MATH 435 ASSIGNMENT 9

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1. Introduction to Module Theory

1.1. 11. Let $A_1, A_2, \cdots A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \cdots, n$. Prove that

$$(A_1 \oplus \cdots \oplus A_n)/(B_1 \oplus \cdots \oplus B_n) \cong (A_1/B_1) \oplus \cdots \oplus (A_n/B_n).$$

proof: All of $A_1 \oplus \cdots \oplus A_n$, $B_1 \oplus \cdots \oplus B_n$ and $(A_1/B_1) \oplus \cdots \oplus (A_n/B_n)$ are $R-$modules, since each component is $R-$module and operation componentwise. Define a map φ,

$$\varphi : A_1 \oplus \cdots \oplus A_n \to (A_1/B_1) \oplus \cdots \oplus (A_n/B_n)$$

$$\varphi : a_1^{A_1} \oplus \cdots \oplus a_1^{A_n} \mapsto (a_1^{A_1} + B_1) \oplus \cdots \oplus (a_1^{A_n} + B_n)$$

Let $a_1^{A_1} \oplus \cdots \oplus a_1^{A_n}, a_2^{A_1} \oplus \cdots \oplus a_2^{A_n} \in A_1 \oplus \cdots \oplus A_n$, $r \in R$.

$$\varphi(r(a_1^{A_1} \oplus \cdots \oplus a_1^{A_n}) + a_2^{A_1} \oplus \cdots \oplus a_2^{A_n}) = \varphi[(ra_1^{A_1} + a_2^{A_1}) \oplus \cdots \oplus (ra_1^{A_n} + a_2^{A_n})]$$

$$= (ra_1^{A_1} + a_2^{A_1} + B_1) \oplus \cdots \oplus (ra_1^{A_n} + a_2^{A_n} + B_n)$$

$$= (ra_1^{A_1} + B_1) \oplus \cdots \oplus (ra_1^{A_n} + B_n) + (a_2^{A_1} + B_1) \oplus \cdots \oplus (a_2^{A_n} + B_n)$$

$$= r\varphi(a_1^{A_1} \oplus \cdots \oplus a_1^{A_n}) + \varphi(a_2^{A_1} \oplus \cdots \oplus a_2^{A_n})$$

Therefore, φ is a R-module homomorphism. Let $b_1^{B_i} \in B_i$ for each $i = 1, ..., n$, then $b_1^{B_1} \oplus \cdots \oplus b_1^{B_n} \in B_1 \oplus \cdots \oplus B_n$. i.e.

$$\varphi(b_1^{B_1} \oplus \cdots \oplus b_1^{B_n}) = (b_1^{B_1} + B_1) \oplus \cdots \oplus (b_1^{B_n} + B_n) = (0 + B_1) \oplus \cdots \oplus (0 + B_n)$$

Therefore, $\ker(\varphi) = B_1 \oplus \cdots \oplus B_n$.
φ is also surjective, since for any element in $(A_1/B_1) \oplus \cdots \oplus (A_n/B_n)$,

$$(a_1^{A_1} + B_1) \oplus \cdots \oplus (a_1^{A_n} + B_n) = \varphi(a_1^{A_1} \oplus \cdots \oplus a_1^{A_n})$$

where $a_1^{A_1} \oplus \cdots \oplus a_1^{A_n} \in A_1 \oplus \cdots \oplus A_n$.
i.e. By the first isomorphism theorem for modules,

$$(A_1 \oplus \cdots \oplus A_n)/(B_1 \oplus \cdots \oplus B_n) \cong (A_1/B_1) \oplus \cdots \oplus (A_n/B_n).$$

1.2. 7. Let N be submodule of M. Prove that if both M/N and N are finitely generated then so is M.
proof: Suppose N is submodule of M and generated by $\{b_1, \cdots, b_n\}$ and M/N is generated by $\{a_1 + N, \cdots, a_n + N\}$. Let $m \in M$, then there exists $r_1, ..., r_n \in R$ such that

$$m + N = r_1(a_1 + N) + \cdots + r_n(a_n + N)$$

$$= r_1a_1 + \cdots + r_na_n + N \implies m - (r_1a_1 + \cdots + r_na_n) \in N$$

Therefore, there exist $r_{n+1}, \cdots, r_{n+m} \in R$ such that

$$m - (r_1a_1 + \cdots + r_na_n) = r_{n+1}b_1 + \cdots + r_{n+m}b_m$$

$$\implies m = r_1a_1 + \cdots + r_na_n + r_{n+1}b_1 + \cdots + r_{n+m}b_m$$

As a result, m is generated by $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$. Hence M is finitely generated.

1.3. 8. Suppose that S is finitely generated as a module over itself, and let $A = \{(a_{i,j})\}_{i=1}^n$ where $S = \langle\{(a_{i,j})\}_{i=1}^n\rangle$. There exists a nonnegative interger M, s.t. $a_{i,m} = 0$ for $\forall m \geq M$. Let

$$e = \{e_1, e_2, ...\}$$

$$e_j = \begin{cases} 1 & j = M \\ 0 & j \neq M \end{cases}$$

Then $e \in S$. Moreover, there exists $\{(r_{i,j})\}_{i=1}^n \in S$ such that

$$e = \sum_{i=1}^{n} r_{i,j}a_{i,j} \implies 1 = \sum_{i=1}^{n} r_{i,M}a_{i,M} = \sum_{i=1}^{n} r_{i,M} \cdot 0 = 0$$

A contradiction $1 \neq 0$. Hence, S is not finitely generated as a module over itself.

1.4. 27*.

1.5. 11. If V is vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, prove that $U \cap W \neq \{0\}$. Generalize.
proof: Suppose V is vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, assume that $U \cap W = \{0\}$, i.e. with dimension 0. Let $\{u_1, u_2, u_3\}$ span U and $\{w_1, w_2, w_3\}$ span W.
Assume

$$a_1u_1 + a_2u_2 + a_3u_3 + b_1w_1 + b_2w_2 + b_3w_3 = 0 \text{ for } a_i, b_i \in F$$

WTS $a_i = 0, b_i = 0$. Then

$$a_1u_1 + a_2u_2 + a_3u_3 = -(b_1w_1 + b_2w_2 + b_3w_3)$$

$$\implies a_1u_1 + a_2u_2 + a_3u_3 = -(b_1w_1 + b_2w_2 + b_3w_3) = 0$$

Since $U \cap W = \{0\}$. Moreover, u_1, u_2, u_3 are nonzero and linearly independent, w_1, w_2, w_3 are nonzero and linearly independent. i.e. $a_1 = a_2 = a_3 = 0$ and $b_1 = b_2 = b_3 = 0$ implies $a_i = 0, b_i = 0$ i.e. $u_1, u_2, u_3, w_1, w_2, w_3$ are nonzero and linearly independent. As a result, $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ span $U \cup W$ and linearly independent set extends to a basis implies basis has size more than or equal to 6. Moreover, $U \cup W \subseteq V$ since both U, V are subspace of V which basis of V has size 5. A contradiction since every bsis has the same size in V. Hence, $U \cap W \neq \{0\}$. The generalization is if V is a vector space over F where U, W are subspaces of V, if $\dim(U) + \dim(W) > \dim(V)$, then $U \cap W \neq \{0\}$.