Math 470 Assignment 14

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7.2.6. Prove that

$$|\sum_{k=1}^{\infty}(1-\cos(\frac{1}{k}))| \leq 2$$

proof: Let $f_k(x) = \frac{1}{k}\cdot\sin\frac{x}{k}$, then by same argument of exercise 7.2.5 $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[0,1]$ by Weierstrass M-Test. Thus

$$\int_0^1 \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty}\int_0^1 f_k(x) = \sum_{k=1}^{\infty}\int_0^1 \frac{1}{k}\cdot\sin\frac{x}{k}dx = \sum_{k=1}^{\infty}(-\cos\frac{x}{k})|_0^1 = \sum_{k=1}^{\infty}(1-\cos(\frac{1}{k}))$$

Since $f_k(x) \leq \frac{x}{k^2}$, then by Theorem 7.14 (ii),

$$|\sum_{k=1}^{\infty}(1-\cos(\frac{1}{k}))| \leq |\sum_{k=1}^{\infty}\int_0^1 \frac{x}{k^2}dx| \leq |\sum_{k=1}^{\infty}\frac{1}{k^2}| = \frac{\pi^2}{6} \leq 2.$$

7.2.7 Suppose that $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on a set E $\subseteq \mathbf{R}$. If g_1 is bounded on E and $g_k(x) \geq g_{k+1}(x) \geq 0$ for all $x \in E$ and $k \in \mathbf{N}$, prove that $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

proof: Choose $M > 0$ so large , such that $g_1(x) \leq M$. Since g is decreasing for all $x \in E$, then $M \geq g_1(x) \geq g_n(x) \geq 0$ for all $x \in E$ and $n \in \mathbf{N}$. Moreover, $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E. By Cauchy Criterion, choose N, such that $\forall n, m \geq N$ implies $\sum_{k=m}^{n} f_k < \frac{\epsilon}{3M}$ for $\forall x \in E$, then

$$\sum_{k=m}^{n} f_k g_k = \sum_{k=m}^{n} f_k g_n + \sum_{k=m}^{n-1}\sum_{j=m}^{k} f_j(g_k - g_{k+1})$$

$$\leq \frac{\epsilon}{3M} \cdot M + M(g_m - g_n) \leq \frac{\epsilon}{3M} \cdot M + M(g_m + g_n) \leq \frac{\epsilon}{3M} \cdot M + \frac{2\epsilon}{3M} \cdot M = \epsilon.$$

Hence $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E. Worked together with Michael and Caleb.

7.2.9. Suppose that $a_k \downarrow 0$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k \sin(kx)$ converges uniformly on any closed interval $[a, b] \subset (0, 2\pi)$.

proof: Let $f_k(x) = \sin(kx)$, $g_k(x) = a_k$ for $k \in \mathbf{N}$. By the trigonometry identity of $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$. Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$, then we can derive the following,

$$s_n(x) = \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \sin(kx) = \frac{\cos(x/2) - \cos((n + 1/2)x)}{2 \sin(x/2)}$$

for $n \in \mathbf{N}$, and $x \in [a, b] \subset (0, 2\pi)$. Hence the partial sum s_n satisfy,

$$|s_n(x)| \leq \frac{1}{|\sin(x/2)|}$$

for $x \in (0, 2\pi)$. If $\delta = \min\{2\pi - b, a\}$ and $x \in [a, b]$, then $\sin(x/2) \geq \sin(\delta/2)$ implies $\frac{1}{\sin(\delta/2)} \geq \frac{1}{\sin(x/2)}$. Therefore by Dirichlet's Test, $\sum_{k=0}^{\infty} a_k \sin(k/x)$ converges uniformly on $[a, b] \subset (0, 2\pi)$.