best separating hyperplane!
- Which separating hyperplane is the best?
  - The maximum margin separator
  - Use a quadratic regularizer on the weights
- What can we do for non-linear data?
  - It's not separable, use slack variables
  - Can we do better?

## Next Time

Kernel Methods and  
Non-Linear Support Vector Machines