Math 470 Assignment 6

Arnold Jiadong Yu

February 7, 2018

6.3.9. Given that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (see Exercise 14.3.7), find the exact value of

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

proof: $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...$ and

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + ...$$

Hence $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$ is the sums of odd terms of series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Then

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

.

*6.3.10. Let $x \leq y$ be any pair of extended real numbers. Prove that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then there is a rearrangement $\sum_{j=1}^{\infty} b_j$ of $\sum_{k=1}^{\infty} a_k$ whose partial sums s_n satisfy

$$\liminf_{n\to\infty} s_n = x \text{ and } \limsup_{n\to\infty} s_n = y$$

proof: Since $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then

$$\sum_{k=1}^{\infty} a_k^+ = \sum_{k=1}^{\infty} a_k^- = \infty$$

by Lemma 6.28. By proof from Theorem 6.29, let k_1 be the smallest integer which satisfies $a_1^+ + a_2^+ + ... + a_{k_1}^+ > y$ where $s_{k_1} = a_1^+ + a_2^+ + ... + a_{k_1}^+$ and $b_1, b_2, ..., b_{k_1} = a_1^+, a_2^+, ..., a_{k_1}^+$. Then $a_1^+ + a_2^+ + ... + a_{k_1-1}^+ \leq y \Rightarrow a_1^+ + a_2^+ + ... + a_{k_1}^+ \leq y + a_{k_1}$. Let r_1 be the smallest integer which satisfies $s_{k_1} - a_1^- - ..... - a_{r_1-k_1}^- < x$ where $b_{k_1+1}, ..., b_{r_1} = -a_1^-, ..., a_{r_1-k_1}^-$ and $s_{r_1} = s_{k_1} - a_1^- - ..... - a_{r_1-k_1}^-$ for $r_1 > k_1 > 0$. Then $s_{r_1} \geq x + b_{r_1}$. Construct a m as an integer such that $r_1 > m > k_1 > 0$. Since b_1 to b_{k_1} are non-negative and b_{k_1+1} to b_{r_1} are non-positive. This implies $x + b_{r_1} \leq s_m \leq y + b_{k_1}$.
Suppose for some $l \geq 1$ that integers $k_1 < r_1 < k_2 < r_2 < k_3 < r_3 < .... < k_l < r_l$ have been chosen such that a partial sum

$$x + \min\{b_1, ..., b_{k_l}\} \leq s_{r_l} \leq y + \max\{b_1, ..., b_{k_l}\}.$$

Since $s_{k_1} > y$ and $y + \max\{b_1, ..., b_{k_l}\} \leq y + \sup_{m \geq k_l} b_m$, this implies

$$y \leq \sup_{k_l \leq m \leq k_{l+1}} s_m \leq y + \sup_{m \geq k_l} b_m \Rightarrow y \leq \sup_{k_l \leq m} s_m \leq y + \sup_{m \geq k_l} b_m.$$

Take limit as $l \to \infty$ on this inequality, we obtain

$$y \leq \lim_{m \to \infty} (\sup_{m \geq k_l} s_m) \leq y + \lim_{m \to \infty} (\sup_{m \geq k_l} b_m).$$

since b_j is rearrangement of a_k, where $a_k \to 0$ as $k \to \infty$. by Convergence Test. Then $b_m \to 0$ as $m \to \infty$. Hence

$$y \leq \limsup_{m \to \infty} s_m \leq y$$

by Comparison Test, it has a limitsup y. The proof for limitinf is similar.