Math 470 Assignment 9

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6.5.1. For each of the following series, let s_n represent its partial sums and s its value. Prove that s is finite and find an n so large that s_n approximates s to an accuracy of 10^{-2}.

a)$\sum_{k=1}^{\infty}(-1)^k(\frac{\pi}{2} - \arctan k)$

proof: Let $f(x) = \frac{\pi}{2} - \arctan x$, then $f'(x) = \frac{1}{1+x^2} < 0$ for all x. Also $f(x) \to 0$ as $x \to \infty$. Thus $f(x) \downarrow 0$ as $x \to \infty$. By Alternating Series Test, $\sum_{k=1}^{\infty}(-1)^k(\frac{\pi}{2} - \arctan k)$ converges. Since

$$|error| \leq \text{ 1st neglectal term} \leq 10^{-2}$$

Thus $f(100) = 0.00999967 < 10^{-2}$. When $n = 100$ terms will estimate with an accuracy of 10^{-2}.

b)$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$

proof: Let $f(x) = \frac{x^2}{2^x}$, then $f'(x) = -\frac{x(\log 2(x)-2)}{2^x} < 0$ for all $x > \frac{2}{\log 2}$, and $\lim_{x\to\infty} f(x) = 0$ by taking L'Hospital's Rule. Thus $f(x) \downarrow 0$ as $x \to \infty$. Therefore, $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$ converges by the Alternating Series Test. Since

$$|error| \leq \text{ 1st neglectal term} \leq 10^{-2}$$

Thus $f(14) = 0.011963, f(15) = 0.006866..$ When $n = 15$ terms will estimate with an accuracy of 10^{-2}.

c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \frac{2\cdot 4 \ldots.(2k)}{1\cdot 3 \ldots (2k-1)}$

proof: Let $a_k = \frac{1}{k^2} \cdot \frac{2 \cdot 4 \ldots . (2k)}{1 \cdot 3 \ldots (2k-1)}$, then

$$\frac{a_{k+1}}{a_k} = \frac{(2k+2)k^2}{(2k+1)(k+1)^2} = \frac{2k^3+2k^2}{2k^3+5k^2+4k+1} < 1$$

for all $k \geq 1$. Also, $a_k = \frac{2}{3}\frac{4}{5} \cdots \frac{2k-2}{2k-1}\frac{2k}{k^2} < \frac{2k}{k^2} = \frac{2}{k} \to 0$ as $k \to \infty$. Thus $a_k \downarrow 0$ as $k \to \infty$. Series converges by the Alternating Series Test. Since

$$|error| \leq \text{ 1st neglectal term} \leq 10^{-2}$$

and $a_k < \frac{2}{k}$. Thus $a_{200} < \frac{2}{200} = 0.01$. When $n = 200$ terms will estimate with an accuracy of 10^{-2}.

6.5.2. a) Find all $p \geq 0$ such that the following series converges:

$$\sum_{k=2}^{\infty} \frac{1}{k \log^p k}$$

b)For each such p, prove that the partial sums of this series s_n and its value s satisfy

$$|s - s_n| \leq \frac{n+p-1}{n(p-1)} \left(\frac{1}{\log^{p-1}(n)} \right)$$

for all $n \geq 2$.

proof:

a) $\sum_{k=2}^{\infty} \frac{1}{k \log^p k}$ converges if and only if $\int_2^{\infty} \frac{1}{k \log^p k} dk < \infty$. Let $u = \log k$, then $du = \frac{1}{k} dk \Rightarrow \int_2^{\infty} \frac{1}{k \log^p k} dk = \int_2^{\infty} \frac{1}{\log^p u} du$. This converges if and only if $\sum_{u=2}^{\infty} \frac{1}{\log^p u}$ converges by Integral Test. By P-series Test, for $p > 1$ it converges.

b) Let $f(x) = \frac{1}{k \log^p k}$, then by Theorem 6.35

$$0 \leq \sum_{k=2}^{n} f(k) + \int_n^{\infty} f(x)dx - \sum_{k=2}^{\infty} f(k) \leq f(n)$$

$$\Rightarrow 0 \leq s_n + \int_n^{\infty} f(x)dx - s \leq f(n)$$

$$\Rightarrow -\int_n^{\infty} f(x)dx \leq s_n - s \leq f(n) - \int_n^{\infty} f(x)dx$$

$$\Rightarrow |s_n - s| = |s - s_n| \leq f(n) + \int_n^\infty f(x)dx$$

$$Since \int_n^\infty f(x)dx = \int_n^\infty \frac{1}{k\log^p k}dk = \frac{1}{(p-1)\log^{p-1}(n)}$$

$$\Rightarrow |s - s_n| \leq \frac{1}{n\log^p(n)} + \frac{1}{(p-1)\log^{p-1}(n)} = \frac{1}{\log^{p-1}(n)}(\frac{1}{n\log n} + \frac{1}{p-1})$$

Since $n\log n \geq n$ for large n, then $\frac{1}{n\log n} \leq \frac{1}{n}$. This implies

$$\frac{1}{\log^{p-1}(n)}(\frac{1}{n\log n}+\frac{1}{p-1}) \leq \frac{1}{\log^{p-1}(n)}(\frac{1}{n}+\frac{1}{p-1}) = \frac{n+p-1}{n(p-1)}\left(\frac{1}{\log^{p-1}(n)}\right)$$

$$\text{Hence } |s - s_n| \leq \frac{n+p-1}{n(p-1)}\left(\frac{1}{\log^{p-1}(n)}\right)$$