MATH 435 ASSIGNMENT 1

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1. Chapter 2 Groups Homework

1.1. Which of the following sets are closed under the given operation?
a. {0, 4, 8, 12} addition mod 16
b. {0, 4, 8, 12} addition mod 15
c. {1, 4, 7, 13} multiplication mod 15
d. {1, 4, 5, 7} multiplication mod 9
proof:
a. Let S be the set $\{0, 4, 8, 12\}$.

S_{16}	0	4	8	12
0	0	4	8	12
4	4	8	12	0
8	8	12	0	4
12	12	0	4	8

It is closed by Cayley table.
b. Let S be the set $\{0, 4, 8, 12\}$.

S_{15}	0	4	8	12
0	0	4	8	12
4	4	8	12	1
8	8	12	1	5
12	12	1	5	9

It is not closed by Cayley table because $1, 5, 9 \notin \{0, 4, 8, 12\}$.
c. Let S be the set $\{1, 4, 7, 13\}$.

S_{15}	1	4	7	13
1	1	4	7	13
4	4	1	13	7
7	7	13	1	1
13	13	7	1	4

It is closed by Cayley table.
d. Let S be the set $\{1,4,5,7\}$.

S_9	1	4	5	7
1	1	4	5	7
4	4	7	2	1
5	5	2	7	8
7	7	1	8	4

It is not closed by Cayley table because $2,8 \notin \{1,4,5,7\}$.

1.2. Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, Z_{11})$.

proof: Let matrix A be $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$. $\det(A) = 10 - 18 = -8 \neq 0$, therefore A has an inverse denoted A^{-1}. Let us try normal matrix A in $GL(2,\mathbb{R})$, then

$$A^{-1} = \frac{1}{-8}\begin{bmatrix} 5 & -6 \\ -3 & 2 \end{bmatrix}$$

Moreover $-8 \mod 11 = 3 \mod 11$, what to find A^{-1} in Z_{11} when $AA^{-1} \mod 11 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since operation in Z_{11} is multiplication and dividing -8 is same as multiple inverse of 3 in Z_{11}. By Cayley table, the inverse of 3 in Z_{11} is 4. That is

$$A^{-1} = 4\begin{bmatrix} 5 & -6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 20 & -24 \\ -12 & 8 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix} \text{ in } Z_{11}$$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix} = \begin{bmatrix} 78 & 66 \\ 77 & 67 \end{bmatrix} \mod 11 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence the inverse element is $\begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix}$ in $GL(2, Z_{11})$

1.3. *. Suppose that in the definition of a group G, the condition that there exists an element e with property $ae = ea = a$ for all a in G is replaced by $ae = a$ for all a in G. Show that $ea = a$ for all a in G.(Thus, a one-sided identity is a two-sided identity.)
proof: By inverse definition of a group G. $\forall a \in G, \exists a^{-1} \in G$, s.t. $aa^{-1} = a^{-1}a = e$. Therefore,

$$aa^{-1} = e \Rightarrow (aa^{-1})a = ea \Rightarrow a(a^{-1}a) = ea \Rightarrow ae = ea \Rightarrow ea = a$$

by associativity.

2. Chapter 3 Finite Groups; Subgroups Homework

2.1. Let S be a subset of a group and let H be the intersection of all subgroups of G that contain S.
a. Prove that $< S > = H$.
b*. If S is nonempty, prove that $< S >= \{s_1^{n_1} s_2^{n_2} ... s_m^{n_m} | m \geq 1, s_i \in S, n_i \in Z\}$.(The s_i terms need not be distinct.)
proof: a. $(\Rightarrow)$. WTS, $< S > \subseteq H$. Let H be the intersection of all subgroups of G that contain S, let K be an arbitrary subgroup of G contain S, then

$$H = \bigcap_{\forall K \leq G \text{ s.t. } S \subseteq K} K$$

Since K is a group itself and $S \subseteq K$, by definition $< S > \leq K$. Since H is the intersection of all those K and $< S >$ is subgroup of all those K, then $< S > \subseteq H$.
$(\Leftarrow)$ WTS: $H \subseteq < S >$. Let $h \in H$, then h is in the intersection of all those K contain S. That is $h \in K$ for $\forall K \leq G$ s.t. $S \subseteq K$. By definition, $< S >$ is the smallest subgroup of G contain S. That is $< S >$ is one of the K. Hence $h \in < S >$, this implies $H \subseteq < S >$. Therefore, $< S >= H$.

3. Chapter 7 Cosets and Lagrange's Theorem Homework

3.1. *Let $\mathbf{C}^*$ be the group of nonzero complex numbers under multiplication and let $H = \{a + bi \in \mathbf{C}^* | a^2 + b^2 = 1\}$. Give a geometric description of the coset $(3 + 4i)H$. Give a geometric description of the coset $(c + di)H$.
proof: By the definition of H, the geometric description of H is a circle with radius 1 center at origin on the complex plane. The geometric description of coset $(3 + 4i)H$ is a circle with radius 5 center at origin on the complex plane. The geometric description of coset $(c + di)H$ is a circle with radius $\sqrt{c^2 + d^2}$ center at origin on the complex plane.

3.2. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of $|H|$?
proof: Let $|H| = a$, then $42 < a < 420$. Moreover by Lagrange's Theorem, $42|a$ and $a|420$. Since $420/42 = 10$, and factors of 10 are $1, 2, 5, 10$. So the possible a could be 84, 210.

3.3. Suppose that H is a subgroup of S_4 and that H contains (12) and (234). Prove $H = S_4$.
proof: Let H be a subgroup of S_4, then $H \leq S_4$. Moreover H contains (12) and (234), $|(12)| = 2$ and $|(234)| = 3$. $(234)(12) = (3142)$ and

$|(3142| = 4$, $(12)(234) = (2341)$ and $|(2341)| = 4$. That is $2, 3, 4$ divides $|H|$ and $|H|$ divides $|S|$ by Lagrange's Theorem. $|H| = 12$ or 24. Moreover, $\#H \geq 2 + 3 + 4 + 4 = 13$. Hence $\#H = 24 = \#S_4$. With $H \leq S_4$, implies $H = S_4$.