高等量子力学第一周作业

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习题 1.1

证明下述内积的定义满足内积定义的基本要求。其中注意到现在是定义在复数域上的矢量空间，关于矢量空间的定义需要适当修改：$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

内积定义为：

$$\langle\phi|\psi\rangle \equiv (\phi,\psi) = \int d\vec{r}\phi^*(\vec{r})\psi(\vec{r})$$

证明：

$$\begin{aligned}
\langle\phi|\psi\rangle &= \int d\vec{r}\phi^*(\vec{r})\psi(\vec{r}) = \left(\int d\vec{r}\psi^*(\vec{r})\phi(\vec{r})\right)^* = \langle\psi|\phi\rangle^*;\\
\langle\phi|\psi_1+\psi_2\rangle &= \int d\vec{r}\phi^*(\vec{r})(\psi_1(\vec{r})+\psi_2(\vec{r})) = \int d\vec{r}\phi^*(\vec{r})\psi_1(\vec{r}) + \int d\vec{r}\phi^*(\vec{r})\psi_2(\vec{r})\\
&= \langle\phi|\psi_1\rangle + \langle\phi|\psi_2\rangle;\\
\langle\phi|\lambda\psi\rangle &= \int d\vec{r}\phi^*(\vec{r})\lambda\psi(\vec{r}) = \lambda\int d\vec{r}\phi^*(\vec{r})\psi(\vec{r}) = \lambda\langle\phi|\psi\rangle;\\
\langle\phi|\phi\rangle &= \int d\vec{r}\phi^*(\vec{r})\phi(\vec{r}) = \int d\vec{r}|\phi(\vec{r})|^2 \geq 0;\\
\langle\phi|\phi\rangle &= 0\text{当且仅当}\phi(\vec{r}) = 0.
\end{aligned}$$

综上所述，上述内积定义满足内积定义的基本要求。

习题 1.2

证明 Schwartz 不等式：对于任意量子态 $|\phi\rangle$ 和 $|\psi\rangle$，

$$\langle\phi|\psi\rangle\langle\psi|\phi\rangle \leq \langle\phi|\phi\rangle\langle\psi|\psi\rangle$$

由于任意量子态模平方大于等于 0，则有：

$$(\langle\phi| + \lambda^*\langle\psi|)(|\phi\rangle + \lambda|\psi\rangle) \geq 0.$$

其中 λ 可以为任意复数。不妨另

$$\lambda = -\frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}.$$

则上述恒成立不等式为：

$$\langle\phi|\phi\rangle - \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\phi|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle} + \frac{\langle\phi|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle} \geq 0.$$

则有:

$$\langle\phi|\,\psi\rangle\,\langle\psi|\,\phi\rangle \le \langle\phi|\,\phi\rangle\,\langle\psi|\,\psi\rangle.$$

习题 1.3

对于下面的三个量子态

$$|\phi_1\rangle = |1\rangle + |2\rangle$$
$$|\phi_2\rangle = |2\rangle + |3\rangle$$
$$|\phi_3\rangle = |1\rangle + |3\rangle$$

其中 $\langle i|j\rangle = \delta_{ij}$ $(i, j \in 1, 2, 3)$。

问题 1: 判断 $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ 是否线型独立;

问题 2: 用施密特正交化的方法构造三个正交归一基矢。

1 选取 $|1\rangle, |2\rangle, |3\rangle$ 作为基矢，三个量子态在该基矢表象下的列向量构成一个矩阵为:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

可以计算矩阵 A 的行列式为 $det(A) = 2 \neq 0$, 即三个列向量线性无关，即 $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ 线型独立。

2

$$\begin{aligned}
|\psi_1\rangle =& \frac{|\phi_1\rangle}{\sqrt{\langle\phi_1|\phi_1\rangle}} = \frac{1}{\sqrt{2}}\,|1\rangle + \frac{1}{\sqrt{2}}\,|2\rangle\,; \\
|\phi_2'\rangle =& |\phi_2\rangle - |\phi_1\rangle\,\langle\phi_1|\phi_2\rangle = -\frac{1}{2}\,|1\rangle + \frac{1}{2}\,|2\rangle + |3\rangle\,, \\
|\psi_2\rangle =& \frac{|\phi_2'\rangle}{\sqrt{\langle\phi_2'|\phi_2'\rangle}} = -\frac{1}{\sqrt{6}}\,|1\rangle + \frac{1}{\sqrt{6}}\,|2\rangle + \frac{2}{\sqrt{6}}\,|3\rangle\,; \\
|\phi_3'\rangle =& |\phi_3\rangle - |\psi_1\rangle\,\langle\psi_1|\,\phi_3\rangle - |\psi_2\rangle\,\langle\psi_2|\,\phi_3\rangle = \frac{2}{3}\,|1\rangle - \frac{2}{3}\,|2\rangle + \frac{2}{3}\,|3\rangle\,, \\
|\psi_3\rangle =& \frac{|\phi_3'\rangle}{\sqrt{\langle\phi_3'|\,\phi_3'\rangle}} = \frac{1}{\sqrt{3}}\,|1\rangle - \frac{1}{\sqrt{3}}\,|2\rangle + \frac{1}{\sqrt{3}}\,|3\rangle\,.
\end{aligned}$$

即三个正交归一基矢分别为:

$$\begin{aligned}
|\psi_1\rangle =& \frac{1}{\sqrt{2}}\,|1\rangle + \frac{1}{\sqrt{2}}\,|2\rangle\,; \\
|\psi_2\rangle =& -\frac{1}{\sqrt{6}}\,|1\rangle + \frac{1}{\sqrt{6}}\,|2\rangle + \frac{2}{\sqrt{6}}\,|3\rangle\,; \\
|\psi_3\rangle =& \frac{1}{\sqrt{3}}\,|1\rangle - \frac{1}{\sqrt{3}}\,|2\rangle + \frac{1}{\sqrt{3}}\,|3\rangle\,.
\end{aligned}$$

习题 1.4

证明厄米算符不同本征值对应的本征态正交。

考虑厄米算符 $\hat{A}$ 的本征值方程:

$$\hat{A}\left|\phi_1\right\rangle = a_1\left|\phi_1\right\rangle;\ \hat{A}\left|\phi_2\right\rangle = a_2\left|\phi_2\right\rangle.$$

其中 $a_1 \neq a_2$，且 a_1, a_2 为实数。
则有:

$$\left\langle\phi_2\right|\hat{A}\left|\phi_1\right\rangle = a_1\left\langle\phi_2\middle|\phi_1\right\rangle = a_2\left\langle\phi_2\middle|\phi_1\right\rangle.$$

即

$$(a_1 - a_2)\left\langle\phi_2\middle|\phi_1\right\rangle = 0.$$

由于 $a_1 \neq a_2$，则有 $\left\langle\phi_2\middle|\phi_1\right\rangle = 0$，即厄米算符不同本征值对应的本征态正交。

习题 1.5

证明如果 P 和 Q 是投影算符，且 $PQ = 0$，那么 $P + Q$ 也是投影算符。

因为 P 和 Q，是投影算符，则有:

$$P^\dagger = P, P^2 = P;\ Q_\dagger = Q, Q^2 = Q.$$

则有:

$$(P+Q)^\dagger = P^\dagger + Q^\dagger = P + Q;\ (P+Q)^2 = P^2 + Q^2 + PQ + QP = P + Q.$$

其中应有 $PQ = QP = 0$。

习题 1.6

证明：$p_i \geq 0$ 并且 $\sum_i p_i = 1$。

$$p_i = \left\langle\psi\right|\hat{P}_i\left|\psi\right\rangle = \left\langle\psi\middle|a_i\right\rangle\left\langle a_i\middle|\psi\right\rangle = \left|\left\langle a_i\middle|\psi\right\rangle\right|^2 \geq 0,$$

$$\sum_i p_i = \left\langle\psi\right|\sum_i \hat{P}_i\left|\psi\right\rangle = \left\langle\psi\middle|\psi\right\rangle = 1.$$

习题 1.7

证明：如果两个量子态相差一个整体相位，即 $\left|\psi\right\rangle = e^{i\theta}\left|\phi\right\rangle$，那么当系统分别处于这两个量子态时，任意可观测量都给出完全一样的测量结果。

考虑任意可观测量对应的厄米算符 $\hat{A}$，其测量结果为:

$$\left\langle\psi\right|\hat{A}\left|\psi\right\rangle = \left\langle\phi\right|e^{-i\theta}\hat{A}e^{i\theta}\left|\phi\right\rangle = \left\langle\phi\right|\hat{A}\left|\phi\right\rangle.$$

即当两个量子态相差一个整体相位时，任意可观测量都给出完全一样的测量结果。

习题 1.8

证明方程:

$$e^{ix'\hat{p}/\hbar}\hat{x}e^{-ix'\hat{p}/\hbar} = \hat{x} + x'.$$

为了方便计算，令 $\hat{A} = x'\hat{p}/\hbar$。则有:

$$e^{i\hat{A}} = 1 + i\hat{A} + \frac{1}{2!}(i\hat{A})^2 + \cdots,$$
$$e^{-i\hat{A}} = 1 - i\hat{A} + \frac{1}{2!}(-i\hat{A})^2 + \cdots,$$

则有

$$\begin{aligned}
e^{i\hat{A}}\hat{x}e^{-i\hat{A}} &= \left(1 + i\hat{A} + \frac{1}{2!}(i\hat{A})^2 + \cdots\right)\hat{x}\left(1 - i\hat{A} + \frac{1}{2!}(-i\hat{A})^2 + \cdots\right) \\
&= \left(\hat{x} + i\hat{A}\hat{x} + \frac{1}{2!}(i\hat{A})^2\hat{x} + \cdots\right)\left(1 - i\hat{A} + \frac{1}{2!}(-i\hat{A})^2 + \cdots\right) \\
&= \hat{x} - i[\hat{x}, \hat{A}] - \frac{1}{2!}[\hat{x}, \hat{A}^2] + \cdots
\end{aligned}$$

带入 $\hat{A} = x'\hat{p}/\hbar$，可以得到

$$[\hat{x}, \hat{A}^n] = \begin{cases} ix', & n = 1; \\ 0, & n \geq 2. \end{cases}$$

则有:

$$e^{ix'\hat{p}/\hbar}\hat{x}e^{-ix'\hat{p}/\hbar} = \hat{x} + x'$$