Math 470 Assignment 15

Arnold Jiadong Yu

March 6, 2018

7.3.1. Find the interval of convergence of each of the following power series.

a) $\sum_{k=0}^{\infty} \frac{kx^k}{(2k+1)^2}$

proof: Let $a_k = \frac{k}{(2k+1)^2}$, then

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} \frac{k(2k+3)^2}{(k+1)(2k+1)^2} = 1$$

by L'Hospital's Rule. Hence $R = 1$. When $x = -1$, the series converges by Alternating Series Test, when $x = 1$, the series diverges by Harmonic Series Test. Therefore, it convergence on the interval $[-1, 1)$.

b) $\sum_{k=0}^{\infty}(2 + (-1)^k)^k x^{2k}$

proof: Let $i = 2k$, then $a_i = (2 + (-1)^{\frac{i}{2}})^{\frac{i}{2}}$, then

$$\limsup_{i\to\infty} |a_i|^{1/i} = \limsup_{i\to\infty}(2 + (-1)^{i/2})^{1/2} = \sqrt{3}$$

thus $R = \frac{\sqrt{3}}{3}$. When $x = -\frac{\sqrt{3}}{3}$ or $\frac{\sqrt{3}}{3}$, $(2 + (-1)^k)^k \frac{1}{3}^k = (\frac{2}{3} + \frac{(-1)^k}{3})^k = 1$ when k is even. Thus diverges by Divergence Test. Hence the series converges on the interval $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.

c) $\sum_{k=0}^{\infty} 3^{k^2} x^{k^2}$

proof: Let $i = k^2$, then $a_i = 3^i$. Thus

$$\limsup_{i\to\infty} |a_i|^{1/i} = \limsup_{i\to\infty} 3^{i^{1/i}} = 3$$

Therefore, $R = \frac{1}{3}$. When $x = -\frac{1}{3}$ or $\frac{1}{3}$, the series diverges by Divergence Test. Hence the series converges on the interval $(-\frac{1}{3}, \frac{1}{3})$.

d)$\sum_{k=0}^{\infty} k^{k^2} x^{k^3}$

proof: Let $i = k^3$, then $a_i = i^{\frac{1}{3} i^{\frac{2}{3}}}$. Thus

$$\limsup_{i\to\infty} |a_i|^{1/i} = \limsup_{i\to\infty} i^{\frac{1}{3} i^{\frac{-1}{3}}} = \limsup_{k\to\infty} k^{1/k} = 1$$

Therefore, $R = 1$. When $x = -1$ or 1, the series diverges since k^{k^2} is increasing. Hence the series converges on the interval $(-1, 1)$.

7.3.2. Find the interval of convergence of each of the following power series.

a) $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

proof: Let $a_k = (\frac{1}{2})^k$, then

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} \frac{1}{2} = \frac{1}{2}$$

Therefore, $R = 2$. When $x = -2$ or 2, the series diverges by Divergence Test. Hence the series converges on the interval $(-2, 2)$.

b)$\sum_{k=0}^{\infty}((-1)^k + 3)^k (x-1)^k$

proof: Let $a_k = ((-1)^k + 3)^k$, then

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} |((-1)^k + 3)| = 4$$

Therefore $R = \frac{1}{4}$, since $x_0 = 1$. When $x = \frac{3}{4}$ or $\frac{5}{4}$, the even term are 1, thus it diverges by Divergence Test. Hence the series converges on the interval $(\frac{3}{4}, \frac{5}{4})$.

c)$\sum_{k=1}^{\infty} \log(\frac{k+1}{k}) x^k$

proof: Let $a_k = \log(\frac{k+1}{k})$, then

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} \log(\frac{k+1}{k}) / \log(\frac{k+2}{k+1}) = 1$$

by L'Hospital's Rule. Therefore, $R = 1$. When $x = 1$, the series diverges by Divergence Test. When $x = -1$, $a_k = (-1)^k \log(\frac{k+1}{k})$. Let $f(x) = \log(\frac{x+1}{x})$, then $f'(x) = \frac{1}{x+1} - \frac{1}{x} < 0$ for $x > 0$. Therefore, it converges by Alternating Series Test. Hence the series converges on the interval $[-1, 1)$.

d) $\sum_{k=1}^{\infty} \frac{1\cdot 3 \ldots (2k-1)}{(k+1)!} x^{2k}$

proof: Let $a_k = \frac{1\cdot 3 \ldots (2k-1)}{(k+1)!}$, then

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} \frac{1\cdot 3 \ldots (2k-1)}{(k+1)!} \Big/ \frac{1\cdot 3 \ldots (2k+1)}{(k+2)!} = \lim_{k\to\infty} \frac{k+2}{2k+1} = \frac{1}{2}$$

Therefore $R^2 = \frac{1}{2}$, $R = \pm\frac{\sqrt{2}}{2}$. When $x = -\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{2}}{2}$, to check the series $\sum_{k=1}^{\infty} \frac{1\cdot 3 \ldots (2k-1)}{(k+1)!} (\frac{1}{2})^k$ converges or not. Let $a_n = \frac{1\cdot 3 \ldots (2n-1)}{(n+1)! 2^n}$

$$|\frac{a_{n+1}}{a_n}| = \frac{2(n+1)-1}{2(n+2)} = \frac{2n+1}{2n+4} = 1 - \frac{3/2}{n+2}$$

since $\frac{3}{2} > 1$, then it converges by Raabe's Test. Hence the series converges on the interval $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$.

7.3.3 Suppose that $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R \in (0, \infty)$.
a) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k x^{2k}$.
b) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k^2 x^k$.
proof: a) Let $x^2 = y$, then $\sum_{k=0}^{\infty} a_k y^k$ has radius of convergence $R \in (0, \infty)$. That is $x^2 = R$ implies $x = \sqrt{R}$. So the radius of convergence is $\sqrt{R}$.
b) Since $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R \in (0, \infty)$, then

$$\limsup_{k\to\infty} |a_k^2|^{1/k} = \limsup_{k\to\infty} (|a_k|^{1/k})^2 = \frac{1}{R^2}$$

Hence the series $\sum_{k=0}^{\infty} a_k^2 x^k$ has a radius of converges R^2.