Math 741 Assignment 1 (Quiz)

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February 10, 2019

5.2.1
solution: Given $L(\theta) = \prod_{i=1}^n p_X(k_i;\theta)$ for discrete. i.e.

$$L(\theta) = \prod_{i=1}^{8} \theta^{k_i}(1-\theta)^{k_i} = \theta^5(1-\theta)^3 = -\theta^8 + 3\theta^7 - 3\theta^6 + \theta^5$$

$$\frac{d}{d\theta}(L(\theta)) = -8\theta^7 + 21\theta^6 - 18\theta^5 + 5\theta^4$$

$$\text{Let } \frac{d}{d\theta}(L(\theta)) = 0 \Rightarrow -8\hat{\theta}^7 + 21\hat{\theta}^6 - 18\hat{\theta}^5 + 5\hat{\theta}^4 = 0$$

$$\Rightarrow -8\hat{\theta}^3 + 21\hat{\theta}^2 - 18\hat{\theta} + 5 = 0 \Rightarrow \hat{\theta} = 0.625 \text{ since } 0 < \theta < 1$$

5.2.3
solution:Given $L(\theta) = \prod_{i=1}^n f_Y(y_i;\theta)$ for continuous. i.e.

$$L(\theta) = \prod_{i=1}^{4} \lambda e^{-\lambda y_i} = \lambda^4 e^{-32.8\lambda}$$

$$L'(\theta) = 4\lambda^3 e^{-32.8\lambda} - 32.8\lambda^4 e^{-32.8\lambda}$$

$$\text{Let } L'(\theta) = 0 \Rightarrow 4\hat{\lambda}^3 e^{-32.8\hat{\lambda}} - 32.8\hat{\lambda}^4 e^{-32.8\hat{\lambda}} = 0$$

$$\Rightarrow 4 - 32.8\hat{\lambda} = 0 \Rightarrow \hat{\lambda} \approx 0.122$$

5.2.4
solution: Given $L(\theta) = \prod_{i=1}^n p_X(k_i;\theta)$ for discrete. i.e.

$$L(\theta) = \prod_{i=0}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum_{i=0}^n k_i} e^{-n\theta^2}}{\prod_{i=0}^n k_i!}$$

$$\ln(L(\theta)) = \ln\theta^{2\sum_{i=0}^{n} k_i} + \ln e^{-n\theta^2} - \ln\prod_{i=0}^{n} k_i! = 2\sum_{i=0}^{n} k_i \ln\theta - n\theta^2 - \sum_{i=0}^{n}(\ln k_i!)$$

$$\frac{d}{d\theta}(\ln(L(\theta))) = \frac{2\sum_{i=0}^{n} k_i}{\theta} - 2n\theta$$

$$\text{Let } \frac{d}{d\theta}(\ln(L(\theta))) = 0 \Rightarrow \frac{2\sum_{i=0}^{n} k_i}{\theta} - 2n\theta = 0 \Rightarrow \theta^2 = \frac{\sum_{i=0}^{n} k_i}{n}$$

$$\Rightarrow \theta = \sqrt{\frac{\sum_{i=0}^{n} k_i}{n}}$$

5.2.6
solution:Given $L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta)$ for continuous. i.e.

$$L(\theta) = \prod_{i=1}^{4} \frac{\theta}{2\sqrt{y_i}} e^{-\theta\sqrt{y_i}} = \frac{\theta^4}{16\cdot\sqrt{6.2}\cdot\sqrt{7.0}\cdot\sqrt{2.5}\cdot\sqrt{4.2}} e^{-\theta(\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2})}$$

$$\begin{aligned} L'(\theta) &= \frac{4\theta^3}{16\cdot\sqrt{6.2}\cdot\sqrt{7.0}\cdot\sqrt{2.5}\cdot\sqrt{4.2}} e^{-\theta(\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2})} \\ &\quad - \frac{(\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2})\theta^4}{16\cdot\sqrt{6.2}\cdot\sqrt{7.0}\cdot\sqrt{2.5}\cdot\sqrt{4.2}} e^{-\theta(\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2})} \end{aligned}$$

$$\text{Let } L'(\theta) = 0 \Rightarrow 4 = (\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2})\theta_e$$

$$\Rightarrow \theta_e = \frac{4}{\sqrt{6.2}+\sqrt{7.0}+\sqrt{2.5}+\sqrt{4.2}} \approx 0.4563$$

5.2.9
solution: (a)Given in the book, $\lambda_e = \frac{\sum_{i=0}^{n} k_i}{n} = \bar{k}$. i.e.

$$\lambda_e = \frac{0(6)+1(19)+2(12)+3(13)+4(9)}{59} = \frac{118}{59} = 2$$

Then

$$p_X(0;\lambda_e) = e^{-2}\frac{2^0}{0!} \approx 0.13534$$

$$p_X(1;\lambda_e) = e^{-2}\frac{2^1}{1!} \approx 0.27067$$

$$p_X(2;\lambda_e) = e^{-2}\frac{2^2}{2!} \approx 0.27067$$

$$p_X(3;\lambda_e) = e^{-2}\frac{2^3}{3!} \approx 0.18045$$

$$p_X(4;\lambda_e) = 1 - (p_X(0;\lambda_e) + p_X(1;\lambda_e) + p_X(2;\lambda_e) + p_X(3;\lambda_e)) \approx 0.14287$$

Hence the expected frequencies are $8, 16, 16, 11, 8$. (Calculations are $0.13534 * 59 \approx 8$, etc.) The expected frequencies are very similar to the observed frequencies with no large difference.
(b) The result is quit surprising because the data followed Poisson distribution and it supposed to have less no-hitters nowadays than 1950s.

5.2.10

solution: Given $L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta)$ for continuous.Then
(a)By observation that $L(\theta) = \frac{1}{\theta^4}$ and $L'(\theta) = -\frac{4}{\theta^5}$. Set the derivative to zero is meaningless since it can never be zero. Therefore, we need to try order statistics. Let us look at the $L(\theta)$ directly. $L(\theta)$ is maximum when θ is the smallest. Moreover the inequality $0 \le y \le \theta$ must hold. Hence $\hat{\theta} = y_{\max} = 14.2$.
(b)By observation that $L(\theta_1, \theta_2) = \frac{1}{(\theta_2-\theta_1)^4}$ and

$$\frac{\partial}{\partial\theta_1}(L(\theta_1, \theta_2)) = \frac{4}{(\theta_2-\theta_1)^3}$$

$$\frac{\partial}{\partial\theta_2}(L(\theta_1, \theta_2)) = -\frac{4}{(\theta_2-\theta_1)^3}$$

By setting them to zero are not meaningful because it can not hold. Therefore, we need to do order statistics.

In order to get $L(\theta_1, \theta_2)$ maximized, $\theta_2 - \theta_1$ must be minimized as well as the inequality $\theta_1 \le y \le \theta_2$ must hold. We want θ_2 as small as possible and θ_1 as large as possible. Hence $\hat{\theta_2} = y_{\max} = 14.2, \hat{\theta_1} = y_{\min} = 1.8$.

5.2.13

solution: Given $L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta)$ for continuous.Then

$$L(\theta) = \prod_{i=1}^{25} \theta k^{\theta}(\frac{1}{y_i})^{\theta+1} = \theta^{25}k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1}$$

$$L'(\theta) = 25\theta^{24}k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1} + \theta^{25}(25\ln(k)k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1} + \ln(\prod_{i=1}^{25}\frac{1}{y_i})k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1}$$

$$= (25 + 25\theta\ln k + \theta\ln(\prod_{i=1}^{25}\frac{1}{y_i}))\theta^{24}k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1}$$

Let $L'(\theta) = 0$, since $\theta^{24}k^{25\theta}(\prod_{i=1}^{25}\frac{1}{y_i})^{\theta+1} \neq 0$. Then

$$25 + 25\hat{\theta}\ln k + \hat{\theta}\ln(\prod_{i=1}^{25}\frac{1}{y_i}) = 0$$

$$\Rightarrow \hat{\theta} = -\frac{25}{25\ln k + \ln(\prod_{i=1}^{25}\frac{1}{y_i})}$$

5.2.15

solution: Given the normal pdf and μ, then

$$L(\sigma^2) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma^2}} = (2\pi\sigma^2)^{-n/2}e^{\frac{1}{2}\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma^2}}$$

$$\ln(L(\sigma^2)) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}(y_i-\mu)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{d}{d\sigma^2}(\ln(L(\sigma^2))) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2$$

Let $\frac{d}{d\sigma^2}(\ln(L(\sigma^2))) = 0$, then

$$-\frac{n}{2}\frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=1}^{n}(y_i-\mu)^2 = 0 \Rightarrow -n\hat{\sigma}^2 + \sum_{i=1}^{n}(y_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i-\mu)^2}{n}$$

They are almost identical. In example 5.2.4, μ is not given. Therefore, In the estimator $\hat{\sigma}^2$, $\hat{\mu}$ is there. In this case, μ is given, we only need to estimate $\hat{\sigma}^2$.

5.2.17

solution: This is a continuous pdf, therefore by definition 5.2.3.

$$E(Y) = \int_{-\infty}^{+\infty} yf_Y(y)dy = \int_0^1(\theta^2+\theta)y^\theta(1-y)dy = (\theta^2+\theta)\int_0^1(y^\theta - y^{\theta+1})dy$$

$$= (\theta^2+\theta)(\frac{y^{\theta+1}}{\theta+1} - \frac{y^{\theta+2}}{\theta+2})\Big|_0^1 = (\theta^2+\theta)(\frac{1}{\theta+1} - \frac{1}{\theta+2}) = \frac{\theta}{\theta+2}$$

Since $E(Y) = \bar{y}$, i.e.

$$\frac{\hat{\theta}}{\hat{\theta}+2} = \bar{y} \Rightarrow \hat{\theta} = \frac{2\bar{y}}{1-\bar{y}}$$

5.2.20

solution:

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} \frac{y}{2\theta_2} dy = \frac{y^2}{4\theta_2}\Big|_{\theta_1-\theta_2}^{\theta_1+\theta_2}$$

$$= \frac{(\theta_1+\theta_2)^2 - (\theta_1-\theta_2)^2}{4\theta_2} = \frac{2\theta_1 \cdot 2\theta_2}{4\theta_2} = \theta_1$$

We only can estimate θ_1 with the first moment, therefore we also need to find the second moment.

$$E(Y^2) = \int_{-\infty}^{+\infty} y^2 f_Y(y) dy = \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} \frac{y^2}{2\theta_2} dy = \frac{y^3}{6\theta_2}\Big|_{\theta_1-\theta_2}^{\theta_1+\theta_2}$$

$$= \frac{(\theta_1+\theta_2)^3 - (\theta_1-\theta_2)^3}{6\theta_2} = \frac{6\theta_1^2\theta_2 + 2\theta_2^3}{6\theta_2} = \theta_1^2 + \frac{1}{3}\theta_2^2$$

Moreover, $E(Y) = \bar{y}, E(Y^2) = \frac{\sum y_i^2}{n}$. Then

$$\theta_1 = \bar{y}, \theta_1^2 + \frac{1}{3}\theta_2^2 = \frac{\sum y_i^2}{n} \Rightarrow \theta_2 = \sqrt{3\frac{\sum y_i^2}{n} - 3\bar{y}^2}$$

$$\Rightarrow \theta_{1e} = \frac{8.3+4.9+2.6+6.5}{4} = 5.575,$$

$$\theta_{2e} = \sqrt{3} \cdot \sqrt{(8.3^2+4.9^2+2.6^2+6.5^2)/4 - 5.575^2} \approx 3.6319$$

5.2.23

solution: $E(Y) = \mu$ and $\text{Var}(Y) = \sigma^2$. Moreover, $\sigma^2 = E(Y^2) - (E(Y))^2$, then

$$E(Y) = \bar{y} \Rightarrow \hat{\mu} = \bar{y}$$

$$E(Y^2) = \frac{1}{n}\sum_{i=1}^n y_i^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n y_i^2 - \hat{\mu}^2 = \frac{1}{n}\sum_{i=1}^n y_i^2 - \bar{y}^2$$

In example 5.2.4

$$\hat{\mu} = \bar{y}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) = \frac{1}{n}(\sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} y_i\bar{y} + n\bar{y}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} y_i^2 - 2\frac{\sum_{i=1}^{n} y_i}{n}\bar{y} + \bar{y}^2) = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - 2\bar{y}^2 + \bar{y}^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2$$

They are identical. Therefore, both MME and MLE give the same estimator of μ and σ^2.