Math 470 Assignment 17

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March 11, 2018

7.3.7. Find a closed form for each of the following series and the largest set on which this formula is valid.

a) $\sum_{k=0}^{\infty} 3x^{3k-1}$

proof: $\sum_{k=0}^{\infty} 3x^{3k-1} = (\sum_{k=0}^{\infty} x^{3k})'$. For $|x| < 1$, this is geometric series. Thus $\sum_{k=0}^{\infty} x^{3k} = \frac{1}{1-x^3}$. Hence

$$\sum_{k=0}^{\infty} 3x^{3k-1} = (\frac{1}{1-x^3})' = \frac{3x^2}{1-x^3}$$

for $|x| < 1$.

b)$\sum_{k=2}^{\infty} kx^{k-2}$

proof: Let $f(x) = \sum_{k=2}^{\infty} kx^{k-2}$, then $xf(x) = \sum_{k=2}^{\infty} kx^{k-1}$. Thus

$$(\sum_{k=2}^{\infty} x^k)' = xf(x) \Rightarrow (\frac{x^2}{1-x})' = xf(x) \Rightarrow$$

$$\frac{2x - x^2}{(1-x)^2} = xf(x) \Rightarrow f(x) = \frac{2-x}{(1-x)^2}$$

for $|x| < 1$.

c)$\sum_{k=1}^{\infty} \frac{2k}{k+1}(1-x)^k$

proof: Let $f(x)=\sum_{k=1}^{\infty}\frac{k}{k+1}(1-x)^k$,then

$$\sum_{k=1}^{\infty}\frac{k}{k+1}(1-x)^k=f(x)\Rightarrow\sum_{k=1}^{\infty}\frac{k}{k+1}(1-x)^{k+1}=(1-x)f(x)\Rightarrow$$

$$(\sum_{k=1}^{\infty}\frac{k}{k+1}(1-x)^{k+1})'=\sum_{k=1}^{\infty}-k(1-x)^k=((1-x)f(x))'\Rightarrow$$

$$\sum_{k=1}^{\infty}-k(1-x)^{k-1}=\frac{((1-x)f(x))'}{1-x}\Rightarrow\int\sum_{k=1}^{\infty}-k(1-x)^{k-1}=\int\frac{((1-x)f(x))'}{1-x}\Rightarrow$$

$$\sum_{k=1}^{\infty}(1-x)^k=\int\frac{((1-x)f(x))'}{1-x}=\frac{1}{x}$$

when $|1-x|<1$ by geometric series test. Then

$$\int_1^x\frac{t-1}{t^2}dt=(1-x)f(x)\Rightarrow\ln t+\frac{1}{t}|_1^x=\ln(x)+\frac{1}{x}-1=(1-x)f(x)\Rightarrow$$

$$f(x)=\frac{\ln(x)+\frac{1}{x}-1}{1-x}$$

SInce $2f(x)=\sum_{k=1}^{\infty}\frac{2k}{k+1}(1-x)^k$, hence $\sum_{k=1}^{\infty}\frac{2k}{k+1}(1-x)^k=\frac{2(\ln(x)+\frac{1}{x}-1)}{1-x}$.

d)$\sum_{k=0}^{\infty}\frac{x^{3k}}{k+1}$

proof: Let $f(x)=\sum_{k=0}^{\infty}\frac{x^{3k}}{k+1}$, then

$$\sum_{k=0}^{\infty}\frac{(x^3)^{k+1}}{k+1}=x^3f(x)\Rightarrow(\sum_{k=0}^{\infty}\frac{(x^3)^{k+1}}{k+1})'=(x^3f(x))'\Rightarrow$$

$$(3x^2)\sum_{k=0}^{\infty}(x^3)^k=(x^3f(x))'\Rightarrow(x^3f(x))'=\frac{3x^2}{1-x^3}$$

when $|x^3|<1$ and by geometric series test. Thus

$$\int_0^x\frac{3t^2}{1-t^3}dt=x^3f(x)\Rightarrow-\log|1-x^3|=x^3f(x)$$

Hence

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{k+1} = f(x) = \frac{-\log|1-x^3|}{x^3}$$

when $|x| < 1$.

7.3.9. Find a closed form.

$$f(x) = \sum_{k=0}^{\infty} (\frac{x}{(-1)^k + 4})^k$$

proof: Let $a_k = \frac{1}{((-1)^k+4)^k}$, then

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} \frac{1}{(-1)^k + 4} = \frac{1}{3}$$

Therefore, $R = 3$. For every $x \in (-3, 3)$, $f(x)$ exists. Thus we can differentiate,

$$f(x) = \sum_{k=0}^{\infty} (\frac{x}{(-1)^k + 4})^k \leq \sum_{k=0}^{\infty} (\frac{x}{3})^k = \frac{1}{1 - \frac{x}{3} = \frac{3}{3-x}} \Rightarrow$$

$$f(x)' \leq (\frac{3}{3-x})' = \frac{3}{(3-x)^2} \Rightarrow |f(x)'| \leq \frac{3}{(3-x)^2}.$$