Math 430 Assignment 10

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4.19

solution:

a) Suppose that there exists some $\mathbf{p} \in \mathbb{R}^m$ for which $\mathbf{p}'\mathbf{A} \geq \mathbf{0}', \mathbf{p}'\mathbf{b} = 0$, and such that $\mathbf{p}'\mathbf{A}_j > 0$. Consider the polyhedron, then

$$\mathbf{p}'\mathbf{b} = 0, \mathbf{A}\mathbf{x} = \mathbf{b} \implies \mathbf{p}'\mathbf{A}\mathbf{x} = \mathbf{p}'\mathbf{b} = 0$$

Since $\mathbf{p}'\mathbf{A} \geq \mathbf{0}'$ and $\mathbf{x} \geq 0$, then $\mathbf{p}'\mathbf{A}_i x_i = 0, \forall i$. Moreover, $\mathbf{p}'\mathbf{A}_j > 0, \mathbf{p}'\mathbf{A}_j x_j = 0$ and $x_j \geq 0$, then $x_j = 0$. Hence x_j is a null variable.

b) Consider a primal and its dual problem, then the primal is

$$\min \mathbf{0}'\mathbf{x}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

the dual is

$$\max \mathbf{q} \cdot \mathbf{b}$$

$$\text{s.t. } \mathbf{q}'\mathbf{A} \leq \mathbf{0}'$$

Let $\mathbf{q} = -\mathbf{p}$, then

$$\max \; -\mathbf{p} \cdot \mathbf{b}$$

$$\text{s.t. } \mathbf{p}'\mathbf{A} \geq \mathbf{0}'$$

By strong duality theorem, there exists some $\mathbf{p} \in \mathbb{R}^m$ for which $\mathbf{p}'\mathbf{A} \geq \mathbf{0}', \mathbf{p} \cdot \mathbf{b} = 0$. Then

$$\mathbf{p}'\mathbf{b} = 0, \mathbf{A}\mathbf{x} = \mathbf{b} \implies \mathbf{p}'\mathbf{A}\mathbf{x} = \mathbf{p}'\mathbf{b} = 0$$

Since $\mathbf{p}^{'}\mathbf{A} \geq \mathbf{0}^{'}$ and $\mathbf{x} \geq 0$, then $\mathbf{p}^{'}\mathbf{A}_i x_i = 0, \forall i$. Moreover, $x_j = 0, \mathbf{p}^{'}\mathbf{A}_j x_j = 0$ and $x_j \geq 0$, then $\mathbf{p}^{'}\mathbf{A}_j \geq 0$. If $\mathbf{p}^{'}\mathbf{A}_j = 0$, then $\mathbf{q}^{'}\mathbf{A}_j = 0 \implies x_j$ is free, a contradiction. Hence $\mathbf{p}^{'}\mathbf{A}_j > 0$. The converse is true.
c) By result and proofs of part a and b, only need to prove that $\mathbf{x}+\mathbf{A}^{'}\mathbf{p} > \mathbf{0}$. Suppose $\mathbf{x}+\mathbf{A}^{'}\mathbf{p} \leq \mathbf{0}$, then

$$\mathbf{A}\mathbf{x}+\mathbf{A}^{'}\mathbf{p} \leq \mathbf{0} \implies \mathbf{b}+\mathbf{p} \leq \mathbf{0}$$

$$\implies \mathbf{b} \leq -\mathbf{p} \implies \mathbf{p}^{'}\mathbf{b} \leq -\mathbf{p}^{'}\mathbf{p} \implies 0 \leq -\sum_{i,j}\mathbf{I}_{ij}$$

A contradiction, i.e. $\mathbf{x}+\mathbf{A}^{'}\mathbf{p} > \mathbf{0}$. Hence, part c proved.

4.26
solution:
Consider the primal problem,

$$\min \mathbf{0}^{'}\mathbf{x}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x} \geq \mathbf{0}$$

and its dual problem

$$\max \mathbf{p}\cdot\mathbf{0}$$

$$\text{s.t. } \mathbf{p}^{'}\mathbf{A} \leq \mathbf{0}^{'}$$

(1) Suppose part a) holds, then $\mathbf{x}$ is the optimal of the primal problem. By strong duality theorem, the dual problem also have optimal. i.e. $\mathbf{p}^{'}\mathbf{A} \leq \mathbf{0}^{'}, \forall \mathbf{p}$. Then part b can not hold.
(2) Suppose part a) doesn't hold, then the primal problem is infeasible, so the dual problem must be infeasible or feasible and unbounded. By observing the dual problem, it must be infeasible. It means the converse of the dual problem must hold. Hence part b holds.

4.27
solution:
Consider the primal problem,

$$\min \mathbf{0}^{'}\mathbf{x}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} \geq \mathbf{0}$$

$$\mathbf{x} \geq \mathbf{0}$$

and its dual problem

$$\max \mathbf{p} \cdot \mathbf{0}$$

$$\text{s.t. } \mathbf{p}^{'}\mathbf{A} \leq \mathbf{0}^{'}$$

$$\mathbf{p} \geq \mathbf{0}$$

$a \implies b$. Prove by contradition, suppose part a and $\neg b$. With $\mathbf{p}^{'}\mathbf{A}_1 = 0 \implies x_1$ is free, which contradicts with part a that $x_1 = 0$ for every $\mathbf{x}$. i.e. part b must hold. Hence $a \implies b$.
$b \implies a$. The converse is trivial, since the dual of the dual problem is the primal problem. Suppose part b, and by strong duality, the primal and dual problem has the same optimal cost. Then the following must hold,

$$\min_{x_1 \geq 0}(-\mathbf{p}^{'}\mathbf{A}_1)x_1 = 0 \implies x_1 = 0$$

since $\mathbf{p}^{'}\mathbf{A}_1 < 0$. i.e. part b $\implies$ part a.

solution:
$a \implies b$. Prove by contradiction. Suppose part a and $\neg b$, i.e. for all $\mathbf{x} \geq 0$, $\mathbf{a}^{'}\mathbf{x} \leq \max_i \mathbf{a}_i^{'}\mathbf{x}$ and $\mathbf{a} > \sum_{i=1}^{m} \lambda_i \mathbf{a}_i$ where $\sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0$. Moreover

$$\mathbf{a}^{'}\mathbf{x} > (\sum_{i=1}^{m} \lambda_i \mathbf{a}_i^{'})\mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{a}_i^{'}\mathbf{x}$$

since $\mathbf{x} \geq 0$, WLOG assme $i = 1$ is the maximal value of $\mathbf{a}_i^{'}\mathbf{x}$, then

$$\mathbf{a}_1^{'}\mathbf{x} \geq \mathbf{a}^{'}\mathbf{x} > (1 - \sum_{i=2}^{m} \lambda_i)\mathbf{a}_1^{'}\mathbf{x} + \sum_{i=2}^{m} \lambda_i \mathbf{a}_i^{'}\mathbf{x}$$

$$\implies \mathbf{a}_1^{'}\mathbf{x} > (1 - \sum_{i=2}^{m} \lambda_i)\mathbf{a}_1^{'}\mathbf{x} + \sum_{i=2}^{m} \lambda_i \mathbf{a}_i^{'}\mathbf{x}$$

$$\implies 0 > \sum_{i=2}^{m} \lambda_i(\mathbf{a}_i^{'}\mathbf{x} - \mathbf{a}_1^{'}\mathbf{x})$$

$$\implies 0 > \sum_{i=2}^{m} \lambda_i(\mathbf{a}_i^{'} - \mathbf{a}_1^{'})\mathbf{x} \leq 0$$

Since it is for all λ_i, and a_i is arbitrary. We will result in $0 > 0$. A contradiction, i.e. part $a \implies b$.
$b \implies a$. Also prove by contradiction. Suppose part b and $\neg a$, i.e. there exists $\sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0$ such that $\mathbf{a} \leq \sum_{i=1}^{m} \lambda_i \mathbf{a}_i$ and for all $\mathbf{x} \geq 0$, $\mathbf{a}^{'}\mathbf{x} > \max_i \mathbf{a}_i^{'}\mathbf{x}$. Moreover

$$\mathbf{a}^{'}\mathbf{x} \leq (\sum_{i=1}^{m} \lambda_i \mathbf{a}_i^{'})\mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{a}_i^{'}\mathbf{x}$$

since $\mathbf{x} \geq 0$, WLOG assme $i = 1$ is the maximal value of $\mathbf{a}_i^{'}\mathbf{x}$, then

$$\mathbf{a}^{'}\mathbf{x} \leq (1 - \sum_{i=2}^{m} \lambda_i)\mathbf{a}_1^{'}\mathbf{x} + \sum_{i=2}^{m} \lambda_i \mathbf{a}_i^{'}\mathbf{x}$$

$$\implies \mathbf{a}^{'}\mathbf{x} - \mathbf{a}_1^{'}\mathbf{x} \leq \sum_{i=2}^{m} \lambda_i(\mathbf{a}_i^{'}\mathbf{x} - \mathbf{a}_1^{'}\mathbf{x})$$

$$\implies 0 < (\mathbf{a}^{'} - \mathbf{a}_1^{'})\mathbf{x} \leq \sum_{i=2}^{m} \lambda_i(\mathbf{a}_i^{'} - \mathbf{a}_1^{'})\mathbf{x} \leq 0 \implies 0 < 0$$

A contradiction, i.e. part $b \implies a$.