Math 470 Assignment 16

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7.3.4. Suppose that $|a_k| \le |b_k|$ for large k. Prove that if $\sum_{k=0}^{\infty} b_k x^k$ converges on an open interval I, then $\sum_{k=0}^{\infty} a_k x^k$ converges on I. Is this result ture if *open* is omitted?

proof: Let the radius of convergence of $\sum_{k=0}^{\infty} b_k x^k$ be R_b, then $I \subseteq (-R_b, R_b)$. Then

$$|a_k| \le |b_k| \Rightarrow |a_k|^{1/k} \le |b_k|^{1/k} \Rightarrow$$

$$\frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} \ge \frac{1}{\limsup_{k\to\infty} |b_k|^{1/k}} = R_b$$

Thus let the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$ be R_a, then $I \subseteq (-R_b, R_b) \subseteq (-R_a, R_a)$. Hence it converges on I.

If *open* is omitted, then it is false. Let $I = [-1, 1)$, $a_k = \frac{(-1)^k}{k}$, $b_k = \frac{1}{k}$, then $|a_k| = |b_k|$ and $\sum_{k=0}^{\infty} b_k x^k$ converges on -1 by Alternating Series Test, but $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{1}{k}$ diverges by Harmonic Series Test.

7.3.5 Suppose that $\{a_k\}_{k=0}^{\infty}$ is bounded sequence of real numbers. Prove that

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

has a postive radius of convergence.

proof: $\{a_k\}_{k=0}^{\infty}$ is bounded sequence of real numbers, then there exist a number $M \in \mathbb{R}$, such that $|a_k| \le M$ for all k. Then

$$|a_k| \le M \Rightarrow |a_k|^{1/k} \le M^{1/k} \Rightarrow \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} \ge \frac{1}{\limsup_{k\to\infty} |M|^{1/k}} = 1$$

Thus $R \ge 1$, hence it has a postive radius of convergence.