Math 470 Assignment 13

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7.2.1. a) Prove that $\sum_{k=1}^{\infty} \sin\frac{x}{k^2}$ converges uniformly on any bounded interval on $\mathbb{R}$.

b) Prove that $\sum_{k=0}^{\infty} e^{-kx}$ converges uniformly on any closed subinterval of $(0,\infty)$.

proof: a) Claim $|\sin\frac{x}{k^2}| \leq \frac{|x|}{k^2}$ for every $x \in [a,b]$ on $\mathbb{R}$. Since

$$(|\sin\frac{x}{k^2}|)' = |\cos(\frac{x}{k^2})\cdot\frac{1}{k^2}| \leq \frac{1}{k^2} = (\frac{|x|}{k^2})'$$

and

$$|\sin\frac{x}{k^2}| = 0 \leq \frac{|x|}{k^2} = 0 \text{ at } x = 0$$

Thus, let $M = \max\{|a|,|b|\}$, then

$$|\sin\frac{x}{k^2}| \leq \frac{|x|}{k^2} \leq \frac{\max\{|a|,|b|\}}{k^2} = \frac{M}{k^2}$$

Since $\sum_{k=1}^{\infty}\frac{M}{k^2} < \infty$. By Weierstrass M-Test, $\sum_{k=1}^{\infty}\sin\frac{x}{k^2}$ converges absolutely and uniformly on $[a,b]$, thus it converges uniformly on any bounded interval on $\mathbb{R}$.

b) Let $0 < a < b \in \mathbb{R}$, such that $[a,b] \subset (0,\infty)$. Then for every $x \in [a,b]$,

$$|e^{-kx}| = (\frac{1}{e^x})^k \leq (\frac{1}{e^a})^k = e^{-ka}$$

Since $|\frac{1}{e^a}| < 1$ for every $a \in \mathbb{R}_{\geq 0}$. Then by Geometric Series Test, $\sum_{k=0}^{\infty} e^{-ka}$ converges. Then $\sum_{k=0}^{\infty} e^{-kx}$ converges uniformly on any closed subinterval of $(0,\infty)$ by Weierstrass M-Test.

7.2.3 Let $E(x) = \sum_{k=0}^{\infty} x^k/k!$.

a) Prove that the series defining $E(x)$ converges uniformly on any closed interval $[a, b]$.

b) Prove that

$$\int_a^b E(x)dx = E(b) - E(a)$$

for all $a, b \in \mathbb{R}$.

c) Prove that the function $y = E(x)$ satisfies the initial value problem

$$y' - y = 0, \;\; y(0) = 1$$

proof: a) Let $a < b \in \mathbb{R}$, choose $M = \max\{|a|, |b|\}$. Then

$$|\frac{x^k}{k!}| \leq \frac{\max\{|a|, |b|\}^k}{k!} = \frac{M}{k!}$$

Let $a_k = \frac{M}{k!}$, then

$$\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \lim_{k\to\infty} \frac{1}{k+1} \to 0 < 1$$

Thus $\sum_{k=0}^{\infty} \frac{M}{k!}$ converges absolutely by Ratio Test. This implies $E(x)$ converges uniformly on any closed interval $[a, b]$.

b) By part a) and Theorem 7.1.4 (ii). Then

$$\int_a^b E(x)dx = \int_a^b \sum_{k=0}^{\infty} \frac{x^k}{k!}dx = \sum_{k=0}^{\infty} \int_a^b \frac{x^k}{k!}dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}|_a^b$$

$$= \sum_{k=0}^{\infty}(\frac{b^{k+1}}{(k+1)!} - \frac{a^{k+1}}{(k+1)!}) = (\sum_{k=0}^{\infty} \frac{b^{k+1}}{(k+1)!} + 1) - (\sum_{k=0}^{\infty} \frac{a^{k+1}}{(k+1)!} + 1)$$

$$= \sum_{k=0}^{\infty}(\frac{b^k}{k!}) - \sum_{k=0}^{\infty}(\frac{a^k}{k!}) = E(b) - E(a)$$

c) By part a) and Theorem 7.1.4 (iii). Then

$$y' = (\sum_{k=0}^{\infty} x^k/k!)' = \sum_{k=0}^{\infty}(x^k/k!)' = \sum_{k=1}^{\infty} x^{k-1}/(k-1)! = \sum_{k=0}^{\infty} x^k/k!$$

Thus

$$y' - y = \sum_{k=0}^{\infty} x^k/k! - \sum_{k=0}^{\infty} x^k/k! = 0$$

and $y(0) = E(0) = \sum_{k=0}^{\infty} 0^k/k! = 1$.

7.2.4. Suppose that

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

Prove that

$$\int_0^{\frac{\pi}{2}} f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

proof:

$$|\frac{\cos(kx)}{k^2}| \leq \frac{1}{k^2} \text{ for all } x \in \mathbb{R}$$

Then $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges implies $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ converges uniformly for all $x \in \mathbb{R}$. Thus $f(x) < \infty$. By Theorem 7.1.4 (ii),

$$\int_0^{\frac{\pi}{2}} f(x) = \int_0^{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{\cos(kx)}{k^2} = \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k^3}$$

When $k = 1, 3, 5, ...$, $\sin(\frac{\pi k}{2}) = (-1)^{k-1}$ and when $k = 2, 4, 6...$, $\sin(\frac{\pi k}{2}) = 0$.

$$\sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k^3} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

7.2.5. Show that

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(\frac{x}{k+1})$$

converges, pointwise on $\mathbb{R}$ and uniformly on each bounded interval in $\mathbb{R}$, to a differentiable function f which satisfies

$$|f(x)| \leq |x| \text{ and } |f'(x)| \leq 1$$

for all $x \in \mathbb{R}$.

proof: By example 7.2.1 a), and let $[a,b] \subset \mathbb{R}$, $M = \max\{|a|,|b|\}$, then

$$|\frac{1}{k}\sin(\frac{x}{k+1})| \leq \frac{1}{k}(\frac{|x|}{k+1}) < \frac{|x|}{k^2} \leq \frac{M}{k^2}$$

$\sum_{k=1}^{\infty} \frac{M}{k^2}$ converges implies $\sum_{k=1}^{\infty} \frac{1}{k}\sin(\frac{x}{k+1})$ converges uniformly by Weierstrass M-Test. Moreover,

$$|f(x)| = |\sum_{k=1}^{\infty} \frac{1}{k}\sin(\frac{x}{k+1})| \leq \sum_{k=1}^{\infty} \frac{|x|}{k(k+1)} = |x|\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1})$$

$$= |x|(1 - \lim_{k\to\infty}\frac{1}{k}) = |x|.$$

Also by Theorem 7.1.4 (iii),

$$|f'(x)| = |\sum_{k=1}^{\infty}(\frac{1}{k}\sin(\frac{x}{k+1}))'| = |\sum_{k=1}^{\infty}\frac{\cos(\frac{x}{k+1})}{k(k+1)}|$$

$$\leq \sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) = 1 - \lim_{k\to\infty}\frac{1}{k} = 1.$$