Math 741 Assignment 10 (Quiz)

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6.4.4. (H)
solution: Given,

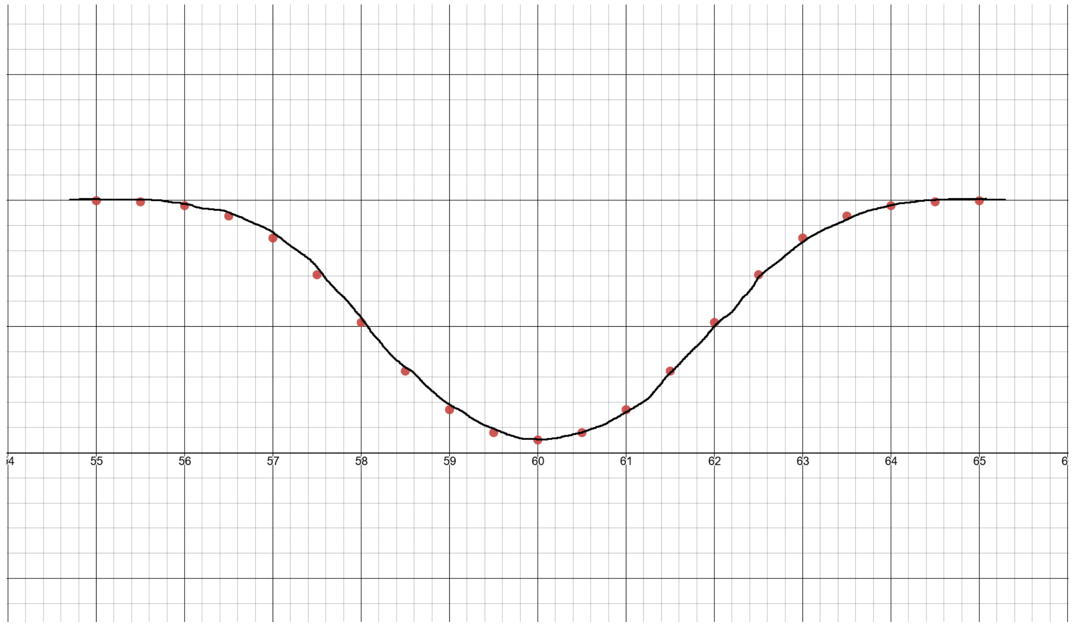
$$H_0 : \mu = 60$$
$$H_1 : \mu \neq 60$$

with $n = 16, \sigma = 4, \alpha = 0.05$. Then the compliment of the critical region is

$$(-z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}+\mu_0, z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}+\mu_0) = (-1.96\cdot\frac{4}{4}+60, 1.96\cdot\frac{4}{4}+60) = (58.04, 61.96)$$

$$\Pi(\beta) = P(\text{ Reject } H_0|H_1 \text{ is true}) = P(\bar{Y} < 58.04) + P(\bar{Y} > 61.96)$$

μ	$\Pi(\beta)$	μ	$\Pi(\beta)$
55	0.998817	65	0.998817
55.5	0.994457	64.5	0.994457
56	0.979325	64	0.979325
56.5	0.938220	63.5	0.938220
57	0.850830	63	0.850830
57.5	0.705406	62.5	0.705406
58	0.515991	62	0.515991
58.5	0.323028	61.5	0.323028
59	0.170066	61	0.1700660
59.5	0.079092	60.5	0.079092



6.4.5.

solution: Given,

$$H_0 : \mu = 240$$

$$H_1 : \mu < 240$$

with $n = 25, \sigma = 50, \alpha = 0.01$ and $\mu_1 = 220$. Then

$$\alpha = P(\bar{X} < \bar{x} | H_0 \text{ is true}) = 0.01$$

$$P(Z < \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}) = 0.01 \Rightarrow \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = -2.326 \Rightarrow \bar{x} \approx 216.74$$

$$\beta = P(\bar{X} > \bar{x} | H_1 \text{ is true}) = P(Z > \frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}})$$

$$= P(Z > \frac{216.74 - 220}{50/\sqrt{25}}) = 1 - P(Z < -0.32635) = 1 - 0.3721 = 0.6279$$

Hence, there is 0.629 of the time that the procedure fail to recognize that μ has dropped to 220.

6.4.6.

solution: Given,

$$H_0 : \mu = 60$$

$$H_1 : \mu \neq 60$$

with $n = 36, \sigma = 8.0, \alpha = 0.07$. Then

$$\alpha = P(60 - \bar{y}^* < \bar{Y} < 60 + \bar{y}^* | H_0 \text{ is true}) = 0.07$$

Since the distribution is normal, then

$$P(\bar{Y} < 60 - \bar{y}^*) = 0.465 \Rightarrow P(Z < \frac{-\bar{y}^*}{\sigma/\sqrt{n}}) = 0.465$$

$$\Rightarrow \frac{-\bar{y}^*}{\sigma/\sqrt{n}} = -0.087845 \Rightarrow \bar{y}^* = 0.11713$$

b) Let $\mu_1 = 62$, then

$$\Pi(\beta) = P(60 - \bar{y}^* < \bar{Y} < 60 + \bar{y}^* | H_1 \text{ is true})$$

$$= P(\frac{60 - \bar{y}^* - \mu_1}{\sigma/\sqrt{n}} < Z < \frac{60 + \bar{y}^* - \mu_1}{\sigma/\sqrt{n}})$$

$$= P(\frac{60 - 0.11713 - 62}{8.0/\sqrt{36}} < Z < \frac{60 + 0.11713 - 62}{8.0/\sqrt{36}}) = 0.02279$$

c) If the critical region had defined the right way, then

$$z_{\alpha/2} = z_{0.035} = 1.812$$

Therefore, $C = \{\bar{y} : \bar{y} < \mu_0 - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\} \cup \{\bar{y} : \bar{y} > \mu_0 + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\}$ Hence, $C = \{\bar{y} : \bar{y} < 57.584\} \cup \{\bar{y} : \bar{y} > 62.416\}$

$$\Pi(\beta) = P(\bar{Y} < 57.584 \text{ or } \bar{Y} > 62.416 | H_1 \text{ is true})$$

$$= P(Z < \frac{57.584 - 62}{8/6} \text{ or } Z > \frac{62.416 - 62}{8/6}) = 1 - P(-3.312 < Z < 0.312)$$
$$= 0.62202$$

6.4.7.
solution: Given,

$$H_0 : \mu = 200$$

$$H_1 : \mu < 200$$

with $\sigma = 15.0, \alpha = 0.10$ and $\Pi(\beta) \geq 0.75, \mu_1 = 197$. Then, the compliment of critical region is

$$(\mu_0 + z_\alpha \cdot \frac{\sigma}{\sqrt{n}}, \infty)$$

and

$$\Pi(\beta) = 1 - P(Z > \frac{\mu_0 + z_\alpha \cdot \frac{\sigma}{\sqrt{n}} - \mu_1}{\frac{\sigma}{\sqrt{n}}})$$

$$\Rightarrow P(Z < \frac{200 + 1.2816 \cdot \frac{15}{\sqrt{n}} - 197}{\frac{15}{\sqrt{n}}}) = 0.75$$

$$\Rightarrow n \approx 95.657$$

Hence, at least $n = 96$ to make the power equal to at least 0.75 when $\mu = 197$.

6.4.9.

solution: Given,

$$H_0 : \mu = 30$$

$$H_1 : \mu > 30$$

with $n = 16, \sigma = 9.0$, and $1 - \beta = 0.85, \mu_1 = 34$. Then

$$P(\bar{X} > \bar{x} | H_1 \text{ is true}) = 0.85$$

$$P(Z > \frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}}) = P(Z > \frac{\bar{x} - 34}{9/4}) = 0.85$$

$$\Rightarrow \bar{x} = 31.668$$

$$\alpha = P(\bar{X} > \bar{x} | H_0 \text{ is true}) = P(Z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}})$$

$$= P(Z > 0.7413443) = 0.2292$$

6.4.10. (H)

solution: Given,

$$f_Y(y) = \begin{cases} \frac{1}{\lambda} e^{-y/\lambda} & y > 0 \\ 0 & o.w. \end{cases}$$

Then, $F_Y(y) = \int_y^\infty \frac{1}{\lambda} e^{-t/\lambda} dt = e^{-y/\lambda}$ Moreover,

$$H_0 : \lambda = 1$$

$$H_1 : \lambda > 1$$

with critical region $[3.20, \infty)$.

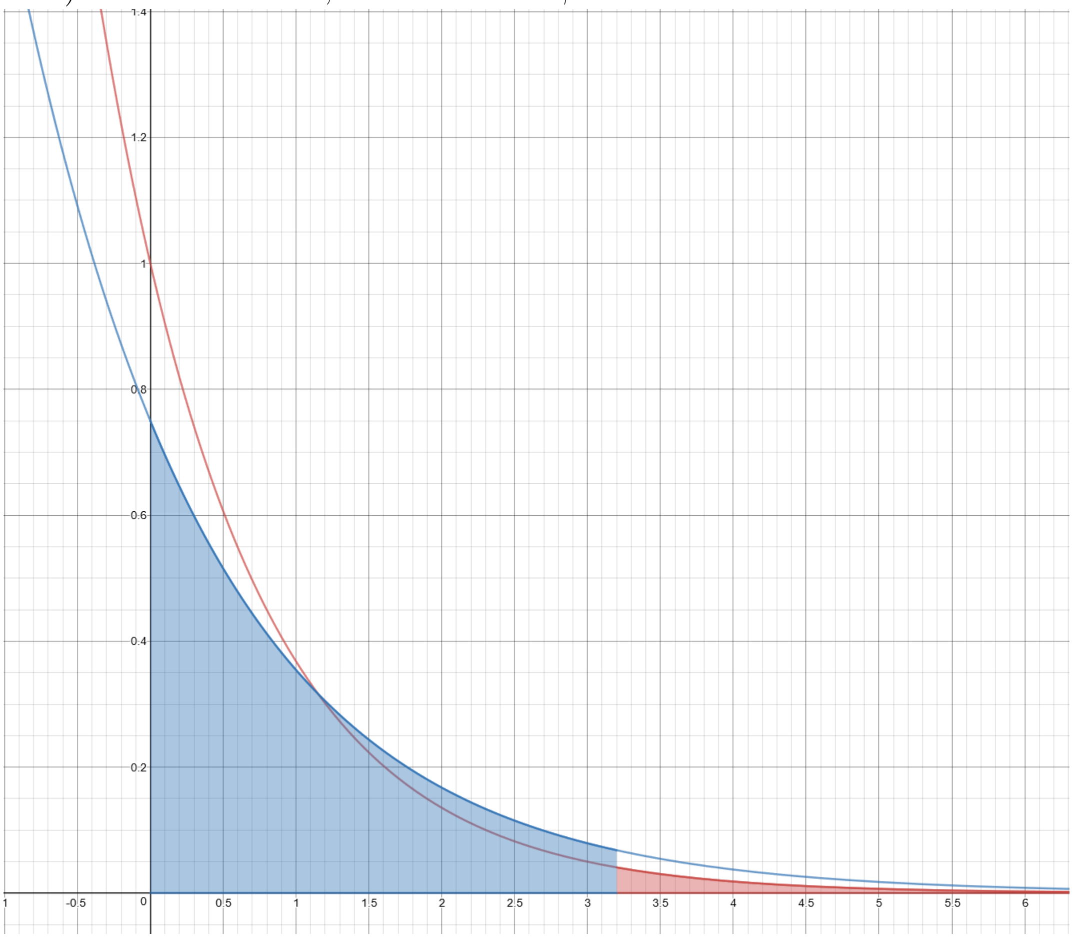
a)

$$\text{Type I error } = P(Y \geq 3.20 | H_0 \text{ is true }) = \int_{3.20}^\infty e^{-y} dy = -e^{-y}|_{3.2}^\infty = 0.04076$$

b)

$$\text{Type II error } = P(Y < 3.20 | H_1 \text{ is true }) = \int_0^{3.20} \frac{3}{4} e^{-3y/4} dy = -e^{-3y/4}|_0^{3.2} = 0.9093$$

c) The red area is α, the blue area is β.



6.4.12.

solution: Given

$$H_0 : \text{exactly half the chips are white}$$

$$H_1 : \text{more than half the chips are white}$$

with $n = 10$. Draw without replacement, three chips and reject H_0 if two or more are white. The pdf of hypergeometric is $\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$,

a) $n_W = 6$,

$$\alpha = P(X < 2|H_0 \text{ is ture}) = \frac{\binom{5}{0}\binom{10-5}{3-0}}{\binom{10}{3}} + \frac{\binom{5}{1}\binom{10-5}{3-1}}{\binom{10}{3}}$$

$$= \frac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} + \frac{\binom{5}{1}\binom{5}{2}}{\binom{10}{3}} = \frac{1}{2}$$

$$\beta = P(X \geq 2|H_1 \text{ is ture}) = \frac{\binom{6}{2}\binom{10-6}{3-2}}{\binom{10}{3}} + \frac{\binom{6}{3}\binom{10-6}{3-3}}{\binom{10}{3}}$$

$$= \frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} + \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} = \frac{2}{3}$$

b) $n_W = 7$,

$$\alpha = P(X < 2|H_0 \text{ is ture}) = \frac{\binom{5}{0}\binom{10-5}{3-0}}{\binom{10}{3}} + \frac{\binom{5}{1}\binom{10-5}{3-1}}{\binom{10}{3}}$$

$$= \frac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} + \frac{\binom{5}{1}\binom{5}{2}}{\binom{10}{3}} = \frac{1}{2}$$

$$\beta = P(X \geq 2|H_1 \text{ is ture}) = \frac{\binom{7}{2}\binom{10-7}{3-2}}{\binom{10}{3}} + \frac{\binom{7}{3}\binom{10-7}{3-3}}{\binom{10}{3}}$$

$$= \frac{\binom{7}{2}\binom{3}{1}}{\binom{10}{3}} + \frac{\binom{7}{3}\binom{3}{0}}{\binom{10}{3}} = \frac{49}{60}$$

6.4.15.
solution: Given

$$H_0 : p = \frac{1}{2}$$

$$H_1 : p > \frac{1}{2}$$

with H_0 rejected if $k = n$. WTF p_1 such that $\beta = 0.05$. Then

$$\beta = 1 - \binom{n}{k} p_1^k (1 - p_1)^{n-k} = 0.05$$

$$\Rightarrow \binom{n}{n} p_1^n (1 - p_1)^0 = 0.95 \Rightarrow p_1 = \sqrt[n]{0.95}$$

6.4.16. (H)

solution: Given X_1 a binomial random variable with $n = 2$, X_2 a binomial random variable with $n = 4$,

$$H_0 : p_{X_1} = p_{X_2} = \frac{1}{2}$$

$$H_1 : p_{X_1} = p_{X_2} > \frac{1}{2}$$

with $X = X_1 + X_2$ and reject H_0 if $k \geq 5$. Then

$$\alpha = P(X \geq 5|H_0 \text{ is true }) = P(X = 5|H_0 \text{ is true }) + P(X = 6|H_0 \text{ is true })$$

$$P(X_1 = 2, X_2 = 3|H_0 \text{ is true }) + P(X_1 = 1, X_2 = 4|H_0 \text{ is true }) + P(X_1 = 2, X_2 = 4|H_0 \text{ is true })$$

$$= P(X_1 = 2|H_0 \text{ is true })P(X_2 = 3|H_0 \text{ is true })$$
$$+P(X_1 = 1|H_0 \text{ is true })P(X_2 = 4|H_0 \text{ is true })$$
$$+P(X_1 = 2|H_0 \text{ is true })P(X_2 = 4|H_0 \text{ is true })$$

$$= \binom{2}{2}(\frac{1}{2})^2(\frac{1}{2})^{2-2} \cdot \binom{4}{3}(\frac{1}{2})^3(\frac{1}{2})^{4-3} + \binom{2}{1}(\frac{1}{2})^1(\frac{1}{2})^{2-1} \cdot \binom{4}{4}(\frac{1}{2})^4(\frac{1}{2})^{4-4}$$

$$+\binom{2}{2}(\frac{1}{2})^2(\frac{1}{2})^{2-2} \cdot \binom{4}{4}(\frac{1}{2})^4(\frac{1}{2})^{4-4} = (\frac{1}{2})^2 \cdot 4 \cdot (\frac{1}{2})^4 + 2 \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^4 + (\frac{1}{2})^2(\frac{1}{2})^4$$

$$= 7 \cdot (\frac{1}{2})^6 = 0.109375$$

Hence, $\alpha = 0.190375$.

6.4.17.

solution: Given,

$$f_Y(y) = \begin{cases} (1+\theta)y^\theta & 0 \leq y \leq 1 \\ 0 & o.w. \end{cases}$$

and

$$H_0 : \theta = 1$$

$$H_1 : \theta < 1$$

with $n = 1$ and reject H_0 if $y \leq \frac{1}{2}$. Then

$$\Pi(\beta) = 1 - \beta = P(\text{ Reject } H_0|H_1 \text{ is true}) = \int_0^{\frac{1}{2}} (1+\theta)y^\theta dy = 2^{-\theta-1}, \theta < 1$$

6.4.20. (H)
solution: Given,

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & o.w. \end{cases}$$

and

$$H_0 : \lambda = 1$$

$$H_1 : \lambda < 1$$

with reject H_0 if $y \geq \ln 10$. Then

$$\beta = P(\text{ Fail to reject } H_0 | H_1 \text{ is true}) = P(Y < \ln 10 | H_1 \text{ is true})$$

$$= \int_0^{\ln 10} \lambda e^{-\lambda y} dy = 1 - e^{-\ln 10 \lambda} = 1 - (\frac{1}{10})^{\lambda}, \lambda < 1$$