MATH 435 ASSIGNMENT 3

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1. Chapter 9 Normal Subgroups and Factor Groups Homework

1.1. The group $Z_4 \oplus Z_{12}/\langle(2,2)\rangle$ is isomorphic to one of $Z_8, Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Determine which one by elimination.
proof: Let G be $Z_4 \oplus Z_{12}$ and $H = \langle(2,2)\rangle$ a normal subgroup of G where $\langle(2,2)\rangle = \{(2,2),(0,4),(2,6),(0,8),(2,10),(0,0)\}$. That is

$$(0,0) + \langle 2,2\rangle = \{(2,2),(0,4),(2,6),(0,8),(2,10),(0,0)\}$$
$$(1,0) + \langle 2,2\rangle = \{(3,2),(1,4),(3,6),(1,8),(3,10),(1,0)\}$$
$$(0,1) + \langle 2,2\rangle = \{(2,3),(0,5),(2,7),(0,9),(2,11),(0,1)\}$$
$$(1,1) + \langle 2,2\rangle = \{(3,3),(1,5),(3,7),(1,9),(3,11),(1,1)\}$$
$$(2,0) + \langle 2,2\rangle = \{(0,2),(2,4),(0,6),(2,8),(0,10),(2,0)\}$$
$$(3,0) + \langle 2,2\rangle = \{(1,2),(3,4),(1,6),(3,8),(1,10),(3,0)\}$$
$$(0,3) + \langle 2,2\rangle = \{(2,5),(0,7),(2,9),(0,11),(2,1),(0,3)\}$$
$$(1,3) + \langle 2,2\rangle = \{(3,5),(1,7),(3,9),(1,11),(3,1),(1,3)\}$$

These are the 8 cosets. We can see that $(1,0) + H$ has an order of 4. $Z_2 \oplus Z_2 \oplus Z_2$ is not isomorphic to G/H because it doesn't have an element with order 4. If G/H is isomorphic to Z_8, then $(1,0)+H$ must map to 2. That is

$$(1,0) + H \mapsto 2$$
$$(2,0) + H \mapsto 4$$
$$(3,0) + H \mapsto 6$$
$$(0,0) + H \mapsto 0$$

the rest of cosets must map to $1,3,5,7$. Since $1,3,5,7$ are generators of Z_8, this means the other 4 cosets are generators of the factor group G/H. But $(1,3) + H$ is not a generator, there is an contradiction. Hence G/H is not isomorphic to Z_8. Therefore, G/H is isomorphic to $Z_4 \oplus Z_2$ by elimination.

1.2. Prove or find a counter example to Berke's question: If H is a normal subgroup of G, does it follow that $G/H \oplus H \cong G$?
proof: Let G be $Z_4 \oplus Z_2$, and H be $Z_2 \oplus Z_2$, then H is a normal subgroup of G.

$$G = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}$$

$$H = \{(0,0),(0,1),(1,0),(1,1)\}$$

Then G/H is $(0,0)+H$ and $(2,0)+H$. Moreover, $(3,0) \in G$ has an order of 4.

$$G/H \oplus H = \{(((0,0)+H),(0,0)),(((0,0)+H),(0,1)),(((0,0)+H),(1,0)),$$
$$(((0,0)+H),(1,1))(((2,0)+H),(0,0)),(((2,0)+H),(0,1)),$$
$$(((2,0)+H),(1,0)),(((2,0)+H),(1,1))\}$$

has no element with order of 4. Therefore, $G/H \oplus H$ is not isomorphic to G.

2. Chapter 12 Introduction to Rings

2.1. Show that $2Z \cup 3Z$ is not a subring of Z.
proof:

$$2Z = \{...,-4,-2,0,2,4,...\}$$

$$3Z = \{...,-6,-3,0,3,6,...\}$$

$$2Z \cup 3Z = \{...,-6,-4,-3,-2,-,2,3,4,6,...\}$$

Notice, $2+3=5 \notin 2Z \cup 3Z$. That is $2Z \cup 3Z$ is not closed under addition, it is not even a group. Therefore, $2Z \cup 3Z$ is not a subring of Z.

2.2. Suppose that a belongs to a ring and $a^4 = a^2$. Prove that $a^{2n} = a^2$ for all $n \geq 1$.
proof: Prove by induction,
Base case: when $n = 1$, $a^{2n} = a^2 = a^2$. When $n = 2$, $a^4 = a^2$ is by assumption.
Suppose $a^{2(n-1)} = a^2$, then

$$a^{2n} = a^{2n-2+2} = a^{2n-2}a^2 = a^2a^2 = a^4 = a^2$$

That is $a^{2n} = a^2$ for all $n \geq 1$.

2.3. * Suppose that R is a ring and that $a^2 = a$ for all a in R. Show that R is commutative.
proof: WTS $ab = ba$ for $a, b \in R$. By assumption, a is arbitrary, then $(2a)^2 = 2a$. i.e.

$$4a^2 = 2a \Rightarrow 4a = 2a \Rightarrow 4a - 3a = 2a - 3a \Rightarrow a = -a$$

Moreover $(a+b)^2 = a + b$, then

$$(a+b)^2 = a + b \Rightarrow a^2 + ab + ba + b^2 = a + b \Rightarrow ab + ba = 0$$

Since $a = -a$, then

$$ab = -ba \Rightarrow ab = ba$$

That is, R is commutative.

3. Chapter 13 Integral Domains

3.1. Let R be the set of all real-valued functions defined for all real numbers under function addition and multiplication.
a) Determine all zero-divisors of R.
b) Determine all nilpotent elements of R.
c) Show that every nonzero element is a zero-divisor or a unit.
proof:
a) Let $f, g \in R$, if f is nowhere zero, and $fg = 0$. Then $g = 0$, i.e. f is not zero-divisor. Consider f is a function intersect x-axis at least once, and g is function such that

$$g(x) = \begin{cases} 0 & f(x) \neq 0 \\ 1 & f(x) = 0 \end{cases}$$

Therefore, $f \neq 0, g \neq 0$ but $fg = 0$. i.e. f is a zero divisor. Hence a zero-divisor of R must have at least one zero value, which means intersect x-axis at least once, but not the zero function.
b) By definition of nilpotent, let $f \in R$ such that $f^n = 0$. i.e.

$$(f(x))^n = 0 \Rightarrow f(x) = 0 \text{ for all } x \in \mathbb{R}$$

Hence zero function is the only nilpotent element of R.
c) By the results of part a and b, we only need to show for a function $f \in R$ has no zero values, then f is a unit. Since R is a ring with unity, because one function($f(x) = 1$ for $\forall x \in \mathbb{R}$) is in R. Let $g = \frac{1}{f}$, such g exists since f is non-zero everywhere. Therefore, $fg = 1$. i.e. f is a unit. Combine the results with part a, then every nonzero element is a zero-divisor or a unit.

3.2. * Let d be a positive integer. Prove that $Q[\sqrt{d}] = \{a + b\sqrt{d} | a, b \in Q\}$ is a field.
proof: