# PIM-TMR convergence

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# Model & objective

- **Output amplitudes $B$ from linear system with transfer matrix $X$ with test entries $A$:**

$$B = |A \cdot X|$$

$$\begin{cases} B \in {\mathbb{R}_*^+}^{N \times m} \\ A \in \mathbb{C}^{N \times n} \\ X \in \mathbb{C}^{n \times m} \end{cases}$$

  - $N$: number of measurements
  - $n$: number of degrees of freedom
  - $m$: number of sensors

- **Retrieve the system transfer matrix $X$ by minimizing using PIM-TMR:**

$$\hat{X} = \min_{X \in \mathbb{C}^{n \times m}} \|B^2 - |A \cdot X|^2\|^2$$

# Validation metric

- **Pearson coefficients statistics between each rows of synthetic measurement matrices:**

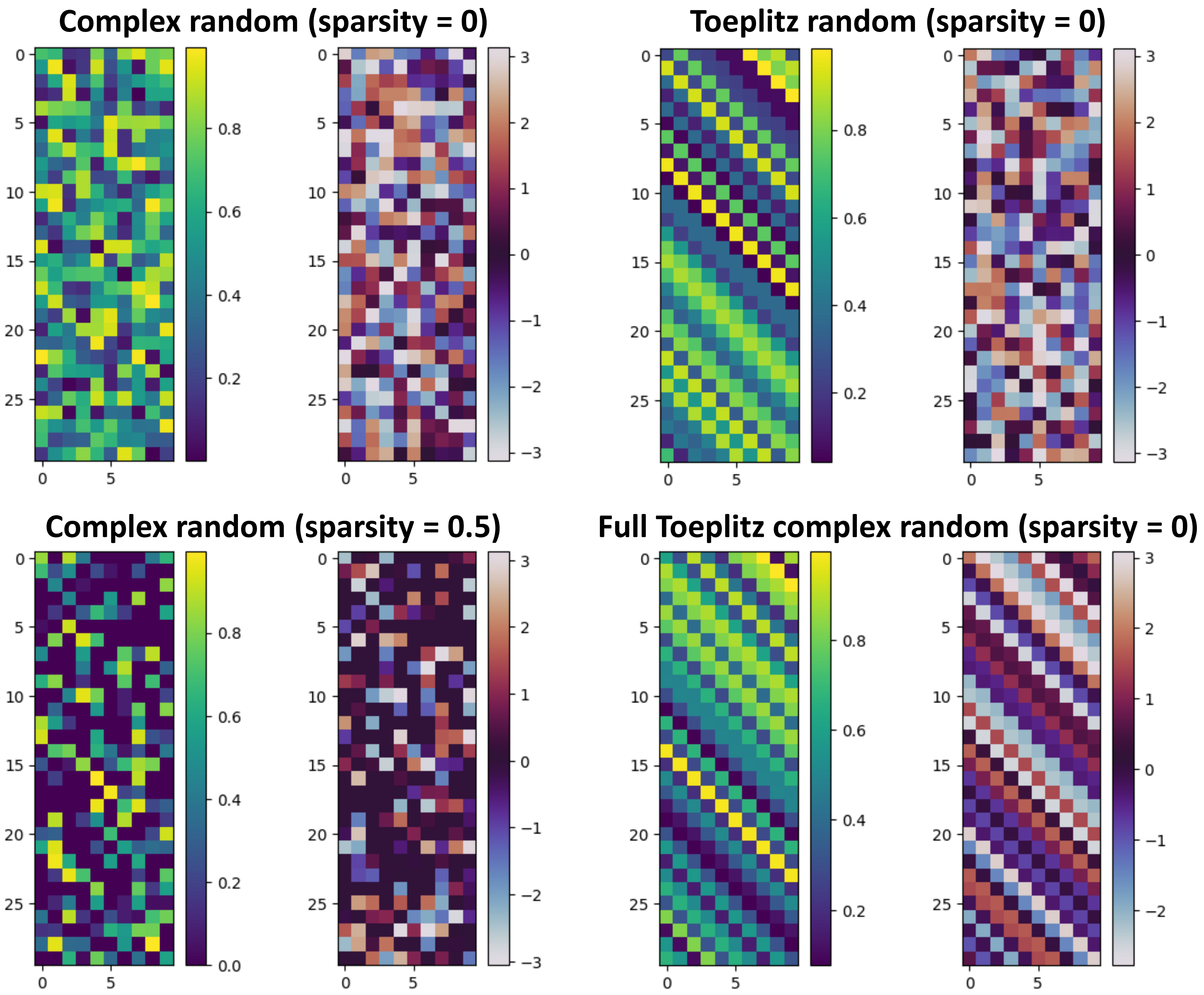
$$\text{Pearson} = \Gamma\left(|A_{val} \cdot X|,^2 \ \left|A_{val} \cdot \hat{X}\right|^2\right)$$

$$\begin{cases} A_{val} \in \mathbb{C}^{N_{val} \times n} \\ X \in \mathbb{C}^{n \times m} \\ \hat{X} \in \mathbb{C}^{n \times m} \end{cases}$$

- $N_{val}$: number of validation tests
- $n$: number of degrees of freedom
- $m$: number of sensors

# Tested system transfer matrices types and examples

- **Random(complex or not, sparsity)**
- **Random_toeplitz(complex or not, sparsity, toeplitz_phase)**
- **Diagonal_random(complex or not)**



# Results without noise

**System hyperparameters:**

$N = 1000$

$n = 30$

$m = 4n$

**Validation hyperparameters:**

$A_{val}$: Complex random

$N_{val} = 10^4$

**Sparsity of $A_{val} = 0$**

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 99,99649 | 0,00143 |
| | Yes | | 99,99897 | 0,00031 |
| | No | 0,9 | 99,99939 | 0,00025 |
| | Yes | | 99,99964 | 0,00014 |
| Diag random | No | 0 | 99,99989 | 0,00008 |
| | Yes | | 99,99984 | 0,00011 |
| Random toeplitz | No | 0 | 99,99792 | 0,00084 |
| | Yes | | 99,99930 | 0,00023 |
| | No | 0,9 | 99,99947 | 0,00021 |
| | Yes | | 99,99955 | 0,00019 |
| Full random toeplitz | Yes | 0 | 99,99967 | 0,00015 |
| | | 0,9 | 99,99965 | 0,00015 |

**Sparsity of $A_{val} = 0,5$**

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 99,99705 | 0,00127 |
| | Yes | | 99,99912 | 0,00029 |
| | No | 0,9 | 99,99963 | 0,00017 |
| | Yes | | 99,99993 | 0,00005 |
| Diag random | No | 0 | 99,99997 | 0,00003 |
| | Yes | | 99,99999 | 0,00001 |
| Random toeplitz | No | 0 | 99,99821 | 0,00081 |
| | Yes | | 99,99934 | 0,00022 |
| | No | 0,9 | 99,99943 | 0,00025 |
| | Yes | | 99,99974 | 0,00014 |
| Full random toeplitz | Yes | 0 | 99,99975 | 0,00013 |
| | | 0,9 | 99,99974 | 0,00012 |

# Adding noise to the sensor measurements

**1) Add noise to normalized sensor measurements:**

$$B^2 = |A \cdot X|^2$$

$$B_\sigma \leftarrow \sqrt{\left|\frac{B^2}{\max(B^2)} + \mathcal{N}(0,\sigma)\right|}$$

$$\begin{cases} B \in \mathbb{R}_*^{+N\times m} \\ A \in \mathbb{C}^{N\times n} \\ X \in \mathbb{C}^{n\times m} \end{cases}$$

- $N$: number of measurements
- $n$: number of degrees of freedom
- $m$: number of sensors
- $\mathcal{N}(0,\sigma)$: Normal distribution with 0 mean and $\sigma$ standard deviation

**2) Retrieve the system transfer matrix $X$ from noisy measurements by minimizing using PIM-TMR:**

$$\hat{X} = \min_{X \in \mathbb{C}^{n\times m}} \|B_\sigma^2 - |A \cdot X|^2\|^2$$

**3) Evaluate Pearson coefficients statistics between each rows of synthetic measurement matrices:**

$$\text{Pearson} = \Gamma\left(|A_{val} \cdot X|,^2\ \left|A_{val} \cdot \hat{X}\right|^2\right)$$

$$\begin{cases} A_{val} \in \mathbb{C}^{N_{val}\times n} \\ X \in \mathbb{C}^{n\times m} \\ \hat{X} \in \mathbb{C}^{n\times m} \end{cases}$$

- $N_{val}$: number of validation tests
- $n$: number of degrees of freedom
- $m$: number of sensors

# Results with noise ($\sigma = 1\%$)

**System hyperparameters:**

$N = 1000$

$n = 30$

$m = 4n$

**Validation hyperparameters:**

$A_{val}$: Complex random

$N_{val} = 10^4$

**Sparsity of $A_{val} = 0$**

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 82,31506 | 5,44311 |
| | Yes | | 91,87208 | 2,12708 |
| | No | 0,9 | 96,31937 | 1,37143 |
| | Yes | | 95,42496 | 1,51860 |
| Diag random | No | 0 | 99,53497 | 0,30716 |
| | Yes | | 99,25404 | 0,41958 |
| Random toeplitz | No | 0 | 81,42050 | 5,64541 |
| | Yes | | 91,52210 | 2,15695 |
| | No | 0,9 | 95,11162 | 1,63732 |
| | Yes | | 96,00484 | 1,47417 |
| Full random toeplitz | Yes | 0 | 91,89604 | 1,98125 |
| | | 0,9 | 95,55366 | 1,51494 |

**Sparsity of $A_{val} = 0,5$**

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 83,35598 | 5,53342 |
| | Yes | | 92,08828 | 2,04858 |
| | No | 0,9 | 97,30383 | 1,16532 |
| | Yes | | 96,89635 | 1,22800 |
| Diag random | No | 0 | 99,75550 | 0,28291 |
| | Yes | | 99,74158 | 0,22156 |
| Random toeplitz | No | 0 | 83,13876 | 5,42849 |
| | Yes | | 91,68044 | 2,08767 |
| | No | 0,9 | 96,72541 | 1,32041 |
| | Yes | | 97,23001 | 1,17062 |
| Full random toeplitz | Yes | 0 | 91,89434 | 1,98159 |
| | | 0,9 | 96,85306 | 1,25997 |

# Results with noise ($\sigma = 5\%$)

**System hyperparameters:**

$N = 1000$

$n = 30$

$m = 4n$

**Validation hyperparameters:**

$A_{val}$: Complex random

$N_{val} = 10^4$

## Sparsity of $A_{val} = 0$

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 85,82103 | 4,27604 |
| | Yes | | 94,33290 | 1,48698 |
| | No | 0,5 | 93,64501 | 1,91524 |
| | Yes | | 94,56029 | 1,49570 |
| | No | 0,9 | 87,16897 | 5,87542 |
| | Yes | | 94,88350 | 1,88161 |
| Diag random | No | 0 | 84,38203 | 13,39941 |
| | Yes | | 94,44914 | 4,58982 |
| Random toeplitz | No | 0 | 85,32503 | 4,47376 |
| | Yes | | 94,37424 | 1,43777 |
| | No | 0,5 | 93,24500 | 2,11597 |
| | Yes | | 94,25724 | 1,54908 |
| | No | 0,9 | 94,02103 | 2,22309 |
| | Yes | | 29,06591 | 26,87297 |
| Full random toeplitz | Yes | 0 | 93,31872 | 1,67630 |
| | | 0,5 | 94,64285 | 1,42927 |
| | | 0,9 | 15,72281 | 13,26665 |

## Sparsity of $A_{val} = 0,5$

| Transfer matrix type | Complex | Sparsity | $\Gamma$ [%] | |
|---|---|---|---|---|
| | | | Avg | Std |
| Random | No | 0 | 87,52097 | 4,12806 |
| | Yes | | 93,32831 | 1,72449 |
| | No | 0,5 | 93,46468 | 2,26313 |
| | Yes | | 94,96606 | 1,41266 |
| | No | 0,9 | 42,92069 | 30,40649 |
| | Yes | | 50,32495 | 29,30658 |
| Diag random | No | 0 | 96,71722 | 5,21080 |
| | Yes | | 95,56515 | 7,37931 |
| Random toeplitz | No | 0 | 86,96411 | 4,30084 |
| | Yes | | 94,48977 | 1,39627 |
| | No | 0,5 | 93,72736 | 2,16414 |
| | Yes | | 95,44176 | 1,30561 |
| | No | 0,9 | 39,35333 | 28,35438 |
| | Yes | | 0,76667 | 6,57408 |
| Full random toeplitz | Yes | 0 | 94,00877 | 1,47650 |
| | | 0,5 | 95,41420 | 1,29226 |
| | | 0,9 | 16,86728 | 16,10311 |

# Conclusion and notes

- **PIM-TMR does not care of system matrix type in the noiseless case !**
- **Sparsity of the validation input matrix does not significantly change results, even in the noisy case**
- **Complex system transfer matrices seem more robust to noise**
- **Extreme sparsity (90%) of the system transfer matrices tends to reduce the metric result**

- **Statistics should be obtained from several draws of transfer matrices within the same matrix type**
- **Hyperparameters should be varied**