Math 741 Assignment 20 (Hand-In)

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10.2.8(H) solution: Given

$$(a+b+c)^n = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{n!}{i!j!(n-i-j)!}a^i b^j c^{n-i-j}$$

$$M_{X_1,X_2,X_3}(t_1,t_2,t_3) = E(e^{t_1X_1+t_2X_2+t_3X_3})$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=n-k_1-k_2}^{n-k_1-k_2} e^{t_1k_1+t_2k_2+t_3k_3}\frac{n!}{k_1!k_2!k_3!}p_1^{k_1}p_2^{k_2}p_3^{k_3}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} e^{t_1k_1+t_2k_2+t_3(n-k_1-k_2)}\frac{n!}{k_1!k_2!(n-k_1-k_2)!}p_1^{k_1}p_2^{k_2}p_3^{n-k_1-k_2}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} e^{t_1k_1+t_2k_2+t_3(n-k_1-k_2)}\frac{n!}{k_1!k_2!(n-k_1-k_2)!}(p_1e^{t_1})^{k_1}(p_2e^{t_2})^{k_2}(p_3e^{t_3})^{n-k_1-k_2}$$

$$= (p_1e^{t_1}+p_2e^{t_2}+p_3e^{t_3})^n$$

10.2.10(H) solution: Consider a positive integer n, and a set of positive real numbers $\mathbf{P} = \{p_1, ..., p_t\}$ such that

$$\sum_{i=1}^{t} p_i = 1, \sum_{i=1}^{t} k_i = n$$

The joint likelihood is

$$L(\mathbf{P}) = n! \cdot \prod_{i=1}^{t}\frac{p_i^{k_i}}{k_i!}$$

$$l(\mathbf{P}) = \log(L(\mathbf{P})) = \log n! + \sum_{i=1}^{t} \log(\frac{p_i^{k_i}}{k_i!})$$

$$\log(L(\mathbf{P})) = \log n! + \sum_{i=1}^{t} k_i \log p_i - \sum_{i=1}^{t} k_i$$

Using auxiliary function of Lagrange Multiplier, (It is similiar as to maximize Shannon Entropy)

$$L(\mathbf{P}, \lambda) = l(\mathbf{P}) + \lambda(1 - \sum_{i=1}^{t} p_i)$$

$$\frac{\partial}{\partial p_i}(L(\mathbf{P}, \lambda)) = \frac{\partial}{\partial p_i}(l(\mathbf{P})) + \frac{\partial}{\partial p_i}[\lambda(1 - \sum_{i=1}^{t} p_i)] = 0$$

$$\implies \frac{\partial}{\partial p_i} \sum_{i=1}^{t} k_i \log p_i - \lambda \frac{\partial}{\partial p_i} \sum_{i=1}^{t} p_i = 0$$

$$\implies \frac{k_i}{\hat{p}_i} = \lambda \implies \hat{p}_i = \frac{k_i}{\lambda}$$

$$\sum_{i=1}^{t} p_i = \sum_{i=1}^{t} \frac{k_i}{\lambda} \implies 1 = \frac{1}{\lambda} \sum_{i=1}^{t} k_i \implies \lambda = n$$

Hence,

$$\hat{p}_i = \frac{k_i}{n}, i = 1, ..., t$$