MATH 435 ASSIGNMENT 7

ARNOLD JIADONG YU

1. Introduction to Module Theory

1.1. 8. An element m of R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\operatorname{Tor}(M) = \{m \in M | rm = 0 \text{ for some nonzero } r \in R\}$$

(a) Prove that if R is an integral domain then $\operatorname{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).
(b) Give an example of a ring R and an R-module M such that $\operatorname{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]
proof: (a) WTS $\operatorname{Tor}(M)$ is submodule of M by the submodule criterion. $\operatorname{Tor}(M)$ is a subset of M. $0 \in \operatorname{Tor}(M)$ since $r \cdot 0 = 0$ for any nonzero $r \in R$. So $\operatorname{Tor}(M)$ is not empty. Let $m_1, m_2 \in \operatorname{Tor}(M)$, then there exists nonzero elements $r_1, r_2 \in R$ s.t. $r_1 m_1 = 0, r_2 m_2 = 0$. Moreover, $r_1 r_2 \in R \backslash \{0\}$ since R is integral domain.

$$r_1 r_2 (m_1 + s m_2) = (r_1 r_2) m_1 + (r_1 r_2)(s m_2)$$

$$= r_2 (r_1 m_1) + (r_1 s)(r_2 m_2) = 0 \text{ for any } s \in R$$

Since R is commutative. i.e. $m_1 + s m_2 \in \operatorname{Tor}(M)$ for all $s \in R$ and for all $m_1, m_2 \in \operatorname{Tor}(M)$. Hence $\operatorname{Tor}(M)$ is a submodule of M by the submodule criterion.

(b)Let R be Z_6 and $M = R$, then $2, 3 \in \operatorname{Tor}(M)$ since $3 \cdot 2 = 0, 2 \cdot 3 = 0$ for $2, 3 \in Z_6$. But $2 + 3 = 5$ and $\nexists r_1 \in Z_6$ s.t. $r_1 \cdot 5 = 0$. Therefore, $\operatorname{Tor}(M)$ is not closed under addition and it is not a subgroup of M. Hence, it is not a submodule of M.

1.2. *15. If M is a finite abelian group then M is naturally a $\mathbb{Z}-$module. Can this action be extended to make M into a $\mathbb{Q}-$module?
proof:

1.3. 21. Let $n \in \mathbb{Z}^+, n > 1$ and let R be the ring of $n \times n$ matrices with entries from a field F. Let M be the set of $n \times n$ matrices with arbitrary elements of F in the first column and zeros elsewhere. Show that M is a submodule of R when R is considered as a left module over itself, but M is not a submodule of R when R is considered as a right R-module.

proof: F is a field, thus $0, 1 \in F$ and it is commutative. Let $A = [a_{i,j}], B = [b_{i,j}] \in M$ and $S = [s_{i,j}] \in R$ where i is row indices from 1 to n and j is column indices from 1 to n, then

$$A + SB = [a_{i,j}] + [\sum_{k=1}^{n} s_{i,k} b_{k,j}]$$

By assumption, when $j \neq 1$, $a_{i,j} = 0, b_{i,j} = 0$. i.e.

$$[a_{i,j}] + [\sum_{k=1}^{n} s_{i,k} b_{k,j}] = [a_{i,1}|0] + [\sum_{k=1}^{n} s_{i,k} b_{k,1}|0] = [a_{i,1} + \sum_{k=1}^{n} s_{i,k} b_{k,1}|0]$$

Except the first column are arbitrary, elsewhere are zeros. $a_{i,1} + \sum_{k=1}^{n} s_{i,k} b_{k,1} \in F$ for all i between 1 and n since F a field. Therefore,

$$A+SB = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 & \dots & 0 \\ b_{21} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + \sum_{k=1}^{n} s_{1k} b_{k1} & 0 & 0 & \dots & 0 \\ a_{21} + \sum_{k=1}^{n} s_{2k} b_{k1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} + \sum_{k=1}^{n} s_{nk} b_{k1} & 0 & 0 & \dots & 0 \end{bmatrix} \in M$$

Then M is a submodule by the submodule criterion.

Consider the right module, then WTS $A + BS \in M$.

$$A + BS = [a_{i,j}] + \sum_{k=1}^{n} [b_{i,k} s_{k,j}] = [a_{i,1}|0] + [b_{i,1} s_{i,j}] \notin M$$

Since $A + BS$ has nonzero columns except first column. To be more percise. For example,

$$BS = \begin{bmatrix} b_{11} & 0 & 0 & \dots & 0 \\ b_{21} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}s_{11} & b_{11}s_{12} & b_{11}s_{13} & \dots & b_{11}s_{1n} \\ b_{21}s_{11} & b_{21}s_{12} & b_{21}s_{13} & \dots & b_{21}s_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1}s_{11} & b_{n1}s_{2} & b_{n1}s_{13} & \dots & b_{n1}s_{1n} \end{bmatrix}$$

Add to A, clearly it is not in M. Hence, M is not a submodule of R when R is considered as a right R-module.