Math 470 Assignment 21

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8.1.9. Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers which satisfy

$$\sum_{k=1}^{\infty} a_k^2 < \infty \text{ and } \sum_{k=1}^{\infty} b_k^2 < \infty$$

Prove that the infinite series $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

proof: Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^k$, and $a_1 = x_1, ...a_k = x_k$, $b_1 = y_1, ...b_k = y_k$ as $k \to \infty$. Then by Cauchy-Schwarz Inequality,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \| \mathbf{x} \| \| \mathbf{y} \| \Rightarrow |\sum_{i=1}^{k} a_i b_i| = \sum_{i=1}^{k} |a_i b_i| \leq |\sum_{i=1}^{k} a_i^2|^{1/2} |\sum_{i=1}^{k} b_i^2|^{1/2}$$

as $k \to \infty$. Moreover $|a_i|^2 = a_i^2$, then

$$|\sum_{i=1}^{k} a_i^2|^{1/2} |\sum_{i=1}^{k} b_i^2|^{1/2} = \left(\sum_{i=1}^{k} |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{k} |b_i|^2\right)^{1/2} < \infty$$

as $k \to \infty$. Hence $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

1. Suppose $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$, prove that

$$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \| \mathbf{x} \| \leq \| \mathbf{x} + t\mathbf{y} \|, \forall t \in \mathbf{R}$$

proof: ($\Rightarrow$) Suppose $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$ and $\mathbf{x} \cdot \mathbf{y} = 0$. Then

$$\| \mathbf{x} + t\mathbf{y} \|^2 = | \mathbf{x} + t\mathbf{y} \| \mathbf{x} + t\mathbf{y} | = \| \mathbf{x} \|^2 + 2t\mathbf{x} \cdot \mathbf{y} + t^2 \| \mathbf{y} \|^2$$

Since $t^2 \geq 0$ for all $t \in \mathbf{R}$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then

$$\| \mathbf{x} \|^2 \leq \| \mathbf{x} \|^2 + t^2 \| \mathbf{y} \|^2 = \| \mathbf{x} \|^2 + 2t\mathbf{x} \cdot \mathbf{y} + t^2 \| \mathbf{y} \|^2 = \| \mathbf{x} + t\mathbf{y} \|^2$$
$$\Rightarrow \| \mathbf{x} \| \leq \| \mathbf{x} + t\mathbf{y} \|, \forall t \in \mathbf{R}$$

($\Leftarrow$) Suppose $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$ and $\| \mathbf{x} \| \leq \| \mathbf{x} + t\mathbf{y} \|, \forall t \in \mathbf{R}$, then

$$\| \mathbf{x} \|^2 \leq \| \mathbf{x} + t\mathbf{y} \|^2 = \| \mathbf{x} \|^2 + 2t\mathbf{x} \cdot \mathbf{y} + t^2 \| \mathbf{y} \|^2$$
$$\Rightarrow -2t\mathbf{x} \cdot \mathbf{y} \leq t^2 \| \mathbf{y} \|^2, \forall t \in \mathbf{R}$$

Since $t^2 \geq 0$, $\mathbf{x} \cdot \mathbf{y} \geq 0$, $\| \mathbf{y} \|^2 \geq 0$, then $-2t\mathbf{x} \cdot \mathbf{y} \leq t^2 \| \mathbf{y} \|^2, \forall t \in \mathbf{R}$ only holds when $\mathbf{x} \cdot \mathbf{y} = 0$. Hence

$$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \| \mathbf{x} \| \leq \| \mathbf{x} + t\mathbf{y} \|, \forall t \in \mathbf{R}$$

for $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$.

2. Suppose $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$, $\| \mathbf{x} \| < 1, \| \mathbf{y} \| < 1$, prove that

$$\sqrt{1 - \| \mathbf{x} \|^2}\sqrt{1 - \| \mathbf{y} \|^2} \leq 1 - | \mathbf{x} \cdot \mathbf{y} |$$

proof: By Cauchy-Schwarz Inequality,

$$| \mathbf{x} \cdot \mathbf{y} | \leq \| \mathbf{x} \| \| \mathbf{y} \| \Rightarrow - | \mathbf{x} \cdot \mathbf{y} | \geq - \| \mathbf{x} \| \| \mathbf{y} \|$$

Since $\| \mathbf{x} \| < 1, \| \mathbf{y} \| < 1$, then

$$\sqrt{1 - \| \mathbf{x} \|^2}\sqrt{1 - \| \mathbf{y} \|^2} \geq 0$$
$$1 - | \mathbf{x} \cdot \mathbf{y} | \geq 0$$

Thus to show

$$\sqrt{1 - \| \mathbf{x} \|^2}\sqrt{1 - \| \mathbf{y} \|^2} \leq 1 - | \mathbf{x} \cdot \mathbf{y} |$$

is the same as to show

$$(1 - \| \mathbf{x} \|^2)(1 - \| \mathbf{y} \|^2) \leq (1 - | \mathbf{x} \cdot \mathbf{y} |)^2$$

Thus

$$(1 - \| \mathbf{x} \|^2)(1 - \| \mathbf{y} \|^2) = 1 - \| \mathbf{x} \|^2 - \| \mathbf{y} \|^2 + \| \mathbf{x} \|^2 \| \mathbf{y} \|^2$$
$$= - \| \mathbf{x} \|^2 + 2 \| \mathbf{x} \| \| \mathbf{y} \| - \| \mathbf{y} \|^2 + 1 - 2 \| \mathbf{x} \| \| \mathbf{y} \| + \| \mathbf{x} \|^2 \| \mathbf{y} \|^2$$
$$= (1 - \| \mathbf{x} \| \| \mathbf{y} \|)^2 - (\| \mathbf{x} \| - \| \mathbf{y} \|)^2 \leq (1 - \| \mathbf{x} \| \| \mathbf{y} \|)^2$$
$$\leq (1 - | \mathbf{x} \cdot \mathbf{y} |)^2$$

by Cauchy-Schwarz Inequality. Hence

$$\sqrt{1 - \| \mathbf{x} \|^2}\sqrt{1 - \| \mathbf{y} \|^2} \leq 1 - | \mathbf{x} \cdot \mathbf{y} |$$