Math 335 Assignment 4

Arnold Jiadong Yu

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(1) Let $n \geq 3$. Show that $\#A_n = \frac{n!}{2}$. (Recall, A_n is the alternating subgroup of S_n, i.e., the subgroup consisting of even permutations.)

proof: Let $\sigma \in S_n$ be an odd permutation namely $\sigma = (1\ 2)$. Pick a map from A_n to odd permutation, then

$$A_n \to \{\text{odd permutation}\}$$

$$f : x \mapsto x(1\ 2)$$

thus for every $x \in A_n$, $f(x) = x(1\ 2) \in \{\text{ odd permutation}\}$. Pick another map from odd permutation to A_n, then

$$\{\text{odd permutation}\} \to A_n$$

$$g : y \mapsto y(1\ 2)$$

thus for every $y \in \{\text{ odd permutation}\}$, $g(y) = y(1\ 2) \in A_n$. Hence

$$(g \circ f)(x) = x(1\ 2)(1\ 2) = x$$

These exists a bijective map from A_n to odd permutation. As a result $\#A_n = \#\{\text{ odd permutation}\}$ and $\#A_n + \#\{\text{ odd permutation}\} = \#S_n = n!$, then $\#A_n = \frac{n!}{2}$.

(2) Let $k \leq n$ be natural numbers. Determine the parity (even or odd) of a cyclic permutation in $(a_1 a_2 ... a_k) \in S_n$. (The answer may depend on k.)

proof: A cyclic permutation of $(a_1 a_2 ... a_k) \in S_n$ can be written in transpositions as

$$(a_1 a_2 ... a_k) = (a_1 a_2)(a_2 a_3)(a_3 a_4)...(a_{k-1} a_k)$$

By Theorem, for every $\sigma \in S_n$, all factorization into transpositions, the parity of the involved transpositions is invariant. Hence the parity of $(a_1 a_2 ... a_k)$ is $k-1$. Thus when k is odd, the parity is even, and when k is even, the parity is odd.

(3) Let n be a natural number.Determine the parity of the permutation $\sigma \in S_n$, defined by $\sigma(i) = n+1-i$ for $i = 1, ..., n$. (The answer may depend on n.)

proof: A permutation $\sigma \in S_n$ defined as $\begin{pmatrix} 1 & 2 & 3 & . & . & . & \text{n} \\ \text{n} & \text{n-1} & \text{n-2} & . & . & . & 1 \end{pmatrix}$
There are two case. When $n = 2k$ is even.

$$\sigma_{2k} = \begin{pmatrix} 1 & 2 & ... & \text{k} & \text{k+1} & ... & \text{2k} \\ \text{2k} & \text{2k-1} & ... & \text{k+1} & \text{k} & ... & 1 \end{pmatrix}$$

When $n = 2k+1$ is odd,

$$\sigma_{2k+1} = \begin{pmatrix} 1 & 2 & ... & \text{k-1} & \text{k} & \text{k+1} & ... & \text{2k+1} \\ \text{2k+1} & \text{2k-1} & ... & \text{k+1} & \text{k} & \text{k -1} & ... & 1 \end{pmatrix}$$

As a result, only when n is odd, k is the image of itself, and it has $\frac{n-1}{2}$ transpositions. When n is even, it also has $\frac{n-1}{2}$ transpositions. Hence σ has partiy even when k is even, σ has parity odd when k is odd.

(4)Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ isomorphic?

proof: No. $\mathbb{Z}_2 \times \mathbb{Z}_4$ does not have a generator, since $(1\ ,\ 1) \to (0\ ,\ 2) \to (1\ ,\ 3) \to (0\ ,\ 0)$. But $\mathbb{Z}_8$ has a generator. By Theorem, if two groups are isomorphic, then a generator must map to a generator. Hence $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ is not isomorphic .

(5)Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$ isomorphic?

proof: Yes. $\#\mathbb{Z}_2 \times \mathbb{Z}_3 = 6 = \#\mathbb{Z}_6$. Pick a map $f : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_6$, such that

$$\begin{array}{rl} (0,0) \to & 0 \\ (1,1) \to & 1 \\ (0,2) \to & 2 \\ (1,0) \to & 3 \\ (0,1) \to & 4 \\ (1,2) \to & 5 \end{array}$$

$< (1,1) >= \mathbb{Z}_2 \times \mathbb{Z}_3$, it generates in the order of $(1,1) \to (0,2) \to (1,0) \to (0,1) \to (1,2) \to (0,0)$. $< 1 >= \mathbb{Z}_6$, it generates in the same order of $< (1,1) >$, $1 \to 2 \to 3 \to 4 \to 5 \to 0$. f is clearly bijective and $f(m(1,1)) = mf(1)$ for every $m \in \mathbb{Z}$. Pick a x and a $y \in \mathbb{Z}_2 \times \mathbb{Z}_4$, then $x = a(1,1)$ and $y = b(1,1)$ for some $a, b \in \mathbb{Z}$. Then $f(x+y) = f((a+b)\cdot(1,1)) = (a+b)f(1) = af(1)+bf(1) = f(a\cdot 1)+f(b\cdot 1) = f(x)+f(y)$. Hence they are isomorphic.

(6)Describe all possible isomorphisms $\mathbb{Z}_3 \to \mathbb{Z}_3$.

proof: Two isomorphisms. Neutral Element is 0 and generators are 1 and 2 in $\mathbb{Z}_3$. Since Neutral Elements must map to Neutral Elements and Generators must map to Generators. Therefore, $\begin{array}{rl} 0 \to & 0 \\ 1 \to & 1 \\ 2 \to & 2 \end{array}$ or $\begin{array}{rl} 0 \to & 0 \\ 1 \to & 2 \\ 2 \to & 1 \end{array}$. Proof of $f(x+y) = f(x) + f(y)$ is similar as question 5. Hence there are two isomorphisms.

(7)Describe all possible isomorphisms $\mathbb{Z}_4 \to \mathbb{Z}_4$.

proof: Two isomorphisms. Neutral Element is 0 and generators are 1 and 3 in $\mathbb{Z}_4$. Since Neutral Elements must map to Neutral Elements and Generators must map to Generators. Therefore, $\begin{array}{rl} 0 \to & 0 \\ 1 \to & 1 \\ 2 \to & 2 \\ 3 \to & 3 \end{array}$ or $\begin{array}{rl} 0 \to & 0 \\ 1 \to & 3 \\ 2 \to & 2 \\ 3 \to & 1 \end{array}$.Proof of $f(x+y) = f(x) + f(y)$ is similar as question 5. Hence there are two isomorphisms.