Math 470 Assignment 8

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6.4.3. Using any test covered in this chapter so far, find out which of the following series converge absolutely, which converge conditionally, and which diverge.

a)$\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{(k+1)!}$

proof: It converges absolutely. Let $a_k = \frac{(-1)^k k^3}{(k+1)!}$, by Ratio Test

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{(k+1)^3}{(k+2)!}}{\frac{k^3}{(k+1)!}} = \lim_{k\to\infty} \frac{(k+1)^3}{k^4+2k^3} = 0 < 1$$

by L'Hospital's Rule. Hence, $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{(k+1)!}$ converges absolutely.

b) $\sum_{k=1}^{\infty} \frac{(-1)(-3)...(1-2k)}{1\cdot 4...(3k-2)}$

proof: It converges absolutely. Let $a_k = \frac{(-1)(-3)...(1-2k)}{1\cdot 4...(3k-2)}$. By Ratio Test,

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} |\frac{1-2(k+1)}{3(k+1)-2}| = lim_{k\to\infty}|\frac{-2k-1}{3k+1}| = \frac{2}{3} < 1$$

Hence, $\sum_{k=1}^{\infty} \frac{(-1)(-3)...(1-2k)}{1\cdot 4...(3k-2)}$ converges absolutely.

c)$\sum_{k=1}^{\infty} \frac{(k+1)^k}{p^k k!}$, $p > e$

proof: It converges absolutely. Let $a_k = \frac{(k+1)^k}{p^k k!}$. By Ratio Test,

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} |\frac{\frac{(k+2)^{k+1}}{p^{k+1}(k+1)!}}{\frac{(k+1)^k}{p^k k!}}| = \lim_{k\to\infty} \frac{1}{p}(\frac{k+2}{k+1})^{k+1} = \frac{e}{p} < 1$$

by $p > e$. Hence $\sum_{k=1}^{\infty} \frac{(k+1)^k}{p^k k!}$ converges absolutely for $p > e$.

d)$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sqrt{k}}{k+1}$

proof: It converges conditionally. Let $f(x) = \frac{\sqrt{x}}{x+1}$, then $f'(x) = \frac{1-x}{2\sqrt{x}(x+1)^2} < 0$ for all $x > 1$. Thus $f(x) \downarrow 0$ as $x \to \infty$, it converges by Alternating Series Test. Let $a_k = \frac{\sqrt{k}}{k+1} > \frac{1}{k}$ for large k, it diverges by Comparison Test. Hence it converges conditionally.

e)$\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k+1}}{\sqrt{k}k^k}$

proof: It converges absolutely. Considering series $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{\sqrt{k}k^k}$ and let $a_k = \frac{\sqrt{k+1}}{\sqrt{k}k^k}$ and $b_k = \frac{1}{k^k}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k}} = 1.$$

Hence$\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k+1}}{\sqrt{k}k^k}$ converges absolutely by Limit Comparison Test.

6.4.4. [Abel's Test] Suppose that $\sum_{k=1}^{\infty} a_k$ converges and that $b_k \downarrow b$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k b_k$ converges.

proof: Suppose that $\sum_{k=1}^{\infty} a_k = A \in \mathbb{R}$, $b_k \downarrow b$ as $k \to \infty$, then $b_k - b \downarrow 0$ as $k \to \infty$. By Dirichlet's Test, $\sum_{k=1}^{\infty} a_k(b_k - b)$ converges. Let $\sum_{k=1}^{\infty} a_k(b_k - b) = B \in \mathbb{R}$, and $\sum_{k=1}^{\infty} a_k(b) = Ab$. Thus $\sum_{k=1}^{\infty} a_k(b) + \sum_{k=1}^{\infty} a_k(b_k - b) = \sum_{k=1}^{\infty} a_k b_k = Ab + B$ converges by Theorem 6.10.

Q: Does $\sum_{k=1}^{\infty}(\frac{\pi}{2} - \arctan k)$ converges or diverges?

proof: Let $\tan\alpha = x \in \mathbb{R}$, then $\arctan x = \alpha$. This implies $\frac{\pi}{2} - \arctan x = \frac{\pi}{2} - \alpha$. By Cofunction Identities of Trigonometry

$$\tan\alpha = \cot(\frac{\pi}{2} - \alpha) \Rightarrow \frac{1}{\tan\alpha} = \frac{1}{\cot(\frac{\pi}{2} - \alpha)} \Rightarrow \tan(\frac{\pi}{2} - \alpha) = \frac{1}{x}.$$

Thus we are looking at convergence or divergence of $\sum_{k=1}^{\infty} \arctan\frac{1}{k}$. Let $f(x) = \arctan\frac{1}{x}$ and $g(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2+1}$ and $g'(x) = \frac{-1}{x^2}$. Thus $f'(x) = g'(x)$ for large x. This implies $\arctan\frac{1}{x}$ has same behavior as $\frac{1}{x}$ for large x. Hence, by Comparison Test and Harmonic Series Test. $\sum_{k=1}^{\infty}(\frac{\pi}{2} - \arctan k)$ diverges.