Math 400 Project 2

Arnold Jiadong Yu

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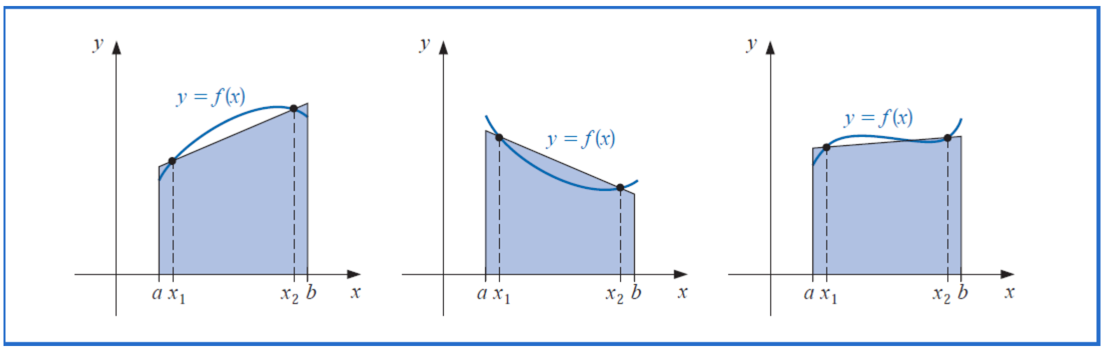
1.

a)



Gaussian quadrature chooses the points or nodes such as $x_0, x_1, x_2, ..., x_n$ for evaluation in an optimal. These points are rather than equally spaced. The nodes $x_0, x_1, x_2...x_n$ in the interval $[a, b]$ and coefficients $c_0, c_1, ..., c_n$ are chosen to minimize the expected error obtained in the approximation

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

To measure this accuracy, we assume the choice of of these values gives the greatest degree of precision. The coefficients are arbitrary, and the nodes are restricted to lying in the interval $[a, b]$. This gives $2n$ unknowns. Consider polynomial with degree $2n - 1$ with $2n$ unknowns. Then this is the largest polynomials which it is reasonable to expect a formula to exact. With the proper choice, exactness can be obtained.

b) Consider

$$\int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1)$$

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

then

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3, f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

Therefore

$$\int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4|_{-1}^{1} = 2a_0 + \frac{2}{3}a_2 \text{ and}$$

$$c_0 f(x_0) + c_1 f(x_1) = (c_0 + c_1)a_0 + (c_0x_0 + c_1x_1)a_1 + (c_0x_0^2 + c_1x_1^2)a_2 + (c_0x_0^3 + c_1x_1^3)a_3$$
$$\Rightarrow 2a_0 + \frac{2}{3}a_2 = (c_0 + c_1)a_0 + (c_0x_0 + c_1x_1)a_1 + (c_0x_0^2 + c_1x_1^2)a_2 + (c_0x_0^3 + c_1x_1^3)a_3$$
$$\Rightarrow c_0 + c_1 = 2, c_0x_0 + c_1x_1 = 0, c_0x_0^2 + c_1x_1^2 = \frac{2}{3}, c_0x_0^3 + c_1x_1^3 = 0 \text{ with } x_0 < x_1$$
$$\Rightarrow c_0 = 1, c_1 = 1, x_0 = -\frac{1}{\sqrt{3}}, x_1 = \frac{1}{\sqrt{3}}$$

Hence

$$\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$

It is exact for all polynomial of degree 3 with $c_0 = 1, c_1 = 1, x_0 = -\frac{1}{\sqrt{3}}, x_1 = \frac{1}{\sqrt{3}}$.

c) Consider

$$\int_{-1}^{1} x^2 f(x)dx = c_0 f(x_0) + c_1 f(x_1)$$

Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

then

$$f(x_0) = a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3, f(x_1) = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3$$

Therefore

$$\int_{-1}^{1}(a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5)dx = \frac{a_0}{3}x^3 + \frac{a_1}{4}x^4 + \frac{a_2}{5}x^5 + \frac{a_3}{6}x^6|_{-1}^{1} = \frac{2}{3}a_0 + \frac{2}{5}a_2 \text{ and}$$
$$c_0 f(x_0) + c_1 f(x_1) = (c_0 + c_1)a_0 + (c_0x_0 + c_1x_1)a_1 + (c_0x_0^2 + c_1x_1^2)a_2 + (c_0x_0^3 + c_1x_1^3)a_3$$
$$\Rightarrow \frac{2}{3}a_0 + \frac{2}{5}a_2 = (c_0 + c_1)a_0 + (c_0x_0 + c_1x_1)a_1 + (c_0x_0^2 + c_1x_1^2)a_2 + (c_0x_0^3 + c_1x_1^3)a_3$$
$$\Rightarrow c_0 + c_1 = \frac{2}{3}, c_0x_0 + c_1x_1 = 0, c_0x_0^2 + c_1x_1^2 = \frac{2}{5}, c_0x_0^3 + c_1x_1^3 = 0 \text{ with } x_0 < x_1$$
$$\Rightarrow c_0 = \frac{1}{3}, c_1 = \frac{1}{3}, x_0 = -\sqrt{\frac{3}{5}}, x_1 = \sqrt{\frac{3}{5}}$$

Hence

$$\int_{-1}^{1} x^2 f(x)dx = \frac{1}{3}f(-\sqrt{\frac{3}{5}}) + \frac{1}{3}f(\sqrt{\frac{3}{5}})$$

It is exact for all polynomial for degree 3 with $c_0 = \frac{1}{3}, c_1 = \frac{1}{3}, x_0 = -\sqrt{\frac{3}{5}}, x_1 = \sqrt{\frac{3}{5}}$.

2)Jacobi's method. The matrix is in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.165 & 0.202 & 0.317 & 0.234 & 0.182 \\ 0 & 27.7 & 0.862 & 0.062 & 0.073 & 0.131 \\ 0 & 0 & 22.35 & 13.05 & 4.42 & 6.001 \\ 0 & 0 & 0 & 11.28 & 0 & 1.11 \\ 0 & 0 & 0 & 0 & 9.85 & 1.684 \end{bmatrix} \mathbf{p} = \begin{bmatrix} h_0 = 51.53 \\ h_1 = 5.20 \\ h_2 = 61.7 \\ h_3 = 149.2 \\ h_4 = 79.4 \\ h_5 = 89.3 \end{bmatrix}$$

Switch rows to obtains the largest diagonal entries.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 27.7 & 0.862 & 0.062 & 0.073 & 0.131 \\ 0 & 0 & 22.35 & 13.05 & 4.42 & 6.001 \\ 0 & 0 & 0 & 11.28 & 0 & 1.11 \\ 0 & 0 & 0 & 0 & 9.85 & 1.684 \\ 0 & 0.165 & 0.202 & 0.317 & 0.234 & 0.182 \end{bmatrix} \mathbf{p} = \begin{bmatrix} 51.53 \\ 61.7 \\ 149.2 \\ 79.4 \\ 89.3 \\ 5.20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 27.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 22.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9.85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.182 \end{bmatrix} \mathbf{p} = - \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.862 & 0.062 & 0.073 & 0.131 \\ 0 & 0 & 0 & 13.05 & 4.42 & 6.001 \\ 0 & 0 & 0 & 0 & 0 & 1.11 \\ 0 & 0 & 0 & 0 & 0 & 1.684 \\ 0 & 0.165 & 0.202 & 0.317 & 0.234 & 0 \end{bmatrix} \mathbf{p} + \begin{bmatrix} 51.53 \\ 61.7 \\ 149.2 \\ 79.4 \\ 89.3 \\ 5.20 \end{bmatrix}$$

The inverse matrix of $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 27.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 22.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9.85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.182 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{10}{277} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{20}{447} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{25}{282} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{20}{197} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{500}{91} \end{bmatrix}$

Therefore

$$\mathbf{p}^{(k+1)} = - \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{431}{1385} & \frac{31}{13850} & \frac{73}{27700} & \frac{131}{27700} \\ 0 & 0 & 0 & \frac{87}{149} & \frac{442}{2235} & \frac{6001}{22350} \\ 0 & 0 & 0 & 0 & 0 & \frac{37}{376} \\ 0 & 0 & 0 & 0 & 0 & \frac{842}{4925} \\ 0 & \frac{165}{182} & \frac{101}{91} & \frac{317}{182} & \frac{9}{7} & 0 \end{bmatrix} \mathbf{p}^{(k)} + \begin{bmatrix} 51.53 \\ \frac{617}{2770} \\ \frac{2984}{447} \\ \frac{1985}{282} \\ \frac{1786}{197} \\ \frac{200}{7} \end{bmatrix}$$

By calculation, within 10^{-6}.

```
The entry of p is : p(0) = 30.024892253845444
The entry of p is : p(1) = 2.170207551107441
The entry of p is : p(2) = -0.014050967268149028
The entry of p is : p(3) = 6.602097356429977
The entry of p is : p(4) = 8.306916575297272
The entry of p is : p(5) = 4.439961837651227
The sum of all p valuse are :51.53002460706321
Total iteration is :147
The different between last adjacent interations: 9.750344710366397E-7
```

Then by Gauss-Seidal method iteration (I did extreme acceleration instead)

```
The entry of p is : p(0) = 30.024887890681264
The entry of p is : p(1) = 2.1702073269302655
The entry of p is : p(2) = -0.014043010033522485
The entry of p is : p(3) = 6.602102574378111
The entry of p is : p(4) = 8.306925640796713
The entry of p is : p(5) = 4.4399195772471725
The sum of all p valuse are :51.53
Total iteration is :22
The different between last adjacent interations: 8.722937569293503E-7
```

The java code is below

```java
package math400p2;
/**
 *
 * @author Arnold Jiadong Yu
 */
public class Math400P2 {
    public static void main(String []args){

        double[] x1 = {0,0,0,0,0,0};
        double[] x2 = {0,0,0,0,0,0};
        double[] sum = {0,0,0,0,0,0};
        double sum_next = 0;
        double tol=0.000001;
        double error = 1.0;
        boolean bool = true;
        int iter = 0;
        while(bool){
            iter++;
            // Jacobi's method

            x2[0] = 51.53 - 1.0*x1[1]-1.0*x1[2]-1.0*x1[3]-1.0*x1[4]-1.0*x1[5];
            x2[1] = 61.7/27.7-0.862/27.7*x1[2]-0.062/27.7*x1[3]-0.073/27.7*x1[4]-0.131/27.7*x1[5];
            x2[2] = 149.2/22.35-13.05/22.35*x1[3]-4.42/22.35*x1[4]-6.0001/22.35*x1[5];
            x2[3] = 79.4/11.28-1.11/11.28*x1[5];
            x2[4] = 89.3/9.85-1.684/9.85*x1[5];
            x2[5] = 5.2/0.182-0.165/0.182*x1[1]-0.202/0.182*x1[2]-0.317/0.182*x1[3]-0.234/0.182*x1[4];


            // Gauss-Seidal method
            // I did extreme acceleration by switching the order to reach max speed.
```

```java
/**
x2[3] = 79.4/11.28-1.11/11.28*x1[5];
x2[4] = 89.3/9.85-1.684/9.85*x1[5];
x2[2] = 149.2/22.35-13.05/22.35*x2[3]-4.42/22.35*x2[4]-6.0001/22.35*x1[5];
x2[1] = 61.7/27.7-0.862/27.7*x2[2]-0.062/27.7*x2[3]-0.073/27.7*x2[4]-0.131/27.7*x1[5];
x2[5] = 5.2/0.182-0.165/0.182*x2[1]-0.202/0.182*x2[2]-0.317/0.182*x2[3]-0.234/0.182*x2[4];
x2[0] = 51.53 - 1.0*x2[1]-1.0*x2[2]-1.0*x2[3]-1.0*x2[4]-1.0*x2[5];
*/

sum_next = x2[0]+x2[1]+x2[2]+x2[3]+x2[4]+x2[5];
for(int i=0;i<6;i++){
    sum[i]= x2[i]-x1[i];
    System.out.println(sum[i]);
}
error = norm(sum)/norm(x2);
if(error < tol)
    bool = false;

for(int i=0;i<6;i++){
    x1[i] = x2[i];
}
}
for(int i = 0; i < 6; i++){
    System.out.println("The entry of p is : p("+ (i) +") = " + x2[i]);
}
System.out.println("The sum of all p valuse are :"+ sum_next);
System.out.println("Total iteration is :" + iter);
System.out.println("The different between last adjacent interations: " + error);
}
```

```java
    public static double norm(double[] n){
        double norm = 0.0;
        for(int i = 0 ; i < 6 ; i++)
            norm += Math.pow(Math.abs(n[i]),2);
        norm = Math.sqrt(norm);
        return norm;
    }
}
```