Math 335 Assignment 9

Arnold Jiadong Yu

April 14, 2018

(1)Let $\mathcal{P}$ be a set of prime numbers, not necessarily finite. Consider the set S of all rational numbers whose denominators in the reduced forms only have prime divisors from $\mathcal{P}$.
(a)Show that S is an integral domain with respect to the sum and product in $\mathbb{Q}$;
(b)Describe the elements in S, which have multiplicative inverses (within S);
(c)Describe the sets $\mathcal{P}$ for which S is a field.

proof:

(a)Pick $\frac{n_1}{p_1}, \frac{n_2}{p_2} \in S$. Then there exists $-\frac{n_1}{p_1}$, s.t. $\frac{n_1}{p_1} + (-\frac{n_1}{p_1}) = 0 \in S$. Also $\frac{n_1}{p_1} + \frac{n_2}{p_2} = \frac{n_1p_2+n_2p_1}{p_1p_2}$. Since $\frac{n_1}{p_1}, \frac{n_2}{p_2}$ are the simplest form, and p_1, p_2 have only prime divisors from $\mathcal{P}$. Let $\frac{n_3}{p_3}$ be the simplest form of $\frac{n_1p_2+n_2p_1}{p_1p_2}$. Then p_3 divides p_1p_2. As a result p_3 only have prime divisor from $\mathcal{P}$. Hence $\frac{n_1p_2+n_2p_1}{p_1p_2} = \frac{n_3}{p_3} \in S$. It is closed under addition. Moreover $\frac{n_1}{p_1} \cdot \frac{n_2}{p_2} = \frac{n_1n_2}{p_1p_2}$ and let $\frac{n_4}{p_4}$ be the simplest from of $\frac{n_1n_2}{p_1p_2}$. Hence p_4 divides p_1p_2 and p_4 only have prime divisors from $\mathcal{P}$, $\frac{n_1n_2}{p_1p_2} = \frac{n_4}{p_4} \in S$. It is closed under multiplication. Moreover, S is a subset of $\mathbb{Q}$, and $\mathbb{Q}$ is a ring w.r.t. sum and product. Hence abelian, associative and distributive property followed from $\mathbb{Q}$. S is a ring w.r.t sum and product. Since $\frac{n_1}{p_1} \cdot \frac{n_2}{p_2} = \frac{n_1n_2}{p_1p_2} = 0$ implies either $n_1 = 0$ or $n_2 = 0$. Therefore, S is an integral domain.

b) For an element in S to have the multiplicative inverses, it means the numerator and denominators get flipped, which means both numerator and denominator need to have prime divisors from $\mathcal{P}$ and they are co-prime to each other.

c) For S to be a field, every non-zero element has an multiplicative inverse. This will force prime number not in $\mathcal{P}$ but is divisors of the numerator to be in $\mathcal{P}$. Therefore, $\mathcal{P}$ is the set of whole prime numbers.

(2) Give an example of a commutative unitary ring R such that $(a+b)^2 = a^2 + b^2$ for all $a, b \in R$.

proof: $(a + b)^2 = a^2 + b^2$ for all $a, b \in R$ implies $a \cdot b = 0$. Which means the ring R have to be symmetric respect to multiplication. Let $R = \mathbb{Z}_2$, it is a commutative unitary ring, $\mathbb{Z}_2$ is abelian group w.r.t. addition and multiplication. Moreover, $\{0, 1\}$ is integers and distributive for integers is true. As a result we can prove $\mathbb{Z}_2$ is a commutative unitary ring. And $(a+b)^2 = a^2 + b^2$ for all $a, b \in R$.

(3) Give an example of a commutative unitary ring R such that $(a+b)^3 = a^3 + b^3$ for all $a, b \in R$.

proof: Let $R = \mathbb{Z}_2$ or $\mathbb{Z}_3$ (both works), $(a + b)^3 = a^3 + b^3$ implies $ab(3a + 3b) = 0$, then $ab = 0$ or $3a + 3b = 0$. For $ab = 0$, then $R = \mathbb{Z}_2$, for $3a + 3b = 0$ implies $3(a + b) = 0$, since $a, b \in R$ then $a + b \in R$. Then R has to have order 3 for every elements in R. Therefore $R = \mathbb{Z}_3$. Similar proof as in question (2). that $(a + b)^3 = a^3 + b^3$ for all $a, b \in R$.

(4) Show that the following subset is field w.r.t. the sum and product in R:

$$\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\} \subset \mathbb{R}.$$

proof: Pick $a', b' \in \mathbb{Q}$, then $(a + b\sqrt{5}) + (a' + b'\sqrt{5}) = (a + a') + (b + b')\sqrt{5} = (a' + b'\sqrt{5}) + (a + b\sqrt{5}) \in \mathbb{Q}(\sqrt{5})$. There exists $-(a + b\sqrt{5}) = (-a) + (-b)\sqrt{5} \in \mathbb{Q}(\sqrt{5})$ such that $(a + b\sqrt{5}) + (-(a + b\sqrt{5})) = 0 \in \mathbb{Q}(\sqrt{5})$. Hence it is closed under addition and abelian w.r.t. $+$. Pick $c, d \in \mathbb{Q}$, then $(a+b\sqrt{5})\cdot(c+d\sqrt{5}) = (ac+5bd)+(ad+bc)\sqrt{5} = (c+d\sqrt{5})\cdot(a+b\sqrt{5}) \in \mathbb{Q}(\sqrt{5})$. It is abelian and closed under multiplication. Since $\mathbb{Q}(\sqrt{5}) \subset \mathbb{R}$ and $\mathbb{R}$ is associative and distributive, then $\mathbb{Q}(\sqrt{5})$ is associative and distributive. Hence $\mathbb{Q}(\sqrt{5})$ is a commutative unitary ring. Suppose $\mathbb{Q}(\sqrt{5})$ has zero-divisor, then $(ac + 5bd) + (ad + bc)\sqrt{5} = 0$ implies $ac + 5bd = 0$ and $ad + bc = 0$ for all $a, b, c, d \in \mathbb{Q}$, then $a = b = c = d = 0$ since $\sqrt{5}$ is irrational. Contradicts with zero-divisor definition. Therefore, $\mathbb{Q}(\sqrt{5})$ doesn't have zero-divisor. As a result, $\mathbb{Q}(\sqrt{5})$ is an integral domain. There exist $x, y \in \mathbb{Q}$ such that $(a+b\sqrt{5})(x+y\sqrt{5}) = 1$ implies $ax + 5by = 1, ay + bx = 0$ since $\sqrt{5}$ is irrational. Then $x = \frac{a}{a^2-5b^2}, y = \frac{-b}{a^2-5b^2} \in \mathbb{Q}$. $a^2 - 5b^2 \neq 0$ because $a + b\sqrt{5} \neq 0$. Hence it is a field.

(5) Show that the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a commutative ring with 1 with respect to the usual (i.e., point-wise) addition and product.

proof: By definition, f is continuous function then it is continuous on every $x \in \mathbb{R}$. Pick an $x \in \mathbb{R}$ and $y_1, y_2, y_3 \in \mathbb{R}$. Define maps,

$$f_1 : x \mapsto y_1, f_2 : x \mapsto y_2, f_3 : x \mapsto y_3$$

Then $-f_1(x) = -y_1 \in \mathbb{R}$, $-f_1(x)$ is continuous at x. This implies $f_1(x) + (-f_1(x)) = y_1 + (-y_1) = 0 \in \mathbb{R}$. Moreover, $f_1(x) + f_2(x) = y_1 + y_2 = y_2 + y_1 = f_2(x) + f_1(x)$. It is closed and abelian w.r.t addition. Since y_1, y_2, y_3 runs in $\mathbb{R}$ and $\mathbb{R}$ is associative and distributive. Then the preimage f_1, f_2, f_3 at x is also associative and distributive. Let $f(x) = 1$, it is a continuous function. Therefore $f(x) \cdot f_1(x) = 1 \cdot f_1(x) = 1 \cdot y_1 = y_1 = y_1 \cdot 1 = f_1(x) \cdot 1 = f_1(x) \cdot f(x)$. Hence the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a commutative ring with 1 with respect to the usual (i.e., point-wise) addition and product.

(6) Show that the set of polynomial functions $f : \mathbb{R} \to \mathbb{R}$ is a commutative ring with 1 with respect to the usual (i.e., point-wise) addition and product.

proof: Since polynomial function is a subset of continuous functions in $\mathbb{R}$. Then it is a commutative unitary ring with respect to the usual addition and product. The proof is trivial as question (5).

(7) Which of the rings in Problems (6) and (7) is an integral domain? Which of them is a field?

proof: (5) is not integral domain, not field. Pick two continuous function f, g, define the map

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \end{cases}$$

$$g(x) = \begin{cases} -x & x \leq 0 \\ 0 & 0 \leq x \end{cases}$$

This impilies $f(x)g(x) = 0$ for $x \neq 0$. Therefore, it is not an integral domain. Therefore, it is not a field.
(6) is an integral domain, but not field. Take two polynomial functions with non-zero coefficient, $f(x) = a_n x^n + ... + a_0, g(x) = b_n x^n + ...b_n$ and let $h(x) = f(x)g(x)$ be a polynomial. Then $h(x) = 0$ implies x is a root of h. Therefore, x is either the root of f or the root of g. As a result, $f(x)g(x) = 0 \Rightarrow f(x) = 0$ or $g(x) = 0$. Hence, it is an integral domain. Any two polynomial functions with a function that has a root. At that point, the product is never 1. Therefore, it is not a field.