Math 741 Assignment 5 (Hand-In)

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5.5.4(H)

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

$$\ln f_Y(y;\theta) = -\ln\theta$$

$$\frac{\partial}{\partial\theta}(\ln f_Y(y;\theta)) = -\frac{1}{\theta}$$

$$\frac{\partial^2}{\partial\theta^2}(\ln f_Y(y;\theta)) = \frac{1}{\theta^2}$$

The second derivative is continuous, but the set of y values depends on θ.

$$E(\frac{1}{\theta^2}) = \frac{1}{\theta^2}$$

Let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(\frac{1}{\theta^2})} = \frac{\theta^2}{n}$$

Thus

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta}dt = \frac{y}{\theta}$$

$$f_{Y_{\max}}(y) = n[F_Y(y)]^{n-1}f_Y(y) = n\left[\frac{y}{\theta}\right]^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$$

$$\text{Var}(\hat{\theta}) = \frac{(n+1)^2}{n^2}\text{Var}(Y_{\max}) = \frac{(n+1)^2}{n^2}(E(Y_{\max}^2) - (E(Y_{\max})^2)$$

$$= \frac{(n+1)^2}{n^2}\left(\int_0^\theta y^2 \frac{ny^{n-1}}{\theta^n}dy - \left(\int_0^\theta y\frac{ny^{n-1}}{\theta^n}dy\right)^2\right)$$

$$= \frac{(n+1)^2}{n^2}\left(\frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2\right) = \frac{(n+1)^2}{n^2}\left(\frac{n\theta^2(n+1)^2 - n^2\theta^2(n+2)}{(n+2)(n+1)^2}\right)$$

$$= \frac{\theta^2}{n(n+2)}$$

The relative efficiency is

$$RE(\hat{\theta}, \alpha) = \frac{\frac{\theta^2}{n}}{\frac{\theta^2}{n(n+2)}} = n+2 > 1, \forall n$$

The variance of unbiased estimator $\hat{\theta}$ is more efficient than CRLB. This should not happen. The error comes from that we can not use CRLB since the set of y values depends on θ.

5.5.6(H)

solution: a) Notice that the distribution is Gamma with parameter $\frac{1}{\theta}$. Therefore,

$$E(Y_i) = E(Y) = r\theta, \text{Var}(Y_i) = \text{Var}(Y) = r\theta^2$$

Let $\hat{\theta} = \frac{1}{r}\bar{Y}$, then

$$E(\hat{\theta}) = E(\frac{1}{r}\bar{Y}) = \frac{1}{r}\cdot\frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{r}\cdot\frac{1}{n}\cdot n\cdot r\cdot\theta = \theta$$

Hence, $\hat{\theta}$ is an unbiased estimator for θ.

b)

$$\text{Var}(\hat{\theta}) = \text{Var}(\frac{1}{r}\bar{Y}) = \frac{1}{r^2}\text{Var}(\bar{Y}) = \frac{1}{r^2}\cdot\frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(Y_i) = \frac{1}{r^2}\cdot\frac{1}{n^2}(nr\theta^2) = \frac{\theta^2}{nr}$$

$$\ln f_Y(y;\theta) = -\ln(r-1)! - r\ln\theta + (r-1)\ln y - \frac{y}{\theta}$$

$$\frac{\partial}{\partial\theta}(\ln f_Y(y;\theta)) = -\frac{r}{\theta} + \frac{y}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}(\ln f_Y(y;\theta)) = \frac{r}{\theta^2} - \frac{2y}{\theta^3}$$

The second derivative is continuous, and the set of y values does not depend on θ. Therefore,

$$E(\frac{r}{\theta^2} - \frac{2y}{\theta^3}) = \frac{r}{\theta^2} - \frac{2}{\theta^3}E(Y) = -\frac{r}{\theta^2}$$

Then let α denote as $CRLB$,

$$\alpha = \frac{1}{-nE(\frac{r}{\theta^2} - \frac{2y}{\theta^3})} = \frac{\theta^2}{nr}$$

The relative efficiency is

$$RE(\hat{\theta}, \alpha) = \frac{\frac{\theta^2}{nr}}{\frac{\theta^2}{nr}} = 1$$

Hence, $\hat{\theta}$ is a minimum-variance estimator for θ.