Math 335 Assignment 3

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(1) True or false: $D_3 = S_3$?

proof: True. $D_n \subseteq S_n$ for $n \geq 3$. Total number elements in S_3 is $3! = 6$ and total number elements in $D_3 = 2 \cdot 3 = 6$. Hence, $D_3 = S_3$.

(2) Let $n \geq 3$ and consider the subset

$$G = \{\text{rotation by} \frac{4k\pi}{n} | k = 0, 1, 2, ...\} \subset D_n.$$

Is G a subgroup of D_n?

proof: G is a subgroup of D_n.
1) When $k = 0$. $G = \{\text{rotation by } 0\} = 1 \subset D_n$ is the identity n-gon.
2) Choose k_1, k_2 as non negative integer, then

$$\{\text{rotation by} \frac{4k_1\pi}{n}\} \text{and} \{\text{rotation by} \frac{4k_2\pi}{n}\} \in G$$

implies

$$\{\text{rotation by} \frac{4k_1\pi}{n} \text{then rotation by} \frac{4k_2\pi}{n}\} = \{\text{rotation by} \frac{4\pi(k_1+k_2)}{n}\} \in G$$

3) $\{\text{rotation by} \frac{4k\pi}{n}\}$ operates with $\{\text{rotation by} \frac{4(n-k)\pi}{n}\} = 1 \in G$, and $\{\text{rotation by} \frac{4k\pi}{n}\}$ and $1 \in G \Rightarrow \{\text{rotation by} \frac{4(n-k)\pi}{n}\} \in G$. Thus the inverse element is in G.

Hence G is a subgroup of D_n.

(3) How many elements does G in Problem (3) have? (Hint: the answer depends on the parity of n: even or odd.)

proof: When n is even and let $n = 2i$ for all $i \in \mathbb{Z}_{\geq 0}$, then $\frac{4k\pi}{n} = \frac{4k\pi}{2i}$. When $k = i$ it equals 2π which rotates back to identity. Thus $k = i = \frac{n}{2}$, there are $\frac{n}{2}$ elements in G. When n is odd and $n = 2i + 1$ for all $i \in \mathbb{Z}_{\geq 0}$, $\frac{4k\pi}{n} = \frac{4k\pi}{2i+1}$. When $k = 2i + 1$, it equals 2π which rotates back to identity. Thus $k = 2i + 1 = n$, there are n elements in G. Hence there are $\frac{n}{2}$ elements in G when n is even and n elements in G when n is odd.

(4) Give an explicit example of two cyclic permutations of length 3 which do not commute.

proof: Let two cyclic permutation of length 3 be $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

Thus, they do not commute.

(5) True or false: every element of S_{24} is a composition of $\underline{disjoint}$ cyclic permutations of lengths 3 and 4.

proof: False.

Suppose every element of S_{24} is a composition of $\underline{disjoint}$ cyclic permutations of lengths 3 and 4. Write $\sigma \in S_{24} = \tau_1\tau_2...\tau_j$ where $\tau_1...\tau_j$ is $\underline{disjoint}$ cyclic permutation of length 3 or 4 and $j \leq 8 \in \mathbb{Z}_{\geq 0}$. Then $\sigma^{12} = (\tau_1\tau_2...\tau_j)^{12} = 1$. Moreover cyclic permutation of length 5 of S_{24} denoted as σ_5 is also a composition of $\underline{disjoint}$ cyclic permutations of lengths 3 and 4. But $\sigma_5^{12} \neq 1$. There is a contradiction. Thus assumption is false.

Not every element of S_{24} is a composition of $\underline{disjoint}$ cyclic permutations of lengths 3 and 4.

(6) Does S_n contain a commutative group with n-elements for every natural number n?

proof: Yes. When $n = 1$, S_1 itself is commutative. When $n = 2$, $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in S_2$ is commutative or $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in S_2$ is also commutative. When $n \geq 3$, D_n is a subgroup of S_n. Let $G = \{\text{rotation } \frac{2k\pi}{n} | k = 0, 1...\}$, thus G is a subgroup of D_n. Choose $G_1 = \{\text{rotation} \frac{2k_1\pi}{n}\}$ and $G_2 = \{\text{rotation} \frac{2k_2\pi}{n}\}$ where $0 \leq k_1, k_2 \leq n \in \mathbb{Z}_{\geq 0}$. Thus $G_1G_2 = \{\text{rotation} \frac{2(k_1+k_2)\pi}{n}\} \in D_n$ and $G_2G_1 = \{\text{rotation} \frac{2(k_2+k_1)\pi}{n}\} \in D_n$. Therefore $G_1G_2 = G_2G_1 \in D_n$ is subgroup of S_n. Hence S_n contain a commutative group with n-elements for every natural number n.