# Stat 154: Elementary Statistics

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Ch 3: **Inferential Statistics**

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# Overview

# Definitions

- **Inferential Statistics**: Drawing conclusioins about a population based on the information from a sample of the population.
- **Experiment**: Conducting an activity in generating data. It could be just gathering information or performing an experiment.
- **Example 3.3**: Toss a fair coin three times and observe the sequences of heads and tails.

$$S = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

- **Sample Sapce (also called Outcome Space)**: The set of all possible outcomes of an experiment.
- **Event**: Any subset or part of a sample space.
- **Simple Event**: Each individual element of a sample space.
- **Compound Event**: Any event including two or more simple events.
- **Probability**: A notion of chance or likelihood is known as probability. The probability values are always between zero and one, inclusive. The probability of the sample space is one and the probability of an unlikely or improbable or impossible event is zero.

# Basic Approaches to Computing Probability

- **Approach # 1**: Subjective Probability $P(A)$ - the probability of event $A$, is estimated by using an individual's personal judgment of the relevant circumstances.
- **Approach # 2**: Relative Frequency Approximation of Probability - Conduct (or observe) a procedure $n$ times, and the number of times event $A$ actually occurs is $s$. Based on these actual results, $P(A)$ is approximated as

$$P(A) \approx \frac{s}{n}.$$

- **Approach # 3**: Classical Approach to Probability (it requires pre-assumed chances of the outcomes) - Assume that a given procedure has $n$ different simple events and that each of those simple events has an equal chance of occuring. If event $A$ can occur in $s$ of these $n$ ways, then

$$P(A) = \frac{s}{n}.$$

# Relationships among Events

- **Subset/ Subevent**: A subset is a portion of a set. $A$ is a subset of $B$ (written as $A \subseteq B$) if and only if every member of $A$ is a member of $B$, which means that the occurrence of $A$ will lead to the occurrence of $B$.

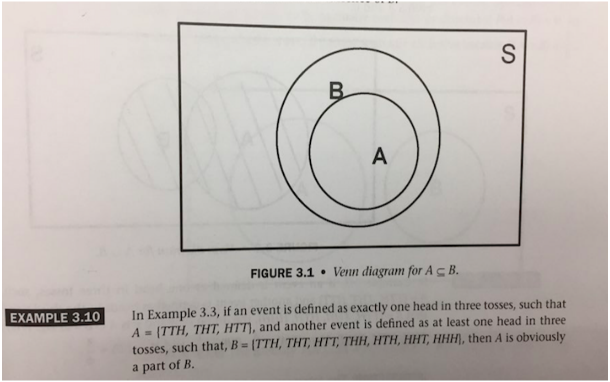



**Figure:** Venn diagram for $A \subseteq B$

# Relationships among Events

- **Complement**: A complement is the opposite event or the event that does not include the elements of the event of interest but includes the rest of the elements of the sample space.
- Let us consider $\overline{A}$ the complement event of $A$.

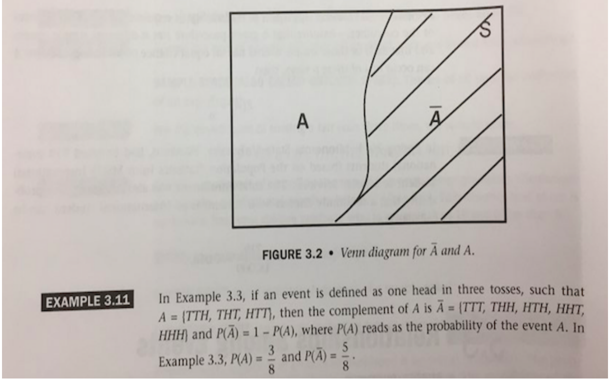



**FIGURE 3.2** • *Venn diagram for $\bar{A}$ and A.*

**EXAMPLE 3.11** In Example 3.3, if an event is defined as one head in three tosses, such that $A = \{TTH, THT, HTT\}$, then the complement of $A$ is $\bar{A} = \{TTT, THH, HTH, HHT, HHH\}$ and $P(\bar{A}) = 1 - P(A)$, where $P(A)$ reads as the probability of the event $A$. In Example 3.3, $P(A) = \frac{3}{8}$ and $P(\bar{A}) = \frac{5}{8}$.

**Figure:** Venn diagram for $\overline{A}$ the complement event of $A$

# Relationships among Events

- **Union**: The union of two events is the event that includes the elements of either of the events or both of the events.
- Let us consider $A \cup B$ to be the union of the events $A$ and $B$.

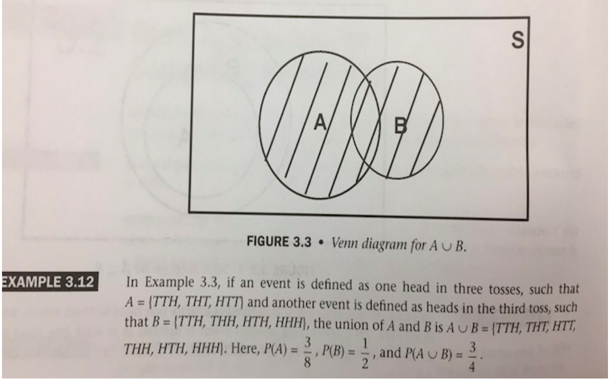



**FIGURE 3.3** • *Venn diagram for $A \cup B$.*

**EXAMPLE 3.12** In Example 3.3, if an event is defined as one head in three tosses, such that $A = \{TTH, THT, HTT\}$ and another event is defined as heads in the third toss, such that $B = \{TTH, THH, HTH, HHH\}$, the union of $A$ and $B$ is $A \cup B = \{TTH, THT, HTT, THH, HTH, HHH\}$. Here, $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$, and $P(A \cup B) = \frac{3}{4}$.

**Figure:** Venn diagram for $A \cup B$

# Relationships among Events

- **Intersection**: The intersection of two events is the event that includes the elements common in both the events.
- Let us consider $A \cap B$ to be the union of the events $A$ and $B$.

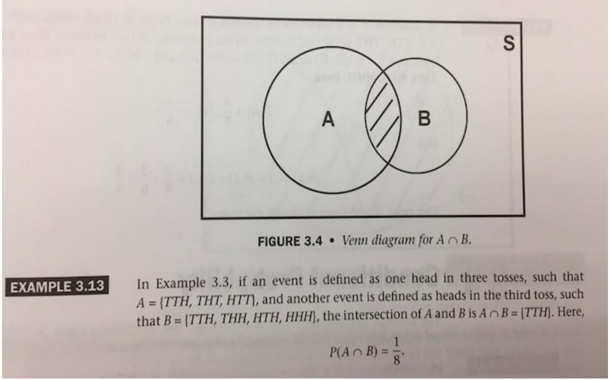



FIGURE 3.4 • *Venn diagram for $A \cap B$.*

**EXAMPLE 3.13** In Example 3.3, if an event is defined as one head in three tosses, such that $A = \{TTH, THT, HTT\}$, and another event is defined as heads in the third toss, such that $B = \{TTH, THH, HTH, HHH\}$, the intersection of $A$ and $B$ is $A \cap B = \{TTH\}$. Here,

$$P(A \cap B) = \frac{1}{8}.$$

**Figure:** Venn diagram for $A \cap B$

# Relationships among Events

- **Mutually Exclusive or Disjoint**: Events are known as mutually exclusive or disjoint when there is no common element(s) among the events.

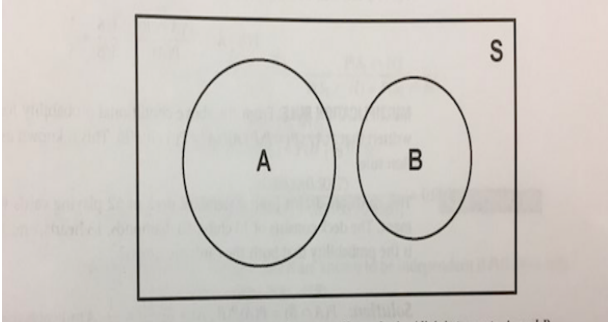



FIGURE 3.5 • *Venn diagram for mutually exclusive/disjoint events A and B.*

**Figure:** Venn diagram for mutually exclusive/disjoint events $A$ and $B$

# Relationships among Events

- **Addition Rule**: The identity $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ is known as the addition rule.
- When the events are mutually exclusive or disjoint, $P(A \cup B) = P(A) + P(B)$.

**EXAMPLE 3.15** In Example 3.3, if an event is defined as one head in three tosses, such that $A = \{TTH, THT, HTT\}$ and another event is defined as two heads in three tosses, such that $C = \{THH, HTH, HHT\}$, the union of $A$ and $C$ is $A \cup C = \{TTH, THT, HTT, THH, HTH, HHT\}$. Here,

$$P(A) = \frac{3}{8},\ P(C) = \frac{3}{8},$$

and

$$P(A \cup C) = P(A) + P(C) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}.$$

This time, $A$ and $C$ are mutually exclusive.

**Figure:** Ex 3-15

# Conditional Probability

- **Conditional**: It reduces or changes the sample space and probabilities are computed based on the reduced or changed sample space.

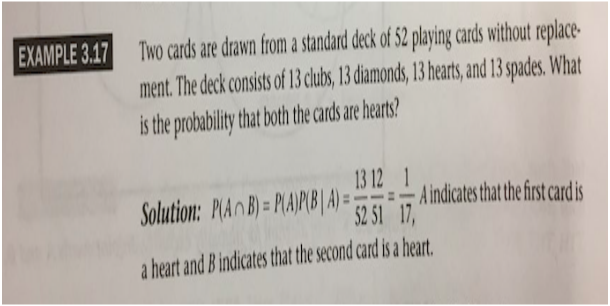
**EXAMPLE 3.16** In Example 3.3 above, the conditional probability of heads in the third toss, given that one head in three tosses is $\frac{1}{3}$. Under the condition that one head in three tosses, the reduced sample space is $A = \{TTH, THT, HTT\}$ and one of these three equally likely simple events of $A$ have head in the third toss. The statement "probability of head in the third toss given that one head in three tosses" can be written as $P(B \mid A)$, where $A = \{TTH, THT, HTT\}$ and $B = \{TTH, THH, HTH, HHH\}$. Then, $P(B \mid A)$ can also be calculated as

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/8}{3/8} = \frac{1}{3}.$$

**Figure:** Ex 3-16

# Conditional Probability

- **Multiplication Rule**: From the above conditional probability formula, it can be written that $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$.



**EXAMPLE 3.17** Two cards are drawn from a standard deck of 52 playing cards without replacement. The deck consists of 13 clubs, 13 diamonds, 13 hearts, and 13 spades. What is the probability that both the cards are hearts?

***Solution:*** $P(A \cap B) = P(A)P(B \mid A) = \frac{13}{52}\frac{12}{51} = \frac{1}{17}$, $A$ indicates that the first card is a heart and $B$ indicates that the second card is a heart.

**Figure:** Ex 3-17

# Conditional Probability

- **Bayes' Rule**: Let us consider that a sample space $S$ is divided into three mutually exclusive partitions $S_1, S_2$ and $S_3$. An event $H$ has occurred, and $P(H)$ can be written as
  $P(H) = P(H \cap S_1) + P(H \cap S_2) + P(H \cap S_3)$.
- $P(S_1|H)$ can be computed as

$$P(S_1|H) = \frac{P(S_1 \cap H)}{P(H)} = \frac{P(S_1 \cap H)}{P(H \cap S_1) + P(H \cap S_2) + P(H \cap S_3)} =$$
$$\frac{P(H|S_1)P(S_1)}{P(H|S_1)P(S_1) + P(H|S_2)P(S_2) + P(H|S_3)P(S_3)}$$

# Bayes' Rule

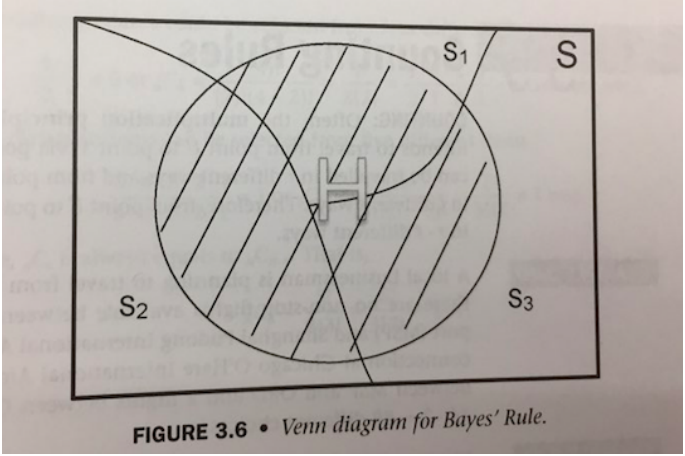



**Figure:** Venn diagram for Bayes' Rule

# Bayes' Rule

**EXAMPLE 3.18** An ultrasonogram machine identifies 50.17% of the babies as boy. A baby is a boy when an ultrasonogram identifies a boy 99.47% of the times and a baby is a girl when the ultrasonogram identifies a girl 99.83% of the times. What is the probability that a baby was predicted to be a boy if it is born a girl?

Here $S_1 \equiv$ Predicted Boy, $S_2 \equiv$ Predicted Girl and $H \equiv$ Born Girl.

The question asks you to find $P(S_1 \mid H)$, the conditional probability of the baby predicted as a boy given that the baby born is a girl.

$P(S_1) = 0.5017$, $P(S_2) = 1 - 0.5017 = 0.4983$, $P(H \mid S_1) = 1 - 0.9947 = 0.0053$ is born a girl when predicted to be a boy; $P(H \mid S_2) = 0.9983$ born a girl when predicted to be a girl.

Then,

$$P(S_1 \mid H) = \frac{P(S_1 \cap H)}{P(H)} = \frac{P(S_1 \cap H)}{P(S_1 \cap H) + P(S_2 \cap H)}$$

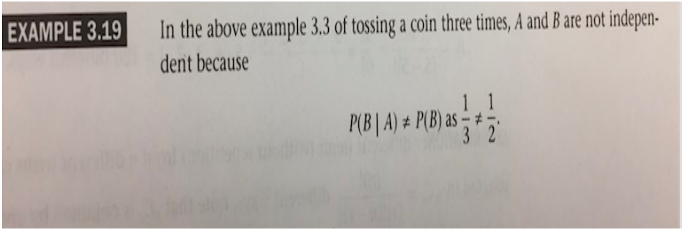
$$= \frac{P(H \mid S_1)P(S_1)}{P(H \mid S_1)P(S_1) + P(H \mid S_2)P(S_2)}$$

$$= \frac{(0.0053)(0.5017)}{(0.0053)(0.5017) + (0.9983)(0.4983)} = 0.00531683.$$

**Figure:** Venn diagram for Bayes' Rule

# Conditional Probability

- **Independence**: Two events $A$ and $B$ are known to be independent if $P(B|A) = P(B)$ or $P(A|B) = P(A)$ or $P(A \cap B) = P(A) \times P(B)$.



**EXAMPLE 3.19** In the above example 3.3 of tossing a coin three times, $A$ and $B$ are not independent because

$$P(B \mid A) \neq P(B) \text{ as } \frac{1}{3} \neq \frac{1}{2}.$$

**Figure:** Ex 3-19

# Counting Rules

- **Counting**: the multiplication principle is used, for example, if one intends to travel from point $U$ to point $V$ via point $W$. From point $U$ to point $W$ can be travelled in $r$ different ways and from point $W$ to point $V$ can be travelled in $t$ different ways. Therefore, from point $U$ to point $V$ via point $W$ can be travelled in $r \times t$ different ways.
- **Permutation**: $r$ different items (without repetition) from $n$ different items can be arranged in $n(n-1)(n-2)...(n-r+1)$ different ways. This way of counting is also known as permuting or arranging $r$ different items from $n$ different items as

$$_nP_r = \frac{(n)!}{(n-r)!}$$

  - where $(n)!$ is called a factorial of $n$ and computed as $n(n-1)(n-2)...2 \times 1$, note that $(0)! = 1$.

# Counting Rules

- **Combination**: $r$ different items (without repetition) from $n$ different items can be selected in

$$_nC_r = \frac{(n)!}{(r)!(n-r)!}$$

different ways, note that $_nC_r$ is obtained by dividing $_nP_r$ by $(r)!$ meaning that the number is lowered by the factor of $(r)!$, the number of ways $r$ different items could be arranged among themselves.

# References

Mezbahur Rahman, Deepak Sanjel, Han Wu. Statistics Introduction, Revised Printing
*KendallHunt*