# Math 741 Assignment 11 (Quiz)

Arnold Jiadong Yu

April 11, 2019

6.5.1. solution:

$$L(p) = \prod_{i=1}^{n} (1-p)^{k_i - 1} p = (1-p)^{\sum_{i=1}^{n} k_i - n} p^n$$

$$L^{'}(p) = (\sum_{i=1}^{n} k_i - n) \cdot (1-p)^{\sum_{i=1}^{n} k_i - n - 1} \cdot (-1) \cdot p^n + (1-p)^{\sum_{i=1}^{n} k_i - n} (np^{n-1})$$

$$= \Big[(1-p)^{\sum_{i=1}^{n} k_i - n - 1} p^{n-1}\Big] \cdot \Big[p(n - \sum_{i=1}^{n} k_i) + n(1-p)\Big]$$

Solve for $p$ for $L^{'}(p) = 0$, then

$$p(n - \sum_{i=1}^{n} k_i) + n(1-p) = 0 \Rightarrow p = \frac{n}{\sum_{i=1}^{n} k_i}$$

Therefore,

$$\max_{\Omega}[L(p)] = \Big(1 - \frac{n}{\sum_{i=1}^{n} k_i}\Big)^{\sum_{i=1}^{n} k_i - n} \Big(\frac{n}{\sum_{i=1}^{n} k_i}\Big)^n$$

$$\max_{\Omega_0}[L(p)] = (1-p_0)^{\sum_{i=1}^{n} k_i - n} p_0^n$$

$$\lambda = \frac{\max_{\Omega_0}[L(p)]}{\max_{\Omega}[L(p)]} = \frac{(1-p_0)^{\sum_{i=1}^{n} k_i - n} p_0^n}{\Big(1 - \frac{n}{\sum_{i=1}^{n} k_i}\Big)^{\sum_{i=1}^{n} k_i - n} \Big(\frac{n}{\sum_{i=1}^{n} k_i}\Big)^n}$$

6.5.2.(H) solution: Given $y_1, ..., y_{10} \sim EXP(\lambda)$, iid. Then

$$f_Y(y;\lambda) = \begin{cases} \lambda e^{-\lambda y} & x > 0, \lambda > 0 \\ 0 & o.w. \end{cases}$$

Therefore,

$$L(\lambda) = \prod_{i=1}^{10} \lambda e^{-\lambda y_i} = \lambda^{10} e^{-\lambda \sum_{i=1}^{10} y_i}$$

$$\ln[L(\lambda)] = 10 \ln \lambda - \lambda \sum_{i=1}^{10} y_i$$

$$\frac{\partial}{\partial \lambda}[\ln[L(\lambda)]] = \frac{10}{\lambda} - \sum_{i=1}^{10} y_i$$

Let $\frac{\partial}{\partial \lambda}[\ln[L(\lambda)]] = 0$, then

$$\frac{10}{\hat{\lambda}} - \sum_{i=1}^{10} y_i = 0 \implies \hat{\lambda} = \frac{10}{\sum_{i=1}^{10} y_i}$$

Therefore,

$$\max_{\Omega}[L(\lambda)] = (\frac{10}{\sum_{i=1}^{10} y_i})^{10} e^{-10}$$

$$\max_{\Omega_0}[L(\lambda)] = \lambda_0^{10} e^{-\lambda_0 \sum_{i=1}^{10} y_i}$$

Let

$$\Lambda = \frac{\lambda_0^{10} e^{-\lambda_0 \sum_{i=1}^{10} y_i}}{(\frac{10}{\sum_{i=1}^{10} y_i})^{10} e^{-10}}$$

Moreover, it is a two-sided test. The integral should be

$$P(0 \leq \Lambda \leq \lambda^* | H_0 \text{ is true}) = 0.05$$

$$= \int_0^{\lambda^*} f_\Lambda(t | H_0 \text{ is true}) dt$$

6.5.3. solution: Given $y_1, ..., y_n \sim N(\mu, 1)$ and

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

We know that MLE is $\mu = \bar{y}$, so

$$L(\lambda) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i - \mu)^2}{2}} = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^{n}(y_i - \mu)^2}{2}}$$

$$\max_{\Omega}[L(\lambda)] = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}}$$

$$\max_{\Omega_0}[L(\lambda)] = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}$$

$$\Lambda = \frac{(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}}{(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}}} = e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}+\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}$$

$$= e^{-\frac{1}{2}\left(\frac{\sum_{i=1}^n (\bar{y}-\mu_0)}{1/\sqrt{n}}\right)^2}$$

The generalized likelihood ratio test is one that rejects the null hypothesis when ever $0 < \lambda \leq \lambda^*$ so that

$$P(0 \leq \Lambda \leq \lambda^* | H_0 \text{ is true}) = \alpha$$

6.5.4.(H) solution: Given $y_1, ..., y_n \sim N(\mu, 1)$ and

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

We know that MLE is $\mu = \bar{y}$, so

$$L(\lambda) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\mu)^2}{2}} = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\mu)^2}{2}}$$

$$\max_{\Omega}[L(\lambda)] = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}}$$

$$\max_{\Omega_0}[L(\lambda)] = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}$$

$$\Lambda = \frac{(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}}{(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}}} = e^{-\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{2}+\frac{\sum_{i=1}^n (y_i-\mu_0)^2}{2}}$$

$$= e^{-\frac{1}{2}n(\bar{y}-\mu_0)^2} = e^{-\frac{1}{2}\left(\frac{(\bar{y}-\mu_0)}{1/\sqrt{n}}\right)^2}$$

where $z = \frac{(\bar{y}-\mu_0)}{1/\sqrt{n}}$. The generalized likelihood ratio test is one that rejects the null hypothesis when ever $0 < \lambda \leq \lambda^*$ so that

$$P(0 \leq \Lambda \leq \lambda^* | H_0 \text{ is true}) = \alpha$$

The likelihood ratio test didn't change in this case.
The critical region depend on the particular value of $\mu_1$ since $\alpha = P(\mu = \mu_1 | \mu = \mu_0)$

6.5.5. solution: