Math 430 Assignment 11

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4.21
solution:
By inspection, we observe the two problems are the dual of each other. i.e. WLOG, assume the primal problem has a feasible solution. If the primal problem is unbounded, then we are done. Assume the primal problem is bounded, then it must have a finite optimal since it also has a feasible solution. By Strong Duality Theorem, the dual problem also has an optimal cost and it is the same as the optimal from the primal problem. i.e. there is at least one feasible solution of the dual problem, denoted $\mathbf{s}$, then there is at least one feasible solution of the dual problem. Now, let $\mathbf{c} = [-1, ..., -1]$. Let $\mathbf{t}$ be an arbitrary feasible solution of the dual problem, then $\mathbf{t}'\mathbf{b} \leq \mathbf{s}'\mathbf{b}$. Define $\mathbf{p} = \mathbf{s} + \lambda\mathbf{t}$ for $\lambda \geq 0$, then $\mathbf{p} \geq 0$. Then $\mathbf{p}'\mathbf{A} \leq (1+\lambda)\mathbf{c}' \leq \mathbf{c}'$ for $\forall\lambda \geq 0$. Therefore, the solutions of the dual problem is unbounded. Hence proved.

5.1
solution:
Consider the original primal problem,

$$\text{minimize } -5x_1 - x_2 + 12x_3$$

$$\text{subject to } 3x_1 + 2x_2 + x_3 = 10$$

$$5x_1 + 3x_2 + x_4 = 16$$

$$x_1, ..., x_4 \geq 0$$

and the optimal solution is $\mathbf{x} = (2, 2, 0, 0)$. By adding a new equality constraint, we reform an auxiliary problem, i.e.

$$\text{minimize } -5x_1 - x_2 + 12x_3 + Mx_5$$

$$\text{subject to } 3x_1 + 2x_2 + x_3 = 10$$
$$5x_1 + 3x_2 + x_4 = 16$$
$$x_1 + x_2 - x_5 = 3$$
$$x_1, ..., x_5 \geq 0$$

Then we run the primal simplex algorithm based on new basis and its inverse, the new basis

$$\mathbf{B} = \begin{bmatrix} 3 & 2 & 0 \\ 5 & 3 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{B}^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -3 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\bar{c} = c - \mathbf{c}_B'\mathbf{B}^{-1}A = \begin{bmatrix} 0 & 0 & 2-2M & 7+M & 0 \end{bmatrix}$$

$$\begin{array}{c} \\ z \\ x_1 \\ x_2 \\ x_5 \end{array} \left[\begin{array}{c|ccccc} z & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 12 & 0 & 0 & 2-2M & 7+M & 0 \\ \hline 2 & 1 & 0 & -3 & 2 & 0 \\ 2 & 0 & 1 & 5 & -3 & 0 \\ 1 & 0 & 0 & 2 & -1 & 1 \end{array} \right] \tag{1}$$

$k = 3, \epsilon = 2, l = 5$, i.e. x_5 goes out and x_3 goes in.

$$\begin{array}{c} \\ z \\ x_1 \\ x_2 \\ x_3 \end{array} \left[\begin{array}{c|ccccc} z & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 11+M & 0 & 0 & 0 & 8 & M-1 \\ \hline 3.5 & 1 & 0 & 0 & 0.5 & 1.5 \\ -0.5 & 0 & 1 & 0 & -0.5 & -2.5 \\ 0.5 & 0 & 0 & 1 & -0.5 & 0.5 \end{array} \right] \tag{2}$$

We stop since all $c_i \geq 0$. Then the optimal cost is -11 and optimal solution is $(3.5, -0.5, 0.5, 0, 0)$.

5.2

solution:

(a) WTS $\mathbf{B}^{-1}\mathbf{b} \geq 0$ for the new basis matrix. By assumption, $\delta\mathbf{E} = \delta e_{11}$. Moreover, let $u = \delta e_1, v = e_1$, then $\mathbf{B} + \delta\mathbf{E} = \mathbf{B} + uv^T$. By Sherman-Morrison Formula, $(\mathbf{B} + uv^T)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}uv^T\mathbf{B}^{-1}}{1+v^T\mathbf{B}^{-1}u} \to \mathbf{B}^{-1}$ as $\delta \to 0$. i.e. $(\mathbf{B} + uv^T)^{-1}\mathbf{b} \geq 0$ as $\delta \to 0$. Hence if δ is small enough, $\mathbf{B} + \delta\mathbf{E}$ is a basis matrix of the new problem.

(b) WTS $(\mathbf{B}+\delta\mathbf{E})\mathbf{x}_B = \mathbf{b}$ where $\mathbf{x}_B = (\mathbf{I}+\delta\mathbf{B}^{-1}\mathbf{E})^{-1}\mathbf{B}^{-1}\mathbf{b}$. After simplifying, it is the same to show $(\mathbf{B}+\delta\mathbf{E})(\mathbf{I}+\delta\mathbf{B}^{-1}\mathbf{E})^{-1} = \mathbf{B}$. By same argument from part a, $(\mathbf{B}+\delta\mathbf{E})(\mathbf{I}+\delta\mathbf{B}^{-1}\mathbf{E})^{-1} \to \mathbf{B}$ as $\delta \to 0$. Hence proved.
(c) WTS $\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}'$ for the new basis matrix. Using the same argument from part a, calculate the inverse of $\mathbf{B}+\delta\mathbf{E}$, then $\mathbf{c}' - \mathbf{c}_B'(\mathbf{B}+\delta\mathbf{E})^{-1}\mathbf{A} \to \mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}'$ as $\delta \to 0$. Hence proved for sufficiently small δ.
(d)

5.3

solution:

By Sherman-Morrison formula we can derive:

$$\mathbf{B}(\delta)^{\mathbf{-1}} = \mathbf{B^{-1}} - \frac{\mathbf{B^{-1}A_0e_1}\delta}{1+\delta\mathbf{e_1'B^{-1}A_0}}$$

Let S be a subset of $[\delta_1, \delta_2]$ where $\mathbf{B}(\delta)$ is an optimal basis, WTS S is closed. In order to check whether $\mathbf{B}(\delta)$ is an optimal basis, two conditions must satisfy, feasibility and optimality. i.e. $\mathbf{B}(\delta)^{-1}\mathbf{b} \geq 0$ and $\bar{\mathbf{c}} = \mathbf{c}' - \mathbf{c_B'}\mathbf{B}(\delta)^{\mathbf{-1}}\mathbf{A} \geq \mathbf{0}$ for $\forall \delta \in S$ must be showed.

Clearly, $0 \in [\delta_1, \delta_2]$ and $\mathbf{B}(0)$ is an optimal basis. Let $\delta^1, \ldots, \delta^n \in S$ be an arbitrary convergence sequence that converges to δ, WTS $\delta \in S$.

$$\mathbf{B}(\delta)^{-1}\mathbf{b} = (\mathbf{B}^{-1} - \frac{\mathbf{B^{-1}A_0e_1}\delta}{1+\delta\mathbf{e_1'B^{-1}A_0}})\mathbf{b} = (\mathbf{B}^{-1} - \frac{\mathbf{B^{-1}A_0e_1}\lim_{n\to\infty}\delta^n}{1+\lim_{n\to\infty}\delta^n\mathbf{e_1'B^{-1}A_0}})\mathbf{b} \geq 0$$

since it is true for every δ^n. Same can be showed for $\bar{\mathbf{c}} = \mathbf{c}' - \mathbf{c_B'}\mathbf{B}(\delta)^{\mathbf{-1}}\mathbf{A} \geq \mathbf{0}$. Therefore, $\delta \in S$. Hence, S is closed since every limit point is in S.