Math 335 Assignment 11

Arnold Jiadong Yu

April 26, 2018

(1) Let R and S be rings and $f, g : R \to S$ be two ring homomorphisms. Is the (point-wise) sum of the two maps $f + g : R \to S$ necessarily a ring homomorphism?

proof: No. Let R and S be rings and $f, g : R \to S$ be two ring homomorphisms, then $x, y \in R$ implies

$$f(x + y) = f(x) + f(y), f(xy) = f(x)f(y)$$

$$g(x + y) = g(x) + g(y), g(xy) = g(x)g(y)$$

but

$$(f + g)(xy) = f(xy) + g(xy) = f(x)f(y) + g(x)g(y)$$

$$\neq (f + g)(x)(f + g)(y) = f(x)f(y) + f(x)g(y) + g(x)f(y) + g(x)g(y)$$

Hence it is not necessarily a ring homomorphism.

(2) Give an example of a ring R and an ideal $I \subset R$, such that R is non-commutative but R/I is a commutative non-zero ring.

proof: Let $R = \mathbb{R} \times S$ where $\mathbb{R}$ is a commutative ring and S is non-commutative ring, then R is clearly non-commutative. Pick a map f, such that

$$f : \mathbb{R} \times S \to \mathbb{R}, (a, b) \mapsto a$$

Let $(a, b), (c, d) \in \mathbb{R} \times S$, $f((a + c, b + d)) = a + c = f((a, b)) + f((c, d))$ and $f((ac, bd)) = a \cdot c = f((a, b))f((c, d))$. Therefore f is homomorphism. $I = \{(0, b)|\forall b \in S\}$. $R/I \cong \text{Im}(f)$. Since $\text{Im}(f)$ is a subring of $\mathbb{R}$ and $\mathbb{R}$ is a field. Then $\mathrm{Im}(f)$ is commutative. Hence the quotient group is commutative.

(3) Give an example of a commutative unitary ring R, which is not an integral domain, and an ideal $I \subset R$, such that R/I is a field (always assumed to be different from 0).

proof: Let $R = \mathbb{Z}_6$, R is a commutative unitary ring, but not an integral domain since $2 \cdot 3 = 0$. The ideal $I = \{0, 3\}$. The quotient ring R/I has elements $I, I+1, I+2$. I is zero of the quotient ring. Therefore, we only consider $I+1, I+2$. $I+1$ is one of the quotient ring. $(I+1)\cdot(I+1) = I+1$ and $(I+2)\cdot(I+2) = I+1$. Hence, R/I is a field by definition.

(4) Give an example of a non-zero ring R and a non-zero ideal $I \subset R$, such that R and R/I are isomorphic rings.

proof: Let R be infinite Cartesian product of $\mathbb{R}$, then $R = \mathbb{R} \times \mathbb{R} \times ...$ and R is a ring since $\mathbb{R}$ is field. Consider the map

$$f : \mathbb{R} \times \mathbb{R} \times ... \to \mathbb{R} \times \mathbb{R} \times ...$$

$$f : (x_0, x_1, x_2, ...) \mapsto (x_1, x_2, x_3, ...)$$

f is clearly surjective, since we shifted one position and everything entity maps to itself. Moreover pick $(x_0, x_1, x_2, ...), (y_0, y_1, y_2, ...) \in \mathbb{R} \times \mathbb{R} \times ...$

$$f((x_0, x_1, x_2, ...) + (y_0, y_1, y_2, ...)) = f((x_0 + y_0, x_1 + y_1, x_2 + y_2, ...)$$

$$= (x_1+y_1, x_2+y_2, ...) = (x_1, x_2, ...)+(y_1, y_2, ...) = f((x_0, x_1, x_2, ...))+f((y_0, y_1, y_2, ...))$$

$$f((x_0, x_1, x_2, ...)(y_0, y_1, y_2, ...)) = f((x_0y_0, x_1y_1, x_2y_2, ...)$$

$$= (x_1y_1, x_2y_2, ...) = (x_1, x_2, ...)\cdot(y_1, y_2, ...) = f((x_0, x_1, x_2, ...))\cdot f((y_0, y_1, y_2, ...))$$

Therefore, f is homomorphism. $\mathrm{Ker}(f) = \mathbb{R} \times 0 \times 0....$ Hence $(\mathbb{R} \times \mathbb{R} \times ...)/(\mathbb{R} \times 0 \times 0...) \cong (\mathbb{R} \times \mathbb{R} \times ...)$. That is $R/I \cong R$.

(5) Give an example of two unitary commutative rings R_1, R_2 and two homomorphisms $f, g : R_1 \to R_2$ of unitary rings, such that $f \neq g$, yet $\ker(f) = \ker(g)$.

proof: Let $R_1 = \mathbb{C}, R_2 = \mathbb{C}$, there are two unitary ring homomorphisms f, g.

$$f : \mathbb{C} \to \mathbb{C}, z \mapsto z, (0 \mapsto 0, 1 \mapsto 1)$$

$$g : \mathbb{C} \to \mathbb{C}, z \mapsto \overline{z}, (0 \mapsto 0, 1 \mapsto 1)$$

f, g are both unitary ring homomorphism. We proved this in class. But $f \neq g$, yet $\ker(f) = \ker(g)$ since $\mathbb{C}$ is a field.

(6) Is there a unitary ring homomorphism $\mathbb{C} \to \mathbb{R}$?

proof: No. Suppose there is a unitary ring homomorphism denoted as f.

$$f : \mathbb{C} \to \mathbb{R}$$

Then $f(1) = 1, f(0) = 0$. $\mathbb{C}$ is a field, the ideal of $\mathbb{C}$ is $\{0\}$ or $\mathbb{C}$. Since $f(1) = 1$, then $I \subset \mathbb{C}$ is $\{0\}$. We proved in class that if the ideal is $\{0\}$, then f is injective. But f can not be injective since $\mathbb{R} \subset \mathbb{C}$. Hence there is an contradiction. Therefore, there doesn't exist a unitary ring homomorphism $\mathbb{C} \to \mathbb{R}$.

(7) Does there exits an example of two integral domains R and S, having different positive characteristics, and a homomorphism $f : R \to S$ of unitary rings?

proof: No. Suppose there is a unitary ring homomorphism f between two integral doamin R and S, both characteristics are prime, we proved in class. Let $\text{char}(R) = p_1$,$\text{char}(S) = p_2$ and $p_1 \neq p_2$. Then p_1 and p_2 are coprime. Moreover, $f(p_1 \cdot 1_R) = p_1 f(1_R) = p_1 \cdot 1_S$. Therefore, $f(p_1 \cdot 1_R) = f(0) = 0 = p_2 \cdot 1_S = p_1 \cdot 1_S$. This implies one is divisor of another one. But this contradicts with p_1 and p_2 are coprime. Hence there doesn't exist one unitray ring homomorphism between two integral domain R and S which have different positive characteristics.

(8) Show that, if F is a finite field of characteristic $p > 0$, then the map $F \to F$, $a \to a^p$, is a ring isomorphism.

proof: We proved in class that the map is ring homomorphism. It is only need to show the map is bijective. Pick $a, b \in F$ and let $a^p = b^p$. Then

$a^p - b^p = (a-b)^p = 0$ ($a^p - b^p = (a-b)^p$ is similar proof in ring homomorphism in class). Since a, b is arbitray, then $(a-b)^p = 0$ implies $a - b = 0 \Rightarrow a = b$. Therefore the map is injective. Moreover, F is finite field, then injection implies bijection. The map is bijective. Hence the map is isomorphism.

(9) Give an example of an integral domain R of characteristic $p > 0$, such that the map $R \to R$, $r \to r^p$, is a non-surjective ring homomorphism.

proof: Let $R = \mathbb{Z}_2[x]$, which is polynomial with integer coefficients comes from $\mathbb{Z}_2$. R is an integral domain, proved in previous homework. Characteristic of R is $2 > 0$. f is a ring homomorphism, it is the similar proof to question 8 by using division algorithm. $x^2 + x + 1 \in R$ that doesn't have a preimage such that $y \in R \Rightarrow y^2 = x^2 + x + 1$. Therefore, it is non-surjective ring homomorphism.