Math 335 Assignment 7

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(1) Show that the multiplicative group of complex numbers of module 1, considered in Problem 5 of Assignment #6 and the quotient group $\mathbb{R}/\mathbb{Z}$ are isomorphic.

proof: Let $G = \{e^{i\theta} \ |\theta \in \mathbb{R}\}$, then let

$$f : x \to e^{2\pi i x} \text{ for } x \in \mathbb{R}$$

be an homomorphism. f maps $\mathbb{R}$ onto the unit circle. Therefore, the kernel is $\mathbb{Z}$. That is G is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$ by the first isomorphism theorem.

(2) Let G_1 and G_2 be two commutative groups. Denote by Hom(G_1;G_2) the set of homomorphisms $G_1 \to G_2$. Show that Hom(G_1;G_2) is a group w.r.t. the operation $\star$ defined by

$$f, g \in \text{Hom}(G_1, G_2), x \in G_1$$

$$(f \star g)(x) = f(x)g(x) \text{ (product in } G_2).$$

Is the same statement true when
(a) G_1 commutative and G_2 is not?
(b) G_1 is arbitrary and G_2 is commutative?

proof: Neutral Element Axiom, g will map $x \in \text{Ker}g \subset G_1$ to $1 \in G_2$, thus $g(x) = 1$ since it is homomorphism. This implies $(f \star g)(x) = f(x)g(x) = f(x) \cdot 1 = f(x)$ and $(g \star f)(x) = g(x)f(x) = 1 \cdot f(x) = f(x)$. Associative Axiom , let $f, g, h \in \text{Hom}(G_1, G_2)$, then $((f \star g) \star h)(x) = (f \star g)(x)h(x) = (f(x)g(x))h(x) = f(x)(g(x)h(x)) = f(x)(g \star h)(x) = (f \star (g \star h))(x)$. Inverse

Axiom, $(f \star g)^{-1}(x) = (f(x)g(x))^{-1} = g^{-1}(x)f^{-1}(x) = f^{-1}(x)g^{-1}(x) \in G_2$, then $(f \star g)^{-1} \in \text{Hom}(G_1, G_2)$. Hence $\text{Hom}(G_1;G_2)$ is a group w.r.t. the operation $\star$.
a) When G_1 commutative and G_2 is not commutative, $\text{Hom}(G_1;G_2)$ is not a group. Because $g^{-1}(x)f^{-1}(x) \neq f^{-1}(x)g^{-1}(x)$.
b) When G_1 is arbitrary and G_2 is commutative, $\text{Hom}(G_1;G_2)$ is a group.

(3) Let G_1, G_2, G_3 be three commutative groups. Show that the following two groups, where we use the notation from the previous problem, are isomorphic:

$$\text{Hom}(G_1, G_2) \times \text{Hom}(G_2, G_3) \text{ and } \text{Hom}(G_1, G_2 \times G_3)$$

proof: Let $f \in \text{Hom}(G_1, G_2)$, $g \in \text{Hom}(G_2, G_3)$ and $h \in \text{Hom}(G_1, G_2 \times G_3)$. Also $x, y \in \{f \times g | \forall f \in \text{Hom}(G_1, G_2), \forall g \in \text{Hom}(G_2, G_3)\}$ and $z \in \{g | \forall g \in \text{Hom}(G_1, G_2 \times G_3)\}$. $a \in G_1, b \in G_2, c \in G_3$. Then $f(a) = b, g(b) = c$ and $h(a) = b \times c$.

Pick a map φ, such that

$$\varphi : \text{Hom}(G_1, G_2) \times \text{Hom}(G_2, G_3) \to \text{Hom}(G_1, G_2 \times G_3)$$

$$\varphi : f(a) \times g(b) \to h(a)$$

is the same as

$$\varphi : b \times c \mapsto b \times c$$

then φ is bijective. WTS, $\varphi(x \star y) = \varphi(x) \star \varphi(y)$.
Let $x = f_1 \times g_1, y = f_2 \times g_2$. Then

$$\varphi(x \star y) = \varphi((f_1(a) \times g_1(b)) \star (f_2(a) \times g_2(b))) = \varphi(h_1(a) \star h_2(a))$$

$$= \varphi(b \times c) = b \times c = h(a) = h_1(a) \star h_2(a) = \varphi(h_1(a)) \star \varphi(h_2(a))$$

$$= \varphi(f_1(a) \times g_1(b)) \star \varphi(f_2(a) \times g_2(b)) = \varphi(x) \star \varphi(y)$$

Hence, bijective and homomorphism implies isomorphism. Therefore, φ is isomorphism map.

(4) Describe all possible homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}_{100}$, i.e., describe the group $\text{Hom}(\mathbb{Q}, \mathbb{Z}_{100})$.

proof: There is only one homomorphism from $\mathbb{Q}$ to $\mathbb{Z}_{100}$ which is the trivial. The group $\text{Hom}(\mathbb{Q}, \mathbb{Z}_{100})$ has only one element which is identity. $\forall \frac{m}{n} \in \mathbb{Q}$, we can write $\frac{m}{n} = 100(\frac{m}{100n})$. By the definition of homomorphism $f \in \text{Hom}(\mathbb{Q}, \mathbb{Z}_{100})$ implies $f(ax) = a \cdot f(x)$ for all $a \in \mathbb{N}$ and $x \in \mathbb{Q}$. Then $f(100 \cdot \frac{m}{100n}) = 100 \cdot f(\frac{m}{100n})$. Since $f(\frac{m}{100n}) \in \mathbb{Z}_{100}$, then $100 \cdot f(\frac{m}{100n}) = 0$ implies $f(100 \cdot \frac{m}{100n}) = 0$. Therefore, $f(\frac{m}{n}) = 0$ for $\forall \frac{m}{n} \in \mathbb{Q}$.

(5) Give an example of a group G and a homomorphism $f : G \to G$ with the property $\text{Ker}(f) = \text{Im}(f)$. (Hint: we did this in class.)

proof: Let $G = \mathbb{Z}_4$, and pick a homomorphism $f : \mathbb{Z}_4 \to \mathbb{Z}_4$. Define f as $2\cdot$ element in $\mathbb{Z}_4$. Thus $\text{Kef}(f) = \{0, 2\} = \text{Im }(f)$.

(6) Which of the following properties transfers from a group to all quotient groups of G:
(a) commutative,
(b) cyclic,
(c) finite,
(d) there are no elements of finite order?

proof: a) commutative. Yes. Pick $x, y \in G$ and H is normal subgroup of G, then $(xH)(yH) = (xy)H = (yx)H = (yH)(xH)$ is commutative.
b) cyclic. Yes. Let $< x >= G$, then both $a, b \in G$ can be generated by x. Then ab can be generate by x, quotient group is $< x > H$ where H is normal subgroup of G.
c) finite. Yes. If $\#G$ is finite, then quotient groups of G is also finite by Lagrange's Theorem, $\#cosets \leq \#G < \infty$.
d) there are no elements of finite order. No. Let $G = \mathbb{Q}$ has no elements of finite order, let $H = \mathbb{Z}$ has no elements of finite order. But the quotient group $\mathbb{Q}/\mathbb{Z}$ has elements of finite order.

(7) Let H be a subgroup of a group G. Assume H has only two left cosets in G. Is H necessarily a normal subgroup of G? If 'yes', what is the familiar group G/H is isomorphic to?

proof: Since H has only two left cosets in G. Call this two left cosets H and $aH = K$, where $a \in G$. H is a subgroup of G. Then $H \cup K = G$. Choose a $x \in H$, then $xH = Hx = H$, H is normal subgroup. Choose a $x \in K$, then $xH = K \neq H$ and $Hx = K \neq H$. Since there are only H and

K, then $xH = Hx$. H is normal subgroup. G/H is isomorphic to $\mathbb{Z}_2$.

(8) What are the answers to the same questions as in the previous problem if 'left' is changed to 'right'?

proof: Same, because H is a normal subgroup of G, Then left cosets are the same as right cosets.