Math 400 Project 1

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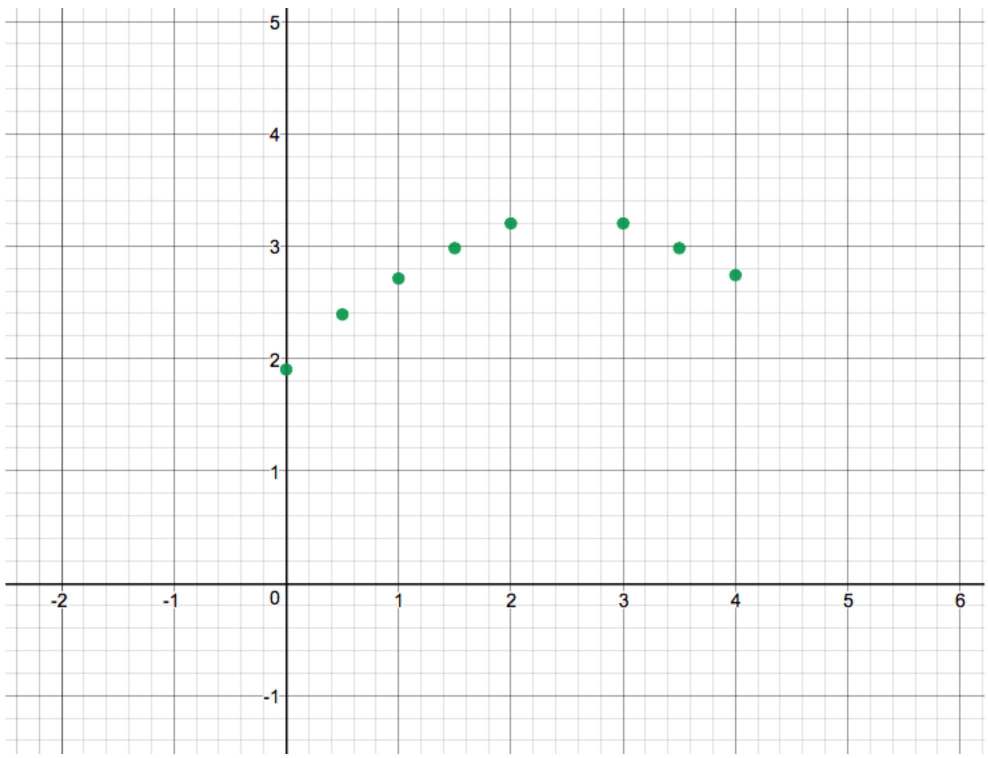
1.
Solution :



3 different polynomials were generated by Matlab. They are degree 5 with 6 points, degree 6 with 7 points and degree 7 with 8 points. Since the derivative at the points were not given, as a result, we only can perform Lagrange Interpolation and Cubic Spline Interpolation. The divided difference method will give same result as Lagrange Interpolation, therefore

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NEWTONS INTERPOLATION POLYNOMIAL

Input data follows:
X(0) =    1.00000000 F(X(0)) =    2.71000000
X(1) =    1.50000000 F(X(1)) =    2.98000000
X(2) =    2.00000000 F(X(2)) =    3.20000000
X(3) =    3.00000000 F(X(3)) =    3.20000000
X(4) =    3.50000000 F(X(4)) =    2.98000000
X(5) =    4.00000000 F(X(5)) =    2.74000000

The coefficients Q(0,0), ..., Q(N,N) are:
  2.71000000
  0.54000000
 -0.10000000
 -0.09666667
  0.03866667
  0.00400000
```

Degree 5

$$f_5(x) = 2.71 + 0.54(x-1) - 0.1(x-1)(x-1.5) - 0.09666667(x-1)(x-1.5)(x-2)$$
$$+0.03866667(x-1)(x-1.5)(x-2)(x-3) + 0.004(x-1)(x-1.5)(x-2)(x-3)(x-3)$$

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NEWTONS INTERPOLATION POLYNOMIAL

Input data follows:
X(0) =   0.50000000 F(X(0)) =   2.39000000
X(1) =   1.00000000 F(X(1)) =   2.71000000
X(2) =   1.50000000 F(X(2)) =   2.98000000
X(3) =   2.00000000 F(X(3)) =   3.20000000
X(4) =   3.00000000 F(X(4)) =   3.20000000
X(5) =   3.50000000 F(X(5)) =   2.98000000
X(6) =   4.00000000 F(X(6)) =   2.74000000

The coefficients Q(0,0), ..., Q(N,N) are:
  2.39000000
  0.64000000
 -0.10000000
  0.00000000
 -0.03866667
  0.02577778
 -0.00622222
```

Degree 6

$$f_6(x) = 2.39 + 0.64(x-0.5) - 0.1(x-0.5)(x-1) + 0 - 0.03866667(x-0.5)(x-1)(x-1.5)(x-2)$$
$$+0.02577778(x-0.5)(x-1)(x-1.5)(x-2)(x-3) - 0.00622222(x-0.5)(x-1)(x-1.5)(x-2)(x-3)(x-3)$$

```
NEWTONS INTERPOLATION POLYNOMIAL

Input data follows:
X(0) =   0.00000000 F(X(0)) =   1.90000000
X(1) =   0.50000000 F(X(1)) =   2.39000000
X(2) =   1.00000000 F(X(2)) =   2.71000000
X(3) =   1.50000000 F(X(3)) =   2.98000000
X(4) =   2.00000000 F(X(4)) =   3.20000000
X(5) =   3.00000000 F(X(5)) =   3.20000000
X(6) =   3.50000000 F(X(6)) =   2.98000000
X(7) =   4.00000000 F(X(7)) =   2.74000000

The coefficients Q(0,0), ..., Q(N,N) are:
  1.90000000
  0.98000000
 -0.34000000
  0.16000000
 -0.08000000
  0.01377778
  0.00342857
 -0.00241270
```

Degree 7

$$f_7(x) = 1.9 + 0.98x - 0.34x(x-0.5) + 0.16x(x-0.5)(x-1) - 0.08x(x-0.5)(x-1)(x-1.5)$$
$$+0.01377778x(x-0.5)(x-1)(x-1.5)(x-2) + 0.0342857x(x-0.5)(x-1)(x-1.5)(x-2)(x-3)$$
$$-0.0024127x(x-0.5)(x-1)(x-1.5)(x-2)(x-3)(x-3)$$

$$f_5(2.5) = 3.28374999625,\ f_6(2.5) = 3.290333335,\ f_7(2.5) = 3.23511908125$$

Natural Spline Interpolation between the interval $(2.0, 3.0)$, thus

```
NATURAL CUBIC SPLINE INTERPOLATION

The numbers X(0), ..., X(N) are:
  0.00000000  0.50000000  1.00000000  1.50000000  2.00000000  3.00000000  3.50000000  4.00000000

The coefficients of the spline on the subintervals are:
for I = 0, ..., N-1
     A(I)          B(I)          C(I)          D(I)
   1.90000000    1.06477571    0.00000000   -0.33910282
   2.39000000    0.81044859   -0.50865423    0.33551411
   2.71000000    0.55342994   -0.00538307   -0.04295360
   2.98000000    0.51583166   -0.06981347   -0.16369971
   3.20000000    0.32324341   -0.31536303   -0.00788038
   3.20000000   -0.33112379   -0.33900416    0.24250347
   2.98000000   -0.48825035    0.02475104   -0.01650069
```

$$S_5(x) = 3.2 + 0.32324341(x-2) - 0.31536303(x-2)^2 - 0.00788038(x-2)^3$$

$$S_5(2.5) = 3.2817959$$

The best estimate is $S_5(2.5) = 3.2817959$. Since Cubic Spline fits not only points, but also the first derivative and second derivative to avoid fluctuation.

2.

$$S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 \text{ on } [x_0, x_1]$$

$$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 \text{ on } [x_1, x_2]$$

(a) $S(x_0) = f(x_0), S(x_1) = f(x_1), S(x_2) = f(x_2)$
(b) $S \in C^1[x_0, x_2], i.e., S_1'(x_1) = S_0'(x_1)$.
Solution : (a) implies $f(x_0) = a_0, f(x_1) = a_1$ and $f(x_2) = a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2$.

$$S_0'(x) = b_0 + 2c_0(x-x_0), S_1'(x) = b_1 + 2c_1(x-x_1)$$

Then (b) implies $a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 = a_1, b_0 + 2c_0(x_1 - x_0) = b_1$. Let $x_1 - x_0 = h_0$ and $x_2 - x_1 = h_1$, then

$$f(x_0) = a_0, f(x_1) = a_1, f(x_2) = a_1 + b_1h_1 + c_1h_1^2$$

$$a_1 = a_0 + b_0h_0 + c_0h_0^2, b_0 + 2c_0h_0 = b_1$$

Since $f(x_0), f(x_1), f(x_2), h_0, h_1$ are known. Therefore, it leads to 5 equations with 6 unknowns, which is not solvable.
If one more condition $S \in C^2[x_0, x_2]$ is added, suppose $S_1''(x_1) = S_0''(x_1)$.

$$S_0''(x) = 2c_0, S_1''(x) = 2c_1$$

This implies $c_0 = c_1$. It provided another distinct equation. As a result, we have 6 equations with 6 unknowns, which makes $a_0, b_0, c_0, a_1, b_1, c_1$ solvable. But $S_1''(x_1) = S_0''(x_1)$ may not be given. Therefore $S \in C^2[x_0, x_2]$ may not lead to a meaningful solution.

3.
Solution: This is a periodic spline interpolation, then

$$S_0'(x_0) = S_n'(x_n), S_0''(x_0) = S_n''(x_n)$$

and

$$S_0'(x) = b_0 + 2c_0(x-x_0) + 3d_0(x-x_0)^2$$

$$S_0''(x) = 2c_0 + 6d_0(x-x_0)$$

$$S_n'(x_n) = b_n$$

$$S_n''(x_n) = 2c_n$$

Thus $b_0 = b_n 2$ and $c_0 = c_n$. Moreover, it is periodic, then $a_0 = a_n$. Simplify with all the conditions given for spline interpolation

$$b_n = \frac{1}{h_n}(a_{n+1} - a_n) - \frac{h_n}{3}(2c_n + c_{n+1})$$

$$d_n = \frac{c_{n+1} - c_n}{3h_n}$$

Since $c_0 = c_n$, then there is no need to solve for c_0, moreover $a_0 = a_n$ and $h_0 = h_n$ for periodic condition, thus the matrix $Mc = r$. Therefore M is

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots & 0 & h_0 \\ h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ h_0 & 0 & 0 & 0 & h_{n-1} & 2(h_{n-1}+h_0) \end{bmatrix}$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}, r = \begin{bmatrix} \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{n-1}}(a_0 - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ \frac{3}{h_0}(a_1 - a_0) - \frac{3}{h_{n-1}}(a_0 - a_{n-1}) \end{bmatrix}$$

All M and r are known, therefore all c can be solved.
By all 7 points are given. Two addition points are given that magnitude of phase -120 is the same as phase 120. We assume the magnitude is a_0 at both phase -120 and 120. We can obtain equations as following

$$a_0 + b_0h_0 + c_0h_0^2 + d_0h_0^3 = a_1, b_0 + 2c_0h_0 + 3d_0h_0^2 = b_1, c_0 + 3d_0h_0 = c_1$$

$$a_1 + b_1h_1 + c_1h_1^2 + d_1h_1^3 = a_2, b_1 + 2c_1h_1 + 3d_1h_1^2 = b_2, c_1 + 3d_1h_1 = c_2$$

$$a_2 + b_2h_2 + c_2h_2^2 + d_2h_2^3 = a_3, b_2 + 2c_2h_2 + 3d_2h_2^2 = b_3, c_2 + 3d_2h_2 = c_3$$

$$a_3 + b_3h_3 + c_3h_3^2 + d_3h_3^3 = a_4, b_3 + 2c_3h_3 + 3d_3h_3^2 = b_4, c_3 + 3d_3h_3 = c_4$$

$$a_4 + b_4h_4 + c_4h_4^2 + d_4h_4^3 = a_5, b_4 + 2c_4h_4 + 3d_4h_4^2 = b_5, c_4 + 3d_4h_4 = c_5$$

$$a_5 + b_5h_5 + c_5h_5^2 + d_5h_5^3 = a_6, b_5 + 2c_5h_5 + 3d_5h_5^2 = b_6, c_5 + 3d_5h_5 = c_6$$

$$a_6 + b_6h_6 + c_6h_6^2 + d_6h_6^3 = a_7, b_6 + 2c_6h_6 + 3d_6h_6^2 = b_7, c_6 + 3d_6h_6 = c_7$$

$$a_7 + b_7h_7 + c_7h_7^2 + d_7h_7^3 = a_0, b_7 + 2c_7h_7 + 3d_7h_7^2 = b_0, c_7 + 3d_7h_7 = c_0$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 40 \\ 30 \\ 40 \\ 50 \\ 30 \\ 10 \end{bmatrix}, \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} -120 \\ -110 \\ -80 \\ -40 \\ -10 \\ 30 \\ 80 \\ 110 \\ 120 \end{bmatrix}, \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} a_0 \\ 7.98 \\ 8.95 \\ 10.71 \\ 11.70 \\ 10.01 \\ 8.23 \\ 7.86 \\ a_0 \end{bmatrix}$$

$$\begin{bmatrix} h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 2(h_3+h_4) & h_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_4 & 2(h_4+h_5) & h_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_5 & 2(h_5+h_6) & h_6 \\ h_7 & 0 & 0 & 0 & 0 & 0 & h_6 & 2(h_6+h_7) \\ 2(h_7+h_0) & h_0 & 0 & 0 & 0 & 0 & 0 & h_7 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \frac{3}{h_2}(a_3-a_2) - \frac{3}{h_1}(a_2-a_1) \\ \frac{3}{h_3}(a_4-a_3) - \frac{3}{h_2}(a_3-a_2) \\ \frac{3}{h_4}(a_5-a_4) - \frac{3}{h_3}(a_4-a_3) \\ \frac{3}{h_5}(a_6-a_5) - \frac{3}{h_4}(a_5-a_4) \\ \frac{3}{h_6}(a_7-a_6) - \frac{3}{h_5}(a_6-a_5) \\ \frac{3}{h_7}(a_0-a_7) - \frac{3}{h_6}(a_7-a_6) \\ \frac{3}{h_0}(a_1-a_0) - \frac{3}{h_7}(a_0-a_7) \end{bmatrix}$$

Put in all the possible values, we can obtain,

$$\begin{bmatrix} 10 & 80 & 30 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 140 & 40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40 & 140 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 140 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40 & 180 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 50 & 160 & 30 \\ 10 & 0 & 0 & 0 & 0 & 0 & 30 & 80 \\ 40 & 10 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} -2.297+0.3a_0 \\ 0.035 \\ -0.033 \\ -0.22575 \\ 0.01995 \\ 0.0698 \\ 0.3a_0 - 2.321 \\ 4.752 - 0.6a_0 \end{bmatrix}$$

Since a_0 is unknown, the goal is to substitute a_0 and reduce one column and one row. As a result we can solve for $c_1, ..., c_7$. Row operation on Row $1, 7$ and 8, we obtain

$$80c_1 + 30c_2 - 30c_6 - 80c_7 = 0.024$$

and substitution to get

$$-2900c_1 + 500c_2 - 800c_3 + 2100c_6 - 1300c_7 = -0.59$$

$$\begin{bmatrix} 30 & 140 & 40 & 0 & 0 & 0 & 0 \\ 0 & 40 & 140 & 30 & 0 & 0 & 0 \\ 0 & 0 & 30 & 140 & 40 & 0 & 0 \\ 0 & 0 & 0 & 40 & 180 & 50 & 0 \\ 0 & 0 & 0 & 0 & 50 & 160 & 30 \\ 80 & 30 & 0 & 0 & 0 & -30 & -80 \\ -2900 & 500 & -800 & 0 & 0 & 2100 & -1300 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} 0.035 \\ -0.033 \\ -0.22575 \\ 0.01995 \\ 0.0698 \\ 0.024 \\ -0.59 \end{bmatrix}$$

By computer calculation it gives

```
LUDec(M,r)
 7.98000000   0.02557633   0.00040119  -0.00000293
 8.95000000   0.03967083   0.00013725  -0.00000036
10.71000000   0.06646983   0.00009372  -0.00002016
11.70000000  -0.03023423  -0.00172035   0.00001775
10.01000000  -0.05245237   0.00040958  -0.00000073
 8.23000000  -0.01613854   0.00030078  -0.00000290
```

```matlab
function  LUDec(M,r)
% Filename: LUDec.m
% Input
% M - n-by-n matrix
% r - n-by-1 right hand vector of Mc=r
% Output
% A,B,C,D as coeffieint of each spline interpolation
N=size(M,1);
L=zeros(N);
U=zeros(N);
% Crout LU Factorization
for k=1:N
    for i=k:N
    L(i,k)=M(i,k)-L(i,1:k-1)*U(1:k-1,k);
    end
U(k,k)=1;
for j=k+1:N
    U(k,j)=(M(k,j)-L(k,1:k-1)*U(1:k-1,j))/L(k,k);
end
end
x=zeros(N,1);
x(1)=r(1)/L(1,1);
for i=2:N
x(i)=(r(i)-L(i,1:i-1)*x(1:i-1))/L(i,i);
end
c=zeros(N,1);
c(N)=x(N)/U(N,N);
for i=N-1:-1:1
c(i)=(x(i)-U(i,i+1:N)*c(i+1:N))/U(i,i);
end
 A = zeros(1,N+1);
 A(1) = 7.98;
 A(2) = 8.95;
 A(3) = 10.71;
 A(4) = 11.70;
A(5) = 10.01;
A(6) = 8.23;
A(7) = 7.86;

H = zeros(1,N);
H(1) = 30;
H(2) = 40;
H(3) = 30;
H(4) = 40;
H(5) = 50;
H(6) = 30;
H(7) = 10;

B = zeros(1,N);
C = zeros(1,N);
D = zeros(1,N);

for I = 1:N
   C(I) = c(I);
end
for I = 1:N-1
B(I) = (A(I+1)-A(I))/H(I) - H(I) * (C(I) + 2.0 * C(I+1)) / 3.0;
D(I) = (C(I+1) - C(I)) / (3.0 * H(I));
end

for I = 1:N-1
fprintf(1, '%13.8f%13.8f%13.8f%13.8f\n',A(I),B(I),C(I),D(I));
end
```