Math 430 Assignment 1

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Exercise 1.4 Consider the problem

$$\text{minimize } 2x_1 + 3|x_2 - 10|$$

$$\text{subject to } |x_1 + 2| + |x_2| \leq 5,$$

and reformulate it as a linear programming problem.

Solution: Let $z_1 = |x_2 - 10|, z_2 = |x_1 + 2|, z_3 = |x_2|$, then

$$z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10$$

$$z_2 \geq x_1 + 3, z_2 \geq -x_1 - 3$$

$$z_3 \geq x_2, z_3 \geq -x_2$$

Therefore, it can be reformulated into

$$\text{minimize } 2x_1 + 3z_1$$

$$\text{subject to } z_2 + z_3 \leq 5 \text{ and}$$

$$z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10$$

$$z_2 \geq x_1 + 2, z_2 \geq -x_1 - 2$$

$$z_3 \geq x_2, z_3 \geq -x_2$$

Furthermore, it can be reduced to Standard form involving $(x_1, x_2, z_1, z_2, z_3)$,

$$\text{minimize } 2x_1 + 3z_1$$

$$\text{subject to } -z_2 - z_3 \geq 5$$

$$z_1 - x_2 \geq -10$$
$$z_1 + x_2 \geq 10$$
$$z_2 - x_1 \geq 2$$
$$z_2 + x_1 \geq -2$$
$$z_3 - x_2 \geq 0$$
$$z_3 + x_2 \geq 0$$

Exercise 1.9 Consider a school district with I neighborhoods, J schools, and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally, the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the fact that numbers of students must be integer.)

Solution: The constraints will be not exceed maximum capacity of each grade in specific school while all students in specific grade and neighbor must enroll in the right grade and overall nearest school. Let x_{ijg} be number of students that from i neighborhood, go to j school, and in g grade. Then the linear programming problem is

$$\text{minimize } \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{g=1}^{G} x_{ijg}d_{ij}$$

$$\text{subject to } \sum_{i=1}^{I} x_{ijg} \leq C_{jg} \text{ for } 1 \leq j \leq J, 1 \leq g \leq G$$

$$\sum_{j=1}^{J} x_{ijg} \geq S_{ig} \text{ for } 1 \leq i \leq I, 1 \leq g \leq G$$

$$x_{ijg} \geq 0 \text{ for } 1 \leq i \leq I, 1 \leq j \leq J, 1 \leq g \leq G$$

Exercise 1.12 (Chebychev center) Consider a set P described by linear inequality constraints, that is, $P = \{\mathbf{x} \in \Re^n | a_i^{'}\mathbf{x} \leq b_i, i = 1, ..., m\}$. A ball with center $\mathbf{y}$ and radius r is defined as the set of all points within (Euclidean) distance r from $\mathbf{y}$. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P. (The center of such a ball is called the *Chebychev center* of P.) Provide a linear programming formulation of this problem.

Solution: The constraint will be all points contained in the ball need to also be contained in set P. Let $B_r(\mathbf{y})$ be such closed ball and $\mathbf{z}_1, \mathbf{z}_2$ be two points in the ball. The linear programming formulation is

$$\text{maximize } r$$

$$\text{subject to } ||\mathbf{z}_1 - \mathbf{y}|| \leq r$$

$$||\mathbf{z}_2 - \mathbf{y}|| \leq r$$

$$||\mathbf{z}_1 - \mathbf{z}_2|| \leq ||\mathbf{z}_1 - \mathbf{y}|| + ||\mathbf{z}_2 - \mathbf{y}||$$

$$a_i^{'}\mathbf{z}_1 \leq b_i, i = 1, ..., m$$

$$a_i^{'}\mathbf{z}_2 \leq b_i, i = 1, ..., m$$

$$r \geq 0$$

$$||\mathbf{z}_1 - \mathbf{y}|| \geq 0$$

$$||\mathbf{z}_2 - \mathbf{y}|| \geq 0$$

$$||\mathbf{z}_1 - \mathbf{z}_2|| \geq 0$$

Exercise 1.14 A company produces and sells two different products. The demand for each product is unlimited, but the company is constrained by cash availability and machine capacity.

Each unit of the first and second product requires 3 and 4 machine hours, respectively. There are 20,000 machine hours available in the current production period. The production costs are \$3 and \$2 per unit of the first and second product, respectively. The selling prices of the first and second product are \$6 and \$5.40 per unit, respectively. The available cash is \$4,000; furthermore, 45% of the sales revenues from the first product and 30% of the sales revenues from the second product will be made available to finance operations during the current period.

(a) Formulate a linear programming problem that aims at maximizing net income subject to the cash availability and machine capacity limitations.

(b) Solve the problem graphically to obtain an optimal solution.

(c) Suppose that the company could increase its available machine hours by 2,000, after spending \$400 for certain repairs. Should the investment be made?

Solution: (a) Let x_1, x_2 be the units of first and second product. The linear programming problem is

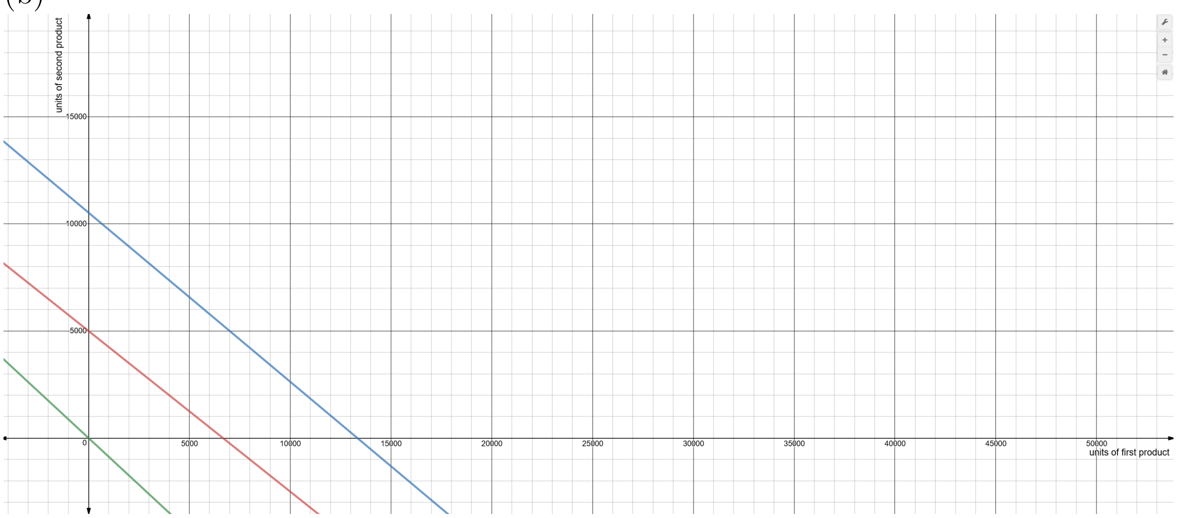
$$\text{minimize} - 3.3x_1 - 3.78x_2$$

$$\text{subject to } 3x_1 + 4x_2 \leq 20000$$

$$3x_1 + 2x_2 \leq 4000 + 2.7x_1 + 1.62x_2$$

$$x_1, x_2 \geq 0$$

(b)



Green line represents the slope of profit, the red line represents constraint on machine capacity and the blue line represents constraints on cash availability. The further point that the green line can move is $(x_1, x_2) = (\frac{20000}{3}, 0)$. Therefore, the maximum profit will be $3.3 \cdot \frac{20000}{3} + 3.78 \cdot 0 = 22000$.

(c) The new linear programming problem will be

$$\text{minimize} - 3.3x_1 - 3.78x_2 + 400$$

$$\text{subject to } 3x_1 + 4x_2 \leq 20000 + 2000$$

$$3x_1 + 2x_2 \leq 4000 + 2.7x_1 + 1.62x_2$$

$$x_1, x_2 \geq 0$$

By using the same method, we can obtain $(x_1, x_2) = (\frac{22000}{3}, 0)$ and maximum profit is $3.3 \cdot \frac{22000}{3} + 3.78 \cdot 0 - 400 = 23800$. The profit increased, therefore the investment should be made.