Name

$M\sharp$

Definitions

- An <u>Ordinary Differential Equation</u> ODE is an equation containing ordinary (not partial) derivatives.
  - We need to find and plot a function $y(t)$ that solves (or approximately solves) the ODE.
  - An ODE describing a physical situation is an <u>ODE model</u>.
  - A constant solution of an ODE model is an <u>equilibrium</u> solution of the ODE.
  - Almost all science and engineering is described by ODE models. They are flexible, cheap and accurate.
  - The ODE model describes how the system changes. They need <u>Initial Conditions</u> IC that say how the system starts.
  - An <u>Initial Value Problem</u> IVP is an ODE with a suitable IC to give a single solution.
  - Almost all interesting ODEs are <u>systems</u> because they involve more than one unknown function.
  - An ODE is <u>autonomous</u> if time t does not appear explicitly.
- The <u>order</u> of an ODE is the highest derivative appearing.
  - ODEs describing water tank mixing are 1st order.
    - Suitable ICs are the state of the tanks at $t = 0$.
  - Most ODEs describing motion are second order. The a in $f = m\,a$ is a second derivative!
    - Suitable ICs are all positions and velocities at $t = 0$.
- An ODE is <u>non-linear</u> in y if there are any nonlinear terms in y and its derivatives e.g. y^3, sin(y), $y\,y'$, y_1^2, etc.
- An ODE is <u>linear</u> in y if there are no nonlinear terms in y and its derivatives. Note: $t^2\,y''$ is linear in y
  - Nonlinear ODE models can be approximated by linear ODE models.
  - Any first order linear ode can be written as $a_1(t)\,y' + a_0(t)\,y = b(t)$
  - Standard form is $y' + p(t)\,y = f$
  - You know how to solve the <u>homogeneous</u> linear ODE $y' + p(t)\,y = 0$.

Models

- Solutions of IVPs can be plotted. Our first two examples of ODE models are
  - Mixing in Tanks https://xkcd.com/2974/
    - $y' = A\,y + f$ with $y(0) = y_0$ given.

- Probes in Earth-Moon system. CR3BP https://orbital-mechanics.space/the-n-body-problem/circular-restricted-three-body-problem.html

    - With $M = 1/82.45$ and $E = 1 - M$ and $r_1(x, y) = \sqrt{(x + M)^2 + y^2}$ and $r_2 = \sqrt{(x - E)^2 + y^2}$

        $$x'' = 2\,y' + x - \frac{E(x+M)}{r_1^3} - \frac{M(x-E)}{r_2^3}$$
        $$y'' = -2\,x' + y - \frac{E\,y}{r_1^3} - \frac{M\,y}{r_2^3}$$
        with $y(0), x(0), x'(0), y'(0)$ given

Linear Algebra Review

- Linear algebra is the language of ODE systems.

    - You need understand what solving a linear system $A\,x = b$ means.

        - You need to be able to verify solutions.

        - You need to understand how to compute <u>all</u> solutions.
          If $A\,x_p = b$ and $\{x_1, x_2\}$ is a basis for Null(A) then $x = c_1\,x_1 + c_2\,x_2 + x_p$

    - You need understand what $A\,v = \lambda\,v$ means.

        - You need to be able to verify partial solutions.

        - You need to be able to compute eigenvalues and eigenvectors.

2.3: First Order Linear ODE

- A <u>separable</u> scalar ODE has the form $y'(t) = g(t)\,h(y)$.

    You solved these in Calculus II by separating variables and integrating
    $$\mathrm{dy}/\mathrm{dt} = g(t)\,h(y) \implies \int\frac{\mathrm{dy}}{h(y)} = \int g(t)\,\mathrm{dt} + C$$
    An initial condition $y(0) = y_0$ would let you solve for the one constant of integration.

- A <u>linear</u> 1st order ODE has the standard form $y'(t) + p(t)\,y(t) = f(t)$.

    - Separable if $f(t) = 0$

        - $\frac{\mathrm{dy}}{\mathrm{dt}} + p(t)\,y = 0 \implies \int\frac{\mathrm{dy}}{y} = -\int p(t)\,\mathrm{dt} \implies \ln(y) = -\int p(t)\,\mathrm{dt} + C \implies y = y_0\,e^{-\int_0^t p(\tau)\,\mathrm{d}\tau}$

    - Multiply by the <u>integrating factor</u> (IF) $v(t) = e^{\int_0^t p(\tau)\,\mathrm{d}\tau}$ satisfies $v'(t) = v(t)\,p(t)$.

        - $v(t)\,y'(t) + v(t)\,p(t)\,y(t) = v(t)\,f(t) \implies v(t)\,y'(t) + v'(t)\,y(t) = v(t)\,f(t) \implies (v(t)\,y(t))' = v(t)\,f(t)$

        - Integrating gives $v(t)\,y(t) = \int v(t)\,f(t)\,\mathrm{dt} + C \implies y(t) = \frac{1}{v(t)}\left(C + \int v(t)\,f(t)\,\mathrm{dt}\right)$

- Summary: To solve $y'(t) + p(t)\,y = f(t)$ compute the IF $v(t) = e^{\int_0^t p(\tau)\,\mathrm{d}\tau}$ and solve the separable ODE $(v\,y)' = v\,f$.