Math 741 Assignment 4 (Hand-In)

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5.4.10

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

Then,

$$E(Y^2) = \int_0^\theta y^2 \frac{1}{\theta} dy = \frac{y^3}{3\theta}\Big|_0^\theta = \frac{\theta^2}{3}$$

Therefore, Y^2 is a biased estimator for θ^2. As a result,

$$E(Y^2) = \frac{\theta^2}{3} \Rightarrow 3E(Y^2) = \theta^2 \Rightarrow E(3Y^2) = \theta^2$$

Hence, $3Y^2$ is an unbiased estimator for θ^2.

5.4.13

solution: Y has a uniform distribution in the interval $[0, \theta]$, then the pdf is

$$f_Y(y;\theta) = \begin{cases} \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

Thus

$$F_Y(y;\theta) = \int_0^y \frac{1}{\theta} dt = \frac{y}{\theta}$$

$$\Longrightarrow F_Y(y;\theta) = \begin{cases} 0 & y < 0 \\ \frac{y}{\theta} & 0 \le y \le \theta \\ 1 & y > \theta \end{cases}$$

$$f_{Y_{\max}}(y) = n[F_Y(y)]^{n-1} f_Y(y) = n\left[\frac{y}{\theta}\right]^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$$

$$\implies f_{Y_{\max}}(y) = \begin{cases} \frac{ny^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & o.w. \end{cases}$$

Since $\hat{\theta} = \frac{n+1}{n} Y_{\max}$, we need to do transofrming

$$f_{\hat{\theta}}(y) = \frac{1}{|\frac{n+1}{n}|} f_{Y_{\max}}\left(\frac{y}{\frac{n+1}{n}}\right) = \frac{n^{n+1}y^{n-1}}{(n+1)^n\theta^n}$$

Let α be the median of the estimator's distribution $f_{\hat{\theta}}$, then

$$\int_0^{\alpha} f_{\hat{\theta}}(y)dy = 0.5$$

Therefore,

$$\int_0^{\alpha} \frac{n^{n+1}y^{n-1}}{(n+1)^n\theta^n}dy = \frac{n^n y^n}{(n+1)^n\theta^n}\Big|_0^{\alpha} = \frac{n^n\alpha^n}{(n+1)^n\theta^n} = 0.5$$

$$\implies \alpha = \sqrt[n]{\frac{(n+1)^n\theta^n}{2n^n}} = \frac{(n+1)}{n\sqrt[n]{2}}\theta$$

If $\alpha = \theta$, then it is unbiased. Since for any arbitrary n, $\alpha \neq \theta$. As a result, $\hat{\theta}$ is not median unbiased.
Let $\alpha = \theta$, then we can slove for n

$$\frac{(n+1)}{n\sqrt[n]{2}} = 1 \implies n+1 = n\sqrt[n]{2} \implies n = 1$$

Hence, it is median unbiased only for $n = 1$.

5.4.16

solution: Let $X_1, ...X_n \sim N(\mu, \sigma^2)$ be iid samples. Then

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$E(\hat{\sigma}^2) = E(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2) = \frac{1}{n}(\sum_{i=1}^{n} E(X_i^2)) - E(\bar{X}^2)$$

The distribution is normal, then

$$E(X_i) = E(X) = \mu, Var(X_i) = Var(X) = \sigma^2$$

$$E(\bar{X}) = E(\frac{X_1 + ... + X_n}{n}) = \mu, \mathrm{Var}(\bar{X}) = \mathrm{Var}(\frac{X_1 + ... + X_n}{n}) = \frac{\sigma^2}{n}$$

$$E(\bar{X}^2) = \mathrm{Var}(\bar{X}) + (E(\bar{X}))^2 = \frac{\sigma^2}{n} + \mu^2$$

$$E(X_i^2) = \mathrm{Var}(X_i) + (E(X_i))^2 = \sigma^2 + \mu^2$$

Therefore,

$$E(\hat{\sigma}^2) = \frac{1}{n}\Big[n \cdot (\sigma^2 + \mu^2)\Big] - (\frac{\sigma^2}{n} + \mu^2) = \frac{n-1}{n}\sigma^2$$

$$\lim_{n\to\infty} E(\hat{\sigma}^2) = \lim_{n\to\infty} \frac{n-1}{n}\sigma^2 = \sigma^2$$

Hence, the maximum likelihood estimator for σ^2 in a normal pdf is asymptotically unbiased.

5.4.22

solution: Given

$$E(W_1) = \mu, \mathrm{Var}(W_1) = \sigma_1^2$$

$$E(W_2) = \mu, \mathrm{Var}(W_2) = \sigma_2^2$$

By example 5.4.3, we can conclude

$$E(cW_1 + (1-c)W_2) = \mu$$

Assume W_1, W_2 are independent, then $\mathrm{Cov}(W_1, W_2) = 0$

$$\mathrm{Var}(cW_1 + (1-c)W_2) = c^2\mathrm{Var}(W_1) + (1-c)^2\mathrm{Var}(W_2) + 2c(1-c)\mathrm{Cov}(W_1, W_2)$$

$$= c^2\sigma_1^2 + (1-c)^2\sigma_2^2$$

We want variance to be as small as possible, then let

$$\frac{d}{dc}(\mathrm{Var}(cW_1 + (1-c)W_2)) = 0 \implies \frac{d}{dc}(c^2\sigma_1^2 + (1-c)^2\sigma_2^2) = 0$$

$$\implies 2c\sigma_1^2 + (2c-2)\sigma_2^2 = 0 \implies c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\frac{d^2}{dc^2}(\mathrm{Var}(cW_1 + (1-c)W_2)) = 2\sigma_1^2 + 2\sigma_2^2 > 0$$

since $\sigma_1 \neq 0, \sigma_2 \neq 0$. Therefore, the c we sloved is minimum.

Hence, for

$$c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

the estimator $cW_1 + (1-c)W_2$ is most efficient.