Name	$M\#$

Definitions

- An Ordinary Differential Equation ODE is an equation containing ordinary (not partial) derivatives.
  - We need to find and plot a function $y(t)$ that solves (or approximately solves) the ODE.
  - An ODE describing a physical situation is an ODE model.
  - A constant solution of an ODE model is an equilibrium solution of the ODE.
  - Almost all science and engineering is described by ODE models. They are flexible, cheap and accurate.
  - The ODE model describes how the system changes. They need Initial Conditions IC that say how the system starts.
  - An Initial Value Problem IVP is an ODE with a suitable IC to give a single solution.
  - Almost all interesting ODEs are systems because they involve more than one unknown function.
  - An ODE is autonomous if time t does not appear explicitly.
- The order of an ODE is the highest derivative appearing.
  - ODEs describing water tank mixing are 1st order.
    - Suitable ICs are the state of the tanks at $t = 0$.
  - Most ODEs describing motion are second order. The a in $f = m\,a$ is a second derivative!
    - Suitable ICs are all positions and velocities at $t = 0$.
- An ODE is non-linear in y if there are any nonlinear terms in y and its derivatives e.g. y^3, sin(y), $y\,y'$, y_1^2, etc.
- An ODE is linear in y if there are no nonlinear terms in y and its derivatives. Note: $t^2\,y''$ is linear in y
  - Nonlinear ODE models can be approximated by linear ODE models.
  - Any first order linear ode can be written as $a_1(t)\,y' + a_0(t)\,y = b(t)$
  - Standard form is $y' + p(t)\,y = f$
  - You know how to solve the homogeneous linear ODE $y' + p(t)\,y = 0$.

Models

- Solutions of IVPs can be plotted. Our first two examples of ODE models are
  - Mixing in Tanks https://xkcd.com/2974/
    - $y' = A\,y + f$ with $y(0) = y_0$ given.

- Probes in Earth-Moon system. CR3BP https://orbital-mechanics.space/the-n-body-problem/circular-restricted-three-body-problem.html
  - With $M = 1/82.45$ and $E = 1 - M$ and $r_1(x, y) = \sqrt{(x+M)^2 + y^2}$ and $r_2 = \sqrt{(x-E)^2 + y^2}$
    $\begin{aligned} x'' &= 2y' + x - \frac{E(x+M)}{r_1^3} - \frac{M(x-E)}{r_2^3} \\ y'' &= -2x' + y - \frac{E\,y}{r_1^3} - \frac{M\,y}{r_2^3} \end{aligned}$ with $y(0)$, $x(0)$, $x'(0)$, $y'(0)$ given

Linear Algebra Review

- Linear algebra is the language of ODE systems.
  - You need understand what solving a linear system $A\,x = b$ means.
    - You need to be able to verify solutions.
    - You need to understand how to compute all solutions.
      If $A\,x_p = b$ and $\{x_1, x_2\}$ is a basis for Null(A) then $x = c_1\,x_1 + c_2\,x_2 + x_p$
  - You need understand what $A\,v = \lambda\,v$ means.
    - You need to be able to verify partial solutions.
    - You need to be able to compute eigenvalues and eigenvectors.

2.3: First Order Linear ODE

- A separable scalar ODE has the form $y'(t) = g(t)\,h(y)$.

  You solved these in Calculus II by separating variables and integrating

  $dy/dt = g(t)\,h(y) \Longrightarrow \int \frac{dy}{h(y)} = \int g(t)\,dt + C$

  An initial condition $y(0) = y_0$ would let you solve for the one constant of integration.
- A linear 1st order ODE has the standard form $y'(t) + p(t)\,y(t) = f(t)$.
  - Separable if $f(t) = 0$
    - $\frac{dy}{dt} + p(t)\,y = 0 \Longrightarrow \int \frac{dy}{y} = -\int p(t)\,dt \Longrightarrow \ln(y) = -\int p(t)\,dt + C \Longrightarrow y = y_0\,e^{-\int_0^t p(\tau)\,d\tau}$
  - Multiply by the integrating factor (IF) $v(t) = e^{\int_0^t p(\tau)\,d\tau}$ satisfies $v'(t) = v(t)\,p(t)$.
    - $v(t)\,y'(t) + v(t)\,p(t)\,y(t) = v(t)\,f(t) \Longrightarrow v(t)\,y'(t) + v'(t)\,y(t) = v(t)\,f(t) \Longrightarrow (v(t)\,y(t))' = v(t)\,f(t)$
    - Integrating gives $v(t)\,y(t) = \int v(t)\,f(t)\,dt + C \Longrightarrow y(t) = \frac{1}{v(t)}\left(C + \int v(t)\,f(t)\,dt\right)$
- Summary: To solve $y'(t) + p(t)\,y = f(t)$ compute the IF $v(t) = e^{\int_0^t p(\tau)\,d\tau}$ and solve the separable ODE $(v\,y)' = v\,f$.

More Definitions

- The general solution of an ODE is a parameterized family of solutions that contains all possible solutions.

- General solutions of 1st order ODEs contain at least one parameter which is fixed by an IC $y(t_0) = y_0$
- General solutions of 2nd order ODEs contain at least two parameter which are fixed by an IC $y(t_0) = y_0$ and $y'(t_0) = v_0$.

- The general solution of the 1st order linear ODE $y' + p(t)\,y = f$ is
$$y(t) = \frac{1}{v(t)}\left(C + \int v(t)\,f(t)\,\mathrm{dt}\right)$$

2.5: Nonlinear ODEs

- 2.5.1 Separable $y' = g(t)\,h(y)$. See above. For a general solution you might need to worry about dividing by zero.
- 2.5.2 A 1st order ODE is <u>exact</u> if it has the form $\frac{\partial f}{\partial t}\,\mathrm{dt} + \frac{\partial f}{\partial y}\,\mathrm{dy} = 0$. The general solution is $f(t, y) = C$.
  - This is only useful if you can tell $M(t, y)\,\mathrm{dt} + N(t, y)\,\mathrm{dy} = 0$ is exact.
    - $M(t, y)\,\mathrm{dt} + N(t, y)\,\mathrm{dy} = 0$ is $\frac{\partial f}{\partial t}\,\mathrm{dt} + \frac{\partial f}{\partial y}\,\mathrm{dy} = 0$ if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$
  - Once you know it is exact you need to compute f from $\frac{\partial f}{\partial t} = M$ and $\frac{\partial f}{\partial y} = N$.
    - $\frac{\partial f}{\partial t} = M \Longrightarrow f(t, y) = \int M(t, y)\,\mathrm{dt} + h(y)$. As far as t is concerned any function $h(y)$ is a constant!
    - $f(t, y) = \int M(t, y)\,\mathrm{dt} + h(y) \Longrightarrow N(t, y) = \frac{\partial}{\partial y}\left[\int M(t, y)\,\mathrm{dt} + h(y)\right] \Longrightarrow h'(y) = N(t, y) - \frac{\partial}{\partial y}\left[\int M(t, y)\,\mathrm{dt}\right]$.
    - Unless you messed up $h'(y)$ is independent of t and integrating gives you $h(y)$. The general solution is $f(t, y) = \int M(t, y)\,\mathrm{dt} + h(y) = C$