MATH 435 ASSIGNMENT 14

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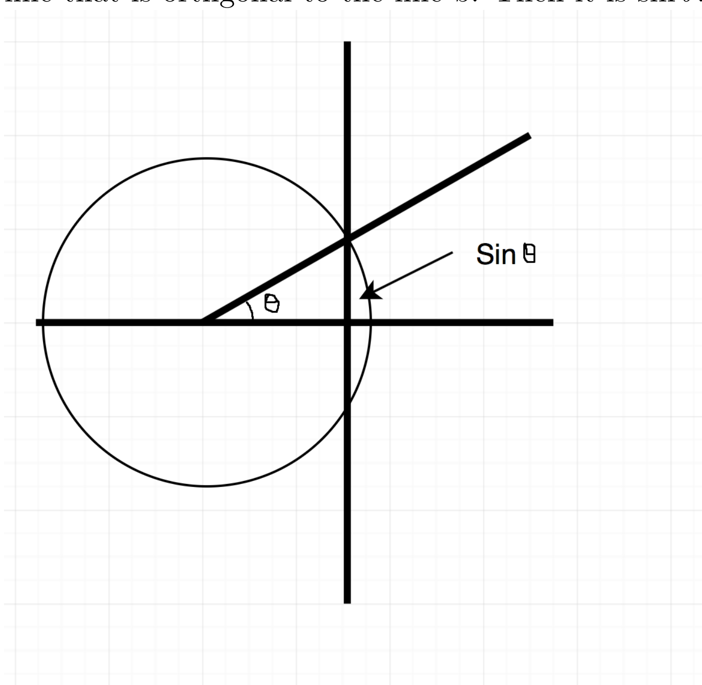
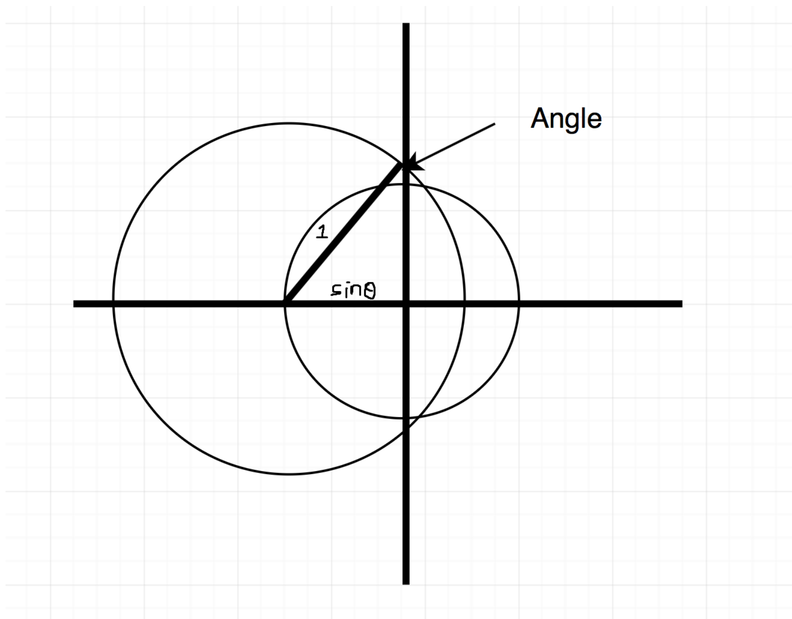
1. Geometric Construction

1.1. 6. Prove that angle θ is constructible if and only if $\sin\theta$ is constructible.
proof: ($\Rightarrow$) Suppose we have angle θ, then we can two line with the intersection angle θ. Call one side a, and other side b. Use the intersection as the origin and set compass with to unit length. Draw the circle, then we have additional two intersections, one on line a and one on line b. Call the intersection on line a, point x. Use x to draw an line that is orthgonal to the line b. Then it is $\sin\theta$.

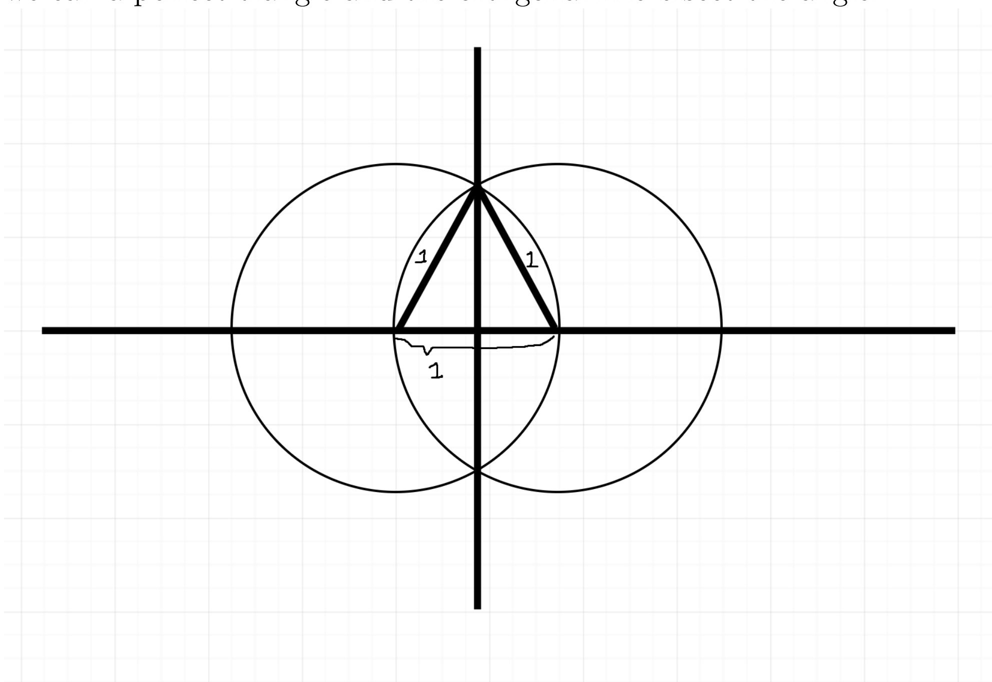


($\Leftarrow$) Suppose we have the length of $\sin\theta$. Draw a straight line a by using a straightedge. Set compass to length $\sin\theta$ and mark two intersections x, y on the line a. Use point y to draw a orthgonal line b to line a. Next, set compass to unit length and use point x as origin and draw a circle. The intersect point on line b denoted z. Then we formed a Right Triangle xyz. xy $= \sin\theta$, yz $= 1$. Therefore, the angle θ is between xy and yz.



1.2. 8. Prove that angle 30° is constructible.
proof: Use stragithedge to draw a straight line. Set the compass to unit length. Pick a point on the straing line and draw a another point unit length away by using compass. Draw two circles with radius unit length and origin of these two points. This will have two intersections on each side of the stragith line. Connect these two points then we can have a line orthgonal to the straight line. Next, connect two points on the straight line to the same point which created by two circles. Then

we can a perfect trangle and the orthgonal line bisect the angle.



1.3. 4. Given that the automorphism group of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$ is isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$, determine the number of subfields of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$ that have degree 4 over Q.
proof: Assume K is a subfield of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$ and $[K : Q] = 4$. Also, $\text{Gal}(Q(\sqrt{2}, \sqrt{5}, \sqrt{7})/Q) \cong Z_2 \oplus Z_2 \oplus Z_2$. i.e. $|\ \text{Gal}(Q(\sqrt{2}, \sqrt{5}, \sqrt{7})/Q)|$ $= 8 = [Q(\sqrt{2}, \sqrt{5}, \sqrt{7}) : Q]$. By Fundamental Theorem of Galois Theory, we only need to decide the subgroups of $Z_2 \oplus Z_2 \oplus Z_2$ with index 4. Assume $H < Z_2 \oplus Z_2 \oplus Z_2$, then $[_2 \oplus Z_2 \oplus Z_2 : H] = 4$. i.e. $|H| = 2$. We only need to consider elements of $Z_2 \oplus Z_2 \oplus Z_2$ with order 2. There are 7 of them expect $(0, 0, 0)$. Hence there are 7 subgroups of $Z_2 \oplus Z_2 \oplus Z_2$ with index 4. i.e. There are 7 subfields of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$.

1.4. 18. Determine the Galois group of $x^3 - 1$ over Q.
proof: By insepction, $x^3 - 1 = (x - 1)(x^2 + x + 1)$. Then the splitting field of $x^3 - 1$ is the splitting field of $x^2 + x + 1$ over Q. Use quardtic formula, we can solve the roots are $\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. Now we look at the field $Q(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2})$. Both $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ is in the form of $a + b(i\sqrt{3})$ where $a, b \in Q$. i.e. $Q(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}) = Q(i\sqrt{3})$. Since $(i\sqrt{3})^2 = -3 \in Q$, then $[Q(i\sqrt{3}) : Q] = 2$. By Fundamental Theorem

of Galois Theory, $|\text{Gal}(Q(i\sqrt{3})/Q)| = 2$. The group with only 2 elements is Z_2.

1.5. 18. Determine the Galois group of $x^3 - 2$ over Q.
proof: The roots of $x^3 - 2$ are $\{\sqrt[3]{2}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}\}$. Then the splitting field is $Q(\sqrt[3]{2}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}})$. Moreover, $\sqrt[3]{2} \cdot e^{i\frac{2\pi}{3}} = \sqrt[3]{2}e^{i\frac{2\pi}{3}}$ and $(e^{i\frac{2\pi}{3}})^2 = e^{i\frac{4\pi}{3}}$. i.e. $Q(\sqrt[3]{2}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}) = Q(\sqrt[3]{2}, e^{i\frac{2\pi}{3}})$. Furthermore, $e^{i\frac{2\pi}{3}} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. i.e. $Q(\sqrt[3]{2}, e^{i\frac{2\pi}{3}}) = Q(\sqrt[3]{2}, i\sqrt{3})$.
$2^{\frac{1}{3}}$ must map to an element with order 3 which have 3 choices. $i\sqrt{3}$ must map to an element with order 2 which have 2 choices. By Fundamental Theorem of Galois Theory, $|\text{Gal}(Q(\sqrt[3]{2}, i\sqrt{3})/Q)| = 6$. The group has 6 elements. There are only two groups Z_6 and S_3 with 6 elements. Moreover, the group of all automorphisms are not commutative. i.e. The Galois group must be S_3.

1.6. 24*.