Math 470 Assignment 32

Arnold Jiadong Yu

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Q: Let X be all sequence of real numbers that are absoltely summable.

$$d((x_1, x_2, ...), (y_1, y_2, ...)) = \sum_{n=1}^{\infty} |x_n - y_n|$$

Suppose $f : X \to \mathbb{R}$ is linear, $f(x+z) = f(x)+f(z)$ and $f(t\cdot x) = t\cdot f(x)$ for $\forall x, z \in X, t \in \mathbb{R}$. Prove the following are equivlent.
1)f is continuous on X.
2)f is continous at $\mathbf{0}$.
3)f is uniformly continous on X.
4) $\sup\{f(\{x_n\}) : \{x_n\} \in X$ and $d(\{x_n\}, \mathbf{0}) \leq 1(\sum_{n=1}^{\infty} |x_n| \leq 1)\} < \infty$.

proof: WTS $4 \Rightarrow 3 \Rightarrow 1 \Rightarrow 2 \Rightarrow 4$. Since $3 \Rightarrow 1 \Rightarrow 2$ is trivial, then only need to show $4 \Rightarrow 3$ and $2 \Rightarrow 4$.

($4 \Rightarrow 3$) Suppose $\sup\{f(\{x_n\}) : \{x_n\} \in X$ and $d(\{x_n\}, \mathbf{0}) \leq 1(\sum_{n=1}^{\infty} |x_n| \leq 1)\} < \infty$, then there exsits a $M \in \mathbb{R}$,s.t. $f(\{x_n\}) < M$ for $\forall\{x_n\} \in X, d(\{x_n\}, \mathbf{0}) \leq 1$. Let $\epsilon > 0$, pick arbitrary $\{x_n\}, \{y_n\} \in X$, s.t. $f(\{x_n\}) < M, f(\{y_n\}) < M$, choose $\delta = \frac{\epsilon}{M+1}$, and $d(\{x_n\}, \{y_n\}) < \delta$ then

$$d(\{x_n\}, \{y_n\}) < \delta \Rightarrow \sum_{n=1}^{\infty} |x_n - y_n| < \delta \Rightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{\delta} < 1 \Rightarrow f(\frac{\{x_n - y_n\}}{\delta}) < M$$

Moreover,

$$\rho(f(\{x_n\}, f(\{y_n\})) = ||f(\{x_n\}) - f(\{y_n\})|| = ||f(\{x_n\} - \{y_n\})|| = ||f(\{x_n - y_n\})||$$

$$= ||f(\delta \cdot \frac{\{x_n - y_n\}}{\delta}|| = \delta \cdot ||f(\frac{\{x_n - y_n\}}{\delta})|| = \frac{\epsilon}{M+1} \cdot ||f(\frac{\{x_n - y_n\}}{\delta})|| < \frac{\epsilon}{M+1} \cdot M < \epsilon$$

That is

$$d(\{x_n\}, \{y_n\}) < \delta \Rightarrow \rho(f(\{x_n\}, f(\{y_n\})) < \epsilon \text{ for } \forall\{x_n\}, \{y_n\} \in X$$

$$\text{s.t. } \sup\{f(\{x_n\}) : \{x_n\} \in X \text{ and } d(\{x_n\}, \mathbf{0}) \leq 1(\sum_{n=1}^{\infty} |x_n| \leq 1)\} < \infty$$

Hence f is uniformly continuous on X.

$(2 \Rightarrow 4)$. Suppose f is continous on at $\mathbf{0} \in X$. Let $\epsilon > 0$ and $\{x_n\}, \mathbf{0} \in X$, there exists $\delta > 0$, s.t. $d(\{x_n\}, \mathbf{0}) < \delta \Rightarrow \rho(f(\{x_n\}) - f(\mathbf{0})) < \epsilon$. Then

$$d(\{x_n\}, \mathbf{0}) < \delta \Rightarrow \sum_{n=1}^{\infty} |x_n| < \delta \Rightarrow \sum_{n=1}^{\infty} \frac{|x_n|}{\delta} < 1$$

And

$$\rho(f(\{x_n\}) - f(\mathbf{0})) = ||f(\{x_n\} - \mathbf{0})|| = ||f(\{x_n\})|| = ||f(\delta \cdot \frac{\{x_n\}}{\delta})||$$

$$= \delta||f(\frac{\{x_n\}}{\delta})|| < \epsilon \Rightarrow ||f(\frac{\{x_n\}}{\delta})|| < \frac{\epsilon}{\delta}$$

That is

$$\sum_{n=1}^{\infty} \frac{|x_n|}{\delta} < 1 \Rightarrow ||f(\frac{\{x_n\}}{\delta})|| < \frac{\epsilon}{\delta}$$

Let $\{a_n\} = \{\frac{x_n}{\delta}\} \in X$, then

$$\sum_{n=1}^{\infty} |a_n| < 1 \Rightarrow ||f(\{a_n\})|| < \frac{\epsilon}{\delta} < \infty$$

Since ϵ, δ is some fixed number. Hence $2 \Rightarrow 4$.

That is all 1,2,3,4 are equivlent.