Math 335 Assignment 13

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May 5, 2018

(1) Show that $\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$.

proof: Pick a map Φ, such that

$$\Phi : \mathbb{R}[X] \to \mathbb{C}$$

$$\Phi : \mathbb{R}[X] \to \mathbb{R}[i] \in \mathbb{C}$$

$$\Phi : a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \mapsto a_n i^n + a_{n-1} i^{n-1} + ... + a_1 i + a_0$$

$$\text{where } a_i \in \mathbb{R} \text{ for } i \in \mathbb{Z}_{\geq 0}$$

a) WTS it is a ring homomorphism.
Pick $f, g \in \mathbb{R}[X]$, then

$$\Phi(f+g) = (a_n + b_n)i^n + (a_{n-1} + b_{n-1})i^{n-1} + ... + (a_1 + b_1)i_1 + (a_0 + b_0)$$

$$= [a_n i^n + a_{n-1} i^{n-1} + ... + a_1 i + a_0] + [b_n i^n + b_{n-1} i^{n-1} + ... + b_1 i + b_0] = \Phi(f) + \Phi(g)$$

$$\Phi(fg) = \sum_{k=0}^{2n} c_k x_k = \Phi(f)\Phi(g), \text{ where } c_k = \sum_{i+j=k} a_i b_j$$

Therefore, Φ is a ring homorphism.
b) WTS it is surjective.
Since $i^2 = -1$ which is real number, and all coefficients are real number. Moreover $i^3 = -i, i^4 = 1$, i.e.

$$\Phi(f) = a_n i^n + a_{n-1} i^{n-1} + ... + a_1 i + a_0 = \underbrace{(a_0 - a_2 + a_4 - a_6 + ...)}_{\text{real part}} + \underbrace{(a_1 - a_3 + a_5 - a_7 + ...)i}_{\text{imagnary part}}$$

Since $a_i \in \mathbb{R}$ for $i \in \mathbb{Z}_{\geq 0}$ is arbitrary, then it can cover the whole complex plane. Hence it is surjective.
c)WTS $\ker(\Phi) = (x^2 + 1)$
Since $i^2+1 = 0$, then $x^2+1 \subseteq \ker(\Phi)$. Pick $f \in \ker(\Phi)$, then $\Phi(f) = f(i) = 0$ implies i is a root. If i is a root, then $-i$ must be a root by Complex Conjugate Theorem. That is $f \in$ ker (Φ) be a function has root $-i, i$, then $x^2 + 1|f$. This mean $\exists g \in \mathbb{R}[X]$ such that $f(x) = (x^2 + 1) \cdot g(x)$ for $\forall f \in \ker(\Phi)$. Therefore, $\ker(\Phi) = (x^2 + 1)$.

Hence, $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$ by Fundamental Homomorphism Theorem of Ring.

(2)Show that $\mathbb{R}[X]/(X^2 + 2) \cong \mathbb{C}$.

proof: Define the map Φ, such that

$$\Phi : \mathbb{R}[X] \to \mathbb{C}$$

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \mapsto a_n(\sqrt{2}i)^n + a_{n-1}(\sqrt{2}i)^{n-1} + ... + a_1(\sqrt{2}i) + a_0$$
$$\text{where } a_i \in \mathbb{R} \text{ for } i \in \mathbb{Z}_{\geq 0}$$

The proof is the same as question 1.

(3)Show that $\mathbb{Z}_3[X]/(X^2 + 1) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

proof: Define the map Φ, such that

$$\Phi : \mathbb{Z}_3[X] \to \mathbb{Z}_3 \times \mathbb{Z}_3$$

$$\Phi : \mathbb{Z}_3[X] \to (\mathrm{Re}(\mathbb{Z}_3[i]), \mathrm{Im}(\mathbb{Z}_3[i]))$$

The map is well defined. Pick $f \in \mathbb{Z}_3[X]$, then

$$\Phi(f) = a_n i^n + a_{n-1}i^{n-1} + ... + a_1 i + a_0 = \underbrace{(a_0 - a_2 + a_4 - a_6 + ...)}_{\text{real part} \in \mathbb{Z}_3} + \underbrace{(a_1 - a_3 + a_5 - a_7 + ...)i}_{\text{imagnary part} \in \mathbb{Z}_3}$$

a) WTS Φ is ring homomorphism.
Pick $f, g \in \mathbb{Z}_3[X]$, then

$$\Phi(f + g) = (a_n + b_n)i^n + (a_{n-1} + b_{n-1})i^{n-1} + ... + (a_1 + b_1)i + (a_0 + b_0)$$
$$= (a_0 + b_0 - a_2 - b_2 + ...) + (a_1 + b_1 - a_3 - b_3 + ...)i$$

$= ([a_0 - a_2 + ...) + (a_1 - a_3 + ...)i] + [b_0 - b_2 + ...) + (b_1 - b_3 + ...)i] = \Phi(f) + \Phi(g)$

$$\Phi(fg) =$$

(4) Are the rings $\mathbb{Z}_2[X]/(X^2+1)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic?

proof:

(5) Show that for any field F, the ring of polynomial $F[X]$ has infinitely many irreducible polynomials.

proof: Suppose F is a field, then every polynomials of degree 1 in $F[X]$ is irreducible. Let p_1 be a irreducible polynomial with degree 1, then $p_2 = p_1 + 1$ is also irreducible. p_1 and p_2 are co-prime polynomial. Let $p_3 = p_1p_2 + 1$, then p_3 is also irreducible. That is for any $n \in \mathbb{N}$, $p_n = p_1p_2...p_{n-1} + 1$ is irreducible. Hence the ring of polynomial $F[X]$ has infinitely many irreducible polynomials.

(6) Let F be a field and $A \subset F[X]$ be the set of polynomials without linear term, i.e. with 0 linear coefficient. Show that A is a subring of $F[X]$.

proof: Let $a_0, a_1, ..., a_n \in F$, then $F[X] = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0$ and $A = a_nx^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_0$.
1) WTS A is a subgroup w.r.t. addition.
a) $0 \in A$. Let $a_0 = a_1 = a_2 = a_3 = ... = a_n = 0$, then $f(x) = 0 \in A$.
b) Pick $f, g \in A$, s.t. $f(x) = a_nx^n + ... + a_2x^2 + a_0, g(x) = b_nx^n + ... + b_2x^2 + b_0$. Then

$$f(x) + g(x) = (a_nx^n + ... + a_2x^2 + a_0) + (b_nx^n + ... + b_2x^2 + b_0)$$

$$= (a_n + b_n)x^n + ... + (a_2 + b_2)x^2 + (a_0 + b_0) \in A$$

c)Let $f(x) = a_nx^n + ... + a_2x^2 + a_0 \in A$. Then

$$-f(x) = (-a_n)x^n + ... + (-a_2)x^2 + (-a_0) \in A$$

Hence it is a subgroup w.r.t. addition.
2) WTS $f, g \in A \Rightarrow fg \in A$. Pick $f, g \in A$, then

$$f(x)g(x) = (a_nx^n + ... + a_2x^2 + a_0) \cdot (b_nx^n + ... + b_2x^2 + b_0)$$

$$= \underbrace{(a_n x^n + ... + a_2 x^2)(b_n x^n + ... + b_2 x^2)}_{\text{all degree is greater than 1}}$$

$$+ \underbrace{a_0(b_n x^n + ... + b_2 x^2)}_{\text{all degree is greater than 1}} + \underbrace{b_0(a_n x^n + ... + a_2 x^2)}_{\text{all degree is greater than 1}} + a_0 b_0 \in A$$

$f(x)g(x)$ doesn't have linear term. Hence A is a subring of $F[X]$.

(7) Does the ring A in Problem 6 has the UFD property?.

proof: No.