Math 741 Assignment 24 (Extra Credit)

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May 23, 2019

11.4.3.(H) solution:

$$Cov(X,Y) = E(XY) - E(X)E(Y), \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$Var(X) = E(X^2) - [E(x)]^2, Var(Y) = E(Y^2) - [E(Y)]^2$$

Therefore,

$$f_X(x) = \int_0^x 8xydy = 4x^3, 0 \le x \le 1$$

$$f_Y(x) = \int_y^1 8xydx = 4y - 4y^3, 0 \le y \le 1$$

$$E(X) = \int_0^1 x \cdot 4x^3 dx = \frac{4}{5}, E(X^2) = \int_0^1 x^2 \cdot 4x^3 dx = \frac{2}{3}$$

$$E(Y) = \int_0^1 y \cdot (4y - 4y^3)dy = \frac{8}{15}, E(Y^2) = \int_0^1 y^2 \cdot (4y - 4y^3)dy = \frac{1}{3}$$

$$Var(X) = \frac{2}{3} - (\frac{4}{5})^2 = \frac{2}{75}, Var(Y) = \frac{1}{3} - (\frac{8}{15})^2 = \frac{11}{225}$$

$$E(XY) = \int_0^1 \int_0^x xy \cdot 8xydydx = \frac{4}{9}$$

$$Cov(X,Y) = \frac{4}{9} - \frac{4}{5} \cdot \frac{8}{15} = \frac{4}{225} = \frac{8}{450}$$

$$\rho(X,Y) = \frac{\frac{8}{450}}{\sqrt{\frac{2}{75}}\sqrt{\frac{11}{225}}} = 0.492366$$

11.4.6.(H) solution:

$$E(X) = \sum_{x=1}^{n} xp(x) = \frac{1}{n}(1 + ... + n) = \frac{n+1}{2}$$

$$E(X^2) = \sum_{x=1}^{n} x^2 p(x) = \frac{1}{n}(1^2 + ... + n^2) = \frac{(n+1)(2n+1)}{6}$$

$$E(Y) = \sum_{y=1^2}^{n^2} yp(y) = \frac{1}{n}(1^2 + ... + n^2) = \frac{(n+1)(2n+1)}{6}$$

$$E(Y^2) = \sum_{y=1^2}^{n^2} y^2 p(y) = \frac{1}{n}(1^4 + ... + n^4) = \frac{1}{5}n^4 + \frac{1}{2}n^3 + \frac{1}{3}n^2 - \frac{1}{30}$$

$$E(XY) = \sum xyp(x, y) = \frac{n(n+1)^2}{4}, P(X = i, Y = j) = \begin{cases} 1/n, j = i^2 \\ 0, o.w. \end{cases}$$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = \frac{n(n+1)^2}{4} - \frac{n+1}{2} \cdot \frac{(n+1)(2n+1)}{6}$$

$$= \frac{n^3 + n^2 - n - 1}{12}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{n^2 - 1}{12}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{16n^4 + 30n^3 - 5n^2 - 30n - 11}{180}$$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$= \frac{\frac{n^3+n^2-n-1}{12}}{\sqrt{\frac{n^2-1}{12}}\sqrt{\frac{16n^4+30n^3-5n^2-30n-11}{180}}}$$

$$\lim_{n\to\infty} \rho(X, Y) = \lim_{n\to\infty} \frac{\frac{n^3+n^2-n-1}{12}}{\sqrt{\frac{n^2-1}{12}}\sqrt{\frac{16n^4+30n^3-5n^2-30n-11}{180}}} = \frac{\sqrt{15}}{4}$$

11.4.7.(H) solution: a)

$$Cov(X + Y, X - Y) = E((X + Y)(X - Y)) - E(X + Y)E(X - Y)$$

$$= E(X^2 - Y^2) - [(E(X) + E(Y))(E(X) - E(Y))$$

$$= E(X^2) - E(Y^2) - [E(X)]^2 + [E(Y)]^2 = Var(X) - Var(Y)$$

b) Suppose $Cov(X, Y) = 0$, then

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) = Var(X) + Var(Y)$$

$$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y) = Var(X) + Var(Y)$$

Moreover,

$$Cov(X + Y, X - Y) = Var(X) - Var(Y)$$

Hence,

$$\rho(X + Y, X - Y) = \frac{Cov(X + Y, X - Y)}{\sqrt{Var(X + Y)}\sqrt{Var(X - Y)}}$$

$$= \frac{Var(X) - Var(Y)}{\sqrt{Var(X) + Var(Y)}\sqrt{Var(X) + Var(Y)}} = \frac{Var(X) - Var(Y)}{Var(X) + Var(Y)}$$

11.4.9.(H) solution:

$$\hat{\beta}_1 = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}$$

$$r = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\sqrt{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}\sqrt{n\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2}}$$

$$= \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \cdot \frac{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}{\sqrt{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}\sqrt{n\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2}}$$

$$= \hat{\beta}_1 \cdot \frac{\sqrt{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}}{\sqrt{n\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2}}$$