Math 470 Assignment 29

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10.4.8

a) Prove that *Cantor's Intersection Theorem* holds for nested compact sets in an arbitrary metric space; that is, if $H_1, H_2 ...$ is a nested sequence of nonempty compact sets in X, then

$$\bigcap_{k=1}^{\infty} H_k \neq \emptyset.$$

b) Prove that $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is closed and bounded but not compact in the metric space $\mathbb{Q}$ introduced in Example 10.5.

c) Show that Cantor's Intersection Theorem does not hold in an arbitrary metric space if *compact* is replaced by *closed and bounded.*

proof: a)Let $H_1, H_2 ...$ be a nested sequence of nonempty compact sets in X such that $H_1 \supseteq H_2 \supseteq H_3 ...$. Suppose $\bigcap_{k=1}^{\infty} H_k = \emptyset$, then $(\bigcap_{k=1}^{\infty} H_k)^c = \emptyset^c = X$. Moreover, $\bigcup_{k=1}^{\infty}(H_k)^c = X$ by DeMorgan's Law. Since H_k is compact, it is closed. Then $(H_k)^c$ is open for all k. Since the complements are open covering X and H_1 is compact, there exists an $N \in \mathbb{N}$ large enough such that $H_1 \subset \cup_{k=1}^{N}(H_k)^c$ by definition of compact. Moreover, $(H_1)^c \subseteq (H_1)^c \subseteq ... \subseteq (H_N)^c \subseteq ...$ is also nested. Then $H_1 \subseteq \cup_{k=1}^{N}(H_k)^c = (H_N)^c$. Therefore, $H_1 \subseteq (H_N)^c$ and $(H_1)^c \subseteq (H_N)^c$. Since H_1 is nonempty, then $(H_N)^c = X$, thus $H_N = \emptyset$. This contradicts that H_N is nonempty. Hence $\bigcap_{k=1}^{\infty} H_k \neq \emptyset$.

b) Let $E = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$. E is bounded since it confined in the interval $(\sqrt{2}, \sqrt{3})$. Let $x_1, x_2, ...$ be convergent sequences in E, since $\sqrt{2}, \sqrt{3}$ are irrationals. Then $x_n \to x \in E$ as $n \to \infty$ and x is rational. By Theorem 10.16, E is closed. Let a_n, b_n be convergent sequences in E such that $a_n \uparrow \sqrt{2}$ and $b_n \downarrow \sqrt{3}$ as $n \to \infty$. Then $(a_n, b_n) \cap \mathbb{Q}$ is a infinite open covering of E which doesn't have finite subcover.

c) Let x_n be convergent sequence in metric space $\mathbb{Q}$ such that $x_n \downarrow \sqrt{3}$ as $n \to \infty$. Then $(\sqrt{3}, x_n)$ is closed and bounded, but not compact by part 2). Moreover, it is nested since $x_1 \geq x_2 \geq ...$ implies $(\sqrt{3}, x_1) \supseteq (\sqrt{3}, x_2) \supseteq (\sqrt{3}, x_3)...$

$$\bigcap_{k=1}^{\infty}(\sqrt{3}, x_n) = \sqrt{3} \notin \mathbb{Q} \Rightarrow \bigcap_{k=1}^{\infty}(\sqrt{3}, x_n) = \emptyset$$

Hence Cantor's Intersection Theorem does not hold in an arbitrary metric space if *compact* is replaced by *closed and bounded.*