Math 335 Assignment 10

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(1) Let $C[0,1]$ be the ring of continuous functions $f : C[0,1] \to \mathbb{R}$ w.r.t. the point-wise sum and product. Let $I \in C[0,1]$ be the subset of functions, such that

$$f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = 0$$

Show that I is an ideal of $C[0,1]$.

proof: 1) To show $I \subset C[0,1]$ is additive subgroup. Pick $f, g \in I$.
a)

$$0 : [0,1] \to \mathbb{R}, x \mapsto 0$$

is the zero map, then $0(1) = 0(\frac{1}{2}) = 0(\frac{1}{3}) = 0$. Hence $0 \in I$.
b) Pick $f, g \in I$. Then $(f+g)(1) = f(1)+g(1) = 0, (f+g)(\frac{1}{2}) = f(\frac{1}{2})+g(\frac{1}{2}) = 0, (f+g)(\frac{1}{3}) = f(\frac{1}{3}) + g(\frac{1}{3}) = 0$, then $f + g \in I$.
c) Pick $h \in I$, s.t.

$$h : [0,1] \to \mathbb{R}, x \mapsto -f(x)$$

Then $(f + h)(x) = f(x) + (-f(x)) = 0$. Moreover $h(1) = -f(1) = 0, h(\frac{1}{2}) = -f(\frac{1}{2}) = 0, h(\frac{1}{3}) = -f(\frac{1}{3})) = 0$ implies h is the inverse map of f and $h \in I$. Hence I is an additive subgroup of $C[0,1]$.
2)To prove absorb product. Pick $f \in I, g \in C[0,1]$, then $(fg)(1) = f(1)g(1) = 0 \cdot g(1) = 0, (fg)(\frac{1}{2}) = f(\frac{1}{2})g(\frac{1}{2}) = 0 \cdot g(\frac{1}{2}) = 0, (fg)(\frac{1}{3}) = f(\frac{1}{3})g(\frac{1}{3}) = 0 \cdot g(\frac{1}{3}) = 0$. As a result, $fg \in I$ implies $Ig \in I$.
By part 1 and 2, I is an ideal of $c[0,1]$.

(2)Let $I \subset C[0,1]$ be as Problem 1. Consider the map:

$$\Psi : C[0,1] \to \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \Psi(g) = \left(g(1), g\left(\frac{1}{2}\right), g\left(\frac{1}{3}\right)\right),$$

where $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the product ring, i.e., the set of ordered triples of real numbers, with component-wise addition and product. Show that:
a)The map Ψ is a *surjective* ring homomorphism,
b)ker(Ψ)=I.

proof: a) Pick $f, g \in \Psi$, W.T.S $\Psi(f+g) = \Psi(f) + \Psi(g), \Psi(fg) = \Psi(f)\Psi(g)$ 1)

$$\Psi(f+g) = ((f+g)(1), (f+g)(\frac{1}{2}), (f+g)(\frac{1}{3})$$

$$= (f(1), f(\frac{1}{2}), f(\frac{1}{3}) + (g(1), g(\frac{1}{2}), g(\frac{1}{3}) = \Psi(f) + \Psi(g)$$

$$\Psi(fg) = ((fg)(1), (fg)(\frac{1}{2}), (fg)(\frac{1}{3})$$

$$= (f(1), f(\frac{1}{2}), f(\frac{1}{3}) \cdot (g(1), g(\frac{1}{2}), g(\frac{1}{3}) = \Psi(f) \cdot \Psi(g)$$

by addition and dot product defined in $\mathbb{R}^n$. Therefore, it is ring homomorphism.
2) W.T.S surjective. Define

$$\Psi^{-1} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to C[0,1], \Psi^{-1}(g(1), g(\frac{1}{2}), g(\frac{1}{3})) = g$$

then pick $f, g \in C[0,1]$. Let $f = g$, then $\Psi^{-1}(f(1), f(\frac{1}{2}), f(\frac{1}{3})) = \Psi^{-1}(g(1), g(\frac{1}{2}), g(\frac{1}{3})) \Rightarrow (f(1), f(\frac{1}{2}), f(\frac{1}{3})) = (g(1), g(\frac{1}{2}), g(\frac{1}{3}))$, then $f(1) = g(1), f(\frac{1}{2}) = g(\frac{1}{2}), f(\frac{1}{3}) = g(\frac{1}{3})$. Therefore, Ψ^{-1} is injective. Hence Ψ is surjective.
b) W.T.S ker(Ψ)$\subset I$, $I \subset$ker(Ψ).
Pick $h \in$ ker(Ψ), then $(h(1), h(\frac{1}{2}), h(\frac{1}{3})) = (0,0,0) \Rightarrow h(1) = h(\frac{1}{2}) = h(\frac{1}{3}) \Rightarrow h \in I$.
Pick $t \in I$, then $t(1) = t(\frac{1}{2}) = t(\frac{1}{3}) = 0 \Rightarrow (t(1), t(\frac{1}{2}), t(\frac{1}{3})) = (0,0,0) \Rightarrow t \in$ ker(Ψ)
Hence, ker(Ψ)=I.

(3) Using the same strategy as in Problem 2, give a full-blown argument for the following claim: for any natural number n, the ring $C[0,1]$ surjects onto the product ring $\underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{n}$ via a ring homomorphism.

proof: Pick a map Φ,

$$\Phi : C[0,1] \to \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{n}, \Phi(f) = (f(1), f(\frac{1}{2}), ..., f(\frac{1}{n}))$$

1) Pick $f, g \in C[0,1]$, then

$$\Phi(f) = (f(1), f(\frac{1}{2}), ..., f(\frac{1}{n})), \Phi(g) = (g(1), g(\frac{1}{2}), ..., g(\frac{1}{n}))$$

$$\Phi(f+g) = ((f+g)(1), (f+g)(\frac{1}{2}), ..., (f+g)(\frac{1}{n}))$$

$$= (f(\frac{1}{2}), ..., f(\frac{1}{n})) + (g(1), g(\frac{1}{2}), ..., g(\frac{1}{n})) = \Phi(f) + \Phi(g)$$

$$\Phi(fg) = ((fg)(1), (fg)(\frac{1}{2}), ..., (fg)(\frac{1}{n}))$$

$$= (f(\frac{1}{2}), ..., f(\frac{1}{n}))(g(1), g(\frac{1}{2}), ..., g(\frac{1}{n})) = \Phi(f)\Phi(g)$$

by addition and dot product rule in $\mathbb{R}^n$. Hence Φ is a ring homomorphism.
2)Same arguments as in question 2, The map can be proved to be surjective. Hence, it is a surjective ring homomorphism.

(4) Is the ideal J, mentioned in Problem 3, unique for a given natural number n?

proof: No. In class, we discussed the map is only well defined when n is finite. Then pick two map,

$$\Phi : C[0,1] \to \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{n}, \Phi(f) = (f(1), f(\frac{1}{2}), ..., f(\frac{1}{k-1}), f(\frac{1}{k})f(\frac{1}{k+1}), ..., f(\frac{1}{n}))$$

$$\Psi : C[0,1] \to \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{n}, \Phi(g) = (g(1), g(\frac{1}{2}), ..., g(\frac{1}{k-1}), g(\frac{1}{k+\frac{1}{2}}), g(\frac{1}{k+1}), ..., g(\frac{1}{n}))$$

for $k \in \mathbb{N}$ and $k < n$. Then $(x-1)(x-\frac{1}{2})...(x-\frac{1}{k-1})(x-\frac{1}{k})(x-\frac{1}{k+1})...(x-\frac{1}{n}) \in$ kerΦ, but it is not in kerΨ. We proved the kernel of the map is the ideal. Hence it is not unique.

(5) True or false: every ring homomorphism $f : F \to R$, where F is a field, is necessarily injective.

proof: False. Pick the trivial map, then f is a ring homomorphism from a field to 0.

$$f : F \to R, x \mapsto 0, \forall x \in F$$

It is clearly not injective.
It will be true for non-trivial map. We proved this in class.