Math 470 Assignment 3

Arnold Jiadong Yu

January 30, 2018

6.2.1. Prove that each of the following series converges.

a)$\sum_{k=1}^{\infty} \frac{2k+5}{3k^3+2k-1}$

proof: Let $k \geq 1$, then $\frac{2k+5}{3k^3+2k-1} < \frac{2k+5}{3k^3} \leq \frac{3k}{3k^3} = \frac{1}{k^2}$ for $\forall k \geq 5$. By p-series test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Hence $\sum_{k=1}^{\infty} \frac{2k+5}{3k^3+2k-1}$ converges by Comparison Test.

b)$\sum_{k=1}^{\infty} \frac{k-1}{k2^k}$

proof: Let $k \geq 1$, then $\frac{k-1}{k2^k} < \frac{k}{k2^k} = \frac{1}{2^k} = (\frac{1}{2})^k$. $|\frac{1}{2}| < 1$, then $\sum_{k=1}^{\infty} (\frac{1}{2})^k$converges by geometric series test. Hence $\sum_{k=1}^{\infty} \frac{k-1}{k2^k}$ converges by Comparison Test.

c)$\sum_{k=1}^{\infty} \frac{logk}{k^p}$, $p > 1$

proof: Let $k \geq 1$, then $logk \leq k$ for large k. And $\frac{logk}{k^p} \leq \frac{k}{k^p} = \frac{1}{k^{p-1}}$, since $p > 1$, choose $p > 2$. Then $\sum_{k=1}^{\infty} \frac{1}{k^{p-1}}$ converges by p-series test. Hence $\sum_{k=1}^{\infty} \frac{logk}{k^p}$ converges by Comparison Test.

d)$\sum_{k=1}^{\infty} \frac{k^3 log^2 k}{e^k}$

proof: Let $k \geq 1$, then $logk \leq k$ for large k. This implies $k^3 log^2 k \leq k^5$. Then $\frac{k^3 log^2 k}{e^k} \leq \frac{k^5}{e^k}$. Since $\lim_{k\to\infty} \frac{k^5}{e^k} = \lim_{k\to\infty} \frac{5k^4}{e^k} = \lim_{k\to\infty} \frac{20k^3}{e^k} = \lim_{k\to\infty} \frac{60k^2}{e^k} = \lim_{k\to\infty} \frac{120k}{e^k} = \lim_{k\to\infty} \frac{120}{e^k} = (120) \lim_{k\to\infty} (\frac{1}{e})^k$. Since $|\frac{1}{e}| < 1$, $\sum_{k=1}^{\infty} (\frac{1}{e})^k$ converges by geometric series test. Thus $(120) \sum_{k=1}^{\infty} (\frac{1}{e})^k$ converges. Hence $\sum_{k=1}^{\infty} \frac{k^3 log^2 k}{e^k}$ converges by Comparison Test.

e)$\sum_{k=1}^{\infty}\frac{\sqrt{k}+\pi}{2+\sqrt[5]{k^8}}$

proof: Let $k \geq 1$, then $\frac{\sqrt{k}+\pi}{2+\sqrt[5]{k^8}} < \frac{\sqrt{k}+\pi}{\sqrt[5]{k^8}} \leq \frac{2\sqrt{k}}{\sqrt[5]{k^8}} = (2)\frac{k^{\frac{1}{2}}}{k^{\frac{8}{5}}} = (2)\frac{1}{k^{\frac{11}{10}}}$ for $\forall k \geq \pi^2$. $\sum_{k=1}^{\infty}\frac{1}{k^{\frac{11}{10}}}$ converges by p-series test. Then $(2)\sum_{k=1}^{\infty}\frac{1}{k^{\frac{11}{10}}}$ converges. Hence $\sum_{k=1}^{\infty}\frac{\sqrt{k}+\pi}{2+\sqrt[5]{k^8}}$ converges by Comparison Test.

f)$\sum_{k=1}^{\infty}\frac{1}{k^{logk}}$

proof: Let $k \geq 1$, then $logk > 2$ for $k > e^2$. Then $k^{logk} > k^2$ implies $\frac{1}{k^{logk}} < \frac{1}{k^2}$. $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-series test. Hence $\sum_{k=1}^{\infty}\frac{1}{k^{logk}}$ converges by Comparison Test.

6.2.2. Prove that each of the following series diverges.

a)$\sum_{k=1}^{\infty}\frac{3k^3+k-4}{5k^4-k^2+1}$

proof: Let $k \geq 1$, then $k-4 > 0$ and $-k^2+1 < 0$ for large k. This implies $\frac{3k^3+k-4}{5k^4-k^2+1} > \frac{3k^3}{5k^4-k^2+1} > \frac{3k^3}{5k^4} = (\frac{3}{5})(\frac{1}{k})$. $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges since it is a harmonic series. Then $(\frac{3}{5})\sum_{k=1}^{\infty}\frac{1}{k}$ also diverges. Hence $\sum_{k=1}^{\infty}\frac{3k^3+k-4}{5k^4-k^2+1}$ diverges by Comparison Test.

b)$\sum_{k=1}^{\infty}\frac{\sqrt[k]{k}}{k}$

proof: Let $k \geq 1$, then $0 < \frac{1}{k} \leq 1 \Rightarrow -\frac{1}{k} < 0 \Rightarrow 1-\frac{1}{k} < 1 \Rightarrow k^{1-\frac{1}{k}} < k \Rightarrow \frac{1}{k^{1-\frac{1}{k}}} < \frac{1}{k}$. Then $\frac{\sqrt[k]{k}}{k} = \frac{k^{\frac{1}{k}}}{k} = \frac{1}{k^{1-\frac{1}{k}}} < \frac{1}{k}$. $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges since it is a harmonic series. Hence $\sum_{k=1}^{\infty}\frac{\sqrt[k]{k}}{k}$ diverges by Comparison Test.

c)$\sum_{k=1}^{\infty}(\frac{k+1}{k})^k$

proof: Let $k \geq 1$, then $\frac{1}{k} > 0 \Rightarrow \frac{k+1}{k} = 1+\frac{1}{k} > 1 \Rightarrow (\frac{k+1}{k})^k > 1^k = 1$. $\sum_{k=1}^{\infty}1$ diverges. Hence $\sum_{k=1}^{\infty}(\frac{l+1}{k})^k$ diverges by Comparison Test.

d)$\sum_{k=2}^{\infty}\frac{1}{klog^pk}$, $p \leq 1$, $p \leq 1$

proof: Let $k \geq 1$ and $f(k) = \frac{1}{klog^pk}$, then $f'(k) = k^{-1}(logk)^{-p} = -\frac{1}{k^2}(\frac{1}{log^pk} + \frac{p}{log^{p+1}k}) < 0$. Then f is decreasing. k is continuous and log^pk is continuous, this implies f is continuous for $k \geq 1$. By Integral Test, want to show $\int_2^\infty f(k)dk$ diverges. Let $u = logk$, then $du = \frac{1}{k}dk$, then $\int_2^\infty f(k)dk > \int_e^\infty f(k)dk = \int_1^\infty \frac{1}{u^p}du = \infty$ by p-series test. Hence $\sum_{k=2}^\infty \frac{1}{klog^p k}$, $p \leq 1$ diverges for $p \leq 1$ by Comparison Test.

6.2.4. Find all $p \geq 0$ such that the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{klog^p(k+1)}$$

.

proof: Let $k \geq 1$, then $log^p(k+1) \geq log^p k$. This implies $\frac{1}{klog^p(k+1)} \leq \frac{1}{klog^p(k)}$. Since $\sum_{k=1}^\infty \frac{1}{klog^p(k)}$ converges when $p > 1$ by Integral Test in 6.2.2(d). Hence $\sum_{k=1}^\infty \frac{1}{klog^p(k+1)}$ converges when $p > 1$ by Comparison Test.

Also $\frac{1}{klog^p(k+1)} > \frac{1}{(k+1)log^p(k+1)}$, but Integral Test $\sum_{k=1}^\infty \frac{1}{(k+1)log^p(k+1)}$ diverges for $p \leq 1$. Hence $\sum_{k=1}^\infty \frac{1}{klog^p(k+1)}$ diverges for $p \leq 1$ by Comparison Test.

6.2.7. Suppose that a_k and b_k are nonnegative for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ converge, then $\sum_{k=1}^\infty a_k b_k$ converges.

proof: Assume $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ converge to $A, B \in \mathbb{R}$. Then $(\sum_{k=1}^\infty a_k)(\sum_{k=1}^\infty b_k) = AB$, it also converges. Since a_k and b_k are nonnegative for all $k \in \mathbb{N}$, then $a_i b_j$ are nonnegative for all $i, j = 1, 2, ..., k \in \mathbb{N}$. $(\sum_{k=1}^\infty a_k)(\sum_{k=1}^\infty b_k) = (a_1 + a_2 + a_3 + ... + a_k)(b_1 + b_2 + b_3 + ... + b_k) = (a_1 + a_2 + a_3 + ... + a_k)b_1 + (a_1 + a_2 + a_3 + ... + a_k)b_2 + (a_1 + a_2 + a_3 + ... + a_k)b_3 + ... + (a_1 + a_2 + a_3 + ... + a_k)b_k > a_1b_1 + a_2b_2 + ...a_kb_k = \sum_{k=1}^\infty a_kb_k$. Hence $\sum_{k=1}^\infty a_kb_k$ converges by Comparison Test.
Worked with Micheal Roark for 6.2.7.

6.2.10. Find all $p \in \mathbb{R}$, such that

$$\sum_{k=2}^{\infty} \frac{1}{(log(logk))^{plogk}}$$

converges.

proof: Let $k \geq 2$, then $logk > 0$. When $p = 0$, $\frac{1}{(log(logk))^{plogk}} = 1$ for $\forall k$. It diverges by Divergence Test. When $p < 0$, $plogk < 0$. This implies $\frac{1}{(log(logk))^{plogk}} = (log(logk))^{-plogk} \nrightarrow 0$, since it is increasing positive base with increasing positive exponent. Hence it diverges by Divergence Test. Therefore, when $p \leq 0$, $\sum_{k=2}^{\infty} \frac{1}{(log(logk))^{plogk}}$ diverges.
When $p > 0$, since $a^b := e^{bloga}$. Then $(log(logk))^{plogk} = e^{plogk(log(log(logk)))}$. $log(log(logk)) \to \infty$ as $k \to \infty$.
Need to find an A, s.t. $A < e^{plogk(log(log(logk)))}$ implies $\frac{1}{A} > \frac{1}{e^{plogk(log(log(logk)))}}$. If $\sum_{k=2)\infty} \frac{1}{A}$ converges, then by Comparison Test $\sum_{k=2}^{\infty} \frac{1}{(log(logk))^{plogk}}$ will converge. This A will help deciding p.