Math 430 Assignment 8

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4.1
solution:

$$\max\ 3p_2 + 6p_3$$
$$\text{s.t. } p_1 \leq 0$$
$$p_2 \geq 0$$
$$p_3 \text{ free}$$
$$2p_1 + 3p_2 - p_3 \geq 1$$
$$3p_1 + p_2 - p_3 \leq -1$$
$$-p_1 + 4p_2 + 2p_3 \leq 0$$
$$p_1 - 2p_2 + p_3 = 0$$

i.e.

$$\max\ 3p_2 + 6p_3$$
$$\text{s.t. } 2p_1 + 3p_2 - p_3 \geq 1$$
$$3p_1 + p_2 - p_3 \leq -1$$
$$-p_1 + 4p_2 + 2p_3 \leq 0$$
$$p_1 - 2p_2 + p_3 = 0$$
$$p_1 \leq 0$$
$$p_2 \geq 0$$
$$p_3 \text{ free}$$

4.2
solution:
The dual problem is

$$\max \mathbf{p}^{'}\mathbf{b}$$
$$\text{s.t. } \mathbf{p}^{'}\mathbf{A} \leq \mathbf{c}^{'}$$
$$\mathbf{p} \geq 0$$

Convert to equivalent minimization problem

$$\min \ -\mathbf{p}^{'}\mathbf{b}$$
$$\text{s.t. } -\mathbf{p}^{'}\mathbf{A} \geq -\mathbf{c}^{'}$$
$$\mathbf{p} \geq 0$$

In order to be identical, then $-\mathbf{p}^{'}\mathbf{b} = \mathbf{c}^{'}\mathbf{x}$ and $\mathbf{b} = -\mathbf{c}$, and $\mathbf{A}$ must be a square matrix to stay in the same dimension. Moreover, $\mathbf{Ax} \Leftrightarrow -\mathbf{p}^{'}\mathbf{A}$ implies $\mathbf{A} = -\mathbf{A}^{'}$. Also, $\mathbf{p}, \mathbf{x}$ have same nonnegative components.
Consider an example of a primal problem

$$\min x_1 + x_2$$
$$\text{s.t. } -x_2 \geq -1$$
$$x_1 \geq -1$$
$$x_1, x_2 \geq 0$$

The equivalent dual problem is

$$\max \ -p_1 - p_2$$
$$\text{s.t. } p_2 \leq 1$$
$$-p_1 \leq 1$$
$$p_1, p_2 \geq 0$$

Convert to minimization problem

$$\min p_1 + p_2$$
$$\text{s.t. } -p_2 \geq -1$$
$$p_1 \geq -1$$

$$p_1, p_2 \geq 0$$

which is identical to the primal problem.

4.4

solution:

Let $\mathbf{A}$ be a symmetric square matrix, i.e. $\mathbf{A} = \mathbf{A}^{'}$ and assume $\mathbf{x}^*$ satisfies $\mathbf{A}\mathbf{x}^* = \mathbf{c}$ and $\mathbf{x}^* \geq 0$, then $\mathbf{x}^*$ is a feasible solution. Moreover, the dual problem is identical to the primal problem by the previous exercise,

$$\max \ \mathbf{p}^{'}\mathbf{c}$$

$$\text{s.t. } \mathbf{p}^{'}\mathbf{A} \leq \mathbf{c}^{'}$$

$$\mathbf{p} \geq 0$$

Since the problem is identical, then let $\mathbf{p}^* = \mathbf{x}^*$. i.e.

$$(\mathbf{p}^*)^{'}\mathbf{A} = \mathbf{A}^{'}\mathbf{p}^* = \mathbf{A}\mathbf{p}^* = \mathbf{A}\mathbf{x}^* = \mathbf{c}$$

Thus $\mathbf{p}^*$ is a feasible solution of the dual problem. Therefore, $(\mathbf{p}^*)^{'}\mathbf{c} = \mathbf{c}\mathbf{x}^*$. As a result both $\mathbf{x}^*, \mathbf{p}^*$ are optimal solutions by Corollary 4.2.