Math 470 Assignment 1

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6.1.0 Let a_x and b_x be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

a) If a_k is strictly decreasing and a$_k \to 0$ as k $\to \infty, then \sum_{k=1}^{\infty} a_k$ converges.

proof: False, example is the harmonic series.

b) If $a_k \neq$ b$_k$ for all k $\in$ N and if $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, then either $\sum_{k=1}^{\infty} a_k$ converges or $\sum_{k=1}^{\infty} b_k$ converges.

proof: False, let $a_k = 1$ and $b_k = -1$, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, but both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.

c) Suppose that $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

proof: True. Suppose $\sum_{k=1}^{\infty}(a_k + b_k)$ converges and equal to C $\in$ R,$\sum_{k=1}^{\infty} a_k$ converges and equal to A $\in$ R.$Thus \sum_{k=1}^{\infty}(-a_k)$ converges and equal to -A $\in$R. C - A = $\sum_{k=1}^{\infty}(a_k + b_k) + \sum_{k=1}^{\infty}(-a_k) = \sum_{k=1}^{\infty}(a_k + b_k - a_k) = \sum_{k=1}^{\infty} b_k$ by Theorem 6.10 and C-A $\in$R.$Hence \sum_{k=1}^{\infty} b_k$ converges. The converse is trivial.

d)If $a_k \to a$ask $\to \infty$, then $\sum\limits_{k=1}^{\infty}(a_k - a_{k+2}) = a_1 + a_2 - 2a$

proof: True. $\sum\limits_{k=1}^{\infty}(a_k - a_{k+2}) = (a_1 - a_3) + (a_2 - a_4) + (a_3 - a_5) + ... + (a_{k-2} - a_k) + (a_{k-1} - a_{k+1}) + (a_k - a_{k+2}) = a_1 + a_2 - a_{k+1} - a_{k+2}$, since $a_k \to aask \to \infty$, then $a_{k+1} = a$ and $a_{k+2} = a$. Hence $\sum\limits_{k=1}^{\infty}(a_k - a_{k+2}) = a_1 + a_2 - 2a.$

6.1.1 Prove that each of the following series converges and find its value.

a) $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{e^{k-1}}$

proof: $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{e^{k-1}} = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{e^{k-1}}(-1)^2 = (-1)^2 \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{e^{k-1}} = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{e^{k-1}} = \sum\limits_{k=0}^{\infty}(-\frac{1}{e})^k$. Since $|-\frac{1}{e}| < 1$ apply Theorem 6.7 (Geometric series). $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{e^{k-1}} = \frac{1}{1+\frac{1}{e}} = \frac{e}{1+e}$.

b) $\sum\limits_{k=0}^{\infty} \frac{(-1)^{k-1}}{\pi^{2k}}$

proof: $\sum\limits_{k=0}^{\infty} \frac{(-1)^{k-1}}{\pi^{2k}} = -\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(\pi^2)^k} = -\sum\limits_{k=0}^{\infty}(-\frac{1}{\pi^2})^k$. Since $|-\frac{1}{\pi^2}| < 1$ apply Theorem 6.7 (Geometric series).$\sum\limits_{k=0}^{\infty} \frac{(-1)^{k-1}}{\pi^{2k}} = -\frac{1}{1+\frac{1}{\pi^2}} = -\frac{\pi^2}{1+\pi^2}$.

c) $\sum\limits_{k=2}^{\infty} \frac{4^{k+1}}{9^{k-1}}$

proof: $\sum\limits_{k=2}^{\infty} \frac{4^{k+1}}{9^{k-1}} = (\frac{4^3}{9}) \sum\limits_{k=2}^{\infty} \frac{4^{k-2}}{9^{k-2}} = (\frac{64}{9}) \sum\limits_{k=0}^{\infty}(\frac{4}{9})^k$, since $|\frac{4}{9}| < 1$ apply Theorem 6.7 (Geometric series). $\sum\limits_{k=2}^{\infty} \frac{4^{k+1}}{9^{k-1}} = (\frac{64}{9})(\frac{1}{1-\frac{4}{9}}) = \frac{64}{5}$.

d) $\sum\limits_{k=0}^{\infty} \frac{5^{k+1}+(-3)^k}{7^{k+2}}$

proof: $\sum\limits_{k=0}^{\infty} \frac{5^{k+1}+(-3)^k}{7^{k+2}} = (\frac{1}{49}) \sum\limits_{k=0}^{\infty} \frac{5^{k+1}+(-3)^k}{7^k} = (\frac{1}{49}) \sum\limits_{k=0}^{\infty}(\frac{5^{k+1}}{7^k} + \frac{(-3)^k}{7^k}) = (\frac{5}{49}) \sum\limits_{k=0}^{\infty}(\frac{5}{7})^k + (\frac{1}{49}) \sum\limits_{k=0}^{\infty}(-\frac{3}{7})^k$, since both $|\frac{5}{7}|$ and $|-\frac{3}{7}|$ are less than 1, then apply Theorem 6.7 (Geometric series). $\sum\limits_{k=0}^{\infty} \frac{5^{k+1}+(-3)^k}{7^{k+2}} = (\frac{5}{49})(\frac{1}{1-\frac{5}{7}}) + (\frac{1}{49})(\frac{1}{1+\frac{3}{7}}) = \frac{5}{14} + \frac{1}{70} = \frac{13}{35}$.

6.1.2. Represent each of the following series as a telescopic series and find its value.

b) $\sum\limits_{k=1}^{\infty} \frac{12}{(k+2)(k+3)}$

proof: $\sum\limits_{k=1}^{\infty} \frac{12}{(k+2)(k+3)} = (12) \sum\limits_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = (12) \sum\limits_{k=1}^{\infty} (\frac{1}{k+2} - \frac{1}{k+3}) = (12)(\frac{1}{3} - \lim_{k\to\infty} \frac{1}{k+2} = 4$ by Theorem 6.6 (Telescopic series).

c) $\sum\limits_{k=2}^{\infty} log(\frac{k(k+2)}{(k+1)^2})$

proof: $\sum\limits_{k=2}^{\infty} log(\frac{k(k+2)}{(k+1)^2}) = \sum\limits_{k=2}^{\infty} log(\frac{\frac{k}{k+1}}{\frac{k+1)}{k+2}}) = \sum\limits_{k=2}^{\infty} log(\frac{k}{k+1} - \frac{k+1}{k+2}) = log(\frac{2}{3}) - \lim_{k\to\infty} log(\frac{k}{k+1}) = log(\frac{2}{3}) - log(1) = log(\frac{2}{3})$ by Theorem 6.6 (Telescopic series).

d) $\sum\limits_{k=1}^{\infty} 2sin(\frac{1}{k} - \frac{1}{k+1})cos(\frac{1}{k} + \frac{1}{k+1})$

proof: By trigonometric formula $sin(a) - sin(b) = 2sin(\frac{a-b}{2})cos(\frac{a+b}{2})$, then $\sum\limits_{k=1}^{\infty} 2sin(\frac{1}{k} - \frac{1}{k+1})cos(\frac{1}{k} + \frac{1}{k+1}) = \sum\limits_{k=1}^{\infty}(sin(\frac{2}{k}) - sin(\frac{2}{k+1})) = sin(2) - \lim_{k\to\infty} sin(\frac{2}{k}) = sin(2) - 0 = sin(2)$ by Theorem 6.6 (Telescopic series).

6.1.3. Prove that each of the following series diverges.

a) $\sum\limits_{k=1}^{\infty} cos(\frac{1}{k^2})$

proof: $\lim_{k\to\infty} cos(\frac{1}{k^2}) = cos(0) = 1 \neq 0$, thus by Theorem 6.5 (Divergence test) $\sum\limits_{k=1}^{\infty} cos(\frac{1}{k^2})$ diverges.

b) $\sum\limits_{k=1}^{\infty} (1 - \frac{1}{k})^k$

proof: Let $\lim_{k\to\infty}(1 - \frac{1}{k})^k = a$, then $Ln(\lim_{k\to\infty}(1 - \frac{1}{k})^k) = \lim_{k\to\infty} Ln(1 - \frac{1}{k})^k) = \lim_{k\to\infty} \frac{Ln(1-\frac{1}{k})}{k^{-1}}$ apply l'hopital's rule $\lim_{k\to\infty} \frac{Ln(1-\frac{1}{k})}{k^{-1}} = \lim_{k\to\infty} \frac{(\frac{1}{1-\frac{1}{k}})(0+k^{-2})}{-k^{-2}} = \lim_{k\to\infty} \frac{-k}{k-1} = -1 = Ln(a), \Rightarrow a = e^{-1}$, since $e^{-1} \neq 0$, thus by Theorem 6.5 (Divergence test) $\sum\limits_{k=1}^{\infty}(1 - \frac{1}{k})^k$ diverges.

c) $\sum\limits_{k=1}^{\infty} \frac{k+1}{k^2}$

proof: Assume $\sum\limits_{k=1}^{\infty} \frac{k+1}{k^2}$ converges to A $\in \mathbb{R}, then$A $= \sum\limits_{k=1}^{\infty} \frac{k+1}{k^2} = \sum\limits_{k=1}^{\infty}(\frac{1}{k} + \frac{1}{k^2})$ and $\frac{1}{k^2}$ is positive. Hence $\sum\limits_{k=1}^{\infty}(\frac{1}{k} + \frac{1}{k^2}) > \sum\limits_{k=1}^{\infty} \frac{1}{k}$, but $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ is harmonic series and it diverges. It contradicts with assumption. That is $\sum\limits_{k=1}^{\infty} \frac{k+1}{k^2}$ diverges.

worked together with Michael

6.1.6. a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded. b) Show that the converse of part a) is false. Namely, show that a series $\sum_{k=1}^{\infty} a_k$ may have bounded partial sums and still diverge.

proof: part a) Suppose $\sum_{k=1}^{\infty} a_k$ converges and equal to$A \in \mathbb{R}$. Since the partial sums $s_1 = a_1, s_2 = a_1 + a_2, ..., s_n = a_1 + a_2 + ... + a_n$, thus construct a sequence s_n. Hence $s_n \to A$ as $n \to \infty$. By Theorem 2.8 Every converges sequence is bounded. Then s_n is bounded.

part b) The series $\sum_{k=1}^{\infty} (-1)^k$ has partial sums either 0 or 1. Then it is bounded by 0 and 1. But it doesn't pass the Divergence test. Hence the converse of part a) is false.