# Math 430 Assignment 5

Arnold Jiadong Yu

October 2, 2018

Exercise 2.12

Solution:

True. Suppose the polyhedron P contains a line, therefore there exists a vector $\mathbf{x} \in P$ and a nonzero $\mathbf{d} \in \mathbb{R}^n$ such that

$$\mathbf{x} + \lambda\mathbf{d} \in P \text{ for all scalars } \lambda$$

Since $\mathbf{d}$ is nonzero, WLOG suppose $d_1 \neq 0$. Moreover each $x_i$ we have either $x_i \geq 0$ or $x_i \leq 0$. WLOG suppose $x_1 \geq 0$, if $d_1 > 0$ we can choose $\lambda < 0$ small enough s.t $x_1^{'} = x_1 + \lambda d_1 < 0$, then the new $\mathbf{x}^{'}$ contain $x_1^{'}$ is not in $P$. If $d_1 < 0$ we can choose $\lambda > 0$ big enough s.t $x_1^{'} = x_1 + \lambda d_1 < 0$, then the new $\mathbf{x}^{'}$ contain $x_1^{'}$ is not in $P$. It is not for all scalars $\lambda$, therefore there is a contradiction. $P$ does not contain a line, then $P$ has at least one extreme point by Theorem 2.6. Hence $P$ has at least one basic feasible solution.

Exercise 2.13

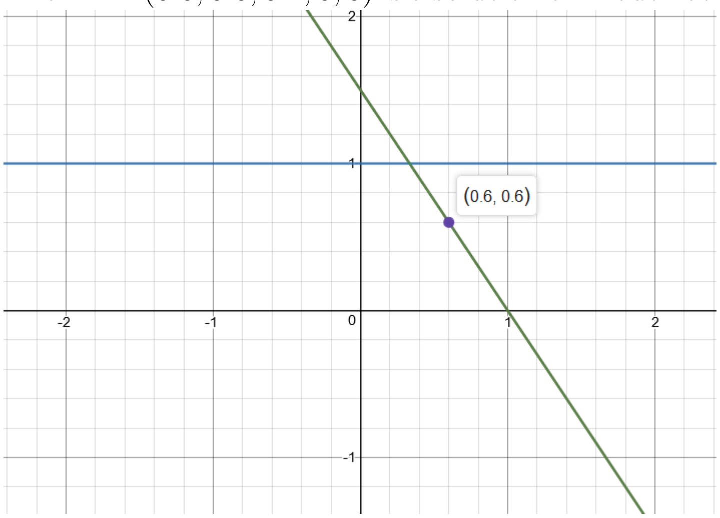
Solution:

(a)By assumption, all basic feasible solutions are nondegenerate. i.e. they are all in exactly $\dim(n-m)$ subspace. Then these feasible solutions has no more than $n$ active constraints and with exactly $m$ positive components and $n-m$ zero components. Assume $\mathbf{x} \in P$ with exactly $m$ positive components. i.e. $\mathbf{x}$ is in $\dim(n-m)$ subspace. Since $\mathbf{x} \in P$. Hence $\mathbf{x}$ is a basic feasible solution.

(b)Consider a polyhedron $P$ has two exactly same constraints such that

$$P = \left\{ \mathbf{x} \in \mathbb{R}^5 \,\middle|\, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \mathbf{x} \geq 0 \right\}$$

Then $\mathbf{x} = (0.6, 0.6, 0.4, 0, 0)$ is a solution of $P$ but not a basic feasible solution.




Exercise 2.15
Solution:
Let $L = \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v} | 0 \leq \lambda \leq 1\}$ and $S = \{\mathbf{z} \in P | \mathbf{a}_i^{'}\mathbf{z} = b_i, i = 1, ..., n-1\}$.
WTS $L = P$.
(1) WTS $L \subseteq S$. Let $\mathbf{x} \in L$, then $\mathbf{x} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ for some $\lambda$.

$$\mathbf{a}_i^{'}\mathbf{x} = \mathbf{a}_i^{'}(\lambda\mathbf{u}+(1-\lambda)\mathbf{v}) = \lambda\mathbf{a}_i^{'}\mathbf{u}+(1-\lambda)\mathbf{a}_i^{'}\mathbf{v} \geq \lambda b_i+(1-\lambda)b_i = b_i \text{ for } i = 1, ..., m$$

This shows $\mathbf{x} \in P$. Moreover

$$\mathbf{a}_i^{'}\mathbf{x} = \mathbf{a}_i^{'}(\lambda\mathbf{u}+(1-\lambda)\mathbf{v}) = \lambda\mathbf{a}_i^{'}\mathbf{u}+(1-\lambda)\mathbf{a}_i^{'}\mathbf{v} = \lambda b_i+(1-\lambda)b_i = b_i \text{ for } i = 1, ..., n-1$$

This shows $\mathbf{x} \in S$. That is $L \subseteq S$.
(2) WTS $S \subseteq L$. Let $\mathbf{z} \in S$, then $\mathbf{z} \in P$ and $\mathbf{a}_i^{'}\mathbf{z} = b_i, i = 1, ..., n-1$. Moreover, $\mathbf{a}_i^{'}\mathbf{z} = b_i = \mathbf{a}_i^{'}\mathbf{u} = \mathbf{a}_i^{'}\mathbf{v}, i = 1, ..., n-1$ where $\mathbf{u}, \mathbf{v}$ are distinct basic feasible solution of $P$. Since $\mathbf{a}_i$ are linear independent for $i = 1, ..., n-1$, then $u_i \neq v_i$ for $i = 1, ..., n-1$ and matrix $A$ contains $\mathbf{a}_i$ for $i = 1, ..., n-1$ is invertible. If $\mathbf{z} = \mathbf{u}$ or $\mathbf{z} = \mathbf{v}$, we are done, $\mathbf{z} \in L$. Assume $\mathbf{z} \neq \mathbf{u}$ and $\mathbf{z} \neq \mathbf{v}$, then

$$\mathbf{z} = \mathbf{a}_i^{-1}b_i = \mathbf{a}_i^{-1}(\lambda\mathbf{a}_i^{'}\mathbf{u} + (1-\lambda)\mathbf{a}_i^{'}\mathbf{v}) = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$$

for $i = 1, ..., n-1$. i.e. $\mathbf{z} \in L$. Since $\mathbf{u}, \mathbf{v}$ are distinct basic feasible solution of $P$, then $u_i = v_i = 0$ for $i = n, ..., m$. With $\mathbf{z} \in P$, $\mathbf{z}$ is also in $L$. Hence $\mathbf{z} \in L$. i.e. $S \subseteq L$.
As a result, $S = L$. i.e. $L = \{\mathbf{z} \in P | \mathbf{a}_i^{'}\mathbf{z} = b_i, i = 1, ..., n-1\}$.

Exercise 2.17

Solution:

Consider the results of exercise 2.5. We can define two affine maps called $f, g$, s.t.

$$f : \mathbf{x}^* \mapsto (\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$$

$$g : (\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*) \mapsto \mathbf{x}^*$$

$f, g$ are bijective and these two polyhedrons are isomorphic. That is $\mathbf{x}^*$ extreme point implies $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is also extreme point. Let $\mathbf{A}$ be $m \times n$ matrix and $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is degenerate, then there are more than $n + m$ constraints are active. i.e. for extreme point $\mathbf{x}$, there are more than $n$ constraints are active. This contradicts with that $\mathbf{x}$ is nondegenerate. Therefore, $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is nondegenerate. Hence $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is a nondegenerate basic feasible solution. (Dr. Beck said it is okay to use exercise 2.5 as a result).