Math 470 Assignment 5

Arnold Jiadong Yu

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6.3.3. For each of the following, find all values of $p \in \mathbb{R}$ for which the given series converges absolutely.

a)$\sum_{k=2}^{\infty} \frac{1}{k \log^p k}$

proof: Let $k \geq 2$ and $a_k = \frac{1}{k \log^p k}$, then $a_k > 0$ for large k and $|a_k| = a_k$. Thus, if $\sum_{k=2}^{\infty} a_k$ converges, it converges absolutely.

$$\int_2^{\infty} \frac{1}{k \log^p k} dk = \lim_{t \to \infty} \int_2^t \frac{1}{k \log^p k} dk$$

Let $u = \log k$, then $du = \frac{1}{k} dk$ and $u = \log 2$ when $k = 2$. Use u substitution.

$$\lim_{t \to \infty} \int_2^t \frac{1}{k \log^p k} dk = \lim_{t \to \infty} \int_{\log 2}^t \frac{1}{u^p} du = \int_{\log 2}^{\infty} \frac{1}{u^p} du$$

Hence, by p-series Test and Integral Test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \Rightarrow \int_{\log 2}^t \frac{1}{u^p} du \Rightarrow \int_2^{\infty} \frac{1}{k \log^p k} dk \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k \log^p k}$$

converges absolutely when $p > 1$.

b)$\sum_{k=2}^{\infty} \frac{1}{\log^p k}$

proof: Let $k \geq 2$ and $a_k = \frac{1}{\log^p k}$, then $a_k > 0$ for large k and $|a_k| = a_k$. When $p \leq 0$, $\sum_{k=2}^{\infty} \frac{1}{\log^p k}$ diverges by Divergent Test. When $p > 0$, $\exists C \in \mathbb{N}$, s.t. $\log k \leq C k^{1/p}$ for large k. This implies $\frac{1}{\log k} \geq \frac{1}{C k^{1/p}}$, then $\frac{1}{\log^p k} \geq \frac{1}{Ck}$

for large k. Hence for $p > 0$, $\sum_{k=2}^{\infty} a_k$ diverges by harmonic series test and Comparison Test. It diverges for all $p \in \mathbb{R}$.

c)$\sum_{k=1}^{\infty} \frac{k^p}{p^k}$

proof: Let $k \geq 1$ and $a_k = \frac{k^p}{p^k}$. When $p = 0$, a_k is undefined. By Ratio Test

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|\frac{(k+1)^p}{p^{k+1}}|}{|\frac{(k)^p}{p^k}|} = \lim_{k\to\infty} \frac{1}{|p|}(\frac{k+1}{k})^p = \frac{1}{|p|}.$$

Then $\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ converges absolutely when $|p| > 1$ and diverges when $|p| < 1$. When $p = 1$, $\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ diverges. When $p = -1$, $\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ converges conditionally.

d)$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k^p-1)}$

proof:Let $k \geq 1$ and $a_k = \frac{1}{\sqrt{k}(k^p-1)}$. When $p = 0$, a_k is undefined. $\sqrt{k} > 0$, then

$$\frac{1}{k^{p+1/2} - \sqrt{k}} \geq \frac{1}{k^{p+1/2}}$$

by p-series Test and Comparison Test. When $p + \frac{1}{2} < 1$, it diverges. Let $b_k = \frac{1}{k^{p+1/2}}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}(k^p-1)}}{\frac{1}{k^{p+1/2}}} = \lim_{k\to\infty} \frac{k^p}{k^p - 1} = 1.$$

Hence by p-series Test and Limit Comparison Test, $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k^p-1)}$ converges for $p > \frac{1}{2}$.

e)$\sum_{k=0}^{\infty}(\sqrt{k^{2p}+1} - k^p)$

proof: Let $k \geq 0$ and $a_k = (\sqrt{k^{2p}+1} - k^p)$. Then

$$a_k = (\sqrt{k^{2p}+1} - k^p) = \frac{1}{\sqrt{k^{2p}+1} + k^p} > 0.$$

Since $\sqrt{k^{2p}+1} < k^p$ for large k. This implies $\frac{1}{\sqrt{k^{2p}+1}+k^p} < \frac{1}{2k^p}$. Hence by p-series Test and Comparison Test, $\sum_{k=0}^{\infty}(\sqrt{k^{2p}+1} - k^p)$ converges absolutely when $p > 1$.

*f)$\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k}$

proof: Let $k \geq 1$ and $a_k = \frac{2^{kp}k!}{k^k} > 0$. By Ratio Test,

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{2^{(k+1)p}(k+1)!}{(k+1)^{k+1}}}{\frac{2^{kp}k!}{k^k}} = \lim_{k\to\infty} 2^p(\frac{k}{k+1})^k = \frac{2^p}{e}$$

by Example 6.3.1. d). Then when $\frac{2^p}{e} < 1$, $\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k}$ converges absolutely and diverges when $\frac{2^p}{e} > 1$. When $\frac{2^p}{e} = 1$, $a_k = \frac{2^k k!}{k^k}$. Since $2 < e$, it diverges. Hence $\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k}$ converges absolutely when $p > \log_2(e)$.

6.3.4. Suppose that $a_k \geq 0$ and that $a_k^{1/k} \to a$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k x^k$ converges absolutely for all $|x| < 1/a$ if $a \neq 0$ and for all $x \in \mathbb{R}$ if $a = 0$.

proof: Suppose that $a_k \geq 0$ and that $a_k^{1/k} \to a$ as $k \to \infty$. Let $a \neq 0$, by Root Test,

$$\limsup_{k\to\infty} |a_k x^k|^{1/k} = \limsup_{k\to\infty} a_k^{1/k}|x| = \limsup_{k\to\infty} a|x| = a|x|$$

then $|x| < 1/a$, $\sum_{k=1}^{\infty} a_k x^k$ converge absolutely. When $a = 0$,

$$\limsup_{k\to\infty} |a_k x^k|^{1/k} = 0 < 1$$

Hence $\sum_{k=1}^{\infty} a_k x^k$ converges absolutely for all $x \in \mathbb{R}$.

6.3.5. Define a_k recursively by $a_1 = 1$ and

$$a_k = (-1)^k(1 + k\sin(\frac{1}{k}))^{-1}a_{k-1}, k > 1.$$

Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

proof: By Ratio Test,

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} |\frac{(-1)^{k+1}}{1 + (k+1)\sin\frac{1}{k+1}}| = \lim_{k\to\infty} |\frac{1}{1 + (k+1)\sin\frac{1}{k+1}}|$$

$$= \lim_{k\to\infty} \left|\frac{\frac{1}{k+1}}{\frac{1}{k+1} + \sin\frac{1}{k+1}}\right| = \lim_{k\to\infty} \left|\frac{-\frac{1}{(k+1)^2}}{-\frac{1}{(k+1)^2} + \cos\frac{1}{k+1}(-\frac{1}{(k+1)^2})}\right|$$

$$= \lim_{k\to\infty} \left|\frac{1}{1 + \cos\frac{1}{k+1}}\right| = \frac{1}{2} < 1$$

by L'Hopital's Rule. Hence it is absolutely convergent.