Math 470 Assignment 7

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6.4.0. Let $\{a_k\}$ and $\{b_k\}$ be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

a) If $a_k \downarrow 0$, as $k \to \infty$, and $\sum_{k=1}^{\infty} b_k$ converges conditionally, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

proof: True, $\sum_{k=1}^{\infty} b_k$ converges conditionally, then partial sum $s_n = \sum_{k=1}^{n} b_k$ is bounded by Theorem 6.11. Hence the statement is true by Dirichlet's Test.

b) If $a_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

proof: False. Let $a_k = (-1)^k/k$, then $\sum_{k=1}^{\infty}(-1)^k a_k = \sum_{k=1}^{\infty}(-1)^{2k} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by Harmonic Series.

c) If $a_k \to 0$ as $k \to \infty$, and $a_k \geq 0$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

proof: False. Let $a_k = \frac{1}{k^2}$ for odd k and $a_k = \frac{2}{k}$ for even k, then

$$\sum_{k=1}^{\infty}(-1)^k a_k = -1 + 1 - \frac{1}{9} + \frac{1}{2} - \frac{1}{25} + \frac{1}{3}.....$$

The series associates, and by only looking at even terms, it is harmonic series which diverges. By only looking at odd term, it is a partial sum of a converge series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Hence it diverges.

d) If $a_k \to 0$ as $k \to \infty$, and $\sum_{k=1}^{\infty}(-1)^k a_k$ converges, then $a_k \downarrow 0$, as $k \to \infty$.

proof: False. Let $a_k = \frac{1}{k^2}$ for odd k and $a_k = \frac{1}{k^3}$ for even k, then $\sum_{k=1}^{\infty}(-1)^k a_k$ is absolute converges by Comparison Test. But the series associates.

6.4.1. Prove that each of the following series converges.

a)$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$, $p > 0$

proof: Let $k \geq 1$ and $a_k = \frac{1}{k^p}$ where $p > 0$, then $a_k \downarrow 0$ as $k \to \infty$. Thus by Alternating Series Test. It converges.

b)$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ $x \in \mathbb{R}$, $p > 0$

proof: By example 6.34, $\sin(kx)$ has a bounded partial sum for all $x \in \mathbb{R}$. Also $\frac{1}{k^p} \downarrow 0$ as $k \to \infty$. By Dirichlet's Test, it converges.

c) $\sum_{k=1}^{\infty} \frac{1-\cos(\frac{1}{k})}{(-1)^k}$

proof: Let $a_k = 1 - \cos(\frac{1}{k})$ and $f(x) = 1 - \cos(\frac{1}{x})$, then $f'(x) = -\frac{sin(\frac{1}{x})}{x^2}$. $f'(x) < 0$ for all x approach ∞. Thus $a_k \downarrow 0$ as $k \to \infty$. It converges by Alternating Series Test.

d) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}k}{3^k}$

proof: $\frac{(-1)^{k+1}k}{3^k} = \frac{(-1)^k(-k)}{3^k}$. Let $a_k = -\frac{k}{3^k}$ and $f(x) = -\frac{x}{3^x}$. then $f'(x) = \frac{3^x - x(\log 3)3^x}{3^{2x}}$. $1 - x\log 3 < 0$ for large x. Thus $a_k \downarrow 0$ as $k \to \infty$. It converges by Alternating Series Test.

e)$\sum_{k=1}^{\infty}(-1)^k(\frac{\pi}{2} - \arctan k)$

proof: Let $a_k = \frac{\pi}{2} - \arctan k$ and $f(x) = \frac{\pi}{2} - \arctan x$, then $f'(x) = -\frac{1}{1+x^2}$. $f'(x) < 0$ for all x. Thus $a_k \downarrow 0$ as $k \to \infty$. It converges by Alternating Series Test.