MATH 435 ASSIGNMENT 2

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1. Chapter 9 Normal Subgroups and Factor Groups Homework

1.1. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a,b,d \in \mathbf{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2,\mathbf{R})$?
proof: No. Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $B \in H$. Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, then $\det(A) = 9 - 8 = 1$ and $A \in GL(2,\mathbf{R})$. $A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Therefore,

$$ABA^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix} \notin H$$

Hence, H is not normal subgroup by Normal Subgroup Test.

1.2. Prove that a factor group of a cyclic group is cyclic.
proof: Let $G =< a >$ be a cyclic group, and $|G| = n$. Moreover, every subgroup of cyclic group is cyclic. A cyclic group is also abelian. That is every subgroup of cyclic group is normal subgroup as well. Let H be such a normal subgroup of G then the factor group is G/H. Since $G/H = \{gH | g \in G\}$ and $G = \{a^n | n \in \mathbb{Z}\}$, that is $g = a^i$ for some integer $i \leq n$. Therefore,

$$gH = a^i H = (aH)^i$$

by multiplication in factor group. Hence $G/H =< aH >$, G/H is cyclic. That is a factor group of a cyclic group is cyclic.

1.3. Let $G = U(16), H = \{1, 15\}$, and $K = \{1, 9\}$. Are H and K isomorphic? Are G/H and G/K isomorphic?
proof: Let $G = U(16)$, then $G = \{1, 3, 5, 7, 9, 11, 13, 15\}$. H, K are normal subgroup of G and both isomorphic to $\mathbb{Z}_2$, therefore $H \cong K$. For G/H, the factor group is

$$1H = \{1, 15\} = 15H$$

$$3H = \{3, 13\} = 13H$$
$$5H = \{5, 11\} = 11H$$
$$7H = \{7, 9\} = 9H$$

For G/K, the factor group is

$$1H = \{1, 9\} = 9H$$
$$3H = \{3, 11\} = 11H$$
$$5H = \{5, 13\} = 13H$$
$$7H = \{9, 15\} = 15H$$

The factor group G/H is cyclic, since $3^2H = 9H = 7H, 3^3H = 11H = 5H, 3^4H = 1H = 15H$. But the factor group G/K is not cyclic, since $3^2H = 9H = 1H, 3^3H = 11H = 3H$, $5^2H = 9H = 1H, 5^3H = 13H = 5H$, $7^2H = 1H = 9H, 7^3H = 15H = 7H$. Therefore, it is not isomorphic between G/H and G/K.

1.4. Let G be a finite group and let H be a normal subgroup of G. Prove that the order of the element gH in G/H must divide the order of g in G.

proof: Let G be a finite group, then every element of G has a finite order. Let $g \in G$, and $|g| = n$. That is the smallest positive integer n s.t. $g^n = e \in G$. Let H be a normal subgroup of G and $|gH| = m$. That is the smallest positive integer m s.t. $(gH)^m = H \in G/H$. Since m is the smallest positive integer s.t. $g^m \in H$ and n is the smallest positive integer s.t. $g^n = e \in H$, then $n \geq m$. By division algorithm, write $n = qm + r$ for some positive integer q and some positive integer r s.t. $0 \leq r \leq m - 1$. Therefore,

$$\begin{gathered} H = eH = g^nH = (gH)^n = (gH)^{qm+r} \\ = (gH)^{qm}(gH)^r = ((gH)^m)^q(gH)^r = H^q(gH)^r = (gH)^r \\ \Rightarrow H = (gH)^r \end{gathered}$$

Moreover, $0 \leq r \leq m - 1$ and m is the smallest positive integer s.t. $(gH)^m = H$ that is $r = 0$. Hence, $n = qm$ and this implies m divides n.