Math 470 Assignment 11

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7.1.2. Prove that the following limits exist and evaluate them.

a) $\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}} dx$

proof: Let $x \in [1,3]$, $f_n(x) = \frac{nx^{99}+5}{x^3+nx^{66}}$ and $f(x) = x^{33}$, let $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{3^{36}-5}{\epsilon}$, then $n \geq N$ implies

$$|f_n(x)-f(x)| = |\frac{nx^{99}+5}{x^3+nx^{66}} - x^{33}| = \frac{|5-x^{36}|}{x^3+nx^{66}} < \frac{|5-x^{36}|}{n} \leq \frac{|5-x^{36}|}{N} \leq \frac{3^{36}-5}{N} < \epsilon$$

thus $f_n(x) \to f(x)$ for $x \in [1,3]$ as $n \to \infty$ converges uniformly. Then by Theorem 7.10,

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}} dx = \int_1^3 x^{33} dx = \frac{x^{34}}{34}\Big|_1^3 = \frac{3^{34}-1}{34}$$

b) $\lim_{n\to\infty} \int_0^2 e^{x^2/n} dx$

proof: Let $x \in [0,2]$, $f_n(x) = e^{x^2/n}$ and $f(x) = 1$. Let $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $0 < e^{4/N} - 1 < \epsilon$, then $n \geq N$ implies $e^{x^2/n} \leq e^{4/n} \leq e^{4/N}$ and

$$|f_n(x) - f(x)| = |e^{x^2/n} - 1| = e^{x^2/n} - 1 \leq e^{4/n} - 1 \leq e^{4/N} - 1 < \epsilon$$

thus $f_n(x) \to f(x)$ for $x \in [0,2]$ as $n \to \infty$ converges uniformly. Then by Theorem 7.10,

$$\lim_{n\to\infty} \int_0^2 e^{x^2/n} dx = \int_0^2 1 dx = x|_0^2 = 2$$

c) $\lim_{n\to\infty}\int_0^3 \sqrt{\sin\frac{x}{n}+x+1}dx$

proof: Let $x\in[0,3]$, $f_n(x)=\sqrt{\sin\frac{x}{n}+x+1}$ and $f(x)=\sqrt{x+1}$, then $\frac{x}{n}\geq 0$. Let $\epsilon>0$, choose $N\in\mathbb{N}$ such that N, then $n\geq N$ and $\sin\frac{x}{n}\leq\frac{x}{n}$ implies

$$|f_n(x)-f(x)|=|\sqrt{\sin\frac{x}{n}+x+1}-\sqrt{x+1}|=\frac{\sin\frac{x}{n}}{\sqrt{\sin\frac{x}{n}+x+1}+\sqrt{x+1}}$$

$$\leq\frac{\frac{x}{n}}{\sqrt{\sin\frac{x}{n}+x+1}+\sqrt{x+1}}\leq\frac{\frac{x}{n}}{2\sqrt{x+1}}\leq\frac{x}{2n}\leq\frac{3}{2n}\leq\frac{3}{2N}<\epsilon$$

thus $f_n(x)\to f(x)$ for $x\in[0,3]$ as $n\to\infty$ converges uniformly. Then by Theorem 7.10,

$$\lim_{n\to\infty}\int_0^3\sqrt{\sin\frac{x}{n}+x+1}dx=\int_0^3\sqrt{x+1}dx=\frac{2}{3}(x+1)^{\frac{3}{2}}|_0^3=\frac{14}{3}$$

7.1.5. Suppose that $f_n\to f$ and $g_n\to g$ as $n\to\infty$, uniformly on some set $E\subseteq\mathbb{R}$.

a) Prove that $f_n+g_n\to f+g$ and $\alpha f_n\to\alpha f$, as $n\to\infty$, uniformly on E for all $\alpha\in\mathbb{R}$.

proof: Suppose that $f_n\to f$ and $g_n\to g$ as $n\to\infty$, uniformly on some set $E\subseteq\mathbb{R}$. Let $\epsilon>0$, there exists N_1 s.t. $n\geq N_1$ implies $|f_n-f|<\frac{\epsilon}{\max\{2,|\alpha|+1\}}$ and there exist N_2 s.t. $n\geq N_2$ implies $|g_n-g|<\frac{\epsilon}{\max\{2,|\alpha|+1\}}$. Choose $N=\max\{N_1,N_2\}$, then $n\geq N$ implies

$$|f_n+g_n-(f+g)|\leq|f_n-f|+|g_n-g|<\frac{2\epsilon}{\max\{2,|\alpha|+1\}}\leq\epsilon$$

and

$$|\alpha f_n-\alpha f|=|\alpha||f_n-f|\leq|\alpha|\frac{\epsilon}{\max\{2,|\alpha|+1\}}<\epsilon$$

Hence $f_n + g_n \to f + g$ and $\alpha f_n \to \alpha f$, as $n \to \infty$, uniformly on E for all $\alpha \in \mathbb{R}$.

b)Prove that $f_n g_n \to fg$ pointwise on E.

proof: Suppose that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, uniformly on some set $E \subseteq \mathbb{R}$. Let $\epsilon > 0$, there exists $N > \sup\{\frac{|f_n(x)|}{\epsilon}, \frac{|g(x)|}{\epsilon} \mid x \in E\}$ such that $|f_n - f| < \frac{\epsilon}{2N}$ and $|g_n - g| < \frac{\epsilon}{2N}$ for all $x \in E$, then $n \geq N$ implies

$$
\begin{aligned}
|f_n g_n - fg| &= |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\
&\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< \frac{N(\epsilon)}{2N} + \frac{N(\epsilon)}{2N} = \epsilon
\end{aligned}
$$

since N depends on f_n , g and ϵ, then it converges pointwise on E.

c) Prove that if f and g are bounded on E, then $f_n g_n \to fg$ uniformly on E.

proof: Suppose that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, uniformly on some set $E \subseteq \mathbb{R}$, and f and g are bounded. Let $\epsilon > 0$, choose $M > 0$ such that $M \geq \sup\{|f(x)|+1, |g(x)|+1 \, |x \in E\}$, there exists N_1 such that $|f_n - f| < \frac{\epsilon}{3M}$ and $|g_n - g| < \frac{\epsilon}{3M}$. Moreover $f_n \to f$ and f is bounded by M there exists N_2 such that $|f_n| \leq 2M$. Choose $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$
\begin{aligned}
|f_n g_n - fg| &= |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\
&\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< \frac{|f_n(x)|(\epsilon)}{3M} + \frac{|g(x)|(\epsilon)}{3M} \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}
$$

Hence if f and g are bounded on E, then $f_n g_n \to fg$ uniformly on E.

d)Show that c) may be false when g is unbounded.

proof: Let $f_n(x) = \frac{1}{n}$ and $f(x) = 0$. Let $g_n(x) = \frac{1}{x}$ and $g = \frac{1}{x}$.Then $f_n \to 0$ uniformly on $\mathbb{R}$ and $g_n \to \frac{1}{x}$ uniformly on $(0, \infty)$. But $f_n g_n = \frac{1}{nx}$ does not converge uniformly on $(0, 1)$ by Example 7.1.1 (b).

7.1.6. Suppose that E is a nonempty subset of $\mathbb{R}$ and that $f_n \to f$ uniformly on E. Prove that if each f_n is uniformly continuous on E, then f is uniformly continuous on E.

proof: Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } |f_n(x) - f(x)| < \frac{\epsilon}{3} \text{ for all } x \in E$$

Since f_N is continuous at $y \in E$, then $\exists \delta > 0$ such that $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Suppose $|x - y| < \delta$ and $x, y \in E$, then

$$|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)|$$

$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Hence f is uniformly continuous on E.

7.1.8. Suppose that $b > a > 0$. Prove that

$$\lim_{n\to\infty} \int_a^b (1 + \frac{x}{n})^n e^{-x} dx = b - a.$$

proof: W.T.S $(1 + \frac{x}{n})^n \to e^x$ uniformly as $n \to \infty$ for $x \in [a, b]$ where $b > a > 0$. Let $\epsilon > 0$, $f_n(x) = (1 + \frac{x}{n})^n$ and $f(x) = e^x$. Then

$$(f_n(x) - f(x))' = (1 + \frac{x}{n})^{n-1} - e^x < (1 + \frac{x}{n})^n - e^x = f_n(x) - f(x)$$

thus $f_n(x) - f(x)$ is decreasing for all $x \in [a, b]$, then $\max\{f_n(x) - f(x)\} = f_n(a) - f(a)$. Choose $N \geq b$, then $n \geq N$ implies $0 < \frac{a}{n} \leq \frac{x}{n} \leq \frac{b}{n} \leq \frac{b}{N} < 1$. By Binomial Series Expansion and Taylor Series of e^x,

$$|f_n(x) - f(x)| \leq |f_n(a) - f(a)| = |(1 + \frac{a}{n})^n - e^a| = |e^a - e^a| = 0 < \epsilon$$

Hence $(1 + \frac{x}{n})^n \to e^x$ uniformly as $n \to \infty$ for all $x \in [a, b]$, then

$$\lim_{n\to\infty} \int_a^b (1 + \frac{x}{n})^n e^{-x} dx = \int_a^b dx = b - a.$$