Math 741 Assignment 6 (Quiz)

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March 5, 2019

5.6.1.
solution: Given the geometric distribution, then

$$E(X_i) = E(X) = \frac{1}{p}, \mathrm{Var}(X_i) = \mathrm{Var}(X) = \frac{1-p}{p^2}$$

Let $\hat{p} = \sum_{i=1}^{n} X_i$, then

$$L(\hat{p}) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p = (1-p)^{\hat{p}-n} p^n = [(1-p)^{\hat{p}-n} p^n] \cdot 1$$

where $g[\hat{p}; p] = (1-p)^{\hat{p}-n} p^n$ and $b(x_1, ..., x_n) = 1$. Hence, $\hat{p}$ is sufficient for p by Second Factorization Theorem.

5.6.2.(H)
solution: In order to show $\hat{p}^*$ is not sufficient for p, we need to find an counter example. Assume $\hat{p}^* = X_1 + 2X_2 + 3X_3 = 3$, then

$$P(X_1 = 1, X_2 = 1, X_3 = 0 | X_1 + 2X_2 + 3X_3 = 3) = \frac{P(X_1 = 1, X_2 = 1, X_3 = 0)}{P(X_1 + 2X_2 + 3X_3 = 3)}$$

$$= \frac{P(X_1 = 1, X_2 = 1, X_3 = 0)}{P(X_1 = 1, X_2 = 1, X_3 = 0) + P(X_1 = 0, X_2 = 0, X_3 = 1)} = \frac{p^2(1-p)}{p^2(1-p) + p(1-p)^2}$$

$$= \frac{p}{p + (1-p)} = p$$

Since the conditional probability depends on p, it is not sufficient for p.

5.6.4.
solution: Let $X_1, ..., X_n \sim N(0, \sigma^2)$, and $\hat{\sigma}^2 = \sum_{i=1}^{n} X_i^2$

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2} x_i^2} = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\hat{\sigma}^2} = [(2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\hat{\sigma}^2}] \cdot 1$$

where $g[h(x_1, ..., x_n); \sigma^2] = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\hat{\sigma}^2}$ and $b(x_1, ..., x_n) = 1$ Hence, $\hat{\sigma}^2$ is sufficient for σ^2 by Second Factorization Theorem.

5.6.5.

solution: Given the pdf, then

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{(r-1)!\theta^r} y_i^{r-1} e^{-y_i/\theta} = \Big(\frac{1}{(r-1)!\theta^r}\Big)^n (\prod_{i=1}^{n} y_i)^{r-1} e^{-\frac{1}{\theta}\sum_{i=1}^n y_i}$$

$$= \Big[\Big(\frac{1}{(r-1)!\theta^r}\Big)^n e^{-\frac{1}{\theta}\sum_{i=1}^n y_i}\Big] \cdot (\prod_{i=1}^{n} y_i)^{r-1}$$

where $g[h(y_1, ..., y_n); \theta] = \Big(\frac{1}{(r-1)!\theta^r}\Big)^n e^{-\frac{1}{\theta}\sum_{i=1}^n y_i}$ and $b(y_1, ..., y_n) = (\prod_{i=1}^n y_i)^{r-1}$. Hence, $\sum_{i=1}^n Y_i$ is sufficient for θ.

5.6.6.(H)

solution: Let $Y_1, ..., Y_n \sim f_Y(y; \theta)$ where

$$f_Y(y;\theta) = \begin{cases} \theta y^{\theta-1} & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

Then,

$$L(\theta) = \prod_{i=1}^{n} \theta y_i^{\theta-1} = \theta^n \Big(\prod_{i=1}^{n} y_i\Big)^{\theta-1}$$

Since $W = \prod_{i=1}^n Y_i$, then

$$\theta^n \Big(\prod_{i=1}^{n} y_i\Big)^{\theta-1} = \theta^n (w)^{\theta-1} = [\theta^n (w)^{\theta-1}] \cdot 1$$

where $g[h(y_1, ..., y_n); \theta] = \theta^n (w)^{\theta-1}$ and $b(y_1, ..., y_n) = 1$. Hence, $W = \prod_{i=1}^n Y_i$ is sufficient statistic for θ.

$$\ln L(\theta) = n \ln\theta + (\theta - 1)(\ln y_1 + ... + \ln y_n)$$

$$\frac{\partial}{\partial\theta} \ln L(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln y_i$$

Let $\frac{\partial}{\partial\theta}\ln L(\theta)=0$, then

$$\frac{n}{\hat{\theta}}+\sum_{i=1}^{n}\ln y_i=0 \implies \hat{\theta}=-\frac{n}{\ln\prod_{i=1}^{n}y_i}=-\frac{n}{\ln w}$$

Therefore, the maximum likelihood estimator of θ is a function of W.

5.6.7.

solution: Given the pdf,

$$f_Y(y;\theta)=\begin{cases}e^{-(y-\theta)} & \theta\le y\\ 0 & o.w.\end{cases}$$

Since the set of Y values where $f_Y(y;\theta)\neq 0$ depends on θ, the likelihood function must be written in a way to include that restriction. We define an indicator function,

$$I_{[\theta,\infty]}(y)=\begin{cases}1 & \theta\le y\\ 0 & o.w.\end{cases}$$

Then we can write $f_Y(y;\theta)=e^{-(y-\theta)}I_{[\theta,\infty]}(y)$ for all y.

$$L(\theta)=\prod_{i=1}^{n}e^{-(y_i-\theta)}I_{[\theta,\infty]}(y_i)=e^{-\sum_{i=1}^{n}y_i}e^{n\theta}\prod_{i=1}^{n}I_{[\theta,\infty]}(y)$$

Since

$$\prod_{i=1}^{n}I_{[\theta,\infty]}(y)=I_{[\theta,\infty]}(y_{\min})$$

Therefore,

$$L(\theta)=e^{n\theta}I_{[\theta,\infty]}(y_{\min})\cdot e^{-\sum_{i=1}^{n}y_i}$$

a) Hence, $\hat{\theta}=Y_{\min}$ is sufficient for θ by Factorization Theorem 2.

b) $\prod_{i=1}^{n}I_{[\theta,\infty]}(y)$ must be include in the function of $r(\hat{\theta},\theta)$ since it contains θ, but $\prod_{i=1}^{n}I_{[\theta,\infty]}(y)\neq I_{[\theta,\infty]}(y_{\max})$. Therefore, $Y_{\max}$ is not sufficient for θ.

5.6.8.

solution: Given uniform distribution,

$$f_Y(y;\theta)=\begin{cases}\frac{1}{\theta} & 0\le y\le\theta\\ 0 & o.w.\end{cases}$$

Since the set of Y values where $f_Y(y;\theta) \neq 0$ depends on θ, the likelihood function must be written in a way to include that restriction. We define an indicator function,

$$I_{[0,\theta]}(y) = \begin{cases} 1 & 0 \leq y \leq \theta \\ 0 & o.w. \end{cases}$$

Then we can write $f_Y(y;\theta) = \frac{1}{\theta} I_{[0,\theta]}(y)$ for all y.

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(y_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{[0,\theta]}(y_i)$$

Since

$$\prod_{i=1}^{n} I_{[0,\theta]}(y) = I_{[0,\theta]}(y_{\max})$$

Therefore,

$$L(\theta) = \frac{1}{\theta^n} I_{[0,\theta]}(y_{\max}) \cdot 1$$

Hence, $\hat{\theta} = Y_{\max}$ is sufficient for θ by Factorization Theorem 2.