Math 430 Assignment 13

Arnold Jiadong Yu

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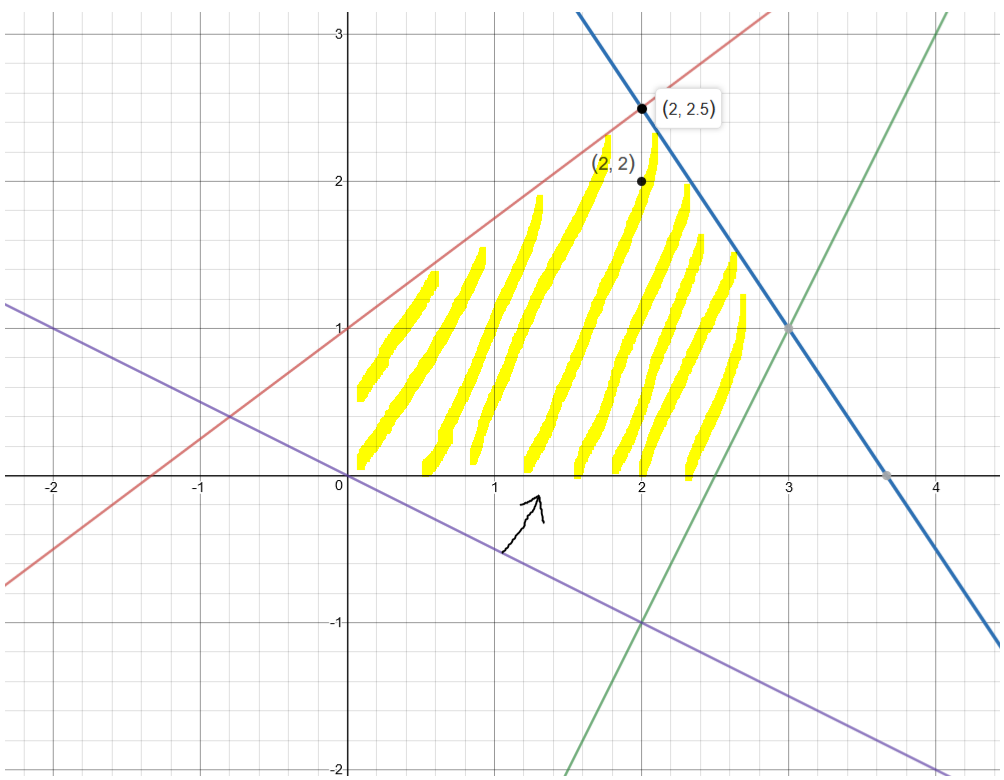
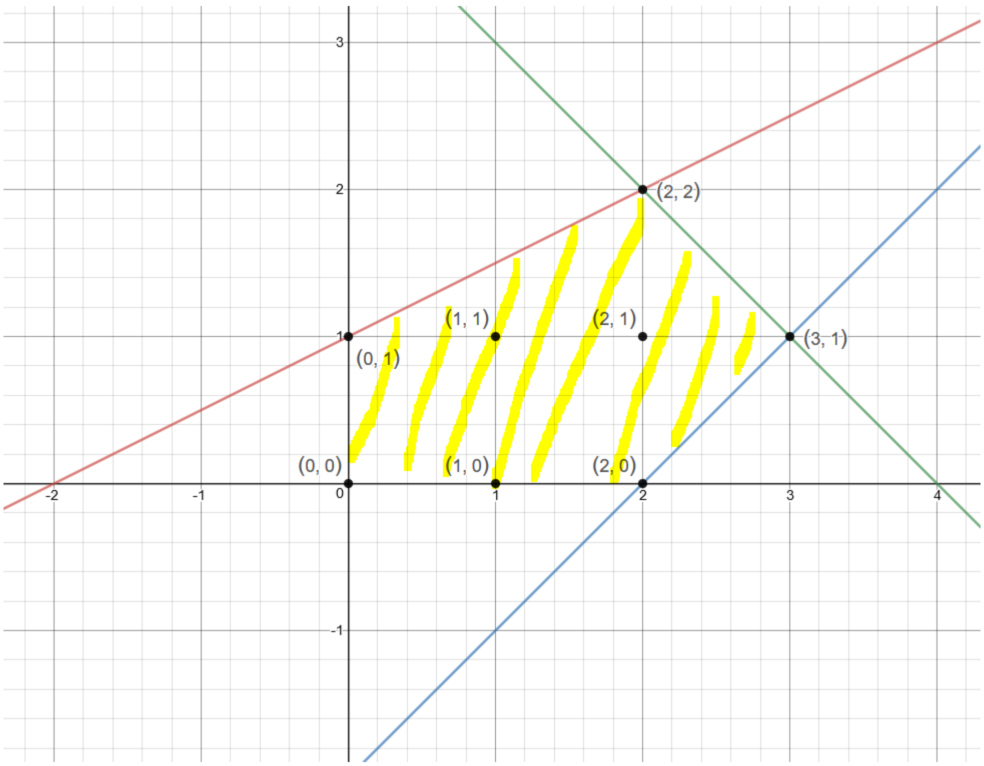
11.1
solution:



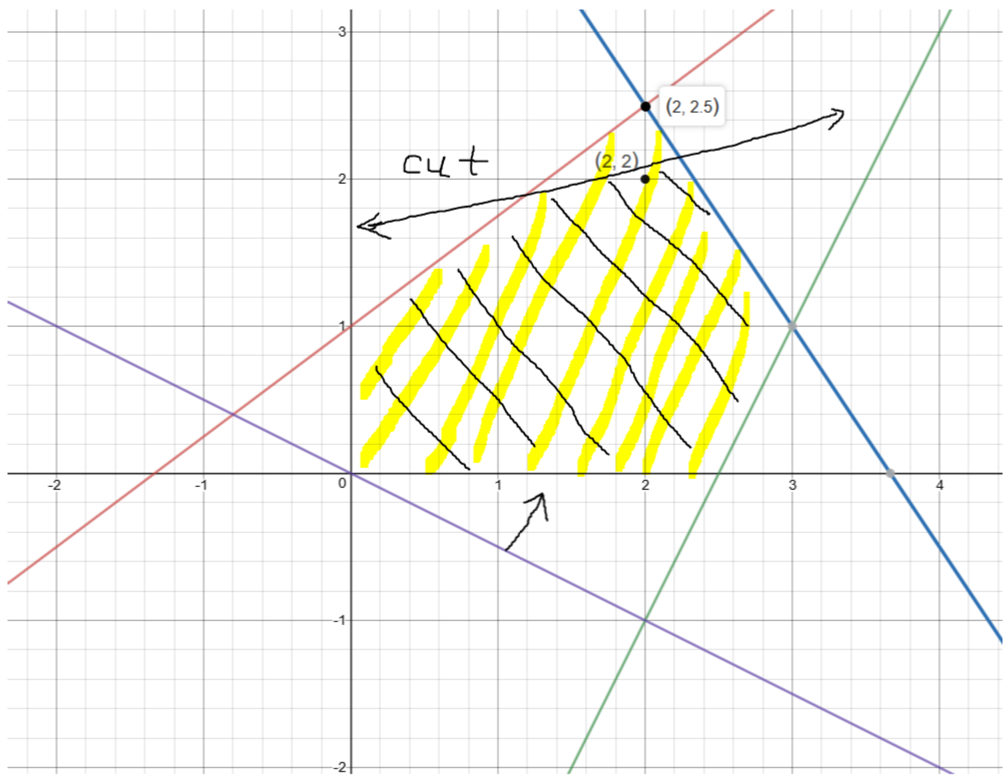
a) From the graph we observed optimal cost is 7 for the LP relaxation with optimal solution $(2, 2.5)$. The optimal cost is 6 for the IP with optimal solution $(2, 2)$.
b) The set of all integer solutions are $(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 1), (2, 2)$.

Therefore, we formed the constraints to satisfy the convex hull of this set of solutions. Let I be the set of solutions above, then

$$\text{conv}(I) = \{(x_1, x_2) | -x_1 + 2x_2 \leq 2, x_1 - x_2 \leq 2, x_1 + x_2 \leq 4, x_1, x_2 \geq 0\}$$



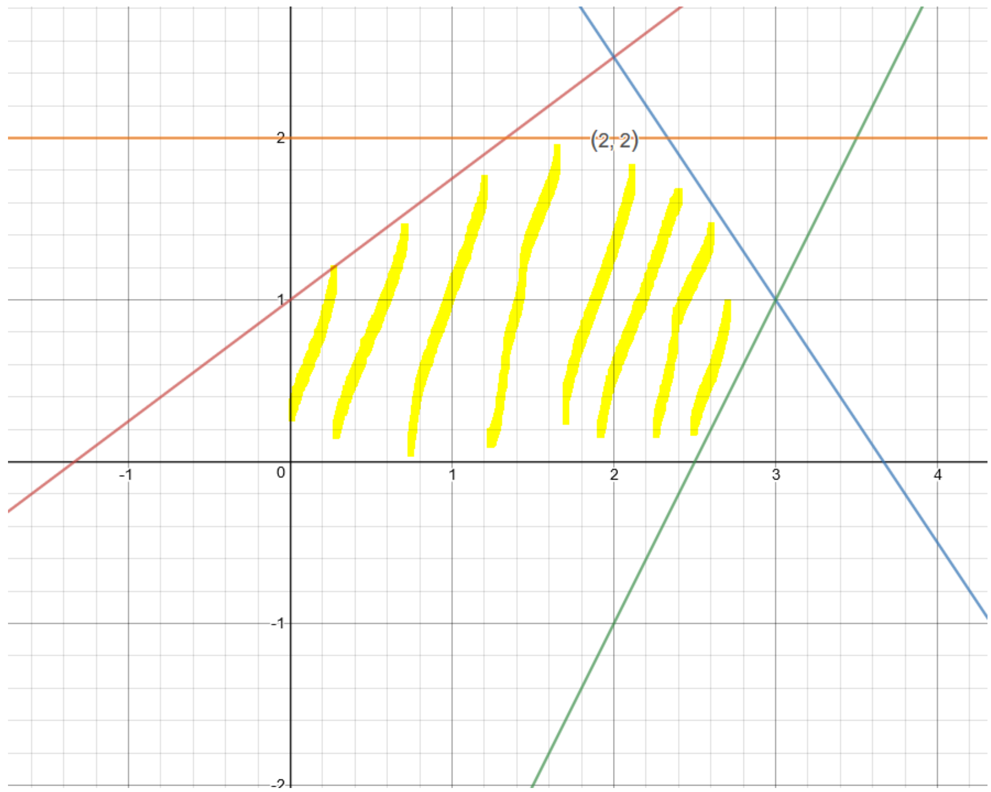
c)The Gomory cutting plan algorithm is to add one new constraint each time by squeezing the polyhedron closer to the conv(I). The cross section will be the new polyhedron below.



Since we know the optimal of LP is $(2, 2.5)$. We need to find the tableau and take the floor of each side. We can obtain the results by calculating $\mathbf{B^{-1}b}, \mathbf{B^{-1}A}, \mathbf{c} - \mathbf{c_B B^{-1}A}$.

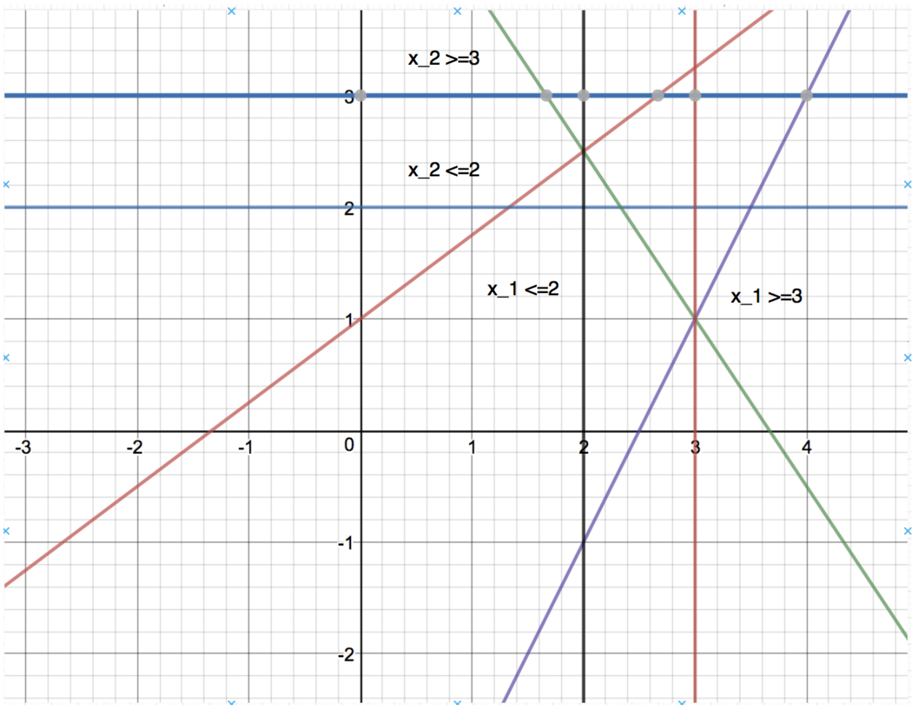
7	0	0	$\frac{2}{9}$	$\frac{5}{9}$	0
$x_1 \to 2$	1	0	$-\frac{1}{9}$	$\frac{2}{9}$	0
$x_2 \to 2.5$	0	1	$\frac{1}{6}$	$\frac{1}{6}$	0
$x_5 \to 3.5$	0	0	$\frac{7}{18}$	$-\frac{5}{18}$	1

i.e $x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \leq \frac{5}{2} \implies x_2 + \lfloor\frac{1}{6}\rfloor x_3 + \lfloor\frac{1}{6}\rfloor x_4 \leq \lfloor\frac{5}{2}\rfloor$. Hence the new constraint is $x_2 \leq 2$. The new graph is



d)

From part a) we get the optimal solution $\mathbf{x}^1 = (2, 2.5)$, then $b(F) = 7$. We create two subproblems, by adding the constraints $x_2 \geq 3$(subproblem F_1), or $x_2 \leq 2$(subproblem F_2). F_1 is infeasible, therefore we delete the subproblem. Now we slove for F_2, we obtain $\mathbf{x}^2 = (\frac{7}{3}, 2)$ and $b(F_2) = \frac{19}{3}$. We create another two subproblems, by adding the constraints $x_1 \geq 3$ (subproblem F_3) or $x_1 \leq 2$ (subproblem F_4). Now we slove F_3, and obtain $\mathbf{x}^3 = (3, 1)$ and $b(F_3) = 5$. i.e. $U = 5$ for now. Now we slove F_4, and obtain $\mathbf{x}^4 = (2, 2)$ an $b(F_4) = 6 \geq U = 5$. Hence the optimal cost is 6 and optimal solution is $(2, 2)$.

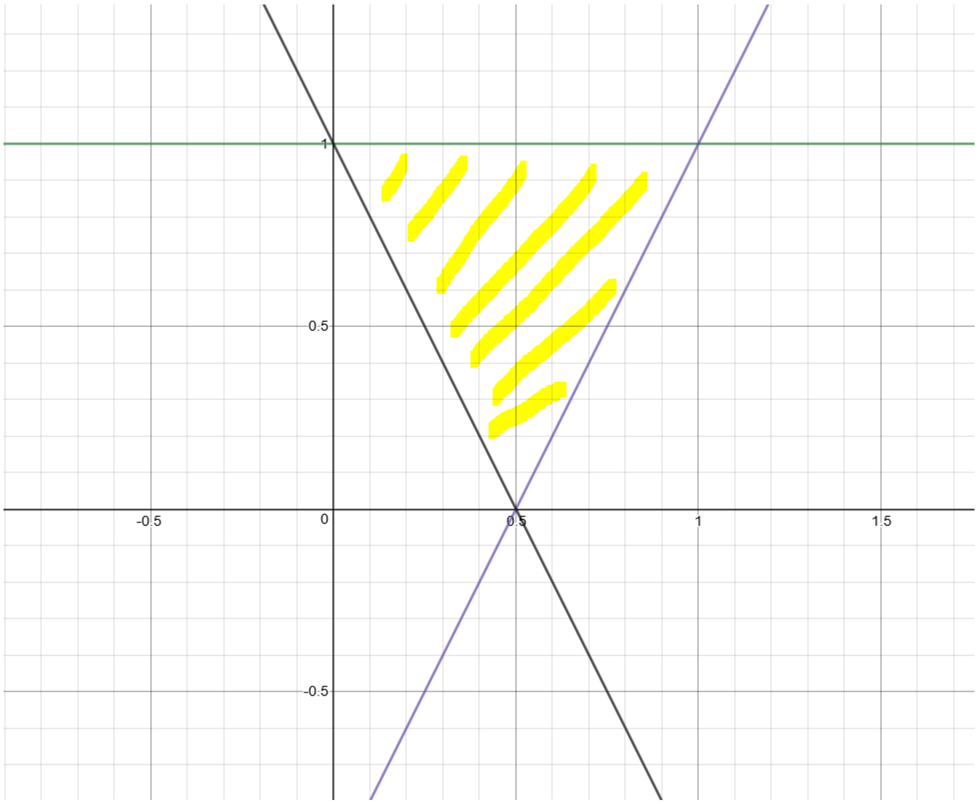


11.2

solution:

a) If the optimal cost is $-\infty$ for the LP relaxation, then the polyhedron is unbounded. Since the IP problem is feasible. Pick an arbitrary integer point $\mathbf{y}$ in the polyhedron and choose a feasible direction $\mathbf{d}$, we change the point to $\mathbf{y}^* = \mathbf{y} + \theta\mathbf{d}$ where $\theta > 0$. Then we separate $\mathbf{y}^*$ into two cases and use branch and bound method. Since the LP has optimal solution $-\infty$, we always can find a feasible direction that to obtain a $\mathbf{y}^*$. i.e. the steps are infinite, therefore IP problem must have optimal cost $-\infty$.

b) No. Here is a counter example. Consider minimize x_2 given the polyhedron below.



Then $Z_{LP} = 0$ and $Z_{IP} = 1$. There doesn't exist $a > 0$ s.t. $1 \leq a \cdot 0$. Hence proved.

11.3

solution:
We consider two cases.
Case (1): $\sum_{j \in J} d_j y_j > b - \lfloor b \rfloor - 1$, then

$$\sum_{j \in N} a_j x_j + b - \lfloor b \rfloor - 1 < \sum_{j \in N} a_j x_j + \sum_{j \in J} d_j y_j \leq b$$

$$\implies \sum_{j \in N} a_j x_j < \lfloor b \rfloor + 1 = \lceil b \rceil \implies \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \sum_{j \in N} a_j x_j < \lceil b \rceil$$

since $\mathbf{x}$ is nonnegative. Moreover,

$$\sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor$$

Since both side are integers.

$$\frac{1}{1-b+\lfloor b\rfloor} \leq 1 \text{ since } b \geq \lfloor b\rfloor \text{ and } \sum_{j\in J^-} d_j y_j \leq 0 \text{ since } d_j < 0, \mathbf{y} \geq 0$$

i.e.

$$\frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq 0 \implies \sum_{j\in N} \lfloor a_j\rfloor x_j + \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq \sum_{j\in N} \lfloor a_j\rfloor x_j$$

$$\implies \sum_{j\in N} \lfloor a_j\rfloor x_j + \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq \lfloor b\rfloor$$

Case (2) : $\sum_{j\in J} d_j y_j \leq b - \lfloor b\rfloor - 1$, then $\sum_{j\in J} d_j y_j \leq b - \lfloor b\rfloor - 1 < 0$. Moreover, $\sum_{j\in J^-} d_j y_j \leq \sum_{j\in J} d_j y_j$ since the negative parts overcome the positive. i.e.

$$\sum_{j\in J^-} d_j y_j \leq \sum_{j\in J} d_j y_j \leq b - \lfloor b\rfloor - 1 = -1 \cdot (1 - b + \lfloor b\rfloor)$$

$$\implies \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq -1$$

Then

$$b \geq \sum_{j\in N} a_j x_j + \sum_{j\in J} d_j y_j \geq \sum_{j\in N} \lfloor a_j\rfloor x_j + \sum_{j\in J^-} d_j y_j$$

$$= \sum_{j\in N} \lfloor a_j\rfloor x_j + \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j + \frac{-b+\lfloor b\rfloor}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j$$

$$\implies b + \frac{b-\lfloor b\rfloor}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \geq \sum_{j\in N} \lfloor a_j\rfloor x_j + \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j$$

$$b + \frac{b-\lfloor b\rfloor}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq b + (b - \lfloor b\rfloor) \cdot (-1) = \lfloor b\rfloor$$

i.e.

$$\sum_{j\in N} \lfloor a_j\rfloor x_j + \frac{1}{1-b+\lfloor b\rfloor} \sum_{j\in J^-} d_j y_j \leq \lfloor b\rfloor$$

Hence statements proved.

11.4

solution:

Claim: In the worst case, it will take 2^n to find the optimal solution by branch and bound.

When n is odd, let $n = 2k+1$ for some integer k, then $2(x_1 + ... + x_n) = 2k + 1 - x_{n+1}$. i.e. x_{n+1} is odd. Consider the worst case, all other x_i where $1 \leq i \leq n$ are fractional variable. First, we branch into two case. First case is $x_1 = 0$, and second case is $x_1 = 1$. i.e. we keep doing this until x_n. In combinatorics, there are two choices for the all variables except x_{n+1}, i.e. it takes $\underbrace{2 \cdot 2 \cdot ... \cdot 2}_{n} \cdot 1 = 2^n$. In graph theory, the worst case is the full binary search tree. It will take exponential time in the worst case. Hence proved.