# MATH 435 ASSIGNMENT 8

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## 1. Introduction to Module Theory

**1.1.** 3. Give an explicit example of a map from one $R$-module to another which is a group homomorphism but not an $R$-module homomorphism.
proof: Let the ground ring be $\mathbb{C}$ and consider $\mathbb{C}$-module, $\mathbb{C}$. Define a map $\phi$,

$$\phi : \mathbb{C} \to \mathbb{C}$$

$$\phi : z \mapsto \overline{z}$$

where $z, \overline{z} \in \mathbb{C}$. Then let $z_1, z_2 \in \mathbb{C}$,

$$\phi(z_1 + z_2) = \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} = \phi(z_1) + \phi(z_2)$$

But

$$\phi(i \cdot 1) = -i \neq i \cdot \phi(1) = i$$

for $1, i \in \mathbb{C}$. Hence it is a group homomorphism but not an $R$-module homomorphism.
We also can use the example in the class

$$R[x] \to R[x]$$

$$f(x) \mapsto f(x^2)$$

**1.2.** 4*.

**1.3.** 5. Exhibit all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$
proof: Since $\mathbb{Z}$-module homomorphism is just group homomorphism. Moreover $\mathbb{Z}/30\mathbb{Z} \cong \mathbb{Z}_{30}$ and $\mathbb{Z}/21\mathbb{Z} \cong \mathbb{Z}_{21}$. Then we only need to consider group homomorphism between $\mathbb{Z}_{30}$ and $\mathbb{Z}_{21}$. Moreover, $\langle 1 \rangle = \mathbb{Z}_{30}$, then we only need to consider where 1 maps to. There exist the trivial map from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{21}$ where every element of $\mathbb{Z}_{30}$ maps to zero of $\mathbb{Z}_{21}$. Therefore, define a map $\varphi$ such that

$$\varphi : \mathbb{Z}_{30} \to \mathbb{Z}_{21}$$

If $\varphi$ is $\mathbb{Z}$-module homomorphisms, then

$$\varphi(30 \cdot 1) = \varphi(0) = 0 = 30 \cdot \varphi(1)$$

where $\varphi(1) \in \mathbb{Z}_{21}$. Moreover $30 \cdot \varphi(1) = 0$ and $30 = 2 \cdot 3 \cdot 5, 21 = 3 \cdot 7$. There are only 3 possible value for $\varphi(1)$ s.t. $30 \cdot \varphi(1) = 0$ which is $0, 7, 14$. As a result, there are 3 $\mathbb{Z}$-module homomorphisms.

$$\overline{1} \mapsto \overline{0}$$
$$\overline{1} \mapsto \overline{7}$$
$$\overline{1} \mapsto \overline{14}$$

If there is another map, assume $1 \mapsto a$ where $7 \nmid a$, then $30a = 0$ which means $21 \mid 30a \Rightarrow 7 \mid a$, which is a contradiciton. Hence there are only 3 maps.

This is related to #6, since we can see $\gcd(30, 21) = 3$ and there are 3 $\mathbb{Z}$-module homomorphisms. This makes $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{30}, \mathbb{Z}_{21}) \cong \mathbb{Z}_3$, which is the same as $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/21\mathbb{Z}) \cong \mathbb{Z}/(30, 21)\mathbb{Z}$. From this problem, we can think the #6, let $z = \gcd(m, n)$, then $m = z \cdot q_1 \cdot ... \cdot q_t, n = z \cdot p_1 \cdot ... \cdot p_s$ where $p_i, q_j$ are all distinct prime numbers. To map $\mathbb{Z}_m$ to $\mathbb{Z}_n$, define a map

$$\phi : \mathbb{Z}_m \to \mathbb{Z}_n$$

Then $m\phi(1) = 0$ implies $\phi(1)$ has to have $p_1 \cdot ... \cdot p_s$ as factors and there will be $z$ copies of them. As a result, there will be $z$ numbers of $\mathbb{Z}$-module homomorphisms maps.

**1.4.** 8. Let $\varphi : M \to N$ be an $R$-module homomorphism. Prove that $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$ (cf. Exercise 8 in Section 1).
proof: Since $\varphi$ is an $R$-module homomorphism then $\varphi(0) = 0$. Let $m \in \mathrm{Tor}(M)$, then there exist a nonzero element $r \in R$, s.t. $rm = 0$. Apply the map to $m$, then

$$r \cdot \varphi(m) = \varphi(r \cdot m) = \varphi(0) = 0$$
$$\Rightarrow r \cdot \varphi(m) = 0 \Rightarrow \varphi(m) \in \mathrm{Tor}(N)$$

for the same nonzero $r \in R$. Hence $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$.

**1.5.** 10. Let $R$ be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and $R$ are isomorphic as rings.
proof: Let $R$ be a commutative ring then $\mathrm{Hom}_R(R, R)$ is a ring. Since we want to show it is ring homomorphism, then 0 must map to 0 and 1 must map to 1. Define $f_r$ is a map defined below, claim $f_r \in \mathrm{Hom}_R(R, R)$.

$$f_r : R \to R$$
$$f_r : a \mapsto ra$$

Let $a, b, r \in R$, then

$$f_r(ra + b) = r(ra + b) = r(ra) + rb = rf_r(a) + f_r(b)$$

i.e. $f_r$ is an $R$-module homomorphism. Moreover, any element has $f_r(a) = af_r(1_R)$ in $\mathrm{Hom}_R(R, R)$ since $f_r$ is $R$-module homomorphism. Define a map $\varphi$, s.t.

$$\varphi : R \to \mathrm{Hom}_R(R, R)$$
$$\varphi : r \mapsto f_r$$

Then for $r, s \in R$.

$$\varphi(r + s) = f_{r+s} = f_r + f_s = \varphi(r) + \varphi(s)$$

Since $\mathrm{Hom}_R(R, R)$ is a ring with addition, $f_{r+s}(x) = (r + s)(x) = rx + sx = f_r(x) + f_s(x)$.

$$\varphi(rs) = f_{rs} = f_r \circ f_s = \varphi(r) \circ \varphi(s)$$

Since $\mathrm{Hom}_R(R, R)$ is a ring with composition, $f_{rs}(x) = (rs)(x) = r(sx) = f_r \circ f_s(x)$. Therefore, it is a ring homomorphism.
WTS $\varphi$ is bijective.
Let $\varphi(r) = \varphi(s)$, then $f_r = f_s \implies r(x) = s(x) \implies r = s$. i.e. $\varphi$ is injective.
Let $g \in$Hom$_R(R, R)$, then $g(1) = r$. Moreover, $g(x) = xg(1) = xr = rx$, since $R$ is commutative. i.e. $g = f_r = \varphi(r)$. This shows $\varphi$ is surjective.
Therefore, $\varphi$ is bijective. With ring homomrphism, $\mathrm{Hom}_R(R, R) \cong R$ as rings.