Math 470 Assignment 18

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7.4.1. Prove that each of the following functions is analytic on $\mathbf{R}$ and find its Maclaurin expansion.

a)$x^2 + \cos(2x)$

proof: $f(x) = x^2 + \cos(2x)$ is infinity differentiable for all $x \in \mathbf{R}$ and $|f^{(n)}(x)| \leq 2^n$ where $2 > 0$ and $n \in \mathbf{N}$. Also $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, then $\cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!}$, then

$$x^2+\cos(2x) = x^2+\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = x^2+\sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!} = 1-x^2+\sum_{k=2}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}$$

Hence it is analytic on $\mathbf{R}$ by Theorem 7.43.

b)$x^2 3^x$

proof: $f(x) = x^2 3^x$ is infinity differentiable for all $x \in \mathbf{R}$. Also $3^x = e^{x \log 3}$ and $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, then

$$3^x = e^{x \ln 3} = \sum_{k=0}^{\infty} \frac{x^k \log^k 3}{k!} \Rightarrow x^2 3^x = \sum_{k=0}^{\infty} \frac{x^{k+2} \log^k 3}{k!} = \sum_{k=2}^{\infty} \frac{x^k \log^{k-2} 3}{(k-2)!}$$

Hence it is analytic on $\mathbf{R}$ by Theorem 7.46.

c)$\cos^2 x - \sin^2 x$

proof: By trigonometry identity, $f(x) = \cos^2 x - \sin^2 x = \cos(2x)$, and it is infinity differentiable in $\mathbf{R}$. Moreover $|f^{(n)}(x)| \leq 2^n$ where $2 > 0$ and

$n \in \mathbf{N}$. Then

$$\cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}$$

Hence it is analytic on $\mathbf{R}$ by Theorem 7.43.

d) $\frac{e^x - 1}{x}$

proof: Let $f(x) = \frac{e^x-1}{x}$ on $\mathbf{R}\backslash\{0\}$ and $f(x) = 0$ for $x = 0$. It is continuous and infinitely differentiable on $\mathbf{R}$. Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, then $e^x - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k!}$. Thus

$$f(x) = \frac{e^x - 1}{x} = \frac{\sum_{k=1}^{\infty} \frac{x^k}{k!}}{x} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$$

Hence it is analytic on $\mathbf{R}$ by Theorem 7.46.