Math 741 Assignment 12 (Quiz)

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7.3.1. solution: To prove that $f_U(u)$ is a probability density function. NTS $\int_0^\infty f_U(u)du = 1$.

$$\int_0^\infty f_U(u)du = \int_0^\infty \frac{1}{2^{m/2}\Gamma(\frac{m}{2})} u^{m/2-1}e^{-u/2}du$$

$$= \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty (\frac{u}{2})^{m/2-1}e^{-u/2}\frac{du}{2}$$

Let $x = u/2, dx = du/2$, then the above equation becomes

$$= \frac{1}{\Gamma(\frac{m}{2})}\cdot\Gamma(\frac{m}{2}) = 1$$

since

$$\int_0^\infty (x)^{m/2-1}e^{-x}dx = \Gamma(\frac{m}{2})$$

Hence proved.

7.3.2. solution: Need to find the moment generating function and computer first, second moment.

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^\infty e^{tx}\frac{1}{2^{m/2}\Gamma(\frac{m}{2})}x^{m/2-1}e^{-x/2}dx$$

$$= \frac{1}{2^{m/2}\Gamma(\frac{m}{2})}\int_{-\infty}^\infty e^{tx}x^{m/2-1}e^{-x/2}dx$$

$$\Rightarrow M_{\chi^2}(t) = (1-2t)^{-m/2}$$

$$M'_{\chi^2}(t) = m(1-2t)^{-m/2-1} = m, t = 0$$

$$M''_{\chi^2}(t) = m(m+2)(1-2t)^{-m/2-2} = m(m+2), t = 0$$

Hence, $E(\chi^2) = m$, $\text{Var}(\chi^2) = m(m+2) - m^2 = 2m$. It is the same to say that $E(\chi^2) = n$, $\text{Var}(\chi^2) = 2n$.

7.3.3. solution: Let $Z_i = \frac{Y_i - 50}{10}$, then $\sum_{i=1}^{3} Z_i^2 \sim \chi^2(3)$. Let $\alpha = 0.05$

$$\sum_{i=1}^{3} Z_i^2 = \sum_{i=1}^{3} (\frac{Y_i - 50}{10})^2 = (\frac{65-50}{10})^2 + (\frac{30-50}{10})^2 + (\frac{55-50}{10})^2 = 6.5$$

$$\text{P-value} = P(0 < \chi_3^2 < 6.5) = 0.9103 > \alpha = 0.05$$

Therefore, it is believalbe.

7.3.4.(H) solution: We know that the variance of a chi square random variable with k df is $2k$. Then

$$\text{Var}(\frac{(n-1)S^2}{\sigma^2}) = 2(n-1)$$

Moreover,

$$\text{Var}(\frac{(n-1)S^2}{\sigma^2}) = (\frac{n-1}{\sigma^2})^2 \text{Var}(S^2) = 2(n-1)$$

Hence,

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

7.3.5. solution: We know that the expect value of a chi square random variable with k df is k. Then

$$E(\frac{(n-1)S^2}{\sigma^2}) = n-1$$

$$E(\frac{(n-1)S^2}{\sigma^2}) = \frac{n-1}{\sigma^2} E(S^2) = n-1 \Rightarrow E(S^2) = \sigma^2$$

Moreover,

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

By chebyshev's inequality, let $\epsilon > 0$.

$$P(|S^2 - \sigma^2| < \epsilon) \geq 1 - \frac{\text{Var}(S^2)}{\epsilon} = 1 - \frac{2\sigma^4}{\epsilon(n-1)}$$

$$\lim_{n\to\infty} P(|S^2-\sigma^2|<\epsilon) \geq \lim_{n\to\infty} 1-\frac{2\sigma^4}{\epsilon(n-1)} = 1$$

Therefore, $\lim_{n\to\infty} P(|S^2-\sigma^2|<\epsilon) = 1$. Hence, S^2 is consistent for σ^2.

7.3.6. solution: From the given information,

$$P(Z \leq \frac{y-200}{\sqrt{2\cdot 200}}) = 0.4$$

Therefore, $y = 194.933$.

7.3.9. solution: a)

$$P(0.109 < F_{4,6} < x) = 0.95 \Rightarrow P(F_{4,6} < x) - P(0 < F_{4,6} < 0.109) = 0.95$$

$$\Rightarrow P(F_{4,6} < x) = 0.975111 \Rightarrow x = 6.2394$$

b)

$$P(0.427 < F_{11,7} < 1.69) = x \Rightarrow x = 0.65083$$

c)

$$P(F_{x,x} > 5.35) = 0.01 \Rightarrow P(0 < F_{x,x} < 5.35) = 0.99 \Rightarrow x = 9$$

d)

$$P(0.115 < F_{3,x} < 3.29) = 0.9 \Rightarrow x = 15$$

e)

$$P(x < \frac{V/2}{U/3}) = 0.25 \Rightarrow P(x < F_{2,3}) = 0.25 \Rightarrow x = 2.2798$$

7.3.11. solution:

$$F = \frac{V/m}{U/n} \Rightarrow \frac{1}{F} = \frac{U/n}{V/m}$$

Therefore, $\frac{1}{F}$ has n and m degree of freedom.

7.3.13.(H) solution: NTS

$$\lim_{n\to\infty} \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})(1+\frac{t^2}{n})^{(n+1)/2}} = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}, -\infty < t < \infty$$

$$\lim_{n\to\infty} \frac{(\frac{n+1}{2}-1)!}{\sqrt{n\pi}(\frac{n}{2}-1)!(1+\frac{t^2}{n})^{(n+1)/2}} = \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!(1+\frac{t^2}{n})^{(n+1)/2}}$$

$$= \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{t^2}{n})^{(n+1)/2}} = \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{t^2/2}{n/2})^{(n+1)/2}}$$

$$= \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot e^{-t^2/2} = e^{-t^2/2} \lim_{n\to\infty} \frac{\sqrt{2\pi\frac{n-1}{2}}(\frac{n-1}{2})^{(n-1)/2} e^{-(n-1)/2}}{\sqrt{n\pi}\sqrt{2\pi\frac{n-2}{2}}(\frac{n-2}{2})^{(n-2)/2} e^{-(n-2)/2}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-t^2/2} \lim_{n\to\infty} \Big[\frac{\sqrt{2\pi\frac{n-1}{2}}}{\sqrt{2\pi\frac{n-2}{2}}} \cdot \frac{(\frac{n-1}{2})^{(n-1)/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} \cdot \frac{e^{-(n-1)/2}}{e^{-(n-2)/2}} \Big]$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-t^2/2} \cdot e^{-1/2} \lim_{n\to\infty} \frac{(\frac{n-1}{2})^{(n-1)/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Since

$$\lim_{n\to\infty} \frac{(\frac{n-1}{2})^{(n-2)/2}(\frac{n-1}{2})^{1/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} = \lim_{n\to\infty} (\frac{n-1}{n-2})^{(n-2)/2} (\frac{n-1}{2n})^{1/2}$$

$$= \lim_{n\to\infty} (1 + \frac{1/2}{(n-2)/2})^{(n-2)/2} (\frac{n-1}{n})^{1/2} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{1/2}$$

7.3.14.(H) solution: Given the student t distribution with 1 df. Then $X \sim t(1)$ and

$$f(x) = \begin{cases} \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)} \frac{1}{1+x^2} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

Since student t distribution is symmetric, and $\int_{-\infty}^{\infty} f(x)dx = 1$

$$2\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\sqrt{\pi}\Gamma(1/2)}{\Gamma(1)} \implies \int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$$