Math 741 Assignment 7 (Hand-In)

Arnold Jiadong Yu

March 8, 2019

5.7.4.(H)
solution: Given an estimator $\hat{\theta}$ such that $\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = 0$.
a) WTS $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$.

$$\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = \lim_{n\to\infty} E[\hat{\theta}_n^2 - 2\theta\hat{\theta}_n + \theta^2] = \lim_{n\to\infty} (E[\hat{\theta}_n^2] - 2\theta E[\hat{\theta}_n] + \theta^2)$$
$$= \lim_{n\to\infty} (E[\hat{\theta}_n^2] - (E[\hat{\theta}_n])^2 + (E[\hat{\theta}_n])^2 - 2\theta E[\hat{\theta}_n] + \theta^2)$$
$$= \lim_{n\to\infty} \left[\mathrm{Var}(\hat{\theta}_n) + (E(\hat{\theta}_n) - \theta)^2 \right] = 0$$

Since $\mathrm{Var}(\hat{\theta}_n) \geq 0, (E(\hat{\theta}_n) - \theta)^2 \geq 0$. Therefore,

$$\lim_{n\to\infty} (E(\hat{\theta}_n) - \theta)^2 = 0 \Rightarrow \lim_{n\to\infty} (E(\hat{\theta}_n) - \theta) = 0$$
$$\implies \lim_{n\to\infty} E(\hat{\theta}_n) = \theta$$

b) WTS $\lim_{n\to\infty} P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) = 1$. Let $\epsilon > 0$, then by Chebyshev's inequality

$$P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) \geq 1 - \frac{E((\hat{\theta}_n - E(\hat{\theta}_n))^2)}{\epsilon^2} = 1 - \frac{E((\hat{\theta}_n - \theta + \theta - E(\hat{\theta}_n))^2)}{\epsilon^2}$$
$$= 1 - \frac{E[(\hat{\theta}_n - \theta)^2 + 2(\hat{\theta}_n - \theta)(\theta - E(\hat{\theta}_n)) + (\theta - E(\hat{\theta}_n))^2]}{\epsilon^2}$$
$$= 1 - \frac{E[(\hat{\theta}_n - \theta)^2] + 2E[(\hat{\theta}_n - \theta)(\theta - E(\hat{\theta}_n))] + E[(\theta - E(\hat{\theta}_n))^2]}{\epsilon^2}$$

Since $\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = 0$ and $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$. then

$$\lim n \to \infty P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) \geq 1 - 0 = 1$$

Hence, $\hat{\theta}_n$ is consistent.