Math 741 Assignment 2 (Hand-In)

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February 10, 2019

5.2.4
solution: Given $L(\theta) = \prod_{i=1}^n p_X(k_i;\theta)$ for discrete. i.e.

$$L(\theta) = \prod_{i=0}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum_{i=0}^n k_i} e^{-n\theta^2}}{\prod_{i=0}^n k_i!}$$

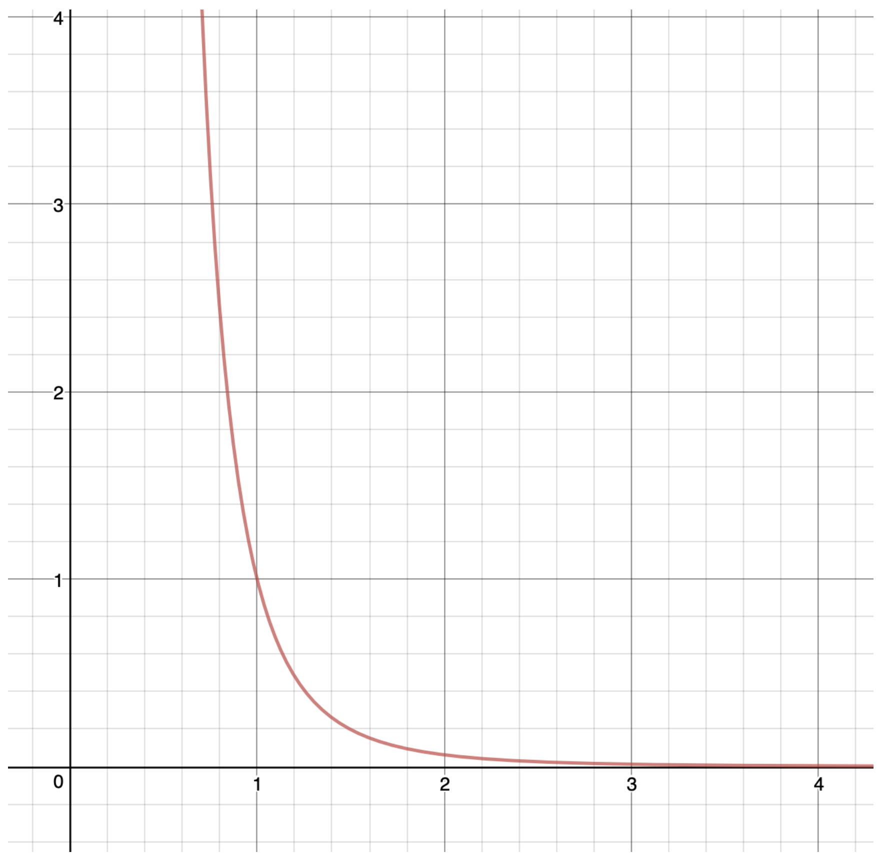
$$\ln(L(\theta)) = \ln \theta^{2\sum_{i=0}^n k_i} + \ln e^{-n\theta^2} - \ln \prod_{i=0}^{n} k_i! = 2\sum_{i=0}^{n} k_i \ln\theta - n\theta^2 - \sum_{i=0}^{n}(\ln k_i!)$$

$$\frac{d}{d\theta}(\ln(L(\theta))) = \frac{2\sum_{i=0}^n k_i}{\theta} - 2n\theta$$

$$\text{Let } \frac{d}{d\theta}(\ln(L(\theta))) = 0 \Rightarrow \frac{2\sum_{i=0}^n k_i}{\hat{\theta}} - 2n\hat{\theta} = 0 \Rightarrow \hat{\theta}^2 = \frac{\sum_{i=0}^n k_i}{n}$$

$$\Rightarrow \hat{\theta} = \sqrt{\frac{\sum_{i=0}^n k_i}{n}}$$

5.2.10
solution: Given $L(\theta) = \prod_{i=1}^n f_Y(y_i;\theta)$ for continuous.Then
(a)By observation that $L(\theta) = \frac{1}{\theta^4}$ and $\theta \geq 0$. The graph is showing below



We can just look at the $L(\theta)$ directly. $L(\theta)$ is maximum when θ is the smallest. Moreover the inequality $0 \leq y \leq \theta$ must hold. Therefore θ must not less than all y. Hence $\hat{\theta} = y_{\max} = 14.2$.
(b)By observation that $L(\theta_1, \theta_2) = \frac{1}{(\theta_2 - \theta_1)^4}$ and $\theta_2 \geq \theta_1 \Rightarrow \theta_2 - \theta_1 \geq 0$. Let $\theta = \theta_2 - \theta_1$, then $L(\theta)$ is the same as part a). Therefore, we can look at the function directly fo find maximum.

In order to get $L(\theta_1, \theta_2)$ maximized, $\theta_2 - \theta_1$ must be minimized as well as the inequality $\theta_1 \leq y \leq \theta_2$ must hold. We want θ_2 as small as possible and θ_1 as large as possible. Hence $\hat{\theta_2} = y_{\max} = 14.2, \hat{\theta_1} = y_{\min} = 1.8$.

5.2.15

solution: Given the normal pdf and μ, then

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma^2}} = (2\pi\sigma^2)^{-n/2} e^{\frac{1}{2}\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma^2}}$$

$$\ln(L(\sigma^2)) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}(y_i-\mu)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{d}{d\sigma^2}(\ln(L(\sigma^2))) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2$$

Let $\frac{d}{d\sigma^2}(\ln(L(\sigma^2))) = 0$, then

$$-\frac{n}{2}\frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=1}^{n}(y_i-\mu)^2 = 0 \Rightarrow -n\hat{\sigma}^2 + \sum_{i=1}^{n}(y_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i-\mu)^2}{n}$$

They are almost identical. In example 5.2.4, μ is not given. Therefore, In the estimator $\hat{\sigma}^2$, $\hat{\mu}$ is there. In this case, μ is given, we only need to estimate $\hat{\sigma}^2$.

5.2.17

solution: This is a continuous pdf, therefore by definition 5.2.3.

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_0^1 (\theta^2+\theta)y^\theta(1-y)dy = (\theta^2+\theta)\int_0^1 (y^\theta - y^{\theta+1})dy$$

$$= (\theta^2+\theta)(\frac{y^{\theta+1}}{\theta+1} - \frac{y^{\theta+2}}{\theta+2})\Big|_0^1 = (\theta^2+\theta)(\frac{1}{\theta+1} - \frac{1}{\theta+2}) = \frac{\theta}{\theta+2}$$

Since $\hat{\mu} = \bar{y}$, therefore

$$\frac{\hat{\theta}}{\hat{\theta}+2} = \bar{y} \Rightarrow \hat{\theta} = (\hat{\theta}+2)\bar{y} \Rightarrow \hat{\theta} - \hat{\theta}\bar{y} = 2\bar{y} \Rightarrow \hat{\theta} = \frac{2\bar{y}}{1-\bar{y}}$$