# Math 470 Assignment 23

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10.1.4. a) Let $a \in X$. Prove that if $x_n = a$ for every $n \in \mathbb{N}$, then $x_n$ converges. What does it converge to?
b) Let $X = \mathbf{R}$ with the discrete metric. Prove that $x_n \to a$ as $n \to \infty$ if and only if $x_n = a$ for large $n$.

proof: a) Let $a \in X$ and $x_n = a$ for every $n \in \mathbb{N}$. Let $\epsilon > 0$, there exists an $N \in \mathbb{N}$, s.t $n \geq N$ implies $d(x_n, a) = 0 < \epsilon$ by positive definite. Therefore, $x_n$ converges to $a$ be definition.
b) ($\Rightarrow$)Let $X = \mathbf{R}$ with the discrete metric and $x_n \to a$ as $n \to \infty$. Let $\epsilon > 0$, there exists an $N \in \mathbb{N}$, s.t $n \geq N$ implies $d(x_n, a) < \epsilon$. Choose $\epsilon = 1$, then $d(x_n, a) < 1$. Since $X$ is discrete metric, then $d(x_n, a) = 0$. Hence $x_n = a$ by positive definite.
($\Leftarrow$)Let $X = \mathbf{R}$ with the discrete metric and $x_n = a$ for large $n$. Let $\epsilon > 0$, there exists an $N \in \mathbb{N}$, s.t $n \geq N$ implies $d(x_n, a) = 0 < \epsilon$ by positive definite. Therefore, $x_n \to a$ as $n \to \infty$.

10.1.6. Let $x_n$ be Cauchy in $X$. Prove that $x_n$ converges if and only if at least one of its subsequences converges.

proof: ($\Rightarrow$) Let $x_n$ be Cauchy in $X$ and converges, then it has at least one of its subsequences converges by Theorem 10.14 (ii).
($\Leftarrow$) Let $x_n$ be Cauchy in $X$ and its subsequence denoted as $x_{n_k}$ converges to $a$ as $k \to \infty$. Let $\epsilon > 0$, then there exists an $N \in \mathbb{N}$, s.t $n, k \geq N$ implies $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$ (Cauchy). Moreover, for $k \geq N$, $d(x_{n_k}, a) < \frac{\epsilon}{2}$ (Subsequence converges). Hence $d(x_n, a) \leq d(x_n, x_{n_k}) + d(x_{n_k}, a) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. By definition, $x_n \to a$ as $n \to \infty$.

10.1.7. Prove that the discrete space $\mathbf{R}$ is complete.

proof: Suppose $x_n$ is a cauchy sequence in the discrete space $\mathbf{R}$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$, s.t. for any $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Since the metric space is discrete, then $d(x_n, x_m) = 0$. This implies $x_n = x_m := a$ for all $n, m \geq N$. By example 10.1.4 a), $x_n \to a$ as $n \to \infty$. Since $a = x_n = x_m$ inside the discrete space. It is complete by definition.

10.1.8. a) Prove that the metric space $\mathcal{C}[a,b]$ in Example 10.6 is complete. c) Prove that the metric space $\mathcal{C}[a,b]$ defined in part b) is not complete.

proof: a) Suppose $f_n$ is a cauchy sequence of continuous function in the metric space $\mathcal{C}[a,b]$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$, s.t. for any $n, m \geq N$ implies $|f_n(x) - f_m(x)| \leq \sup_{x \in [a,b]} |f_n(x) - f_m(x)| = ||f_n - f_m|| < \epsilon$ for $\forall x \in [a,b]$. Then by Uniform Cauchy Criterion, $f_n$ converges uniformly on $\mathcal{C}[a,b]$. Moreover, $f_n \to f$ as $n \to \infty$ and $f \in \mathcal{C}[a,b]$. Hence it is complete by definition.
c)Let $a = 0$ and $b = 1$, $f_n = x^n$ be sequences of functions in the metric space $\mathcal{C}[0,1]$. Then $||f_n||_1 = \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}|_0^1 = \frac{1}{n+1} \to 0$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$, and $n, m \geq N$ s.t $||f_n - f_m||_1 \to 0 < \epsilon$. It is a converges cauchy sequence, let it converges to $f$. But $f$ may not be continuous function. $f(x) = 1$ when $x = 1$ and $f(x) = 0$ when $0 \leq x < 1$. Therefore, it is not in the metric space. Hence, it is not complete.