Math 741 Assignment 20 (Quiz)

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10.2.1 solution: Let X_i denoted the number of times that students fall into the i score category, $i = 1, 2, 3, 4, 5$.

$$P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 2, X_5 = 3) =$$

$$\frac{6!}{0!0!1!2!3!}(0.116)^0(0.325)^0(0.236)^1(0.211)^2(0.112)^3 = 0.000885691001$$

10.2.2 solution:

$$P(1,1,1,1) = \frac{4!}{1!1!1!1!}(9/16)^1(3/16)^1(3/16)^1(1/16)^1 = 0.0296631$$

10.2.4 solution: Let X_i denoted the number of times that fall into the class i IQ category, $i = 1, 2, 3$ and p_i are the probability.

$$p_1 = P(Z < \frac{90-100}{16}) = 0.2660$$

$$p_2 = P(\frac{90-100}{16} < Z < \frac{110-100}{16}) = 0.4680$$

$$p_3 = 1 - p_1 - p_2 = 0.2660$$

$$P(2,4,1) = \frac{7!}{2!4!1!}(0.2660)^2(0.4680)^4(0.2660)^1 = 0.09480$$

10.2.5 solution: p_i denote four category covered the pipeline.

$$p_1 = \int_{-60}^{-20} \frac{60+y}{3600} dy = 2/9, p_2 = \int_{-20}^{0} \frac{60+y}{3600} dy = 5/18$$

$$p_3 = \int_0^{20} \frac{60-y}{3600} dy = 5/18, p_4 = \int_{20}^{60} \frac{60-y}{3600} dy = 2.9$$

$$P(0,2,4,0) = \frac{6!}{0!2!4!0!}(2/9)^0(5/18)^2(5/18)^4(2/9)^0 = 0.006891$$

10.2.7 solution:

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

$$F_Y(y) = \begin{cases} y^3 & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

$$p_1 = 1/64, p_2 = 7/64, p_3 = 19/64, p_4 = 37/64$$

a)

$$f_{X_1,X_2,X_3,X_4}(3,7,15,25) =$$

$$\frac{50!}{3!7!15!25!}(1/64)^3(7/64)^7(19/64)^{15}(37/64)^{25} = 0.0004880$$

b) $\mathrm{Var}(X_3) = np_3(1-p_3) = 50 \cdot (19/64)(45/64) = 10.437$

10.2.8(H) solution: Given

$$(a+b+c)^n = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{n!}{i!j!(n-i-j)!} a^i b^j c^{n-i-j}$$

$$M_{X_1,X_2,X_3}(t_1,t_2,t_3) = E(e^{t_1X_1+t_2X_2+t_3X_3})$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=n-k_1-k_2}^{n-k_1-k_2} e^{t_1k_1+t_2k_2+t_3k_3} \frac{n!}{k_1!k_2!k_3!} p_1^{k_1}p_2^{k_2}p_3^{k_3}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} e^{t_1k_1+t_2k_2+t_3(n-k_1-k_2)} \frac{n!}{k_1!k_2!(n-k_1-k_2)!} p_1^{k_1}p_2^{k_2}p_3^{n-k_1-k_2}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} e^{t_1k_1+t_2k_2+t_3(n-k_1-k_2)} \frac{n!}{k_1!k_2!(n-k_1-k_2)!} (p_1e^{t_1})^{k_1}(p_2e^{t_2})^{k_2}(p_3e^{t_3})^{n-k_1-k_2}$$

$$= (p_1e^{t_1} + p_2e^{t_2} + p_3e^{t_3})^n$$

10.2.10(H) solution: Consider a positive integer n, and a set of positive real numbers $\mathbf{P} = \{p_1, ..., p_t\}$ such that

$$\sum_{i=1}^{t} p_i = 1, \sum_{i=1}^{t} k_i = n$$

The joint likelihood is

$$L(\mathbf{P}) = n! \cdot \prod_{i=1}^{t} \frac{p_i^{k_i}}{k_i!}$$

$$l(\mathbf{P}) = \log(L(\mathbf{P})) = \log n! + \sum_{i=1}^{t} \log(\frac{p_i^{k_i}}{k_i!})$$

$$\log(L(\mathbf{P})) = \log n! + \sum_{i=1}^{t} k_i \log p_i - \sum_{i=1}^{t} k_i$$

Using auxiliary function of Lagrange Multiplier, (It is similiar as to maximize Shannon Entropy)

$$L(\mathbf{P}, \lambda) = l(\mathbf{P}) + \lambda(1 - \sum_{i=1}^{t} p_i)$$

$$\frac{\partial}{\partial p_i}(L(\mathbf{P}, \lambda)) = \frac{\partial}{\partial p_i}(l(\mathbf{P})) + \frac{\partial}{\partial p_i}[\lambda(1 - \sum_{i=1}^{t} p_i)] = 0$$

$$\implies \frac{\partial}{\partial p_i} \sum_{i=1}^{t} k_i \log p_i - \lambda \frac{\partial}{\partial p_i} \sum_{i=1}^{t} p_i = 0$$

$$\implies \frac{k_i}{\hat{p}_i} = \lambda \implies \hat{p}_i = \frac{k_i}{\lambda}$$

$$\sum_{i=1}^{t} p_i = \sum_{i=1}^{t} \frac{k_i}{\lambda} \implies 1 = \frac{1}{\lambda} \sum_{i=1}^{t} k_i \implies \lambda = n$$

Hence,

$$\hat{p}_i = \frac{k_i}{n}, i = 1, ..., t$$