Math 470 Assignment 30

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10.6.6 Suppose that H is nonempty compact subset of X and Y is Euclidean space.
a) If $f : H \to Y$ is continuous, prove that

$$||f||_H = \sup_{x\in H} ||f(x)||_Y$$

is finite and there exists an $x_0 \in H$ such that $||f(x_0)||_Y = ||f||_H$.
b) A sequence of functions $f_k : H \to Y$ is said to converge uniformly on H to a function $f : H \to Y$ if and only if given $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k \geq N \text{ and } x \in H \text{ imply } ||f_k(x) - f(x)|| < \epsilon$$

Show that $||f_k - f||_H \to 0$ as $k \to \infty$ if and only if $f_k \to f$ uniformly on H as $k \to \infty$.
c) Prove that a sequence of functions f_k converge uniformly on H if and only if, given $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that

$$k, j \geq N \text{ implies } ||f_k - f_j||_H < \epsilon$$

proof: a)Since H is compact and f is continuous on H, then $f(x)$ is continuous on the image of a compact set which is compact by Theorem 10.61. Moreover, H is closed and bounded, then $f(x)$ is closed and bounded for all $x \in H$. By Extreme Value Theorem (Theorem 10.63), $||f||_H = \sup_{x\in H} ||f(x)||_Y$ is finite and here exists an $x_0 \in H$ such that $||f(x_0)||_Y = ||f||_H$.
b) ($\Rightarrow$)Suppose $||f_k - f||_H \to 0$ as $k \to \infty$. Let $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $||f_k - f||_H < \epsilon$ for $k \geq N$. Therefore, $\sup_{x\in H} ||f_k(x) - f(x)||_Y < \epsilon$ for $k \geq N$ by part a). Thus,$||f_k(x) - f(x)||_Y \leq \sup_{x\in H} ||f_k(x) - f(x)||_Y < \epsilon$ for all $x \in H$ and $k \geq N$. Hence $f_k \to f$ uniformly on H as $k \to \infty$.
($\Leftarrow$) Suppose $f_k \to f$ uniformly on H as $k \to \infty$. Let $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $||f_k(x) - f(x)||_Y < \epsilon$ for $k \geq N$ and $x \in H$. Therefore, $\sup_{x\in H} ||f_k(x) - f(x)||_Y < \epsilon$. This implies $||f_k - f||_H < \epsilon$ for $k \geq N$ by part a). Hence $||f_k - f||_H \to 0$ as $k \to \infty$.
c) ($\Rightarrow$)Suppose $f_k \to f$ uniformly on H as $k \to \infty$. Let $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that

$$||f_k - f||_H < \frac{\epsilon}{2} \text{ and } ||f_j - f||_H < \frac{\epsilon}{2} \text{ for } k, j \geq N$$

Therefore

$$||f_k - f_j||_H \leq ||f_k - f||_H + ||f_j - f||_H = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } k, j \geq N$$

($\Leftarrow$) Let $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that $k, j \geq N$ implies $||f_k - f_j||_H < \epsilon$, then $\sup_{x \in H} ||f_k(x) - f_j(x)|| < \epsilon$. It is Cauchy in Y. Since Y is Euclidean Space, then it converges to some function in H by Cauchy Criterion. This implies $f_k \to$ some function f in H. It converges uniformly by part b).

Q: Let $X = \{$bounded function from $\mathbb{R}$ to $\mathbb{R}\}$

$$d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)|$$

a) Show that (X, d) is a metric space.
b) Show that (X, d) is not separable.
proof: a)(1) If $d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)| = 0$, then $|f(t) - g(t)| \leq 0$. Since $|f(t) - g(t)| \geq 0$, then $|f(t) - g(t)| = 0$. Therefore, $f(t) = g(t)$ for $t \in \mathbb{R}$
(2)$|f(t) - g(t)| = |g(t) - f(t)|$ for $t \in \mathbb{R}$. This implies $\sup_{t \in \mathbb{R}} |f(t) - g(t)| = \sup_{t \in \mathbb{R}} |g(t) - f(t)|$. Therefore, $d(f, g) = d(g, f)$.
(3) Let $h(t)$ be a bounded function from $\mathbb{R}$ to $\mathbb{R}$. Then

$$|f(t) - g(t)| = |f(t) - h(t) + h(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)| \text{ for every } t \in \mathbb{R}$$

Therefore

$$\sup_{t \in \mathbb{R}} |f(t) - g(t)| \leq \sup_{t \in \mathbb{R}} |f(t) - h(t)| + \sup_{t \in \mathbb{R}} |h(t) - g(t)|$$
$$\Rightarrow d(f, g) \leq d(f, h) + d(h, g)$$

Hence, (X, d) is a metric space by (1),(2), and (3).
b) Suppose (X, d) is separable, let C be an countable dense subset of X. Choose f_k be a sequence of bounded functions in C and $f \in X$, such that $f_k \to f$ as $k \to \infty$. Let $\epsilon > 0$, then $d(f_k, f) = \sup_{t \in \mathbb{R}} |f_k(t) - f(t)| < \epsilon$ as $k \to \infty$. $|f_k|$ is bounded by $|f|$ for all k, $f \notin C$, $f \in \overline{C}$. Since f_k is range in $\mathbb{R}$ and bounded by f. There could be g_k is range in $\mathbb{R}$ and bounded by g. Then we can keeping adding sequences of sequences to the coubtable dense subset of C, where forces $C = X$. This contradicts that C is an countable dense subset of X. Hense (X, d) is not separable.