Math 470 Assignment 2

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5.4.1. Evaluate the following improper integrals.

a) $\int_1^\infty \frac{1+x}{x^3}dx$

proof: $\int_1^\infty \frac{1+x}{x^3}dx = \lim_{t\to\infty}\int_1^t \frac{1+x}{x^3}dx = \lim_{t\to\infty}\int_1^t(\frac{1}{x^3}+\frac{1}{x^2})dx = \lim_{t\to\infty}[(-\frac{1}{2x^2})|_1^t + (-\frac{1}{x})|_1^t] = \lim_{t\to\infty}[\frac{1}{2}-\frac{1}{2t^2}+1-\frac{1}{t}] = \frac{3}{2}$.

5.4.2. For each of the following, find all values of p $\in \mathbb{R}$ for which f is improperly integrable on I .

c) $f(x) = \frac{1}{xlog^p x}, I = (e,\infty)$

proof: Suppose f is improperly integrable on I, then $\int_e^\infty f(x)dx = \lim_{t\to\infty}\int_e^t \frac{1}{xlog^p x}dx = \lim_{t\to\infty}\int_e^t \frac{1}{x}(\frac{1}{logx})^p dx = \lim_{t\to\infty}\int_e^t \frac{dx}{x}(\frac{1}{logx})^p$. Let $u = logx$, then $du = \frac{dx}{x}$ when $x = e, u = 1$. Hence $\lim_{t\to\infty}\int_e^t \frac{dx}{x}(\frac{1}{logx})^p = \lim_{t\to\infty}\int_1^t \frac{1}{u^p}du$. By the result of "For which p does $\sum_{k=1}^\infty \frac{1}{k^p}dx$ converges?" example. In this case, for $p > 1$, f is improperly integrable on I.

d) $f(x) = \frac{1}{1+x^p}, I = (0,\infty)$

proof: Suppose f is improperly integrable on I, then $\int_0^\infty \frac{1}{1+x^p}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^p}dx$. Since $x \in I$ and x is positive implies $0 < x^p < x^p+1$ for any p. Thus $\frac{1}{1+x^p} < \frac{1}{x^p}$ for any p. Then $0 \le \int_0^\infty \frac{1}{1+x^p}dx = \int_0^1 \frac{1}{1+x^p}dx + \int_1^\infty \frac{1}{1+x^p}dx < \int_0^1 \frac{1}{1+x^p}dx + \int_1^\infty \frac{1}{x^p}dx$. $\sum_{k=1}^\infty \frac{1}{k^p}$ converges for $p > 1$ implies $\sum_{k=1}^\infty \frac{1}{1+k^p}$ also converges for $p > 1$, and $\int_0^1 \frac{1}{1+x^p}dx$ is finite for any p. Thus its sum is finite and converges implies for p > 1, f is improperly integrable on I by integral test.

For $p = 1$, $\int_0^\infty \frac{1}{1+x}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x}dx = \lim_{t\to\infty} log(1+x)|_0^t = \lim_{t\to\infty} log(1+t)$. It diverges.

For p < 1, $\int_0^\infty \frac{1}{1+x^p}dx = \int_0^1 \frac{1}{1+x^p}dx + \int_1^\infty \frac{1}{1+x^p}dx > \int_1^\infty \frac{1}{1+x^p}dx$. When $x \geq 1$ and $p < 1$, then $0 < x^p < 1$. It implies $\int_1^\infty \frac{1}{1+x^p}dx > \int_1^\infty \frac{1}{2x^p}dx > \infty$. Hence f diverges for $p < 1$.

For $p > 1$, f is improperly integrable on I.

e)$f(x) = \frac{log^a x}{x^p}$, where $a > 0$ is fixed, and $I = (1, \infty)$

proof: Suppose $x \in I$, then $x > 1$. Thus $a > 0$ and $x > 1$ implies $log^a x > 0$. Since $log^a x$ is increasing, thus choose a constant C that is so large s.t. $log^a x \geq 1$. Then $\frac{log^a x}{x^p} \geq \frac{1}{x^p}$. Function $\frac{1}{x^p}$ is known that when $p > 1$, $\frac{1}{x^p}$ is improper integral on I. Therefore, choose a t, s.t. $\frac{1}{x^{p-t}} > \frac{log^a x}{x^p} \geq \frac{1}{x^p}$. When $p - t > 1$, $\frac{1}{x^{p-t}}$ is improper integrable on I for any $x \geq C$.

6.2.0 Let $\{a_k\}$ and $\{b_k\}$ be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

a)If $\sum_{k=1}^\infty a_k$ converges and $\frac{a_k}{b_k} \to 0$ as $k \to \infty$, then $\sum_{k=1}^\infty b_k$ converges.

proof : False. Let $a_k = \frac{1}{k^2}$. It converges to $\frac{\pi^2}{6}$. Let $b_k = k^2$. $\frac{a_k}{b_k} = \frac{1}{k^4} \to 0$ as $k \to \infty$. But $\sum_{k=1}^\infty b_k = \sum_{k=1}^\infty k^2$ diverges.

b)Suppose that $0 < a < 1$. If $a_k \geq 0$ and $\sqrt[k]{a_k} \leq a$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^\infty a_k$ converges.

proof: True. By hypothesis, $0 \leq a_k \leq a^k < 1$ for all $k \in \mathbb{N}$. Since $0 < a < 1$, then $\sum_{k=0}^\infty a^k$ converges by geometric series test. Hence, $0 < \sum_{k=1}^\infty a_k \leq \sum_{k=1}^\infty a^k < \sum_{k=0}^\infty a^k$. Then $\sum_{k=1}^\infty a_k$ converges by Comparison Test.

c)Suppose that $a_k \to 0$ as $k \to \infty$. If $a_k \geq 0$ and $\sqrt{a_k + 1} \leq a_k$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^\infty a_k$ converges.

proof: True. Suppose $a_k \to 0$ as $k \to \infty$, and $a_k \geq 0$ and $\sqrt{a_{k+1}} \leq a_k$, then $0 \leq a_{k+1} \leq a_k^2$. Choose Nth position in series $\sum_{k=1}^\infty a_k$, s.t. $a_N < \frac{1}{3}$.

Thus $a_{N+1} \leq a_N^2 \leq \frac{1}{9}$, this implies $0 \leq \sum_{k=N}^{\infty} a_k \leq \sum_{k=1}^{\infty} \frac{1}{3^k}$. By Geometric Series Test, $\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges. And by Comparison Test, $\sum_{k=N}^{\infty} a_k \leq \sum_{k=1}$ converges. Also $\sum_{k=N}^{\infty} a_k \leq \sum_{k=1}$ is bounded, $\sum_{k=N}^{\infty} a_k \leq \sum_{k=1}$ is the partial sum of $\sum_{k=1}^{\infty} a_k$. Hence by Theorem 6.11 $\sum_{k=1}^{\infty} a_k$ converges .

d)Suppose that $a_k = f(k)$ for some continuous function $f : [1, \infty) \to [0, \infty)$ which satisfies $f(x) \to 0$ as $x \to \infty$. If $\sum_{k=1}^{\infty} a_k$ converges, then $\int_1^{\infty} f(x)$ converges.

proof: False. Let $f(k) = a^k$, a is slightly less than 1. $\sum_{k=1}^{\infty} a_k$ converges since $a < 1$, it is a geometric series. $\int_1^{\infty} f(k) = \sum_{t=1}^{\infty} \int_t^{t+1} f(k)dk$. Since a is so close to 1, $\int_t^{t+1} f(k)dk \approx 1$, thus $\int_1^{\infty} f(k) \approx \infty$, it diverges.