Math 430 Assignment 4

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Exercise 2.7
Solution:
Let $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i'\mathbf{x} \geq b_i, i = 1, ..., m\}$ and $Q = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{g}_i'\mathbf{x} \geq h_i, i = 1, ..., k\}$ where they are the same nonempty polyhedron. Suppose $\mathbf{a}_1, ..., \mathbf{a}_m$ span $\mathbb{R}^n$, then there are n many of $\mathbf{a}_i$ linearly independent in the set $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$. After reordering, assume the n linearly independent $\mathbf{a}_i$ are $\mathbf{a}_1, ..., \mathbf{a}_n$, then $m \geq n$. P is polyhedron in this form and by Theorem 2.6 (a), P has at least one extreme point, denoted $\mathbf{x}$. Moreover, $P = Q$ implies x is also an extreme point of Q. Also by Theorem 2.6 (a) and (c), There exist n vectors out of the family $\mathbf{g}_i, ..., \mathbf{g}_k$, which are linearly independent. By linear algebra, the vectors $\mathbf{g}_1, ..., \mathbf{g}_k$ must span $\mathbb{R}^n$.

Exercise 2.8
Solution:
Let $P = \{\mathbf{x} | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, $\mathbf{A}$ is a $m \times n$ matrix and the rows of the matrix $\mathbf{A}$ are linearly independent, then matrix A has full rank (rank($\mathbf{A}$) $= m$). Let $\mathbf{x}$ be a basic solution and let $J = \{i | x_i \neq 0\}$. Then to find a basic solution, matrix $\mathbf{A}$ must reduce to a $m \times m$ matrix, i.e. there are $n - m$ many of $x_i = 0$. Moreover, those m columns of $\mathbf{A}$ are also linearly independent corresponding to $x_i \neq 0$, it also forms a basis which span $\mathbb{R}^m$. WLOG, assume the first m columns of $\mathbf{A}$ are linearly independent. Then $\mathbf{A}_{(n-m)\times m}$ will multiply $\mathbf{x}_i = 0$

to get zeros which does not effect the unique solution of the system.

$$\begin{bmatrix}\mathbf{A}_{m\times m} & | & \mathbf{A}_{(n-m)\times m}\end{bmatrix}\begin{bmatrix}x_1\\ \dots\\ x_m\\ -\\ 0\\ \dots\\ 0\end{bmatrix} = \mathbf{b}$$

has a unique solution.
($\Rightarrow$) If a basis is associated with the basic solution $\mathbf{x}$, this basis in columns of $\mathbf{A}$ is linearly independent. Moreover, it is $\mathbf{A}_{i\in J}$. i.e. every column A_i, $i \in J$ is in the basis.
($\Leftarrow$) If every column $\mathbf{A}_i, i \in J$ is in the basis. i.e. all $\mathbf{A}_{i\in J}$ are linearly independent. After matrix multiplication, it can form a basic solution $\mathbf{x}$. Therefore, it is associated with the basic solution $\mathbf{x}$.

Exercise 2.9
Solution:
(a) Let B_1, B_2 be two distinct bases lead to the same basic solution, denoted $\mathbf{x}$. i.e. there are n constraints are active in B_1 and B_2. Since B_1, B_2 are distinct, then there is at lease one $b \in B_1$ that is not in B_2. This makes the active constraints of polyhedron at least $n+1$. By definition 2.10, $\mathbf{x}$ is said to be degenerate.
(b) No. Consider a polyhedron with these constraints

$$P = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \leq 1, x_1 + x_2 \leq 1 x_1, x_2 \geq 0\}$$

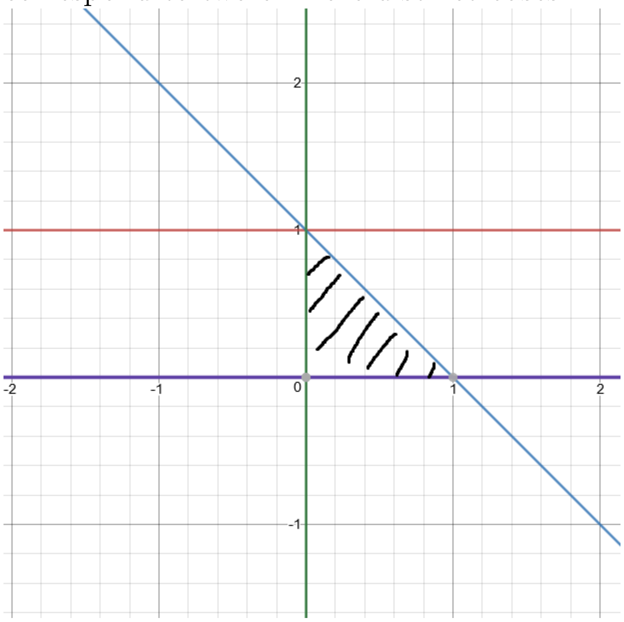
Transform into standard form, we obtain

$$P = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_2 + x_3 = 1, x_1 + x_2 + x_4 = 1, x_1, x_2, x_3, x_4 \geq 0\}$$

This will form a matrix

$$\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

We can find out $(0, 1, 0, 0)$ is the degenerate basic feasible solution and corresponds to $(0, 1)^T$ and $(1, 1)^T$, $(1, 1)^T$ and $(1, 0)^T$. This is the same basis of $(0,1)^T$ and $(1,0)^T$. Moreover, this basis span $\mathbb{R}^2$. Therefore, it is false to correspond to two or more distinct bases.



(c) No. Consider the same polyhedron in part b. The adjacent basic solution are $(0,0)$ and $(1,0)$. $(0,0)$ corresponds to exact two active constraints, $(1,0)$ also corresponds to exact two active constraints. By definition of degenerate, the two points $(0,0)$ and $(1,0)$ are not degenerate points. Moreover, it is also easy to get the result from observe the graph.

Exercise 2.10

Solution:

(a) True. $\mathbf{A}_{m\times n}$ has full rank and $n = m+1$, let S_0 be the set contain basic solutions of $\mathbf{A}_{m\times m}\mathbf{x} = \mathbf{b}$. Then let S be the set contain basic solutions of $\mathbf{A}_{m\times(m+1)}\mathbf{x} = \mathbf{b}$. We conclude that $S = S_0 + \mathbf{x}_0 = \{\mathbf{x} + \mathbf{x}_0 | \mathbf{x} \in S_o\}$, where $\mathbf{x}_0$ is the zero part of $\mathbf{x}$ that will cancel the additional column of matrix $\mathbf{A}$ in order to get basic solutions. As a result, S is affine subspace of $\mathbb{R}^m$. Moreover, $\mathbf{A}$ has full rank, i.e. S has dimension $n - m = 1$. Hence, the solutions in S lives in one dimension, i.e. a line. As a result, it has at most two basic feasible solution.

(b) False. Consider a $P = \{\mathbf{x} \in R^2 | \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}, \mathbf{x} \geq 0\}$. Then the optimal solution is $[0, \infty)$ which is not bounded.

(c) False. Since every optimal solution does not have to be basic feasible solutions of P, it could have more than m variables to be positive. Consider $m \times (n-m)$ part of matrix $\mathbf{A}$ be zero. Then $1 \times (n-m)$ part of $\mathbf{x}$ could be any non-negative values. Therefore, it could have more than m variable to be positive.

(d) True. Let f be the cost function and, $\mathbf{x}, \mathbf{y}$ are two distinct optimal solution, i.e. $f(\mathbf{x}) = f(\mathbf{y})$. Since f has optimal solution, it could either be convex or concave. WLOG, assume it is convex. Then for $\lambda \in [0,1]$,

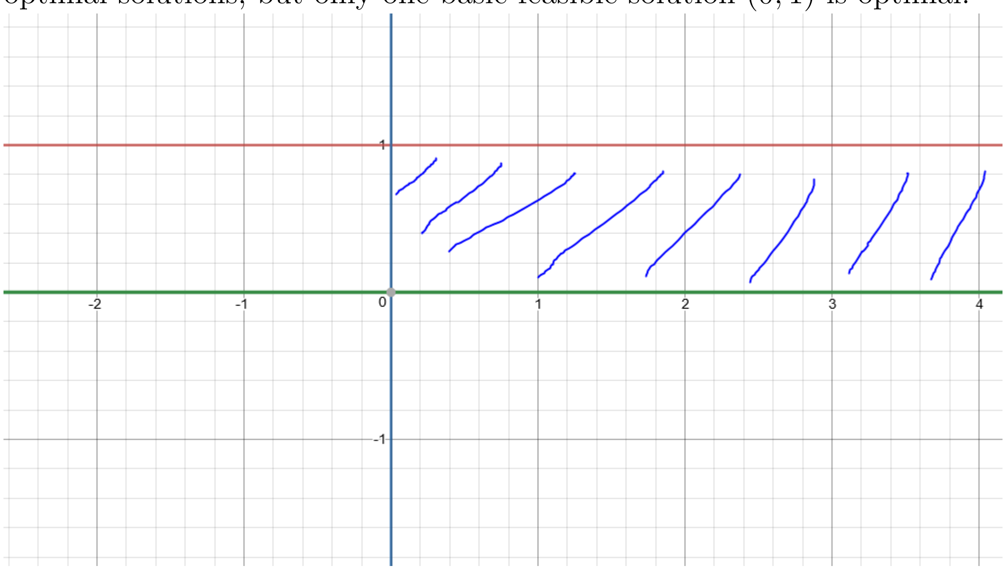
$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) = f(\mathbf{x})$$

Since it is convex, then $f(\mathbf{x})$ is minimum. i.e.

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = f(\mathbf{x}) \Rightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \text{ is also optimal solution}$$

Since $\lambda \in [0,1]$ is arbitrary, there are uncountably many optimal solutions. It is trivial for cost function to be concave. If there are more than two distinct optimal solution since we can construct uncountably many optimal solutions between any two distinct optimal solutions.

(e) False. Consider a polyhedron $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \leq 1, x_1, x_2 \geq 0\}$, and let optimal solutions be the line $x_2 = 1$, then there are uncountably many optimal solutions, but only one basic feasible solution $(0,1)$ is optimal.



(f) False. Consider a polyhedron $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$. The problem is minimizing $\max |x_1 - 0.4|$ is the same of minimizing $\max\{-x_1 + 0.4, x_1 - 0.4\}$. The optimal solution is $(0, 0.4)$ but it is not a basic feasible solution of P. It can also be observed from the graph below.