Math 430 Assignment 7

Arnold Jiadong Yu

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3.3

solution:
($\Rightarrow$) Let $\mathbf{A}$ is $m \times n$ matrix with full rank and $\mathbf{x}$ is a basic feasible solution. Suppose a vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction at $\mathbf{x}$. i.e there exist a positive scalar θ s.t. $\mathbf{x} + \theta\mathbf{d} \in P$. Thus, $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{b}$ and $\mathbf{x} + \theta\mathbf{d} \geq 0$. Therefore,

$$\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{A}\mathbf{x} + \theta\mathbf{A}\mathbf{d} = \mathbf{b} + \theta\mathbf{A}\mathbf{d} = \mathbf{b} \Rightarrow \mathbf{A}\mathbf{d} = 0$$

Since $\mathbf{x}$ is a basic feasible solution of P, after reordering $x_i = 0$ for $m < i \leq n$. Moreover

$$\mathbf{x} + \theta\mathbf{d} \geq 0 \Rightarrow x_i + \theta d_i \geq 0 \Rightarrow d_i \geq 0$$

Since $\theta > 0$. i.e. $d_i \geq 0$ for every i such that $x_i = 0$ to ensure $\mathbf{x} + \theta\mathbf{d} \geq 0$. As a result, if a vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction at $\mathbf{x}$, then $\mathbf{A}\mathbf{d} = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.
($\Leftarrow$) Let $\mathbf{A}$ is $m \times n$ matrix with full rank and $\mathbf{x}$ is a basic feasible solution. Suppose $\mathbf{A}\mathbf{d} = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$. WTS $\mathbf{d}$ is a feasible direction at $\mathbf{x}$, then we need to construct a positive scalar θ such that $\mathbf{x} + \theta\mathbf{b} \in P$.
Let

$$\theta = \min\{-\frac{x_j}{d_j} | d_j < 0, x_j > 0\}$$

i.e. every component of $x_j + \theta b_j$ is positive for x_j is a basic variable, $x_i + \theta b_i = 0$ for every i such that $x_i = 0$. Hence, there exists a positive scalar θ such that $\mathbf{x} + \theta\mathbf{b} \geq 0 \Rightarrow \mathbf{x} + \theta\mathbf{b} \in P$. i.e. a vector $\mathbf{d}$ is a feasible direction at $\mathbf{x}$.

3.6

solution:

(1) Suppose the reduced cost of every nonbasic variable is positive, and let $\mathbf{y}$ be an arbitrary feasible solution with $\mathbf{y} \neq \mathbf{x}$, define $\mathbf{d} = \mathbf{y} - \mathbf{x}$. This implies $\mathbf{Ax} = \mathbf{Ay} = \mathbf{b}$. Therefore $\mathbf{Ad} = \mathbf{0}$. We can rewrite the equality into

$$\mathbf{B}\mathbf{d}_B + \sum_{i \in N} \mathbf{A}_i d_i = 0$$

where N is the set of indices corresponding to the nonbasic variables. Since $\mathbf{B}$ is invertible, we obtain

$$\mathbf{d}_B = -\sum_{i \in N} \mathbf{B}^{-1}\mathbf{A}_i d_i$$

$$\mathbf{c}^{'}\mathbf{d} = \mathbf{c}_B^{'}\mathbf{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N}(c_i - \mathbf{c}_B^{'}\mathbf{B}^{-1}\mathbf{A}_i)d_i = \sum_{i \in N} \bar{c}_i d_i > 0$$

by assumption. i.e. $\mathbf{c}^{'}(\mathbf{y} - \mathbf{x}) = \mathbf{c}^{'}\mathbf{d} > 0$, and since $\mathbf{y}$ is arbitary feasible solution, whenever you move toward that feasible solution the cost increases. Hence $\mathbf{x}$ is the unique optimal solution.

(2) Suppose $\mathbf{x}$ is the unique optimal solution and is nondegenerate. $\mathbf{x}$ is nondegenerate, then the basic variables are positive and nonbasic varibles are zeros. Suppose one nonbasic variable in reduced cost function is nonpositive. i.e.

$$\bar{c}_m \leq 0 \text{ for some } m \in N$$

Let $\mathbf{d}$ be the mth feasible direction at $\mathbf{x}$, then there exists a positive scalar θ for which $\mathbf{x} + \theta\mathbf{d} \in P$ since $\mathbf{x}$ is nondegenerate. Therefore,

$$\mathbf{c}^{'}\mathbf{d} = \mathbf{c}_B^{'}\mathbf{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N}(c_i - \mathbf{c}_B^{'}\mathbf{B}^{-1}\mathbf{A}_i)d_i = \sum_{i \in N} \bar{c}_i d_i = \bar{c}_m \leq 0$$

choose $d_m = 1$ and $d_j = 0$ for $j \in N$ and $j \neq m$. Moreover,

$$\mathbf{c}^{'}(\mathbf{x} + \theta\mathbf{d}) \leq \mathbf{c}^{'}\mathbf{x}$$

Contradict to $\mathbf{x}$ is a unique optimal solution. Hence, the reduced cost function of every nonbasic variable is positive.

3.10

solution:

If $n - m = 2$, then there are two nonbasic variables, and there are at most two components are nonzero for the reduced cost function. We only care about these two nonzero components of the cost function. Therefore, the cost function is a line in $\mathbb{R}^2$. There are two cases since simplex method can only terminate at two places. First consider the polyhedron is unbounded. Then the cost function will terminate and $-\infty$ will be the optimal solution, or will terminate at a vertex since it is a line moving inside the polyhedron. Second consider the polyhedron is bounded. Then it also will hit a vertex with an optimal solution and can not move anymore since the cost function is a line. As result, when $n-m=2$, the simplex method will also terminate.

3.11

solution:
My idea for this problem is since $n-m=3$, then there are three nonbasic bariables, and there are at most three components are nonzero for the reduced cost function. Therefore, the cost function is a hyperplane in $\mathbb{R}^3$. This may cause cycle if the hyperplane overlaps one face of the polyhedron. Since one face of polyhedron may contains multiple vertex, thus it will cycle between these vertexes.
But I was not able to find an example.