Math 430 Assignment 9

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4.10
solution:
Consider the primal problem

$$\min \mathbf{c}^{'}\mathbf{x}$$
$$\text{s.t } \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

then the dual problem is

$$\max \mathbf{p}^{'}\mathbf{b}$$
$$\text{s.t. } \mathbf{p}^{'}\mathbf{A} \leq \mathbf{c}^{'}$$

Let

$$L(\mathbf{x}, \mathbf{p}) = \mathbf{c}^{'}\mathbf{x} + \mathbf{p}^{'}(\mathbf{b} - \mathbf{A}\mathbf{x})$$

($\Rightarrow$)Suppose $(\mathbf{x}^*, \mathbf{p}^*)$ is an equilibrium, then

$$L(\mathbf{x}^*, \mathbf{p}) \leq L(\mathbf{x}^*, \mathbf{p}^*) \leq L(\mathbf{x}, \mathbf{p}^*) \ \forall \mathbf{x} \geq \mathbf{0}, \forall \mathbf{p}$$

$$\implies ((\mathbf{p}^*)^{'} - \mathbf{p}^{'})(\mathbf{b} - \mathbf{A}\mathbf{x}^*) \geq 0, (\mathbf{c}^{'} - (\mathbf{p}^*)^{'}\mathbf{A})(\mathbf{x} - \mathbf{x}^*) \geq 0 \ \forall \mathbf{x} \geq \mathbf{0}, \forall \mathbf{p}$$

Consider the the primal and dual problem, then $\mathbf{b} - \mathbf{A}\mathbf{x} \geq 0, \forall \mathbf{x} \geq \mathbf{0}$ and $(\mathbf{c}^{'} - (\mathbf{p})^{'}\mathbf{A} \geq 0, \forall \mathbf{p}$ i.e.

$$((\mathbf{p}^*)^{'} - \mathbf{p}^{'})(\mathbf{b}) \geq 0, (\mathbf{c}^{'})(\mathbf{x} - \mathbf{x}^*) \geq 0 \ \forall \mathbf{x} \geq \mathbf{0}, \forall \mathbf{p}$$

This implies that $\mathbf{x}^*$ and $\mathbf{p}^*$ are optimal to the primal and dual, respectively.
($\Leftarrow$) The converse is trivial. Suppose that $\mathbf{x}^*$ and $\mathbf{p}^*$ are optimal to the primal and dual, then

$$((\mathbf{p}^*)^{'} - \mathbf{p}^{'})(\mathbf{b}) \geq 0, (\mathbf{c}^{'})(\mathbf{x} - \mathbf{x}^*) \geq 0 \ \forall \mathbf{x} \geq \mathbf{0}, \forall \mathbf{p}$$

Consider the the primal and dual problem, then $\mathbf{b}-\mathbf{A}\mathbf{x}\geq 0, \forall \mathbf{x}\geq \mathbf{0}$ and $(\mathbf{c}^{'}-(\mathbf{p})^{'}\mathbf{A}\geq 0, \forall \mathbf{p}$ i.e.

$$\implies ((\mathbf{p}^{*})^{'}-\mathbf{p}^{'})(\mathbf{b}-\mathbf{A}\mathbf{x}^{*})\geq 0, (\mathbf{c}^{'}-(\mathbf{p}^{*})^{'}\mathbf{A})(\mathbf{x}-\mathbf{x}^{*})\geq 0 \ \forall \mathbf{x}\geq \mathbf{0}, \forall \mathbf{p}$$

$$\implies L(\mathbf{x}^{*},\mathbf{p})\leq L(\mathbf{x}^{*},\mathbf{p}^{*})\leq L(\mathbf{x},\mathbf{p}^{*}) \ \forall \mathbf{x}\geq \mathbf{0}, \forall \mathbf{p}$$

4.13a

solution:

Consider the primal problem, matrix $\mathbf{A}$ is $m\times n$,

$$\min \mathbf{c}^{'}\mathbf{x}$$

$$\text{s.t } \mathbf{A}\mathbf{x}=\mathbf{b}$$

$$\mathbf{x}\geq \mathbf{0}$$

and its dual,

$$\max \mathbf{p}^{'}\mathbf{b}$$

$$\text{s.t. } \mathbf{p}^{'}\mathbf{A}\leq \mathbf{c}^{'}$$

Suppose that the primal problem has a nondegenerate and unique optimal solution denoted $\mathbf{x}^{*}$ and corresponding matrix is $\mathbf{B}_{m\times m}$. $\mathbf{x}^{*}$ only has $n-m$ zeros. By Strong duality theorem, the dual problem also has a optimal denoted as $\mathbf{p}^{*}$, then

$$\mathbf{c}^{'}\mathbf{x}^{*}=\mathbf{p}^{*}\mathbf{b}$$

Moreover, by Complementary slackness,

$$p_i(\mathbf{b}_i^{'}\mathbf{x}^{*}-b_i)=0, \forall i,$$

$$(c_j-(\mathbf{p}^{*})^{'}\mathbf{B}_j)x_j=0, \forall, j.$$

$\mathbf{p}^{*}$ is just $\widetilde{\mathbf{c}^{'}}\mathbf{B}^{-1}$. Since $\mathbf{B}$ is unique, then $\mathbf{p}^{*}$ is unique by above. equations. $\mathbf{p}^{*}$ is also nondegenerate by number of equation above.

The dual of the dual problem is the primal problem, hence the converse is trivial.

4.15

solution:

The primal problem is

$$\text{minimize } x_2$$

$$\text{subject to } x_2 = 1$$

$$x_1 \geq 0, x_2 \geq 0$$

The dual problem is

$$\text{maximize } p_1$$

$$\text{subject to } p_1 \leq 1$$

Do a substitution, let $p_1 = x_1 - x_2 + x_3$, then the new dual problem can be

$$\text{maximize } x_1 - x_2 + x_3$$

$$\text{subject to } x_1 - x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

The optimal solution of the primal problem is unique and it is nondegenerate, the optimal cost is 1. The optimal cost of the dual problem is also 1, but it is not unique. The reason why this occurs is we are just minimizing a constant, which does not really make sense. The primal problem is kind of like minimizing 1 subject to nothing when transforms to its dual.

4.16

solution:

Consider the primal problem

$$\min\ 2x_1 + 2x_2 + x_3 + x_4$$

$$\text{s.t. } x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + x_4 = 1$$

$$x_1, x_2, x_3 \geq 0, x_4 \leq 0$$

then the minimal cost is 2 and it could be $(0, 1, 0, 0), (1, 0, 0, 0), (1/2, 1/2, 0, 0)$

The dual problem is

$$\max\ p_1 + p_2$$

$$\text{s.t. } p_1 + p_2 \leq 2$$

$$p_1 + p_2 \leq 2$$

$$p_1 \leq 1$$

$$p_2 \geq 1$$

$$p_1, p_2 \text{ free}$$

then the maximal cost is 2 and it could be $(1, 1), (0, 2)$.