Math 470 Assignment 19

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7.4.2. Prove that each of the following functions is analytic on $(-1, 1)$ and find its Maclaurin expansion.

a) $\frac{x}{x^5+1}$

proof: Let $f(x) = \frac{x}{x^5+1}$, then $f(x)$ is define on $\mathbf{R}$. Thus when $|x| < 1$,

$$f(x) = \frac{x}{x^5+1} = (x)\frac{1}{1-(-x^5)} = x\sum_{k=0}^{\infty}(-x^5)^k = \sum_{k=0}^{\infty}(-1)^k x^{5k+1}$$

by Geometric Seires Test, thus $f(x)$ converges. Hence it is analytic on $(-1, 1)$.

b) $\frac{e^x}{1+x}$

proof: Let $f(x) = \frac{e^x}{1+x}$, $f(x)$ is defined on the interval $(-1, 1)$ and ∞ differentiable on the interval $(-1, 1)$. Thus when $|x| < 1$,

$$f(x) = \frac{e^x}{1+x} = e^x\frac{1}{1-(-x)} = \sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{k=0}^{\infty}(-1)^k x^k$$

$$= [1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...][1 - x + x^2 - x^3 + ...] = \sum_{k=0}^{\infty}(\sum_{i=0}^{k}\frac{(-1)^{k-i}}{i!})x^k$$

by Geometric Series Test. Hence it is analytic on $(-1, 1)$.

c) $\log(\frac{1}{|x^2-1|})$

proof: Let $f(x) = \log(\frac{1}{|x^2-1|})$, then for all $x \in (-1,1)$

$$f(x) = -\log(1-x^2) = \int_0^x \frac{2t}{1-t^2}dx = \int_0^x 2t\sum_{k=0}^{\infty} t^{2k}dx$$

$$= \int_0^x 2\sum_{k=0}^{\infty} t^{2k+1}dx = \sum_{k=0}^{\infty} \frac{x^{2k+2}}{k+1} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k}$$

by Geometric Series Test. Hence it is analytic on $(-1,1)$.