Math 470 Assignment 12

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7.1.9. Let f, g be continuous on a closed bounded interval $[a,b]$ with $|g(x)| > 0$ for $x \in [a,b]$. Suppose that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, uniformly on $[a,b]$.
a) Prove that $\frac{1}{g_n}$ is defined for large n and $\frac{f_n}{g_n} \to \frac{f}{g}$ uniformly on $[a,b]$ as $n \to \infty$.
b) Show that a) is false if $[a,b]$ is replaced by (a,b).

proof: a) Let f, g be continuous on a closed bounded interval $[a,b]$ with $|g(x)| > 0$ for $x \in [a,b]$. Suppose that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, uniformly on $[a,b]$. Then $\lim_{n\to\infty} g_n(x) = g(x) \neq 0$ for large n, thus $g_n(x) \neq 0$ for large n implies $\frac{1}{g_n}$ is defined.

Then by Extreme Value Theorem, g is bounded on $[a,b]$ implies there $\exists x_m, x_M \in [a,b]$ where $f(x_m) = m, f(x_M) = M$ such that $< m < |g(x)| < M$ for all $x \in [a,b]$, then $\frac{1}{m} > \frac{1}{g(x)} > \frac{1}{M}$.

Moreover, $g_n \to g$ uniformly on $[a,b]$ implies for $\forall \epsilon > 0$, there exist $N \in \mathbb{N}$ such that for $n \geq N$,

$$|g_n(x) - g(x)| < \frac{\epsilon m^2}{2}$$

then

$$||g_n(x)| - |g(x)|| \leq |g_n(x) - g(x)| < \frac{\epsilon m^2}{2} \Rightarrow$$

$$\frac{m}{2} < m - \frac{\epsilon m^2}{2} < |g_n(x)| < \frac{\epsilon m^2}{2} + |g(x)| < 2M$$

thus $\frac{2}{m} > \frac{1}{|g_n(x)|} > \frac{1}{2M}$. Therefore

$$|\frac{1}{g_n(x)} - \frac{1}{g(x)}| = \frac{|g_n(x) - g(x)|}{|g_n(x)||g(x)|} < (\frac{\epsilon m^2}{2})(\frac{2}{m})(\frac{1}{m}) = \epsilon$$

Hence $\frac{1}{g_n} \to \frac{1}{g}$ uniformly on $[a, b]$ as $n \to \infty$. Since $f_n \to f$ is also uniformly on $[a, b]$ as $n \to \infty$. By example 7.1.5 c) $\frac{f_n}{g_n} = f_n(\frac{1}{g_n}) \to f(\frac{1}{g}) = \frac{f}{g}$ uniformly on $[a, b]$ as $n \to \infty$.

b)Let $f_n(x) = \frac{1}{n}$ and $g_n(x) = x$, then $f_n \to 0$ uniformly on $\mathbb{R}$ and $g_n(x) \to x$ uniformly on $(0, \infty)$. But $\frac{f_n}{g_n} = \frac{1}{nx}$ doesn't converge uniformly by example 7.1.1. b).

7.1.10. Let E be a nonempty subset of $\mathbb{R}$ and f be a real-valued function defined on E. Suppose that f_n is a sequence of bounded functions on E which converges to f uniformly on E. Prove that

$$\frac{f_1(x) + ... + f_n(x)}{n} \to f(x)$$

proof: Let E be a nonempty subset of $\mathbb{R}$ and f be a real-valued function defined on E. Suppose that f_n is a sequence of bounded functions on E which converges to f uniformly on E. Let $\epsilon > 0$, choose N_1 such that $n \geq N_1$ implies $|f_n(x) - f| < \frac{\epsilon}{2}$. Since $f_n \to f$ uniformly on E and f_n is bounded, then $f_n - f$ is bounded on E implies $\sum_{k=1}^{N_1} |f_k(x) - f(x)|$ is bounded. Choose N_2 such that $n \geq N_2$ implies $\frac{1}{n} \cdot \sum_{k=1}^{N_1} |f_k(x) - f(x)| < \frac{\epsilon}{2}$, choose $N = \max\{N_1, N_2\}$ then $n \geq N$ implies

$$|\frac{f_1(x) + ... + f_n(x)}{n} - f(x)| \leq$$

$$\frac{1}{n} \cdot |f_1(x) - f(x) + ... + f_{N_1} - f(x)| + |\frac{f_{N_1+1} - f(x) + ... + f_n(x) - f(x)}{n}|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot \frac{n - (N_1 + 1)}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $\frac{f_1(x)+...+f_n(x)}{n} \to f(x)$.