Math 470 Assignment 24

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10.2.1. Find all cluster points of each of the following sets.

a)$E = \mathbf{R}\backslash\mathbf{Q}$

proof: Let $\delta > 0$ and $x \in \mathbf{R}$. By the density of irrational, then there are infinitely many points in the $B_\delta(x) = (x-\delta, x+\delta) \cap \mathbf{R}\backslash\mathbf{Q}$. By definition, every $x \in \mathbf{R}$ is a cluster point of $\mathbf{R}\backslash\mathbf{Q}$.

b)$E = [a, b), a, b \in \mathbf{R}, a < b$

proof: Let $\delta > 0$ and $x \in [a, b]$. Let $c = \max\{x - \delta, a\}$ and $d = \min\{x + \delta, b\}$. Therefore $(x - \delta, x + \delta) \cap [a, b) \subset (c, d)$ is itself a nondegenerate interval. Every point $x \in [a, b]$ is a cluster point of $[a, b)$. If any point $x \notin [a, b]$, then x is not a cluster point of $[a, b)$.

c)$E = \{(-1)^n n : n \in \mathbf{N}\}$

proof: Let $\delta > 0$ and $x \in \mathbf{R}, E = \{-1, 2, -3, 4...\} \subset \mathbf{Z}$. Pick $\delta = 1$, then $B_1(x) \cap E$ has at most one point. Therefore, E has no cluster point by definition.

d)$E = \{x_n : n \in \mathbf{N}\},$ where $x_n \to x$ as $n \to \infty$

proof: E has no cluster points when E is finite, same argument as part c), pick δ less than the distance between two closest point in E. Assume E is infinite, then let $\delta > 0$, there exists an $N \in \mathbf{N}$ such that for $n \geq N$ implies $x_n \in B_\delta(x)$. As a result $B_\delta(x) \cap E$ has infinitely many points, then x is a cluster point of E. For another point $y \neq x$. $B_\delta(y) \cap E$ has finitely many points. Hence x is the only cluster point of E.

e)$E = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ...\}$

proof: E has no cluster point, pick $\delta = \frac{1}{2}$. Similar argument as part c).

10.2.2. a) A point a in a metric space X is said to be isolated if and only if there is an $r > 0$ so small that $B_r(a) = \{a\}$. Show that a point $a \in X$ is not a cluster point of X if and only if a is isolated.
b) Prove that the discrete space has no cluster points.

proof:
a) $(\Rightarrow)$ Let $a \in X$, it is not a cluster point of X, then let $\delta > 0$. $B_\delta(x) \cap X$ contains finitely many points, denoted as $x_1, x_2, ..., x_N$ where $N \in \mathbf{N}$. Choose $r = \min\{\delta, d(x_1, a), ..., d(x_N, a)\}$, then $B_r(a) = \{a\}$. Hence it is isolated.
$(\Leftarrow)$ Suppose $a \in X$ is isolated, then let $\delta > 0$, $B_\delta(a) = \{a\}$, do not contain infinitely many points. Hence it is not a cluster point of X by definition.

b)Let $a \in \mathbf{R}$. Pick $r < 1$, then $B_r(a) = \{a\}$. Hence by part a), a is not a cluster point. Every points in the discrete space are isolated.