Math 335 Assignment 2

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(1) Let G be a group and $x \in G$. Show that any bracketing of the sequence $xxxx$, understood as the corresponding product, leads to a same element of G.
There are five ways to bracket the sequence $xxxx$.
1. $((xx)x)x$
2. $(x(xx))x$
3. $(xx)(xx)$
4. $x((xx)x)$
5. $x(x(xx))$

Since $x \in G$, then by associative $x(xx) = (xx)x \in G$. Then sequence $1 = 2$ and $4 = 5$. Let $xx = a \in G$, then $x(ax) = (xa)x$ which implies sequence $2 = 4$. And $(xx)(xx) = (xx)a = x(xa)$ =sequence 5. Thus all sequences are equivalent. Hence they all leads to a same element of G.

(2) Let G be a *not necessarily finite* group and $x \in G$ be an element. Consider the map $G \to G$, defined by $z \mapsto x^{-1}zx$.
(a) Is this map injective?
(b) Is this map surjective?

(a) This map is injective. Define a function f such that $f(z_1) = x^{-1}z_1x$ and $f(z_2) = x^{-1}z_2x$. If $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$, then the map is injective. Suppose $f(z_1) = f(z_2)$, then

$$x^{-1}z_1x = x^{-1}z_2x \Rightarrow x(x^{-1}z_1x) = x(x^{-1}z_2x) \Rightarrow$$

$$(xx^{-1})(z_1x) = (xx^{-1})(z_2x) \Rightarrow 1(z_1x) = 1(z_2x) \Rightarrow (1 \cdot z_1)x = (1 \cdot z_2)x \Rightarrow$$

$$z_1x = z_2x \Rightarrow (z_1x)x^{-1} = (z_2x)x^{-1} \Rightarrow z_1(xx^{-1}) = z_2(xx^{-1}) \Rightarrow z_1 \cdot 1 = z_2 \cdot 1 \Rightarrow z_1 = z_2$$

Since $x \in G$, $z \in G$ and $x^{-1}z_1x \in G$, by associative, neutral element and inverse element axiom. We have $z_1 = z_2$. Thus, this map is injective.
(b) This map is surjective. Define a function g such that $g(z) = x^{-1}zx = y \in G$, then $g(y) = x^{-1}yx \in G$. This implies $y = xg(y)x^{-1} = x^{-1}zx \Rightarrow z = xxg(y)x^{-1}x^{-1} = g(y)$ by associative. Then $z \mapsto y \mapsto z$ is an identity map by function g. Thus it is bijective. Since it is bijective, it has to be surjective.

(3) Let G_1 and G_2 be groups, and $H_1 \subset G_1$ and $H_2 \subset G_2$ be subgroups. Is $H_1 \times H_2$ a subgroup of $G_1 \times G_2$?

$H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

Let G_1 and G_2 be groups, and $H_1 \subset G_1$ and $H_2 \subset G_2$ be subgroups. Then $e_1 \in H_1$, $e_2 \in H_2$, $a_1, b_1 \in H_1 \Rightarrow a_1 * b_1 \in H_1$, $a_2, b_2 \in H_2 \Rightarrow a_2 * b_2 \in H_2$ and $a_1 \in H_1 \Rightarrow a_1^{-1} \in H_1$, $a_2 \in H_2 \Rightarrow a_2^{-1} \in H_2$ by definition.

1. $e_1 \in H_1$, $e_2 \in H_2 \Rightarrow (e_1, e_2) \in H_1 \times H_2$.

2. $(a_1, a_2), (b_1, b_2) \in H_1 \times H_2 \Rightarrow (a_1, a_2) \times (b_1, b_2) = (a_1 * b_1, a_2 * b_2) \in H_1 \times H_2$ since $a_1, b_1 \in H_1 \Rightarrow a_1 * b_1 \in H_1$, $a_2, b_2 \in H_2 \Rightarrow a_2 * b_2 \in H_2$

3. $(a_1, a_2) \in H_1 \times H_2 \Rightarrow (a_1^{-1}, a_2^{-1}) \in H_1 \times H_2$ since $a_1 \in H_1 \Rightarrow a_1^{-1} \in H_1$, $a_2 \in H_2 \Rightarrow a_2^{-1} \in H_2$,

Hence, $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$ (e_1, e_2 are netural elements in each subgroups and $*$ is operation).

(4) List all subgroups of $\mathbb{Z}_5$.

All subgroups of $\mathbb{Z}_5$ are $\{0\}$ and $\{\mathbb{Z}_5\}$.

All the subgroups are the trivial subgroups. Since subgroup must be a subset of $\mathbb{Z}_5$. Assume a subgroup G of $\mathbb{Z}_5$ and G is not the trivial subgroup. Then there are at least two elements in G, choose $G = \{0, 1\}$. Since the inverse of 1 is not in G, contradicts with definition. Then G is not a subgroup of $\mathbb{Z}_5$. Same concepts can be repeated for non-trivial subgroups, and can prove all subgroups of $\mathbb{Z}_5$ are the trivial subgroups.

$\mathbb{Z}_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(5) Let $H \subset \mathbb{R}^*$ be a subgroup and $G \subset GL_2(\mathbb{R})$ be the subset of matrices whose determinants belong to H. Is G a subgroup of $GL_2(\mathbb{R})$.

G is a subgroup of $GL_2(\mathbb{R})$.

Let $H \subset \mathbb{R}^*$ be a subgroup and $G \subset GL_2(\mathbb{R})$ be the subset of matrices whose determinants belong to H. Then $1 \in \mathbb{R}^*$, $a, b \in H \Rightarrow ab \in H$ and $a \in \mathbb{R}^* \Rightarrow a^{-1} \in H$ by definition.

1. $\det I = 1 \in H \Rightarrow I \in G$.

2. $\det A, \det B \in H \Rightarrow \det A \det B \in H$. Since $\det A \det B = \det AB$, then $\det AB \in H \Rightarrow AB \in G$. Hence $A, B \in G \Rightarrow AB \in G$.

3. $\det A \in H \Rightarrow (\det A)^{-1} \in H$. Since $(\det A)^{-1} = \frac{1}{\det A} = \det A^{-1}$, then $\det A^{-1} \in H \Rightarrow A^{-1} \in G$. Hence $A \in G \Rightarrow A^{-1} \in G$.

Hence, G is a subgroup of $GL_2(\mathbb{R})$.

(6) Is $\mathbb{Z}_5 \times \mathbb{Z}_5$ a cyclic group?

$\mathbb{Z}_5 \times \mathbb{Z}_5$ is not a cyclic group.

There are a total 25 pairs, and separates them into two groups for discussion. One group is (x, y) where $x = y$. Another group is (x, y) where $x \neq y$. When group one operates, it only generates (x, y) where $x = y$. When group two operates, it only generates (x, y) where $x \neq y$. This means not all elements in $\mathbb{Z}_5 \times \mathbb{Z}_5$ can be generated by either the same pair or distinct pair. Thus $\mathbb{Z}_5 \times \mathbb{Z}_5$ is not a cyclic group.

$\mathbb{Z}_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3