Math 335 Assignment 6

Arnold Jiadong Yu

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(1) Are all proper subgroups of the dihedral group D_3 cyclic?

proof: Yes. $\#D_3 = 6$. All proper subgroups of D_3 are order of 1, 2 or 3. The Flip in D_3 is order 2, and Rotation is order 3. 2 and 3 are prime number, then they are all cyclic.

(2) Are all proper subgroups of the permutation group S_4 cyclic?

proof: No. D_3 is a proper subgroup of S_4 and it is not cyclic. Therefore, not all proper subgroups of S_4 is cyclic.

(3) Does S_{12} have a subgroup, isomorphic to $\mathbb{Z}_{13}$?

proof: No. Suppose there is a subgroup $H \subset S_{12}$ is isomorphic to $\mathbb{Z}_{13}$. Since 13 is prime number, thus every element except identity of $\mathbb{Z}_{13}$ is cyclic. By the definition of isomorphic, choose $x \in H$, then $|x| = 13$. But for every $a \in S_{12}$, $|a| \neq 13$. There is a contradiction. Therefore, S_{12} does not have a subgroup, isomorphic to $\mathbb{Z}_{13}$.

(4) Let G be the set of real matrices with determinant 1 or -1. Show that G is a subgroup of $GL_2(\mathbb{R})$ and describe the left and right cosets in $GL_2(\mathbb{R})$.

proof: $\det(I) = 1$, thus $I \in G$. Choose $A, B \in G$, then $\det(A) = 1$ or -1, $\det(B) = 1$ or -1 implies $\det(AB) = \det(A)\det(B) = 1$ or -1, then $AB \in G$. Choose $A \in G$, $\det(A^{-1}) = (\det(A))^{-1} = 1$ or -1, thus $A^{-1} \in G$. Therefore, G is a subgroup. The left cosets equal to the right cosets. Choose

$B \in GL_2(\mathbb{R})$, then $\det(BG) = \det(B)\det(G) = \det(G)\det(B) = \det(GB)$. That is the left cosets are the same as the right cosets.

(5) Let $\Gamma \subset \mathbb{C}$ be the set of complex numbers whose module is 1. Show that Γ is a subgroup of the multiplicative group $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and describe the cosets in $\mathbb{C}^*$.

proof: Let $\Gamma = \{a + bi | a, b \in \mathbb{R} \text{ and } a^2 + b^2 = 1\}$. $1 \in \Gamma$. Choose $x = a + bi, y = c + bi \in \Gamma$, then $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, $xy = (a+bi)(c+di) = (ac - bd) + (ad + bc)i$. Since $(ac - bd)^2 + (ad - bc)^2 = 1$, then $xy \in \Gamma$. Choose $x = a + bi \in \Gamma$, since $xx^{-1} = 1$ implies $x^{-1} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$, then $|x^{-1}| = 1$ implies $x^{-1} \in \Gamma$. Hence Γ is a subgroup of $\mathbb{C}^*$. Since any complex number with modulus 1 is just one circle with radius 1. Thus all cosets in $\mathbb{C}^*$ is just all boundary of circles with radius $\mathbb{R}\backslash\{0\}$.

(6) How many cosets does $\mathbb{Z}$ have in $\mathbb{Q}$?

proof: Infinitely many. $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$. $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}\}$, then $a\mathbb{Z} = \frac{m}{n} + a = \frac{m+an}{n} \in \mathbb{Q}$ where $\frac{m}{n} \in \mathbb{Q}$ and $a \in \mathbb{Z}$. This is a map that every element in $\mathbb{Z}$ shifts $\frac{m}{n}$ units. Since there are infinitely many $\frac{m}{n} \in \mathbb{Q}$. The cosets are infinitely many.

(7) Let $p \geq 2$ be a prime number. How many subgroups does $\mathbb{Z}_p$ have?

proof: Since $p \geq 2$ is a prime number. Then $\#\mathbb{Z}_p = p$. By Lagrange Theorem, denoted H as subgroup, then p must divide by $\#H$. Thus all subgroups of $\mathbb{Z}_p$ is trivial subgroups, and they are

$$\{0\}, \mathbb{Z}_p.$$

(8) Let $p \geq 2$ be a prime number. How many subgroups does $\mathbb{Z}_{p^2}$ have?

proof: Since $p \geq 2$ is a prime number. Also by Lagrange Theorem, all subgroups of $\mathbb{Z}_{p^2}$ is $\{0\}, \mathbb{Z}_p$, and $\mathbb{Z}_{p^2}$.