Math 741 Assignment 7 (Quiz)

Arnold Jiadong Yu

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5.7.1.
solution: Given the normal distribution with $\mu = 18$ and $\sigma = 5.0$, we want to solve for n such that the following equation hold,

$$P(16 \le \bar{Y}_n \le 20) = 0.9$$

$$P(\frac{16-18}{\sigma/\sqrt{n}} \le Z \le \frac{20-18}{\sigma/\sqrt{n}}) = 0.9$$

$$P(Z \le \frac{-2}{5/\sqrt{n}}) = 0.05 \Rightarrow -1.645 \ge \frac{-2}{5/\sqrt{n}}$$

$$\Rightarrow n \ge (1.645 * 5/2)^2 = 19.613$$

Hence, $n = 17$.
5.7.2.
solution: Let $Y_1, ..., Y_n \sim N(0, \sigma^2)$, iid and $S_n^2 = \frac{1}{n}\sum_{i=1}^n Y_i^2$.

$$f_Y(y; 0, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}y^2}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}y^2} dy = \sigma^2$$

$$E(Y^4) = \int_{-\infty}^{\infty} y^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}y^2} dy = 3\sigma^4$$

$$\text{Var}(Y^2) = 2\sigma^4, E(Y_i^2) = E(Y^2), \text{Var}(Y_i^2) = \text{Var}(Y^2)$$

$$E(S_n^2) = \frac{1}{n} \cdot nE(Y^2) = \sigma^2$$

$$\text{Var}(S_n^2) = \frac{1}{n^2} \cdot nVar(Y^2) = \frac{2\sigma^4}{n}$$

Let $\epsilon > 0$, then by Chebyshev's inequality

$$P(|S_n^2 - E(S_n^2)| < \epsilon) \geq 1 - \frac{\text{Var}(S_n^2)}{\epsilon^2} = 1 - \frac{2\sigma^4}{n\epsilon^2}$$

$$\lim_{n\to\infty} P(|S_n^2 - E(S_n^2)| < \epsilon) \geq \lim_{n\to\infty} (1 - \frac{2\sigma^4}{n\epsilon^2}) = 1$$

$$\implies \lim_{n\to\infty} P(|S_n^2 - \sigma^2)| < \epsilon) = 1$$

Hence, S_n^2 is a consistent estimator for σ^2.

5.7.3.

solution: Let $Y_1, ..., Y_n \sim Exp(\lambda)$ and iid, then $E(Y) = \frac{1}{\lambda}, \text{Var}(Y) = \frac{1}{\lambda^2}$. i.e.

$$E(Y_i) = E(Y), \text{Var}(Y_i) = \text{Var}(Y)$$

Therefore, $E(Y_1) = \frac{1}{\lambda}, \text{Var}(Y_1) = \frac{1}{\lambda^2}$. Let $\epsilon > 0$, then

$$P(|\hat{\lambda}_n - \lambda| < \epsilon) = \int_{\lambda-\epsilon}^{\lambda+\epsilon} \lambda e^{-\lambda y} dy = -e^{-\lambda y}\Big|_{\lambda-\epsilon}^{\lambda+\epsilon} = e^{\lambda\epsilon-\lambda^2} - e^{-\lambda\epsilon-\lambda^2}$$

$$= \frac{e^{\lambda\epsilon} - e^{-\lambda\epsilon}}{e^{\lambda^2}}$$

Let $\epsilon = \frac{\lambda}{2}$, then

$$P(|\hat{\lambda}_n - \lambda| < \epsilon) = P(|\hat{\lambda}_n - \lambda| < \frac{\lambda}{2}) = e^{-\lambda^2/2} - e^{-3\lambda^2/2} < 1 \text{ for all } \lambda$$

Since it is a function doesn't contain n,

$$\lim_{n\to\infty} P(|\hat{\lambda}_n - \lambda| < \epsilon) < 1 \text{ for } \epsilon = \frac{\lambda}{2}$$

It doesn't hold for all $\epsilon > 0$. Hence, $\lim_{n\to\infty} P(|\hat{\lambda}_n - \lambda| < \epsilon) < 1$. $\hat{\lambda}_n$ is not sufficient for λ.

b)

$$P(|\hat{\lambda}_n - \lambda| < \epsilon) = P(|\sum_{i=1}^{n} Y_i - \lambda| < \epsilon) = P(-\epsilon + \lambda < \sum_{i=1}^{n} Y_i < \epsilon + \lambda)$$

$$< P(-\epsilon + \lambda < Y_1 < \epsilon + \lambda)$$

since $\forall y > 0$. Let $\epsilon = \frac{\lambda}{2}$, then

$$P(|\hat{\lambda}_n - \lambda| < \frac{\lambda}{2}) < P(|Y_1 - \lambda| < \frac{\lambda}{2}) < 1$$

Therefore,

$$\lim_{n\to\infty} P(|\hat{\lambda}_n - \lambda| < \epsilon) < 1 \text{ for } \epsilon = \frac{\lambda}{2}$$

It doesn't hold for all $\epsilon > 0$. Hence, $\lim_{n\to\infty} P(|\hat{\lambda}_n - \lambda| < \epsilon) < 1$. $\hat{\lambda}_n$ is not sufficient for λ.

5.7.4.(H)

solution: Given an estimator $\hat{\theta}$ such that $\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = 0$.
a) WTS $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$.

$$\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = \lim_{n\to\infty} E[\hat{\theta}_n^2 - 2\theta\hat{\theta}_n + \theta^2] = \lim_{n\to\infty} (E[\hat{\theta}_n^2] - 2\theta E[\hat{\theta}_n] + \theta^2)$$

$$= \lim_{n\to\infty} (E[\hat{\theta}_n^2] - (E[\hat{\theta}_n])^2 + (E[\hat{\theta}_n])^2 - 2\theta E[\hat{\theta}_n] + \theta^2)$$

$$= \lim_{n\to\infty} \left[\mathrm{Var}(\hat{\theta}_n) + (E(\hat{\theta}_n) - \theta)^2\right] = 0$$

Since $\mathrm{Var}(\hat{\theta}_n) \geq 0, (E(\hat{\theta}_n) - \theta)^2 \geq 0$. Therefore,

$$\lim_{n\to\infty} (E(\hat{\theta}_n) - \theta)^2 = 0 \Rightarrow \lim_{n\to\infty} (E(\hat{\theta}_n) - \theta) = 0$$

$$\implies \lim_{n\to\infty} E(\hat{\theta}_n) = \theta$$

b) WTS $\lim_{n\to\infty} P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) = 1$. Let $\epsilon > 0$, then by Chebyshev's inequality

$$P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) \geq 1 - \frac{E((\hat{\theta}_n - E(\hat{\theta}_n))^2)}{\epsilon^2} = 1 - \frac{E((\hat{\theta}_n - \theta + \theta - E(\hat{\theta}_n))^2)}{\epsilon^2}$$

$$= 1 - \frac{E[(\hat{\theta}_n - \theta)^2 + 2(\hat{\theta}_n - \theta)(\theta - E(\hat{\theta}_n)) + (\theta - E(\hat{\theta}_n))^2]}{\epsilon^2}$$

$$= 1 - \frac{E[(\hat{\theta}_n - \theta)^2] + 2E[(\hat{\theta}_n - \theta)(\theta - E(\hat{\theta}_n))] + E[(\theta - E(\hat{\theta}_n))^2]}{\epsilon^2}$$

Since $\lim_{n\to\infty} E[(\hat{\theta}_n - \theta)^2] = 0$ and $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$. then

$$\lim n \to \infty P(|\hat{\theta}_n - E(\hat{\theta}_n)| < \epsilon) \geq 1 - 0 = 1$$

Hence, $\hat{\theta}_n$ is consistent.

5.7.5.

solution: $f_{Y_{\max}}(y) = \frac{ny^{n-1}}{\theta^n}$ for $0 \leq y \leq \theta$.

$$E[(Y_{\max} - \theta)^2] = \int_0^\theta (y-\theta)^2 \frac{ny^{n-1}}{\theta^n} dy = \frac{2\theta^2}{n^2+3n+2} \text{ by caculator}$$

$$\lim_{n\to\infty} E[(Y_{\max} - \theta)^2] = \lim_{n\to\infty} \frac{2\theta^2}{n^2+3n+2} = 0$$

Hence, $\hat{\theta}_n$ is squared-error consistent.