Math 470 Assignment 33

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10.5.8 A set E in metric space is called *clopen* if it is both open and closed.
a)Prove that every metric space has at least two clopen sets.
b)Prove that a metric space is connected if and only if it contains exactly two clopen sets.

proof: a) Let (X,d) be a metric space, then $\emptyset, X \subseteq X$. By 10.11 Remark, $\emptyset, X$ is both open and closed. Hence every metric space has at least two clopen sets.
b)($\Rightarrow$). Prove by contrapositive. Suppose a metric space (X,d) contains more than two clopen sets, WTS (X,d) is disconnected. Let $A \neq \emptyset$ and $A \subset X$ be a clopen set. That is A and A^c are open and non-empty. $A \cap A^c = \emptyset$ and $A \cup A^c = X$ by definition. Hence (X,d) is disconnected. That is if a metric space is connected, it contains no more than two clopen sets. Also by part a), if a metric space is connected, it contains exactly two clopen sets.
($\Leftarrow$). Also by contrapositive. Suppose a metric space is disconnected, WTS it contains more than two clopen sets. Proof is trivial.

10.5.11 Suppose that $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets in a metric space X such that $\cap_{\alpha\in A}E_\alpha \neq \emptyset$. Prove that

$$E = \bigcup_{\alpha\in A} E_\alpha$$

is connected.

proof: Prove by contrapositive. Suppose $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets in a metric space X and $E = \bigcup_{\alpha\in A} E_\alpha$ is disconnected, WTS $\cap_{\alpha\in A}E_\alpha = \emptyset$.
Since E is disconnected, then there exsits nonempty open subsets S and T of E, such that $S \cap T = \emptyset$ and $S \cup T = E$. Moreover, $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets, then there $\nexists S_\alpha, T_\alpha$, s.t. $S_\alpha \cap T_\alpha = \emptyset$ and $S_\alpha \cup T_\alpha = E_\alpha$. i.e. $S = \cup_{\beta\in A}E_\beta$,$T = \cup_{\gamma\in A}E_\gamma$ and $\beta \cap \gamma = \emptyset$, $\beta \cup \gamma = A$. Therefore, by Demorgan's Law

$$S \cup T = E \Rightarrow (S \cup T)^c = E^c = \emptyset \Rightarrow (S^c \cap T^c) = \emptyset$$

$$\Rightarrow (\cap_{\beta\in A}E_\beta) \cap (\cap_{\gamma\in A}E_\gamma) = \cap_{\alpha\in A}E_\alpha = \emptyset$$

Hence $E = \bigcup_{\alpha\in A} E_\alpha$ is disconnected implies $\cap_{\alpha\in A}E_\alpha = \emptyset$. That is suppose that $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected sets in a metric space X such that $\cap_{\alpha\in A}E_\alpha \neq \emptyset$, then $E = \bigcup_{\alpha\in A} E_\alpha$is connected.