Math 741 Assignment 4 (Quiz)

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February 25, 2019

5.4.1
solution: Count each individual.

$$P(|\hat{\theta} - 3| > 1.0) = 2/10 = 0.2$$

5.4.4
solution: Given $\sigma = 10, \mu = 20, n = 16$, and the distribution is normal. Therefore,

$$P(19.0 < \bar{Y} < 21.0) = P(\frac{19.0 - 20}{10/\sqrt{16}} < Z < \frac{21.0 - 20}{10/\sqrt{16}})$$

$$= P(-0.4 < Z < 0.4) = 0.3108$$

5.4.6
solution:

$$E(Y_{\min}) = \int_0^\theta y f_{Y_{\min}}(y;\theta) dy$$

where

$$F_{Y_{\min}} = P(Y_{\min} \le y) = 1 - P(Y_{\min} \ge y) = 1 - P(Y_1 \ge y, Y_2 \ge y, ..., Y_n \ge y)$$

$$= 1 - P(Y_1 \ge y) \cdot ... \cdot P(Y_n \ge y) = 1 - P(Y \ge y)^n = 1 - (1 - F_Y(y))^n$$

Since $f_Y(y) = \frac{1}{\theta}$, then $F_Y(y) = \int_0^y \frac{1}{\theta} dt = \frac{y}{\theta}$. Therefore,

$$F_{Y_{\min}} = 1 - (1 - \frac{y}{\theta})^n \implies f_{Y_{\min}} = -n(1 - \frac{y}{\theta})^{n-1}(-\frac{1}{\theta}) = \frac{n}{\theta}(1 - \frac{y}{\theta})^{n-1}$$

$$E(Y_{\min}) = \int_0^\theta y \frac{n}{\theta}(1 - \frac{y}{\theta})^{n-1} dy = \frac{\theta}{n+1} (\text{ used computer })$$

$$(n+1)E(Y_{\min}) = \theta \Rightarrow E((n+1)Y_{\min}) = \theta$$

Hence, unbiased estimator for θ based on $Y_{\min}$ is $(n+1)Y_{\min}$.

5.4.8

solution: Y has a uniform distribution in the interval $[0,\theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

Then

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta} dt = \frac{y}{\theta}$$

The rth order statistic is given by

$$f_{Y_r'}(y_r') = \frac{n!}{(r-1)!(n-r)!}[F_Y(y)]^{r-1}[1-F_Y(y)]^{n-r} f_Y(y)$$

Therefore,

$$f_{Y_3'}(y_3') = \frac{4!}{(3-1)!(4-3)!}[F_Y(y)]^{3-1}[1-F_Y(y)]^{4-3} f_Y(y)$$

$$= \frac{4!}{2!}\left[\frac{y}{\theta}\right]^2\left[1-\frac{y}{\theta}\right]\frac{1}{\theta} = \frac{12y^2}{\theta^3}(1-\frac{y}{\theta}) = \frac{12y^2}{\theta^3} - \frac{12y^3}{\theta^4}$$

$$E(Y_3') = \int_0^\theta y(\frac{12y^2}{\theta^3} - \frac{12y^3}{\theta^4})dy = \int_0^\theta \frac{12y^3}{\theta^3} - \frac{12y^4}{\theta^4} dy$$

$$= (\frac{12y^4}{4\theta^3} - \frac{12y^5}{5\theta^4})\Big|_0^\theta = 3\theta - 2.4\theta = 0.6\theta$$

$$E(Y_3') = 0.6\theta \Rightarrow \frac{5}{3}E(Y_3') = \theta \Rightarrow E(\frac{5}{3}Y_3') = \theta$$

Hence, $\frac{5}{3}Y_3'$ is an unbiased estimator for θ.

Order the sample, we see the third order is 18. i.e. $\hat{\theta} = \frac{5}{3} \cdot 18 = 30$. It is possible that we would know that an estimate for θ based on $Y_3^{'}$ was incorrect, because $30 >> 21$. If θ is 30, then we suppose have sample close to 30.

5.4.9

solution: Given the pdf, then

$$E(Y) = \int_0^{\frac{1}{\theta}} y f_Y(y;\theta)dy = \int_0^{\frac{1}{\theta}} 2y^2\theta^2 dy = \frac{2}{3\theta}$$

Since Y_1, Y_2 are random sample drawn from this pdf, i.e. $E(Y_1) = E(Y_2) = E(Y) = \frac{2}{3\theta}$. Therefore,

$$E(c(Y_1 + 2Y_2)) = cE(Y_1) + 2cE(Y_2) = \frac{2c}{3\theta} + \frac{4c}{3\theta} = \frac{1}{\theta}$$

$$\Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$$

5.4.10

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

Given $n = 1$, then

$$E(Y^2) = \int_0^\theta y^2 \frac{1}{\theta} dy = \frac{y^3}{3\theta}\Big|_0^\theta = \frac{\theta^2}{3}$$

Therefore, Y^2 is a biased estimator for θ^2. As a result,

$$E(Y^2) = \frac{\theta^2}{3} \Rightarrow 3E(Y^2) = \theta^2 \Rightarrow E(3Y^2) = \theta^2$$

Hence, $3Y^2$ is an unbiased estimator for θ^2.

5.4.13

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

Thus

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta} dt = \frac{y}{\theta}$$

$$f_{Y_{\max}}(y) = n[F_Y(y)]^{n-1} f_Y(y) = n\left[\frac{y}{\theta}\right]^{n-1} \frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$$

$$f_{\hat{\theta}}(y) = \frac{n}{\theta^n}(\frac{n+1}{n}y)^{n-1} = \frac{(n+1)^{n-1}}{\theta^n n^{n-2}} y^{n-1}$$

Let α be the median of the estimator's distribution $f_{\hat{\theta}}$, then

$$\int_0^{\alpha} f_{\hat{\theta}}(y)dy = 0.5$$

Therefore,

$$\int_0^{\alpha} \frac{(n+1)^{n-1}}{\theta^n n^{n-2}} y^{n-1} dy = \frac{(n+1)^{n-1}}{\theta^n n^{n-3}} y^n \Big|_0^{\alpha} = \frac{(n+1)^{n-1}}{\theta^n n^{n-3}} \alpha^n = 0.5$$

$$\implies \alpha = \sqrt[n]{\frac{\theta^n n^{n-3}}{2(n+1)^{n-1}}}$$

If $\alpha = \theta$, then it is unbiased. Since for any arbitrary n, $\alpha \neq \theta$. As a result, $\hat{\theta}$ is not median unbiased.
Let $\alpha = \theta$, then we can slove for n

$$\sqrt[n]{\frac{n^{n-3}}{2(n+1)^{n-1}}} = 1 \implies 2(n+1)^{n-1} - n^{n-3} = 0$$

$$\implies n = 1$$

Hence, it is median unbiased only for $n = 1$.
5.4.14
solution: Given the pdf,

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y > 0 \\ 0 & \text{o.w.} \end{cases}$$

then

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta} e^{-t/\theta} dt = -e^{-y/\theta} + 1$$

and

$$f_{Y_{\min}} = n[1 - F_Y(y)]^{n-1} f_Y(y) = \frac{n}{\theta} e^{-ny/\theta}$$

$$E(nY_{\min}) = nE(Y_{\min}) = n \int_0^{+\infty} y \frac{n}{\theta} e^{-ny/\theta} dy = \int_0^{+\infty} \frac{n^2 y}{\theta} e^{-ny/\theta} dy$$

Let $u = \frac{n^2 y}{\theta}$ and $dv = e^{-ny/\theta}dy$, then $du = \frac{n^2}{\theta}dy$ and $v = -\frac{\theta}{n}e^{-ny/\theta}$. Therefore,

$$E(nY_{\min}) = uv - \int_0^{+\infty} v du = -nye^{-ny/\theta}\Big|_0^{+\infty} + \int_0^{+\infty} ne^{-ny/\theta} dy$$

$$= 0 - \theta e^{-ny/\theta}\Big|_0^{+\infty} = \theta$$

Hence, $nY_{\min}$ is an unbiased estimator for θ.
Since the pdf is exponential with parameter $\frac{1}{\theta}$ and samples are iid. Therefore, $E(Y_i) = E(Y) = \theta$.

$$E(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n} \cdot n\theta = \theta$$

Hence, $\frac{1}{n}\sum_1^n Y_i$ is also an unbiased estimator for θ.

5.4.16

solution: Let $X_1, ...X_n \sim N(\mu, \sigma^2)$ be iid samples. Then

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$E(\hat{\sigma}^2) = E(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2) = \frac{1}{n}(\sum_{i=1}^{n} E(X_i^2)) - E(\bar{X}^2)$$

The distribution is normal, then

$$E(X_i) = \bar{x}, Var(X_i) = \hat{\sigma}^2$$

$$E(\bar{X}) = E(\frac{X_1 + ... + X_n}{n}) = \bar{x}, \mathrm{Var}(\bar{X}) = \mathrm{Var}(\frac{X_1 + ... + X_n}{n}) = \frac{\hat{\sigma}^2}{n}$$

$$E(\bar{X}^2) = \mathrm{Var}(\bar{X}) + (E(\bar{X}))^2 = \frac{\hat{\sigma}^2}{n} + \bar{x}^2$$

$$E(X_i^2) = \mathrm{Var}(X_i) + (E(X_i))^2 = \hat{\sigma}^2 + \bar{x}^2$$

Therefore,

$$E(\hat{\sigma}^2) = \frac{1}{n}\Big[n \cdot (\hat{\sigma}^2 + \bar{x}^2)\Big] - (\frac{\hat{\sigma}^2}{n} + \bar{x}^2) = \frac{n-1}{n}\hat{\sigma}^2$$

$$\lim_{n\to\infty} E(\hat{\sigma}^2) = \lim_{n\to\infty} \frac{n-1}{n}\hat{\sigma}^2 = \hat{\sigma}^2$$

Hence, the maximum likelihood estimator for σ^2 in a normal pdf is asymptotically unbiased.

5.4.17

solution:(a) $E(X_i) = 1(p) + 0(1-p) = p$, then $E(\hat{p}_1) = E(X_1) = p$.

$$E(\hat{p}_2) = E(\frac{X_1 + ... + X_n}{n}) = \frac{1}{n}(E(X_1) + ... + E(X_n)) = \frac{1}{n} \cdot np = p$$

Therefore, both $\hat{p}_1$ and $\hat{p}_2$ are unbiased estimators for p.

(b) The distribution is Bernoulli, therefore

$$\text{Var}(X_i) = p(1-p) = \text{Var}(\hat{p}_1)$$

$$\text{Var}(\hat{p}_2) = \text{Var}(\frac{1}{n}(X_1 + ... + X_n)) = \frac{1}{n^2}(\text{Var}(X_1) + ... + \text{Var}(X_n)) = \frac{p(1-p)}{n}$$

$$RE(\hat{p}_2 \text{ with respect to } \hat{p}_1) = \text{Var}(\hat{p}_2)/\text{Var}(\hat{p}_1) = \frac{1}{n} < 1 \text{ for } n > 1$$

Hence, $\hat{p}_2$ is a better estimator than $\hat{p}_1$.

5.4.19

solution: a) Given the exponential pdf, i.e.

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta}e^{-y/\theta} & y > 0 \\ 0 & o.w. \end{cases}$$

then

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta}e^{-t/\theta}dt = -e^{-y/\theta} + 1$$

and

$$f_{Y_{\min}} = n[1 - F_Y(y)]^{n-1} f_Y(y) = \frac{n}{\theta}e^{-ny/\theta}$$

$$E(\hat{\theta}_3) = E(nY_{\min}) = nE(Y_{\min}) = n\int_0^{+\infty} y\frac{n}{\theta}e^{-ny/\theta}dy = \int_0^{+\infty} \frac{n^2 y}{\theta}e^{-ny/\theta}dy$$

Let $u = \frac{n^2 y}{\theta}$ and $dv = e^{-ny/\theta}dy$, then $du = \frac{n^2}{\theta}dy$ and $v = -\frac{\theta}{n}e^{-ny/\theta}$. Therefore,

$$E(\hat{\theta}_3) = uv - \int_0^{+\infty} vdu = -nye^{-ny/\theta}\Big|_0^{+\infty} + \int_0^{+\infty} ne^{-ny/\theta}dy$$

$$= 0 - \theta e^{-ny/\theta}\Big|_0^{+\infty} = \theta$$

Hence, $\hat{\theta}_3 = nY_{\min}$ is an unbiased estimator for θ.
Since the pdf is exponential with parameter $\frac{1}{\theta}$ and samples are iid. Therefore, $E(Y_i) = E(Y) = \theta$ and $E(\hat{\theta}_1) = E(Y_1) = \theta$

$$E(\hat{\theta}_2) = E(\bar{Y}) = E(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n} \cdot n\theta = \theta$$

Hence, $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is also an unbiased estimator for θ. As a result, $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ are all unbiased estimators for θ.
b)Var(Y_i) =Var$(Y) = \hat{\sigma}^2$, then

$$\text{Var}(\hat{\theta}_1) = E(\hat{\theta}_1^2) - (E(\hat{\theta}_1))^2 = \hat{\sigma}^2 = \theta^2$$

Then

$$\text{Var}(\hat{\theta}_2) = \frac{1}{n}\text{Var}(\hat{\theta}_1) = \frac{\hat{\sigma}^2}{n} = \frac{\theta^2}{n}$$

$$\text{Var}(\hat{\theta}_3) = E(\hat{\theta}_3^2) - (E(\hat{\theta}_3))^2$$

$$E(\hat{\theta}_3^2) = n^2 E(Y_{\min}^2) = \int_0^{+\infty} y^2 \frac{n^3}{\theta} e^{-ny/\theta} dy = 2\theta^2 \text{ (by computer)}$$

i.e.

$$\text{Var}(\hat{\theta}_3) = 2\theta^2 - \theta^2 = \theta^2$$

c)

$$RE(\hat{\theta}_1 \text{ with respect to } \hat{\theta}_3) = \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_3)} = 1$$

$$RE(\hat{\theta}_2 \text{ with respect to } \hat{\theta}_3) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_3)} = \frac{1}{n} < 1 \text{ for } n > 1$$

Hence, $\hat{\theta}_2$ is more efficiency.

5.4.20
solution: Given $X_1, ..., X_n \sim Poi(\lambda)$ iid, then $E(X_i) = E(X) = \lambda, \text{Var}(X_i) = \text{Var}(X) = \lambda$.

$$E(\hat{\lambda}_1) = E(X_1) = E(X) = \lambda$$

$$E(\hat{\lambda}_2) = E(\bar{X}) = E(\frac{X_1 + ... + X_n}{n}) = \frac{1}{n}(E(X_1) + ... + E(X_n)) = \frac{1}{n}(n\lambda) = \lambda$$

$$\text{Var}(X_i) = \text{Var}(X) = \lambda$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}(\frac{X_1 + ... + X_n}{n}) = \frac{1}{n^2}(\mathrm{Var}(X_1) + ... + \mathrm{Var}(X_n)) = \frac{\lambda}{n}$$

$$RE(\hat{\lambda}_2 \text{ with respect to } \hat{\lambda}_1) = \frac{\mathrm{Var}(X_i)}{\mathrm{Var}(\bar{X})} = \frac{1}{n} < 1 \text{ for } n > 1$$

5.4.22

solution: Given

$$E(W_1) = \mu, \mathrm{Var}(W_1) = \sigma_1^2$$

$$E(W_2) = \mu, \mathrm{Var}(W_2) = \sigma_2^2$$

By example 5.4.3, we can conclude

$$E(cW_1 + (1-c)W_2) = \mu$$

Assume W_1, W_2 are independent, then $\mathrm{Cov}(W_1, W_2) = 0$

$$\mathrm{Var}(cW_1 + (1-c)W_2) = c^2\mathrm{Var}(W_1) + (1-c)^2\mathrm{Var}(W_2) + 2c(1-c)\mathrm{Cov}(W_1, W_2)$$

$$= c^2\sigma_1^2 + (1-c)^2\sigma_2^2$$

We want variance to be as small as possible, then let

$$\frac{d}{dc}(\mathrm{Var}(cW_1 + (1-c)W_2)) = 0 \implies \frac{d}{dc}(c^2\sigma_1^2 + (1-c)^2\sigma_2^2) = 0$$

$$\implies 2c\sigma_1^2 + (2c-2)\sigma_2^2 = 0 \implies c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\frac{d^2}{dc^2}(\mathrm{Var}(cW_1 + (1-c)W_2)) = 2\sigma_1^2 + 2\sigma_2^2 > 0$$

since $\sigma_1 \neq 0, \sigma_2 \neq 0$. Therefore, the c we sloved is minimum.

Hence, for

$$c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

the estimator $cW_1 + (1-c)W_2$ is most efficient.