Math 470 Assignment 4

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February 2, 2018

6.3.0. Let $\{a_k\}$ and $\{b_k\}$ be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

a)Suppose that $0 < \alpha < \infty$. If $|a_k|^{\alpha/k} \to a_0$, where $a_0 < 1$, then $\sum_{k=1}^{\infty} a_k^{\alpha}$ is absolutely convergent.

proof: False. Suppose that $0 < \alpha < \infty$, for a real sequence $\{a_k\}$,

$$|a_k^{\alpha}| \leq |a_k|^{\alpha}$$

Then

$$\limsup_{k\to\infty} |a_k^{\alpha}|^{1/k} = \limsup_{k\to\infty} |a_k|^{\alpha/k} = a_0 < 1$$

by 6.22(iii). Hence $\sum_{k=1}^{\infty} a_k^{\alpha}$ is absolutely convergent by Root Test.

b)If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent and $a_k \downarrow 0$ as $k \to \infty$, then

$$\limsup_{k\to\infty} |a_k|^{1/k} < 1.$$

proof: False. Let $a_k = \frac{1}{k^2}$, then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent and $a_k \downarrow 0$ as $k \to \infty$. Suppose $\lim_{k\to\infty} |\frac{1}{k^2}|^{1/k} = \lim_{k\to\infty} |k^{-2/k}| = a$ for large k.Then

$$loga = log \lim_{k\to\infty} |k^{-2/k}| = \lim_{k\to\infty} logk^{-2/k} = \lim_{k\to\infty} \frac{-2logk}{k}$$

by L'Hopital's Rule, $loga = 0$ implies $a = 1$. Thus by 6.22(iii)

$$\limsup_{k\to\infty} |\frac{1}{k^2}|^{1/k} = 1.$$

c)If $a_k \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ is absolutely convergent, then $\sum_{k=1}^{\infty} a_k$ converges.

proof: False. Let $a_k = -2$ and $b_k = \frac{1}{k^2}$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} b_k$ is absolutely convergent, but $\sum_{k=1}^{\infty} a_k$ diverges.

d)If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then $\sum_{k=1}^{\infty} a_k^2$ is absolutely convergent.

proof: True. Suppose $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then $a_k \to 0$ as $k \to \infty$. This implies $|a_k| \leq 1$ for large k. Then $|a_k|^2 \leq |a_k|$. Hence $\sum_{k=1}^{\infty} a_k^2$ is absolutely convergent by Comparison Test.

6.3.1. Prove that each of the following series converges.

a)$\sum_{k=1}^{\infty} \frac{1}{k!}$

proof: Let $k \geq 1$, then $k! > k^2$ for large k. This implies $\frac{1}{k!} < \frac{1}{k^2}$. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-series Test. Hence by Comparison Test $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges.

b)$\sum_{k=1}^{\infty} \frac{1}{k^k}$

proof: Let $k \geq 1$, then $\limsup_{k\to\infty} |\frac{1}{k^k}|^{1/k} = \limsup_{k\to\infty} |\frac{1}{k}|$, since $\frac{1}{k} \to 0$ as $k \to \infty$. By 6.22(iii), $\limsup_{k\to\infty} |\frac{1}{k^k}|^{1/k} = 0$. Hence by Root Test $\sum_{k=1}^{\infty} \frac{1}{k^k}$ is absolute convergent.

c)$\sum_{k=1}^{\infty} \frac{\pi^k}{k!}$

proof: Let $k \geq 1$, then

$$\limsup_{k\to\infty} \frac{|\frac{\pi^{k+1!}}{(k+1)!}|}{|\frac{\pi^k}{k!}|} = \limsup_{k\to\infty} |\frac{\pi}{k+1}|$$

Since $\frac{\pi}{k+1} \to 0$ as $k \to \infty$, then

$$\limsup_{k\to\infty} |\frac{\pi}{k+1}| = 0$$

by 6.22(iii). Hence by Ratio Test, $\sum_{k=1}^{\infty} \frac{\pi^k}{k!}$ is absolutely convergent.

d)$\sum_{k=1}^{\infty} (\frac{k}{k+1})^{k^2}$

proof: Let $k \geq 1$, then

$$\limsup_{k\to\infty} |(\frac{k}{k+1})^{k^2}|^{1/k} = \limsup_{k\to\infty} \frac{k}{k+1}^k = \limsup_{k\to\infty} (1 - \frac{1}{k+1})^k.$$

Since $0 < 1 - \frac{1}{k+1} < 0$ and increasing, then $(1 - \frac{1}{k+1})^k$ is decreasing and bounded below by 0, then limit of sequence $(1 - \frac{1}{k+1})^k$ exists. Suppose $\lim_{k\to\infty}(1 - \frac{1}{k+1})^k = a \in \mathbb{R}$, then

$$loga = log \lim_{k\to\infty} (1 - \frac{1}{k+1})^k = \lim_{k\to\infty} log(\frac{k}{k+1})^k = \lim_{k\to\infty} klog\frac{k}{k+1} = \lim_{k\to\infty} \frac{log\frac{k}{k+1}}{\frac{1}{k}}$$

by L'Hopital's Rule,

$$\lim_{k\to\infty} \frac{log\frac{k}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{(log\frac{k}{k+1})'}{(\frac{1}{k})'} = \lim_{k\to\infty} \frac{\frac{1}{k} - \frac{1}{k+1}}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{-k^2}{k^2+k} = -1$$

then $loga = -1$ implies $a = \frac{1}{e}$, by 6.22(iii)

$$\limsup_{k\to\infty} |(\frac{k}{k+1})^{k^2}|^{1/k} = \frac{1}{e} < 1$$

Hence by Root Test, $\sum_{k=1}^{\infty} (\frac{k}{k+1})^{k^2}$ is absolutely convergent.

6.3.2. Decide, using results covered so far in this chapter, which of the following series converge and which diverge.

a)$\sum_{k=1}^{\infty} \frac{k^3}{(k+1)^{logk}}$

proof: Convergent. Let $k \geq 1$, then $logk > 5$ for large k. This implies $(k+1)^{logk} > (k+1)^5 \Rightarrow \frac{1}{(k+1)^5} > \frac{1}{(k+1)^{logk}} \Rightarrow \frac{k^3}{(k+1)^{logk}} < \frac{k^3}{(k+1)^5} < \frac{k^3}{(k)^5} = \frac{1}{k^2}$ for large k. Hence $\sum_{k=1}^{\infty} \frac{k^3}{(k+1)^{logk}}$ converges by Comparison Test.

b)$\sum_{k=1}^{\infty} \frac{k^{100}}{e^k}$

proof: Convergent. Let $k \geq 1$, then

$$\limsup_{k\to\infty} \frac{|\frac{(k+1)^{100}}{e^{k+1}}|}{|\frac{k^{100}}{e^k}|} = \limsup_{k\to\infty} (\frac{k+1}{k})^{100}\frac{1}{e}$$

since $(\frac{k+1}{k})^{100}\frac{1}{e} \to \frac{1}{e}$ as $k \to \infty$. Then by 6.22(iii)

$$\limsup_{k\to\infty} (\frac{k+1}{k})^{100}\frac{1}{e} = \frac{1}{e} < 1$$

Hence by Ratio Test, $\sum_{k=1}^{\infty} \frac{k^{100}}{e^k}$ is absolutely convergent.

c)$\sum_{k=1}^{\infty} (\frac{k+1}{2k+3})^k$

proof: Convergent. Let $k \geq 1$, then

$$\limsup_{k\to\infty} |(\frac{k+1}{2k+3})^k|^{1/k} = \limsup_{k\to\infty} \frac{k+1}{2k+3}$$

since $\frac{k+1}{2k+3} \to \frac{1}{2}$ as $k \to \infty$, then by 6.22(iii)

$$\limsup_{k\to\infty} \frac{k+1}{2k+3} = \frac{1}{2} < 1$$

Hence by Root Test, $\sum_{k=1}^{\infty} (\frac{k+1}{2k+3})^k$ is absolutely convergent.

d)$\sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{(2k)!}$

proof: Convergent. Let $k \geq 1$ and $a_k = \frac{1\cdot 3\cdots(2k-1)}{(2k)!}$ then

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \limsup_{k\to\infty} \frac{\frac{1\cdot 3\cdots(2k+1)}{(2k+2)!}}{\frac{1\cdot 3\cdots(2k-1)}{(2k)!}} = \limsup_{k\to\infty} \frac{(2k+1)}{(2k+1)(2k+2)} = \limsup_{k\to\infty} \frac{1}{2k+2}$$

since $\frac{1}{k+2} \to 0$ as $k \to \infty$. By 6.22(iii)

$$\limsup_{k\to\infty} \frac{1}{2k+2} = \frac{1}{2} < 1$$

Hence by Ratio Test $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{(2k)!}$ is absolutely convergent.

e)$\sum_{k=1}^{\infty}(\frac{(k-1)!}{k!+1})^k$

proof: Convergent. Let $k \geq 1$ and $a_k = (\frac{(k-1)!}{k!+1})^k$. Then

$$\limsup_{k\to\infty} |(\frac{(k-1)!}{k!+1})^k|^{1/k} = \limsup_{k\to\infty} \frac{(k-1)!}{k!+1}$$

Since $\frac{(k-1)!}{k!+1} < \frac{(k-1)!}{k!} = \frac{1}{k} \to 0$ as $k \to \infty$. By 6.22(iii)

$$\limsup_{k\to\infty} \frac{(k-1)!}{k!+1} = 0 < 1$$

Hence $\sum_{k=1}^{\infty}(\frac{(k-1)!}{k!+1})^k$ is absolutely convergent by Root Test.

f)$\sum_{k=1}^{\infty}(\frac{3+(-1)^k}{5})^k$

proof: Convergent. Let $k \geq 1$ and $a_k = (\frac{3+(-1)^k}{5})^k$. Then

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} \frac{3+(-1)^k}{5} = \lim_{k\to\infty}(\sup \frac{3+(-1)^k}{5}) = \lim_{k\to\infty} \frac{4}{5} = \frac{4}{5} < 1$$

Hence $\sum_{k=1}^{\infty}(\frac{3+(-1)^k}{5})^k$ is absolutely convergent by Root Test.

g)$\sum_{k=1}^{\infty} \frac{(3-(-1)^k)^k}{\pi^k}$

proof: Convergent. Let $k \geq 1$ and $a_k = \frac{(3-(-1)^k)^k}{\pi^k}$. Then

$$\limsup_{k\to\infty} |a_k| = \limsup_{k\to\infty} |\frac{(3-(-1)^k)^k}{\pi^k}|^{1/k} = \limsup_{k\to\infty} \frac{3-(-1)^k}{\pi} = \lim_{k\to\infty}(\sup \frac{3-(-1)^k}{\pi})$$

$$= \lim_{k\to\infty} \frac{4}{5} = \frac{4}{5} < 1$$

Hence $\sum_{k=1}^{\infty} \frac{(3-(-1)^k)^k}{\pi^k}$ is absolutely convergent by Root Test.