Math 470 Assignment 26

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10.3.4. Suppose that $A \subseteq B \subseteq X$. Prove that $\overline{A} \subseteq \overline{B}$ and $A^0 \subseteq B^0$.

proof: Suppose $A \subseteq B \subseteq X$, then $A^0 \subseteq A \subseteq B$ and $A \subseteq B \subseteq \overline{B}$ by Theorem 10.34 i). By Theorem 10.34 ii), $A^0 \subseteq B^0$. By Theorem 10.34 iii) $\overline{A} \subseteq \overline{B}$.

10.3.5 Show that if E is nonempty and closed in X and $a \notin E$, then $\inf_{x \in E} d(x, a) > 0$.

proof: Suppose E is nonempty and closed in X and $a \notin E$. Let $\epsilon > 0$, then $B_\epsilon(a) \cap E = \emptyset$, then pick any $x \in E$, $d(x, a) \geq \epsilon$. Choose the closest $x \in E$ to a, this implies $\inf_{x \in E} d(x, a) \geq \epsilon > 0$.

10.3.9. Let $f : \mathbf{R} \to \mathbf{R}$. Prove that f is continuous on $\mathbf{R}$ if and only if $f^{-1}(I)$ is open on $\mathbf{R}$ for every open interval I.

proof: ($\Rightarrow$). Suppose f is continuous on $\mathbf{R}$ and $I = (a, b)$, then let $\epsilon > 0$, $f(x) \in I$ implies $(f(x) - \epsilon, f(x) + \epsilon) \subset I$. Moreover, f is continuous on $\mathbf{R}$, it is continuous on I. There exists an $\delta > 0$, for $x, y \in I$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then

$$x - \delta < y < x + \delta \Rightarrow f(x) - \epsilon < f(y) < f(x) + \epsilon$$

Hence $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset I$ implies $y \in (x - \delta, x + \delta) \subset f^{-1}(I)$. As a result, f^{-1} is open in $\mathbf{R}$ by definition.
($\Leftarrow$) Let $\epsilon > 0$, pick $y \in \mathbf{R}$. Suppose $I = (f(y) - \epsilon, f(y) + \epsilon)$, then $y \in f^{-1}(I)$. Suppose $f^{-1}(I)$ is open on $\mathbf{R}$. Let $\delta > 0$, then $(y - \delta, y + \delta) \subset f^{-1}(I)$. Pick an $x \in (y - \delta, y + \delta)$, then $f(x) \in (f(y) - \epsilon, f(y) + \epsilon)$. Therefore

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Hence f is continuous by definition.