Math 741 Assignment 12 (Hand-In)

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7.3.4.(H) solution: We know that the variance of a chi square random variable with k df is $2k$. Then

$$\text{Var}(\frac{(n-1)S^2}{\sigma^2}) = 2(n-1)$$

Moreover,

$$\text{Var}(\frac{(n-1)S^2}{\sigma^2}) = (\frac{n-1}{\sigma^2})^2\text{Var}(S^2) = 2(n-1)$$

Hence,

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

7.3.13.(H) solution: NTS

$$\lim_{n\to\infty} \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})(1+\frac{t^2}{n})^{(n+1)/2}} = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}, -\infty < t < \infty$$

$$\lim_{n\to\infty} \frac{(\frac{n+1}{2}-1)!}{\sqrt{n\pi}(\frac{n}{2}-1)!(1+\frac{t^2}{n})^{(n+1)/2}} = \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!(1+\frac{t^2}{n})^{(n+1)/2}}$$

$$= \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{t^2}{n})^{(n+1)/2}} = \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{t^2/2}{n/2})^{(n+1)/2}}$$

$$= \lim_{n\to\infty} \frac{(\frac{n-1}{2})!}{\sqrt{n\pi}(\frac{n-2}{2})!} \cdot e^{-t^2/2} = e^{-t^2/2} \lim_{n\to\infty} \frac{\sqrt{2\pi\frac{n-1}{2}}(\frac{n-1}{2})^{(n-1)/2}e^{-(n-1)/2}}{\sqrt{n\pi}\sqrt{2\pi\frac{n-2}{2}}(\frac{n-2}{2})^{(n-2)/2}e^{-(n-2)/2}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-t^2/2} \lim_{n\to\infty} \Big[\frac{\sqrt{2\pi \frac{n-1}{2}}}{\sqrt{2\pi \frac{n-2}{2}}} \cdot \frac{(\frac{n-1}{2})^{(n-1)/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} \cdot \frac{e^{-(n-1)/2}}{e^{-(n-2)/2}} \Big]$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-t^2/2} \cdot e^{-1/2} \lim_{n\to\infty} \frac{(\frac{n-1}{2})^{(n-1)/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Since

$$\lim_{n\to\infty} \frac{(\frac{n-1}{2})^{(n-2)/2}(\frac{n-1}{2})^{1/2}}{\sqrt{n}(\frac{n-2}{2})^{(n-2)/2}} = \lim_{n\to\infty} (\frac{n-1}{n-2})^{(n-2)/2}(\frac{n-1}{2n})^{1/2}$$

$$= \lim_{n\to\infty} (1 + \frac{1/2}{(n-2)/2})^{(n-2)/2} (\frac{n-1}{n})^{1/2} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{1/2}$$

7.3.14.(H) solution: Given the student t distribution with 1 df. Let $X \sim t(1)$ and

$$f(x) = \begin{cases} \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)} \frac{1}{1+x^2} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

Since student t distribution is symmetric, and $\int_{-\infty}^{\infty} f(x)dx = 1$

$$2\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\sqrt{\pi}\Gamma(1/2)}{\Gamma(1)} \implies \int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$$