Math 430 Assignment 6

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October 7, 2018

Exercise 2.18
Solution:
In order to find the existence of such $\overline{\mathbf{b}}$ to satisfy property (i), we need to explicit construct such $\overline{\mathbf{b}}$ based on the original polyhedron. Since a polyhedron is just intersection of halfspaces. Therefore, let

$$P = \bigcap_{i \in A} H_i$$

where H_i represent halfspace and A is a set contain the indices. Since we may have more active constrains then needed, which means some halfspaces are excessive. This will result in degenerate basic feasible solution. If we take away the extra halfspaces, we still have the same polyhedron. Let B denotes the set contains all the indices of excessive halfspaces. i.e. $H_j = \{\mathbf{x}|\mathbf{a}_j'\mathbf{x} \geq b_j\}$ will be the excessive halfspaces for all $j \in B$. Let new halfspace $H_k = \{\mathbf{x}|\mathbf{a}_j'\mathbf{x} \geq b_j - \epsilon\}$ for all $k \in B$. Therefore, we let

$$\overline{\mathbf{b}} = \begin{cases} b_i & \text{if } i \in A \setminus B \\ b_i - \epsilon & \text{if } i \in B \end{cases}$$

By this construction, we find $\overline{\mathbf{b}}$ such that $|\mathbf{b} - \overline{\mathbf{b}}| \leq \epsilon$. Moreover, we construct a new polyhedron,

$$\overline{P} = \bigcap_{i \in A \setminus B} H_i \bigcap_{k \in B} H_k = \{\mathbf{x}|\mathbf{A}\mathbf{x} \geq \overline{\mathbf{b}}\}$$

Since we changed all excessive halfspaces, now all the halfspaces are essential to construct $\overline{P}$. i.e. no degenerate basic feasible solution. Hence, property (ii) satisfied.

Exercise 3.1

Solution:

Prove by contradiction. Assume there exists a $\mathbf{y} \in S$, such that it is a globally optimal i.e. $f(\mathbf{y}) < f(\mathbf{x}^*)$ where $\mathbf{x}^*$ is the local minima. By definition of convex set,

$$\lambda \mathbf{y} + (1-\lambda)\mathbf{x}^* \in S$$

By definition of convex function,

$$f(\lambda \mathbf{y} + (1-\lambda)\mathbf{x}^*) \leq \lambda f(y) + (1-\lambda) f(\mathbf{x}^*)$$

Since $f(\mathbf{y}) < f(\mathbf{x}^*)$, then $\lambda f(y) - \lambda f(\mathbf{x}^*) < 0$. i.e.

$$f(\lambda \mathbf{y} + (1-\lambda)\mathbf{x}^*) \leq \lambda f(y) + (1-\lambda) f(\mathbf{x}^*) = \lambda f(y) - \lambda f(\mathbf{x}^*) + f(\mathbf{x}^*) \leq f(\mathbf{x}^*)$$

Choose $\epsilon \geq \lambda ||\mathbf{y} - \mathbf{x}^*||$, then

$$f(\lambda \mathbf{y} + (1-\lambda)\mathbf{x}^*) \leq f(\mathbf{x}^*) \Rightarrow \lambda \mathbf{y} + (1-\lambda)\mathbf{x}^* \text{ is the local minima}$$

This contradict with $\mathbf{x}^*$ is the local minima.

Hence $\mathbf{x}^*$ is also globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

Exercise 3.2

Solution:

(a) ($\Rightarrow$) Suppose a feasible solution $\mathbf{x}$ is optimal and $\mathbf{d}$ be a feasible direction at $\mathbf{x}$. Then

$$\mathbf{c}'\mathbf{x} \leq \mathbf{c}'(\mathbf{x} + \theta \mathbf{d})$$

for some positive scalar θ. i.e. $0 \leq \theta \mathbf{c}'\mathbf{d}$ implies $\mathbf{c}'\mathbf{d} \geq 0$.

($\Leftarrow$) Suppose $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at vertex $\mathbf{x}$. Then for some positive scalar θ, $\mathbf{x} + \theta \mathbf{d} \in P$ where $\mathbf{x} + \theta \mathbf{d}$ is an arbitrary point in P. Therefore,

$$\mathbf{c}'(\mathbf{x} + \theta \mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta \mathbf{c}'\mathbf{d} \geq \mathbf{c}'\mathbf{x}$$

As a result, $\mathbf{x}$ is optimal solution. Hence a feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

(b)This part is almost identical to part a.($\Rightarrow$) Suppose a feasible solution $\mathbf{x}$ is the unique optimal solution, then $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$ for any $\mathbf{y} \in P$. Let $\mathbf{d}$ is a nonzero feasible direction at $\mathbf{x}$. i.e. there exsits a postive scalar θ, s.t. $\mathbf{x} + \theta \mathbf{d} \in P$ and $\mathbf{x} \neq \mathbf{x} + \theta \mathbf{d}$ since $\mathbf{d} \neq 0$. Therefore,

$$\mathbf{c}'\mathbf{x} < \mathbf{c}'(\mathbf{x} + \theta \mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta \mathbf{c}'\mathbf{d}$$

$$\Rightarrow \theta \mathbf{c}^{'}\mathbf{d} > 0 \Rightarrow \mathbf{c}^{'}\mathbf{d} > 0$$

Since $\theta > 0$.
($\Leftarrow$) Suppose $\mathbf{c}^{'}\mathbf{d} \geq 0$ for every nonzero feasible direction $\mathbf{d}$ at vertex $\mathbf{x}$. Then $\mathbf{x} + \theta\mathbf{d} \in P$ is an arbitrary point for some positive scalar θ and $\mathbf{x} \neq \mathbf{x} + \theta\mathbf{d}$ since $\mathbf{d}$ is nonzero. Therefore,

$$\mathbf{c}^{'}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}^{'}\mathbf{x} + \theta\mathbf{c}^{'}\mathbf{d} > \mathbf{c}^{'}\mathbf{x}$$

Hence $\mathbf{x}$ is a unique feasible optimal solution. Hence proved.
Exercise 3.5
Solution:
By definition, a vector $\mathbf{d} \in \mathbb{R}^3$ is said to be feasible direction at $\mathbf{x}$, if there exists a positive scalar θ for which $\mathbf{x} + \theta\mathbf{d} \in P$. Consider the vector $\mathbf{x} = (0, 0, 1)$, then $\mathbf{x} + \theta\mathbf{d} = (\theta d_1, \theta d_2, \theta d_3 + 1)$. Moreover, $\mathbf{x} + \theta\mathbf{d} \in P$ implies $\theta d_1 + \theta d_2 + \theta d_3 + 1 = 1$ and $\theta d_1 \geq 0, \theta d_2 \geq 0, \theta d_3 + 1 \geq 0$. i.e. $d_1 + d_2 + d_3 = 0$. Since $\theta > 0$, thus $d_1 \geq 0, d_2 \geq 0, d_3 = -d_1 - d_2, d_1 + d_2 \leq \frac{1}{\theta}$. Let S be the set of all feasible directions, then

$$S = \{(d_1, d_2, -d_1 - d_2) | d_1 \geq 0, d_2 \geq 0\}$$