MATH 435 ASSIGNMENT 13

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1. Algebraic Extensions

1.1. **Theorem 21.2 Uniqueness Property.** If a is algebraic over a field F, then there is a unique monic irreducible polynomial $p(x)$ in $F[x]$ such that $p(a) = 0$.
proof: The existence was proved in the book by Theorem 21.1. Therefore, we only show monic and uniqueness here. There exists an irreducible polynomial in the form $f(x) = a_n x^n + ... + a_1 x + a_0$ where $a_i \in F$. i.e. $a_i^{-1} \in F$. Let $p(x) = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + ... + \frac{a_1}{a_n} x + \frac{a_0}{a_n}$. If $f(a) = 0$ for some a, then $p(a) = 0$. Hence, $p(x)$ is the monic irreducible polynomial in $F[x]$ such that $p(a) = 0$.
Now we want to prove uniqueness. Let f, g be two monic irreducible polynomials in $F[x]$ such that $f(a) = 0 = g(a)$ and $f \neq g$. Then f, g have the minimal degree and deg(f) =deg(g). Define $h = f - g$, then $h(a) = f(a) - g(a) = 0$, i.e. a is also a root of polynomial h. Since f, g have the same degree and leading coefficient is one, then $f - g$ will cancel the first term of the polynomials, which means deg(h) <deg(f). This contradicts with f is minimal degree and f is irreducible since it can factor out h. Hence assumption is wrong. There is a unique monic irreducible polynomial $p(x)$ in $F[x]$ such that $p(a) = 0$.

1.2. 3*.

1.3. 8. Find the degree and a basis for $Q(\sqrt{3}+\sqrt{5})$ over $Q(\sqrt{15})$. Find the degree and a basis for $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2})$ over Q.
proof: Consider $Q(\sqrt{3} + \sqrt{5})$ is an extension over $Q(\sqrt{15})$. WTS $Q(\sqrt{15}) \subset Q(\sqrt{3}, \sqrt{5}) = Q(\sqrt{3} + \sqrt{5})$.
$Q(\sqrt{15}) \subset Q(\sqrt{3}, \sqrt{5})$ since $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$.
$Q(\sqrt{3}, \sqrt{5}) \subseteq Q(\sqrt{3} + \sqrt{5})$ since $(\sqrt{3} + \sqrt{5})^{-1} \in Q(\sqrt{3} + \sqrt{5}) \implies -\sqrt{3} + \sqrt{5} \in Q(\sqrt{3} + \sqrt{5})$, thus

$$[(\sqrt{3} + \sqrt{5}) + (-\sqrt{3} + \sqrt{5})]/2 = \sqrt{5} \in Q(\sqrt{3}, \sqrt{5})$$

$$[(\sqrt{3} + \sqrt{5}) - (-\sqrt{3} + \sqrt{5})]/2 = \sqrt{3} \in Q(\sqrt{3}, \sqrt{5})$$

$Q(\sqrt{3},\sqrt{5}) \supseteq Q(\sqrt{3}+\sqrt{5})$ since $\sqrt{3},\sqrt{5} \in Q(\sqrt{3},\sqrt{5})$, thus $\sqrt{3}+\sqrt{5} \in Q(\sqrt{3},\sqrt{5})$.
Therefore, $Q(\sqrt{15}) \subset Q(\sqrt{3},\sqrt{5}) = Q(\sqrt{3}+\sqrt{5})$.
Moreover, $Q(\sqrt{3},\sqrt{5}) = Q(\sqrt{3},\sqrt{15})$, since $\sqrt{3}\cdot\sqrt{5} = \sqrt{15}$ and $\sqrt{15}\cdot(\sqrt{3})^{-1} = \sqrt{5}$. As a result, $Q(\sqrt{3}+\sqrt{5})$ over $Q(\sqrt{15})$ is the same as $Q(\sqrt{3},\sqrt{15})$ over $Q(\sqrt{15})$. We observe $\sqrt{3} \in Q(\sqrt{3},\sqrt{15})$ but $\sqrt{3} \notin Q(\sqrt{15})$. The irreducible polynomial is x^2-3 by Eisenstein Criterion where the prime is 3. Hence, the degree is 2 and the basis is $\{1,\sqrt{3}\}$.
Consider $Q(\sqrt{2},\sqrt[3]{2},\sqrt[4]{2})$ is an extension over Q. We observe that $(\sqrt[4]{2})^2 = \sqrt{2}$, thus $Q(\sqrt{2},\sqrt[3]{2},\sqrt[4]{2}) = Q(\sqrt[3]{2},\sqrt[4]{2})$.

$$[Q(\sqrt[3]{2},\sqrt[4]{2}):Q] = [Q(\sqrt[3]{2},\sqrt[4]{2}):Q(\sqrt[3]{2}][Q(\sqrt[3]{2}:Q]$$
$$[Q(\sqrt[3]{2},\sqrt[4]{2}):Q] = [Q(\sqrt[3]{2},\sqrt[4]{2}):Q(\sqrt[4]{2}][Q(\sqrt[4]{2}:Q]$$

$[Q(\sqrt[4]{2}:Q] = 4$ and $[Q(\sqrt[3]{2}:Q] = 3 \implies 3|[Q(\sqrt[3]{2},\sqrt[4]{2}):Q]$ and $4|[Q(\sqrt[3]{2},\sqrt[4]{2}):Q]$. Therefore, the minimal degree is 12 and the basis is $\{1,2^{\frac{1}{12}},2^{\frac{1}{6}},2^{\frac{1}{4}},2^{\frac{1}{3}},2^{\frac{5}{12}},2^{\frac{1}{2}},2^{\frac{7}{12}},2^{\frac{2}{3}},2^{\frac{3}{4}},2^{\frac{5}{6}},2^{\frac{11}{12}}\}$.

1.4. 14. Find the minimal polynomial for $\sqrt{-3}+\sqrt{2}$ over Q.
proof: Consider $Q(\sqrt{-3}+\sqrt{2})$, then $(\sqrt{-3}+\sqrt{2})^{-1} = \frac{1}{5}(\sqrt{2}-\sqrt{-3}) \in Q(\sqrt{-3}+\sqrt{2})$. Thus $\sqrt{2}-\sqrt{-3} \in Q(\sqrt{-3}+\sqrt{2})$. Claim $Q(\sqrt{-3}+\sqrt{2}) = Q(\sqrt{-3},\sqrt{2})$.
$Q(\sqrt{-3}+\sqrt{2}) \subseteq Q(\sqrt{-3},\sqrt{2})$ since $\sqrt{-3},\sqrt{2} \in Q(\sqrt{-3},\sqrt{2}) \implies \sqrt{-3}+\sqrt{2} \in Q(\sqrt{-3},\sqrt{2})$
$Q(\sqrt{-3}+\sqrt{2}) \supseteq Q(\sqrt{-3},\sqrt{2})$ since

$$[(\sqrt{2}+\sqrt{-3})+(\sqrt{2}-\sqrt{-3})]/2 = \sqrt{2} \in Q(\sqrt{-3},\sqrt{2})$$
$$[(\sqrt{2}+\sqrt{-3})-(\sqrt{2}-\sqrt{-3})]/2 = \sqrt{-3} \in Q(\sqrt{-3},\sqrt{2})$$

i.e. $Q(\sqrt{-3}+\sqrt{2}) = Q(\sqrt{-3},\sqrt{2})$. Moreover

$$[Q(\sqrt{-3},\sqrt{2}):Q] = [Q(\sqrt{-3},\sqrt{2}):Q(\sqrt{2})][Q[\sqrt{2}:Q] = 2\cdot 2 = 4$$

since x^2+3 is irreducible in $Q(\sqrt{2})$ and x^2-2 is irreducible in Q. i.e. the minimal degree is 4.
Let $x = \sqrt{-3}+\sqrt{2}$, then $5/x = \sqrt{2}-\sqrt{-3}$. $x+5/x = 2\sqrt{2} \implies (x+5/x)^2 = 8$. i.e.

$$(x+5/x)^2-8 = 0 \implies x^2+10+25/x^2-8 = 0$$
$$\implies x^4+2x^2+25 = 0$$

Hence the minimal polynomial is x^4+2x^2+25.