Math 335 Assignment 5

Arnold Jiadong Yu

March 10, 2018

(1) Let G_1 and G_2 be isomorphic groups. Show that:
(a) if G_1 is commutative then so is G_2,
(b) if G_1 is cyclic then so is G_2.

proof: a) Suppose G_1 and G_2 be isomorphic groups and G_1 is commutative. Then pick $u, v \in G_1$ and $x, y \in G_2$, an isomorphism

$$f : G_1 \to G_2$$

such that $f(u) = x$ and $f(v) = y$. Hence,

$$xy = f(u)f(v) = f(uv) = f(vu) = f(v)f(u) = yx$$

Therefore, G_2 is commutative.

b) Suppose G_1 and G_2 be isomorphic groups and G_1 is cyclic, then $G_1 =< a >$ for some $a \in G_1$ and $< a >= \{a^k | k \in \mathbb{Z}\}$ .Pick an isomorphism

$$f : G_1 \to G_2$$

then $f(a) \in G_2$

$$< f(a) >= \{f^k(a)|k \in \mathbb{Z}\} = \{f(a^k)|k \in \mathbb{Z}\} = \{f(x)|x \in G1\} = G_2$$

Hence G_2 is cyclic.

(2) Consider the set T of all 2×2 matrices of the form

$$\begin{pmatrix} 1 & \mathrm{a} \\ 0 & 1 \end{pmatrix}$$

where a is real number.

(a) Is T a subgroup of $GL_2(\mathbb{R})$ (the group of invertible 2×2-matrices over$\mathbb{R}$)?

(b) If the answer to (a) is 'Yes', is the map

$$T \to \mathbb{R}, \begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix} \mapsto a$$

an isomorphism? (Here, $\mathbb{R}$ is the usual additive group of reals.)

proof: a) Yes. Identity hold when $a = 0$, $I \in T$. Inverse hold, there exist $\begin{pmatrix} 1 & \text{-a} \\ 0 & 1 \end{pmatrix} \in T$ such that $\begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \text{-a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$. Closure under multiplication, choose $a, b \in \mathbb{R}$, then $\begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \text{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \text{a+b} \\ 0 & 1 \end{pmatrix} \in T$. Hence T a subgroup of $GL_2(\mathbb{R})$ (the group of invertible 2×2-matrices over $\mathbb{R}$).

b) The map denoted f is isomorphism. There exists an inverse map f^{-1} such that

$$T \to \mathbb{R} \to T, \begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix} \mapsto a \mapsto \begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix}$$

Thus the map is bijective. Choose $a, b \in \mathbb{R}$, then $f\left(\begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \text{b} \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & \text{a+b} \\ 0 & 1 \end{pmatrix}\right) = a + b = f\left(\begin{pmatrix} 1 & \text{a} \\ 0 & 1 \end{pmatrix}\right) f\left(\begin{pmatrix} 1 & \text{b} \\ 0 & 1 \end{pmatrix}\right)$ Hence $T \to \mathbb{R}$ is isomorphism.

(3) Consider the set L of all 2×2 matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \text{a} & 1 \end{pmatrix}$$

where a is real number. What are the answers to the questions on L, similar to (a, b) in Problem 2?

proof: L is a subgroup of $GL_2(\mathbb{R})$ (the group of invertible 2×2-matrices over$\mathbb{R}$). Proof is trivial. L is isomorphism to $\mathbb{R}$, the proof is similar to question 2.

(4) If T and L are both subgroups of $GL_2(\mathbb{R})$, are they isomorphic?

proof: Yes. Since $T \cong \mathbb{R}$ and $L \cong \mathbb{R}$, then $T \cong L$ by Theorem. Theorem proved in class.

(5) Let $n \geq 2$ be a natural number. Show that the order $|\sigma|$ of any element $\sigma \in S_n \backslash \{1\}$ divides $n!$

proof: Let σ be any element of $S_n \backslash \{1\}$, then σ can be written as a product of disjoint cyclic permutations. Then $\sigma = \tau_{i_1}\tau_{i_2}...\tau_{i_j}$ where $i, j < n$, $i_1 + i_2 + ... + i_j \leq n$ and $i_1, ...i_j$ represent the length of the cyclic permutations. By the definition of order, the order of $|\sigma|$ is the least common multiple of $i_1, i_2, ..., i_j$. Since $\text{lcm}\{4, 5, 5\} = \text{lcm}\{4, 5\}$. Then let $m_1, m_2, ..., m_k$ be distinct numbers from the set $\{i_1, i_2, ..., i_j\}$. Then $\text{lcm}\{m_1, m_2, ..., m_k\}$ divides $m_1 \cdot m_2 \cdot ... \cdot m_k$. Moreover every $m_1, m_2, ..., m_k \leq n$, thus the product divides $n!$. Therefore $\text{lcm}\{m_1, m_2, ..., m_k\}$ divides $n!$. Hence the order of $|\sigma|$ divides $n!$.

(6) Determine the set of all possible orders $|\sigma|$ where $\sigma \in D_5 \backslash \{1\}$. ($D_5$ is the 5-th dihedral group).

proof: Let R denoted as rotation and F denoted as flip, then $D_5 = \{1, R, R^2, R^3, R^4, F, FR, FR^2, FR^3, FR^4\}$. By the definition of order, when σ is R, R^2, R^3, R^4, the order is 5. When σ is F, FR, FR^2, FR^3, FR^4, the order is 2. Since $F^2 = 1$ for D_5. Hence the possible orders are 2 and 5.

(7) Does the group $(\mathbb{R}^*, \cdot)$ contain an element of order n for every natural n?

proof: No. Suppose the group $(\mathbb{R}^*, \cdot)$ contain an element of order n for every natural n, choose $n = 3$ and $x \in \mathbb{R}^*$. Then $x^3 = 1$ implies $x = 1$. But $x = 1$ is not in consideration. There is a contradiction. Hence the group $(\mathbb{R}^*, \cdot)$ does not contain an element of order n for every natural n.

(8) Does there exist an infinite group in which every element is of finite order?

proof: Yes. Pick an infinite group in the form $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where $\theta = \frac{\pi k}{2}$ for all $k \in \mathbb{Z}$. Then $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus any element in the group to the order 2 is identity. Hence there exist an infinite group which every element is of order 2.

(9) How many generators does $\mathbb{Z}_{64}$ have? Describe them explicitly.

proof: Since $2^6 = 64$, then all the elements in $\mathbb{Z}_{64}$ has a factor of 2 can not be a generator. Thus there are $64 \cdot (1 - \frac{1}{2}) = 32$ generators, and there are all the odd number element in $\mathbb{Z}_{64}$.