Math 741 Assignment 5 (Quiz)

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5.5.1

solution: Given the pdf of the distribution and random samples,

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta}e^{-y/\theta} & y > 0 \\ 0 & o.w. \end{cases}$$

It is exponential distribution with parameter $\frac{1}{\theta}$. Therefore, $E(Y_i) = E(Y) = \theta$ and $\mathrm{Var}(Y_i) = \mathrm{Var}(Y) = \theta^2$. Then

$$E(\hat{\theta}) = E(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \theta$$

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(Y_i) = \frac{\theta^2}{n}$$

i.e. $\hat{\theta}$ is an unbiased estimator for θ. Since the second derivative of $f_Y(y;\theta)$ is continuous,

$$\ln(f_Y(y;\theta) = -\ln\theta - \frac{y}{\theta}$$

$$\frac{\partial}{\partial\theta}(\ln(f_Y(\theta)) = -\frac{1}{\theta} + \frac{y}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}(\ln(f_Y(\theta)) = \frac{1}{\theta^2} - \frac{2y}{\theta^3}$$

$$E(\frac{1}{\theta^2} - \frac{2y}{\theta^3}) = \frac{1}{\theta^2} - \frac{2}{\theta^2}E(Y) = -\frac{1}{\theta^2}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(\frac{1}{\theta^2} - \frac{2y}{\theta^3})} = \frac{\theta^2}{n}$$

The relative efficiency is

$$RE(\hat{\theta}, \alpha) = \frac{\theta^2/n}{\theta^2/n} = 1$$

Hence, $\bar{Y}$ is a best estimator for θ.

5.5.2

solution: Given the Poisson distribution and random samples, then

$$E(X_i) = E(X) = \lambda, \mathrm{Var}(X_i) = \mathrm{Var}(X) = \lambda$$

Then

$$E(\hat{\lambda}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \lambda$$

i.e. $\hat{\lambda}$ is an unbiased estimator for λ.

$$\mathrm{Var}(\hat{\lambda}) = \mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{\lambda}{n}$$

$$\ln p_X(x; \lambda) = -\lambda + x\ln\lambda - \ln x!$$

$$\frac{\partial}{\partial\lambda}(\ln p_X(x; \lambda)) = -1 + \frac{x}{\lambda}$$

$$\frac{\partial^2}{\partial\lambda^2}(\ln p_X(x; \lambda)) = -\frac{x}{\lambda^2}$$

The second derivative is continuous, then

$$E(-\frac{X}{\lambda^2}) = -\frac{1}{\lambda^2}E(X) = -\frac{1}{\lambda}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(-\frac{X}{\lambda^2})} = \frac{\lambda}{n}$$

The relative efficiency is

$$RE(\hat{\lambda}, \alpha) = \frac{\lambda/n}{\lambda/n} = 1$$

Hence, $\hat{\lambda}$ is an efficient estimator for λ.

5.5.3

solution: Let $Y_1, ..., Y_n \sim N(\mu, \sigma^2)$ where σ^2 is known. Then

$$E(Y_i) = E(Y) = \mu, \mathrm{Var}(Y_i) = \mathrm{Var}(Y) = \sigma^2$$

Therefore,

$$E(\bar{Y}) = \mu, \mathrm{Var}(\bar{Y}) = \frac{\sigma^2}{n}$$

$$\ln f_Y(y; \mu) = \ln(\frac{1}{\sqrt{2\pi}\sigma}) - \frac{1}{2\sigma^2}(Y - \mu)^2$$

$$\frac{\partial}{\partial \mu} = -\frac{1}{2\sigma^2}(y - u)$$

$$\frac{\partial^2}{\partial \mu^2} = -\frac{1}{2\sigma^2}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(-\frac{1}{2\sigma^2})} = \frac{\sigma^2}{n}$$

The relative efficiency is

$$RE(\hat{\lambda}, \alpha) = \frac{\lambda/n}{\lambda/n} = 1$$

Hence, $\bar{Y}$ is a best estimator for μ.

5.5.4(H)

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y; \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq y \leq \theta \\ 0 & \text{o.w.} \end{cases}$$

$$\ln f_Y(y; \theta) = -\ln \theta$$

$$\frac{\partial}{\partial\theta}(\ln f_Y(y;\theta)) = -\frac{1}{\theta}$$
$$\frac{\partial^2}{\partial\theta^2}(\ln f_Y(y;\theta)) = \frac{1}{\theta^2}$$
$$E(\frac{1}{\theta^2}) = \frac{1}{\theta^2}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(\frac{1}{\theta^2})} = \frac{\theta^2}{n}$$

Thus

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta} dt = \frac{y}{\theta}$$

$$f_{Y_{\max}}(y) = n[F_Y(y)]^{n-1} f_Y(y) = n\left[\frac{y}{\theta}\right]^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$$

$$\mathrm{Var}(\hat{\theta}) = \frac{(n+1)^2}{n^2}\mathrm{Var}(Y_{\max}) = \frac{(n+1)^2}{n^2}(E(Y_{\max}^2) - (E(Y_{\max})^2)$$
$$= \frac{(n+1)^2}{n^2}(\int_0^\theta y^2 \frac{ny^{n-1}}{\theta^n}dy - (\int_0^\theta y\frac{ny^{n-1}}{\theta^n}dy)^2)$$
$$= \frac{(n+1)^2}{n^2}(\frac{n\theta^2}{n+2} - (\frac{n\theta}{n+1})^2) = \frac{(n+1)^2}{n^2}(\frac{n\theta^2(n+1)^2 - n^2\theta^2(n+2)}{(n+2)(n+1)^2})$$
$$= \frac{\theta^2}{n(n+2)}$$

The relative efficiency is

$$RE(\hat{\theta},\alpha) = \frac{\frac{\theta^2}{n}}{\frac{\theta^2}{n(n+2)}} = n+2 > 1, \forall n$$

The variance of unbiased estimator $\hat{\theta}$ is more efficient than CRLB.
This should not happen. The error comes from that we can not use CRLB since the set of y values depend on θ.

5.5.5

solution:

$$E(\bar{X}) = \theta, \mathrm{Var}(\bar{X}) = \frac{\theta(\theta-1)}{n}$$

$$\ln f_X(x;\theta) = (x-1)\ln(\theta-1) - x\ln\theta$$
$$\frac{\partial}{\partial\theta}(\ln f_X(x;\theta)) = \frac{x-1}{\theta-1} - \frac{x}{\theta}$$
$$\frac{\partial^2}{\partial\theta^2}(\ln f_X(x;\theta)) = -\frac{x-1}{(\theta-1)^2} + \frac{x}{\theta^2}$$
$$E(-\frac{x-1}{(\theta-1)^2} + \frac{x}{\theta^2}) = \frac{E(X)}{\theta^2} - \frac{E(X)-1}{(\theta-1)^2} = \frac{1}{\theta} - \frac{1}{\theta-1} = -\frac{1}{\theta(\theta-1)}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(-\frac{x-1}{(\theta-1)^2} + \frac{x}{\theta^2})} = \frac{\theta(\theta-1)}{n}$$

The relative efficiency is

$$RE(\bar{X}, \alpha) = \frac{\frac{\theta(\theta-1)}{n}}{\frac{\theta(\theta-1)}{n}} = 1$$

Hence, $\bar{X}$ is efficient estimator for θ.

5.5.6(H)

solution: a) Notice that the distribution is Gamma with parameter $\frac{1}{\theta}$. Therefore,

$$E(Y_i) = E(Y) = r\theta, \mathrm{Var}(Y_i) = \mathrm{Var}(Y) = r\theta^2$$

Let $\hat{\theta} = \frac{1}{r}\bar{Y}$, then

$$E(\hat{\theta}) = E(\frac{1}{r}\bar{Y}) = \frac{1}{r}\cdot\frac{1}{n}\sum_{i=1}^{n} E(Y_1) = \frac{1}{r}\cdot\frac{1}{n}\cdot n\cdot r\cdot\theta = \theta$$

Hence, $\hat{\theta}$ is an unbiased estimator for θ.

b)

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\frac{1}{r}\bar{Y}) = \frac{1}{r^2}\mathrm{Var}(\bar{Y}) = \frac{1}{r^2}\cdot\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(Y_i) = \frac{1}{r^2}\cdot\frac{1}{n^2}(nr\theta^2) = \frac{\theta^2}{nr}$$

$$\ln f_Y(y;\theta) = -\ln(r-1)! - r\ln\theta + (r-1)\ln y - \frac{y}{\theta}$$

$$\frac{\partial}{\partial\theta}(\ln f_Y(y;\theta)) = -\frac{r}{\theta} + \frac{y}{\theta^2}$$
$$\frac{\partial^2}{\partial\theta^2}(\ln f_Y(y;\theta)) = \frac{r}{\theta^2} - \frac{2y}{\theta^3}$$

The second derivative is continuous, and

$$E(\frac{r}{\theta^2} - \frac{2y}{\theta^3}) = \frac{r}{\theta^2} - \frac{2}{\theta^3}E(Y) = -\frac{r}{\theta^2}$$

Then let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(\frac{r}{\theta^2} - \frac{2y}{\theta^3})} = \frac{\theta^2}{nr}$$

The relative efficiency is

$$RE(\hat{\theta}, \alpha) = \frac{\frac{\theta^2}{nr}}{\frac{\theta^2}{nr}} = 1$$

Hence, $\hat{\theta}$ is a minimum-variance estimator for θ.