Math 470 Assignment 31

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Q1: Let $X = \{(a_1, a_2, ...) : a_k \in \mathbb{R}, \sum_{k=1}^{\infty} |a_k| < \infty\}$.

$$d((a_1, a_2, ...), (b_1, b_2, ...)) = \sum_{k=1}^{\infty} |a_k - b_k|$$

Show (X, d) is separable. (l_1 norm)

proof: Let $\mathbf{x} = (x_1, x_2, ...) \in X$. Let $E = \{\mathbf{x}^n = (x_1^{(n)}, x_2^{(n)}, ...) : x_i^{(n)} = x_i \text{ for } i \leq n \text{ and } x_i^{(n)} = 0 \text{ for } i > n\}$ and $E \subset X$. Then $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$. Let $\epsilon > 0$, then

$$d(\mathbf{x}, \mathbf{x}^n) = \sum_{k>n}^{\infty} |x_k| < \frac{\epsilon}{2} \text{ as } n \to \infty$$

Therefore, $\mathbf{x}$ is limit point of $\mathbf{x}^n$. WTS E is countable.
Since $\mathbb{Q} \in \mathbb{R}$ is countable. Every entry can be approximated by a sequence of rational numbers. Therefore, let $y_k^{(n),(i)} \to x_i^{(n)}$ as $k \to \infty$, where $y_k^{(n),(i)} \in \mathbb{Q}$ for all $k \in \mathbb{N}$ and . Fix k, then $y_k^{(n),(i)} \in E$ for all k, call it $\mathbf{y}^{(n),(i)}$. That is $(y_k^{(n),(i)} - x_i^{(n)}) \to 0$ for large k. Moreover, $d(y_k^{(n),(i)}, x_i^{(n)})) < \frac{\epsilon}{2k}$. Hence

$$d(\mathbf{x}, \mathbf{y}^{(x),(i)}) \leq d(\mathbf{x}, \mathbf{x}^n) + d(\mathbf{x}^n, \mathbf{y}^{(n),(i)}) < \frac{\epsilon}{2} + k \cdot \frac{\epsilon}{2k} < \epsilon$$

E is countable and every $\mathbf{x}$ is a limit point of E. That is (X, d) is separable.

Q2: let X = set of all bounded sequence of real numbers.

$$d((a_1, a_2, ...), (b_1, b_2, ...)) = \sup_{k \in \mathbb{N}} |a_k - b_k|$$

Show that (X, d) is not separable. (l_∞ norm)

proof: Let $E = \{(a_1, a_2, ...) : a_i = 1 \text{ or } 0\}$ (binary sequence), pick $\mathbf{x}, \mathbf{y} \in E$ and $\mathbf{x} \neq \mathbf{y}$, then

$$d(\mathbf{x}, \mathbf{y}) = 1$$

Moreover, $E \subset X$ since every element in E is bounded. Therefore

$$\bigcap_{\mathbf{x} \in E} B_{\frac{1}{2}}(\mathbf{x}) = \emptyset$$

Suppose $Y \subset X$ is dense, then $\overline{Y} = X$. This means every $\mathbf{x} \in E$ must be in $\overline{Y}$. Then $Y \cap B_\epsilon(\mathbf{x}) \neq \emptyset$ for each $\epsilon > 0$. Since there are uncoutable many binary sequence, then Y must be uncountable. That is (X, d) is not separable.