Math 741 Assignment 1

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(*Central Limit Theorem*) Let $W_1, W_2, ...$ be an infinite sequence of independent random variables, each with the same distribution. Suppose that the mean μ and the variance σ^2 of $f_W(w)$ are both finite. For any number a and b,

$$\lim_{n\to\infty} P\Big(a \leq \frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} \leq b\Big) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-z^2/2} dz$$

Lemma: Let $W_1, W_2, ...$ be the set of random variables such that

$$\lim_{n\to\infty} M_{W_n}(t) = M_W(t)$$

for all t in some interval about 0. Then

$$\lim_{n\to\infty} F_{W_n}(w) = F_W(w) \text{ for all } -\infty < w < \infty.$$

proof: To prove the central limit theorem using moment-generating functions requires showing that

$$\lim_{n\to\infty} M_{(W_1+...+W_n-n\mu)/(\sqrt{n}\sigma)}(t) = M_Z(t) = e^{t^2/2}$$

For notational simplicity, let

$$\frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} = \frac{S_1 + ... + S_n}{\sqrt{n}}$$

where $S_i = (W_i - \mu)/\sigma$. Notice that $E(S_i) = 0$ and $\mathrm{Var}(S_i) = 1$. Moreover, from Theorem 3.12.3, Since E(aW+b) = aE(W) + b, μ and σ are given. We can see them as constant, therefore $E(S_i) =$ E($(W_i - \mu)/\ \sigma) = (E(W_i) - \mu)/\sigma = (\mu - \mu)/\sigma =$ 0. Since Var$(W) = \sigma^2 = E(W^2) - \mu^2$. Then Var$(S_i) = E(S_i^2) - [E(S_i)]^2 = E(S_i^2) = (E(W_i^2) - 2\mu E(W_i) + \mu^2)/\sigma^2 = \sigma^2/\sigma^2 = 1$.

$$M_{(S_1+\ldots+S_n)/\sqrt{n}}(t) = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

By Theorem 3.12.3 part a), we can get $M_{S_i/\sqrt{n}}(t) = M(\frac{t}{\sqrt{n}})$. Since they are all independent, by part b), we can obtain the equation above.

where $M(t)$ denotes the moment-generating function common to each of the $S_i's$.

By virtue of the way the $S_i's$ are defined, $M(0) = 1, M^{(1)}(0) = E(S_i) = 0$, and $M^{(2)}(0) =$ Var$(S_i) = 1$. Applying Taylor's theorem, then, to $M(t)$, we can write

$$M(t) = 1 + M^{(1)}(0)t + \frac{1}{2}M^{(2)}(r)t^2 = 1 + \frac{1}{2}t^2 M^{(2)}(r)$$

Taylor's Theorem with remainder shows $M(t-0) = M(0) + M^{(1)}(0)(t-0) + \frac{1}{2!}M^{(2)}(r)(t-0)^2$. After simplify, the equation above can be obtained.

for some number r, $|r| < |t|$. Thus

$$\lim_{n\to\infty}\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n = \lim_{n\to\infty}\left[1 + \frac{t^2}{2n}M^{(2)}(s)\right]^n, |s| < \frac{|t|}{\sqrt{n}}$$

After substitute t with $\frac{t}{\sqrt{n}}$. The equation above can be obtained

$$= \exp \lim_{n\to\infty} n \ln\left[1 + \frac{t^2}{2n}M^{(2)}(s)\right]$$

$$= \exp \lim_{n\to\infty} \frac{t^2}{2} \cdot M^{(2)}(s) \cdot \frac{\ln\left[1 + \frac{t^2}{2n}M^{(2)}(s)\right] - \ln(1)}{\frac{t^2}{2n}M^{(2)}(s)}$$

The existence of $M(t)$ implies the existence of all its derivatives. In particular, $M^{(3)}(t)$ exists, so $M^{(2)}(t)$ is continuous. Therefore, $\lim_{t\to 0} M^{(2)}(t) = M^{(2)}(0) = 1$. Since $|s| < |t|/\sqrt{n}, s \to 0$ as $n \to \infty$, so

$M^{(3)}(t)$ exists, then $M^{(2)}(t)$ is continuous as well as limit at 0 exists. i.e. $\lim_{t\to 0} M^{(2)}(t) = M^{(2)}(0) = 1$

$$\lim_{n\to\infty} M^{(2)}(s) = M^{(2)}(0) = 1$$

Also, as $n \to \infty$, the quantity $(t^2/2n)M^{(2)}(s) \to 0 \cdot 1 = 0$, so it plays the role of "Δx" in the definition of the derivative. Hence we obtain

$$\lim_{n\to\infty}\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n = \exp\frac{t^2}{2}\cdot 1 \cdot \ln^{(1)}(1) = e^{(1/2)t^2}$$

$Ln^{(1)}(1) = \lim_{n\to\infty} \frac{\ln\left[1+\frac{t^2}{2n}M^{(2)}(s)\right]-\ln(1)}{\frac{t^2}{2n}M^{(2)}(s)}$ by the definition of derivative.

Since this last expression is the moment-generating function for a standard normal random variable, the theorem is proved.