Math 470 Assignment 20

Arnold Jiadong Yu

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8.1.1 Let $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{R}^n$.

a)If $\| \mathbf{x} - \mathbf{z} \| < 2$ and $\| \mathbf{y} - \mathbf{z} \| < 3$, prove that $\| \mathbf{x} - \mathbf{y} \| < 5$.

proof: By triangle inequality,

$$\| \mathbf{x} - \mathbf{y} \| = \| \mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y} \| \leq \| \mathbf{x} - \mathbf{z} \| + \| \mathbf{z} - \mathbf{y} \|$$

$$= \| \mathbf{x} - \mathbf{z} \| + \| \mathbf{y} - \mathbf{z} \| < 2 + 3 = 5$$

Hence $\| \mathbf{x} - \mathbf{y} \| < 5$.

b)If $\| \mathbf{x} \| < 2, \| \mathbf{y} \| < 3$ and $\| \mathbf{z} \| < 4$, prove that $| \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} | < 14$.

proof: By triangle inequality, Cauchy-Schwarz inequality and vector algebra.

$$| \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} | = | \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) | \leq \| \mathbf{x} \| \| \mathbf{y} - \mathbf{z} \| \leq \| \mathbf{x} \| \left(\| \mathbf{y} \| + \| \mathbf{z} \| \right)$$

$$< 2 \cdot (3 + 4) = 14$$

Hence $| \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} | < 14$.

c)If $\| \mathbf{x} - \mathbf{y} \| < 2$ and $\| \mathbf{z} \| < 3$, prove that $| \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) | < 6$.

proof: By triangle inequality, Cauchy-Schwarz inequality and vector algebra.

$$| \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) | = | \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} - \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{z} |$$

$$\leq\mid \mathbf{z}\cdot(\mathbf{y}-\mathbf{x})\mid \leq\parallel \mathbf{z}\parallel (\parallel \mathbf{y}-\mathbf{x}\parallel) =\parallel \mathbf{z}\parallel (\parallel \mathbf{x}-\mathbf{y}\parallel) < 3\cdot 2 = 6$$

Hence $\mid \mathbf{x}\cdot(\mathbf{y}-\mathbf{z})-\mathbf{y}\cdot(\mathbf{x}-\mathbf{z})\mid < 6$.
d)If $\parallel 2\mathbf{x}-\mathbf{y}\parallel < 2$ and $\parallel \mathbf{y}\parallel < 1$, prove that $\mid\parallel \mathbf{x}-\mathbf{y}\parallel^2 - \mathbf{x}\cdot\mathbf{x}\mid < 2$.

proof: By triangle inequality, Cauchy-Schwarz inequality and vector algebra.

$$\mid\parallel \mathbf{x}-\mathbf{y}\parallel^2 - \mathbf{x}\cdot\mathbf{x}\mid = \mid (\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y}) - \mathbf{x}\cdot\mathbf{x}\mid = \mid \mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{x}\mid$$

$$= \mid -2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}\mid = \mid \mathbf{y}(-2\mathbf{x}+\mathbf{y})\mid \leq \parallel \mathbf{y}\parallel\parallel -2\mathbf{x}+\mathbf{y}\parallel$$
$$= \parallel \mathbf{y}\parallel\parallel \mathbf{y}-\mathbf{2x}\parallel = 2\cdot 1 = 2$$

Hence $\mid\parallel \mathbf{x}-\mathbf{y}\parallel^2 - \mathbf{x}\cdot\mathbf{x}\mid < 2$.

8.1.2 Let $B := \{x\in\mathbf{R}^n : \parallel \mathbf{x}\parallel \leq 1\}$.

a)If $\mathbf{a},\mathbf{b},\mathbf{c}\in B$ and

$$\mathbf{v} := \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a}}{3},$$

prove that $\mathbf{v}$ belongs to B.

proof: WTS $\parallel \mathbf{v}\parallel \leq 1$. $\mathbf{a},\mathbf{b},\mathbf{c}\in B$ implies $\parallel \mathbf{a}\parallel \leq 1$, $\parallel \mathbf{b}\parallel \leq 1$, $\parallel \mathbf{c}\parallel \leq 1$. By triangle inequality, Cauchy-Schwarz inequality and vector algebra.

$$\parallel 3\mathbf{v}\parallel = \parallel (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a}\parallel \leq \parallel (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\parallel + \parallel (\mathbf{a}\cdot\mathbf{c})\mathbf{b}\parallel + \parallel (\mathbf{c}\cdot\mathbf{b})\mathbf{a}\parallel$$

$$= \mid \mathbf{a}\cdot\mathbf{b}\mid\parallel \mathbf{c}\parallel + \mid \mathbf{a}\cdot\mathbf{c}\mid\parallel \mathbf{b}\parallel + \mid \mathbf{c}\cdot\mathbf{b}\mid\parallel \mathbf{a}\parallel \leq 3\parallel \mathbf{a}\parallel\parallel \mathbf{b}\parallel\parallel \mathbf{c}\parallel \leq 3$$
$$\Rightarrow \parallel \mathbf{v}\parallel \leq 1$$

Hence $\mathbf{v}$ belongs to B.

b)If $\mathbf{a},\mathbf{b}\in B$, prove that

$$\mid \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d}\mid \leq \parallel \mathbf{b}-\mathbf{c}\parallel + \parallel \mathbf{a}-\mathbf{d}\parallel$$

for all $\mathbf{c},\mathbf{d}\in\mathbf{R}^n$.

proof: $\mathbf{a},\mathbf{b} \in B$ implies $\| \mathbf{a} \| \leq 1$, $\| \mathbf{b} \| \leq 1$. By triangle inequality, Cauchy-Schwarz inequality and vector algebra.

$$| \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{d} | = | \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{d} |$$

$$= | \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} - \mathbf{d}) | \leq | \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) | + | \mathbf{b} \cdot (\mathbf{a} - \mathbf{d}) |$$
$$\leq \| \mathbf{a} \| \| \mathbf{c} - \mathbf{b} \| + \| \mathbf{b} \| \| \mathbf{a} - \mathbf{d}) \| = \| \mathbf{a} \| \| \mathbf{b} - \mathbf{c} \| + \| \mathbf{b} \| \| \mathbf{a} - \mathbf{d}) \|$$
$$\leq \| \mathbf{b} - \mathbf{c} \| + \| \mathbf{a} - \mathbf{d} \|$$

for all $\mathbf{c},\mathbf{d} \in \mathbf{R}^n$.