高等量子力学第十二次作业

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1. Prove that the d'Alembert operator $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is invariant under the Lorentz transform.

   Proof. d'Alembert 可以写作:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \partial_\mu \partial^\mu = g_{\mu\nu}\partial^\mu \partial^\nu.$$

   经过洛伦兹变换后，有:

$$\Box' = L^\mu_\lambda g_{\mu\nu} L^\nu_\rho L^\lambda_\mu \partial^\mu L^\rho_\nu \partial^\nu = g_{\lambda\rho}\partial^\lambda \partial^\rho = \partial_\lambda \partial^\lambda = \Box.$$

   即 d'Alembert 算符保持 Lorentz 不变。

   $\Box$

2. Starting from the Dirac equation, verify the equation is consistent with KG equation. We can get KG by multiplying the Dirac equation

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \sum_{k=1}^{3} \alpha^k \frac{\partial \psi}{\partial x_k} + i\frac{mc}{\hbar}\beta\psi = 0.$$

   by the operator

$$\frac{1}{c}\frac{\partial}{\partial t} - \sum_{l=1}^{3} \alpha^l \frac{\partial}{\partial x_l} - i\frac{mc}{\hbar}\beta.$$

   Proof. 可以计算:

$$\begin{aligned}
&\left(\frac{1}{c}\frac{\partial}{\partial t} - \sum_{l=1}^{3} \alpha^l \frac{\partial}{\partial x_l} - i\frac{mc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x_k} + i\frac{mc}{\hbar}\beta\right) \\
=&\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x_k} + i\frac{mc}{\hbar}\beta\right)^2 \\
&+ \frac{1}{c}\frac{\partial}{\partial t}\left(\sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x_k} + i\frac{mc}{\hbar}\beta\right) - \left(\sum_{l=1}^{3} \alpha^l \frac{\partial}{\partial x_l} + i\frac{mc}{\hbar}\beta\right)\frac{1}{c}\frac{\partial}{\partial t}.
\end{aligned}$$

   其中，由于

$$\left[\frac{\partial}{\partial t}, \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x_k} + i\frac{mc}{\hbar}\beta\right] = 0.$$

   后两项和等于 0。

   利用

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = \mathbf{I}.$$

可以进一步计算得到:

$$
\begin{aligned}
&\left(\frac{1}{c}\frac{\partial}{\partial t}-\sum_{l=1}^{3}\alpha^l\frac{\partial}{\partial x_l}-i\frac{mc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t}+\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial x_k}+i\frac{mc}{\hbar}\beta\right)\\
=&\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\left(\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial x_k}+i\frac{mc}{\hbar}\beta\right)^2\\
=&\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{m^2c^2}{\hbar^2}.
\end{aligned}
$$

即有:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{m^2c^2}{\hbar^2}\right)\psi=0.$$

即得到了 Klein-Gordon 方程。

□

3. Prove $[H,\boldsymbol{\Sigma}\cdot\boldsymbol{p}]=0$.

Proof. 哈密顿量为:

$$H=-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+mc^2\beta.$$

则可以计算:

$$[H,\boldsymbol{\Sigma}\cdot\boldsymbol{p}]=-i\hbar c[\boldsymbol{\alpha}\cdot\boldsymbol{\nabla},\boldsymbol{\Sigma}\cdot\boldsymbol{p}]=-i\hbar c\begin{pmatrix}0 & [\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},\boldsymbol{\sigma}\cdot\boldsymbol{p}]\\ [\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},\boldsymbol{\sigma}\cdot\boldsymbol{p}] & 0\end{pmatrix}$$

□

显然有:

$$[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},\boldsymbol{\sigma}\cdot\boldsymbol{p}]=-\frac{1}{i\hbar}[\boldsymbol{\sigma}\cdot\boldsymbol{p},\boldsymbol{\sigma}\cdot\boldsymbol{p}]=0.$$

则有:

$$[H,\boldsymbol{\Sigma}\cdot\boldsymbol{p}]=0.$$