Math 335 Assignment 1

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(1) Show that, for a finite set S and a map $f : S \to S$, the following conditions are equivalent:
(i) f is injective.
(ii) f is surjective.
(iii) f is bijective.

proof: (i)Suppose a finite set S has n $\in \mathbb{N}$ elements, a finite set S_1 is the same as S, and f is injective. Thus $f : S \to S$ is the same as $f : S \to S_1$, S_1 has at least n elements. By assumption, S_1 has at most n elements, thus all elements in S_1 is mapped from S. Hence it is also surjective.

(ii)Suppose a finite set S has n elements, a finite set S_1 is the same as S, and f is surjective. Thus, $f : S \to S$ is the same as $f : S \to S_1$. By assumption, $\forall y \in S_1$, $\exists$ a $x \in S$ s.t. $y = f(x) \in S_1$. S_1 has n elements, thus S has at least n elements. Since S has only n elements. All elements in S is mapped to S_1. Hence, all elements in S is mapped to all elements in S_1, it is also injective.

(iii)f is both injective and surjective, then it is bijective. The converse is trivial. If f is bijective, then it must be both injective and surjective.

Hence, for a finite set S and a map $f : S \to S$, injective, surjective and bijective are equivalent.

(2) Show that, for any infinite set S, there exists:
(ii) a surjective but non-injective map $f : S \to S$.
(iiii) an injective but non-surjective map $g : S \to S$.

proof: (ii) For any infinite set S, $\exists$ a infinite countable subset M s.t. $M \subseteq S$, and M has the same cardinality as $\mathbb{N}$. Then construct a function f, s.t. there is a surjective $f : S \to \mathbb{N}$. Since every elements in $\mathbb{N}$ was mapped from the elements in S, but S is not countable. Thus there is more elements in S than in $\mathbb{N}$. Hence, it is non-injective.

(iiii) For any infinite set S, $\exists$ a infinite countable subset M s.t. $M \subseteq S$, and M has the same cardinality as $\mathbb{N}$. Then construct a function g, s.t. there is a injective $g : \mathbb{N} \to S$. Since every elements in $\mathbb{N}$ mapped to the elements in S, but S is not countable. Thus there is more elements in S than in $\mathbb{N}$. Hence, it is non-surjective.

(3) Give an example of a set X and a binary operation $X \times X \to X$, satisfying the associativity and the neutral element axioms, but not the inverse element axiom.

proof: Let a set X be non-negative real number, $X = \mathbb{R}_{\geq 0}$ and binary operation is addition. The sum of every two non-negative real number is also non-negative real number, which satisfying $X \times X \to X$. It also satisfies associativity and neutral element axioms where the neutral element is 0. The inverse element will be the negative of that real number, but it is not in the set X.

(4) Let G be a group. Consider the map $G \to G.x \mapsto x^{-1}$ Is this map bijective?

proof: True. Given G is a group, and binary operation is multiplication. Thus G satisfies associativity, the neutral element axioms($e = 1$), and the inverse element axiom. Hence $\forall x \in G$, $\exists! x^{-1} \in G$ s.t. $xx^{-1} = 1$. Since $G \to G \to G$, $x \mapsto x^{-1} \mapsto x$ is identity map. Then every elements on the left mapped to its own inverse, and inverse is unique. Inverse elements mapped to its inverse, which is itself. Hence $x \mapsto x^{-1}$ is bijective.

(5) Give an explicit example of a group G for which the map, considered in Problem (4), a fixed an element, different from the neutral element.

proof: Let a group G be $\mathbb{Z}_2 = \{0, 1\}$, binary operation is modular addition. $0 +_2 0 = 0$ and $1 +_2 0 = 1$, hence 0 is the neutral element. But $1 +_2 1 = 0$, means 1 is its own inverse, followed the map in question (4), different from the neutral element.