HOMEWORK OF AQM

JIANYU DONG 202328000807038

1. Calculate all the C-G coefficients for the case with $j_1 = 1$ and $j_2 = 1/2$.

Proof. Utilizing the properties of the C-G coefficients, most of the coefficients are 0. For the case with $j_1 = 1$ and $j_2 = 1/2$, we could make the list as below.

		$\frac{3}{2}$ $+\frac{3}{2}$	$\frac{3}{2}$ $+\frac{1}{2}$	$\frac{1}{2}$ $+\frac{1}{2}$	$\frac{3}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{3}{2}$ $-\frac{3}{2}$
$+1$	$+\frac{1}{2}$		0	0	0	0	0
$+1$	$-\frac{1}{2}$	0			0	0	0
0	$+\frac{1}{2}$	0			0	0	0
0	$-\frac{1}{2}$	0	0	0			0
-1	$+\frac{1}{2}$	0	0	0			0
-1	$-\frac{1}{2}$	0	0	0	0	0	

Then, utilizing the normalization condition, which means every column of the table is normalized, we could easily get the first and the last element of the table are 1.

Next, we only need to calculate the elements in the middle block.

$$
\begin{aligned}
|j_1 j_2; jm\rangle &= \sum_{m_1 m_2} |j_1 m_1; j_2 m_2\rangle \langle j_1 m_1; j_2 m_2| j_1 j_2; jm\rangle \\
&= \sum_{m_1 m_2} C^{jm}_{j_1 m_1; j_2 m_2} |j_1 m_1; j_2 m_2\rangle . \\
\hat{J}_+ |j_1 j_2; jm\rangle &= \sqrt{(j-m)(j+m+1)}\, |j_1 j_2; j(m+1)\rangle \\
&= \sum_{m_1 m_2} C^{jm}_{j_1 m_1; j_2 m_2} (\hat{J}_{1+} + \hat{J}_{2+}) |j_1 m_1; j_2 m_2\rangle \\
&= \sum_{m_1 m_2} C^{jm}_{j_1 m_1; j_2 m_2} (\sqrt{(j_1 - m_1)(j_1 + m_1 + 1)}\, |j_1 (m_1+1); j_2 m_2\rangle \\
&\quad + \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)})\, |j_1 m_1; j_2 (m_2+1)\rangle .
\end{aligned}
$$

Then project the state to $\langle j_1 m_1'; j_2 m_2'|$, we get

$$
\begin{aligned}
&\sqrt{(j-m)(j+m+1)}\, C^{j(m+1)}_{j_1 m_1'; j_2 m_2'} \\
=&\sum_{m_1 m_2} C^{jm}_{j_1 m_1; j_2 m_2} (\sqrt{(j_1 - m_1)(j_1 + m_1 + 1)}\, \delta_{m_1', (m_1+1)} \delta_{m_2' m_2} \\
&+ \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)}\, \delta_{m_1' m_1} \delta_{m_2', (m_2+1)}) \\
=&C^{jm}_{j_1 (m_1'-1); j_2 m_2'} \sqrt{(j_1 - m_1' + 1)(j_1 + m_1')} + C^{jm}_{j_1 m_1'; j_2 (m_2'-1)} \sqrt{(j_2 - m_2' + 1)(j_2 + m_2')}.
\end{aligned}
$$

Then we could let $j=\frac{3}{2}$, $m=\frac{1}{2}$, $m_1'=1$, $m_2'=\frac{1}{2}$, there is

$$\sqrt{3}C^{\frac{3}{2}\frac{3}{2}}_{11;\frac{1}{2}\frac{1}{2}}=\sqrt{2}C^{\frac{3}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}+C^{\frac{3}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}.$$

Utilizing the normalization condition $|C^{\frac{3}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}|^2+|C^{\frac{3}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}|^2=1$, we get

$$C^{\frac{3}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}=\sqrt{\frac{1}{3}};\quad C^{\frac{3}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}=\sqrt{\frac{2}{3}}.$$

Let $j=\frac{3}{2}$, $m=-\frac{3}{2}$, $m_1'=0$, $m_2'=-\frac{1}{2}$ or $j=\frac{3}{2}$, $m=-\frac{3}{2}$, $m_1'=-1$, $m_2'=\frac{1}{2}$, we could get the next two equations

$$\sqrt{3}C^{\frac{3}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}}=\sqrt{2}C_{1-1;\frac{1}{2}-\frac{1}{2}};\quad \sqrt{3}C^{\frac{3}{2}-\frac{1}{2}}_{1-1;\frac{1}{2}\frac{1}{2}}=C^{\frac{3}{2}-\frac{3}{2}}_{1-1;\frac{1}{2}-\frac{1}{2}}.$$

Utilizing the condition $C^{\frac{3}{2}-\frac{3}{2}}_{1-1;\frac{1}{2}-\frac{1}{2}}=1$, we could determine that

$$C^{\frac{3}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}}=\sqrt{\frac{2}{3}};\quad C^{\frac{3}{2}-\frac{1}{2}}_{1-1;\frac{1}{2}\frac{1}{2}}=\sqrt{\frac{1}{3}}.$$

Let $j=\frac{1}{2}$, $m=-\frac{1}{2}$, $m_1'=1$, $m_2'=-\frac{1}{2}$ or $j=\frac{1}{2}$, $m=-\frac{1}{2}$, $m_1'=0$, $m_2'=\frac{1}{2}$, we could get the next two equations

$$C^{\frac{1}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}=\sqrt{2}C^{\frac{1}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}};\quad C^{\frac{1}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}=\sqrt{2}C^{\frac{1}{2}-\frac{1}{2}}_{1-1;\frac{1}{2}\frac{1}{2}}+C^{\frac{1}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}}.$$

Utilizing the normalization condition

$$|C^{\frac{1}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}|^2+|C^{\frac{1}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}|^2=1;\quad |C^{\frac{1}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}}|^2+|C^{\frac{1}{2}-\frac{1}{2}}_{1-1;\frac{1}{2}\frac{1}{2}}|^2=1.$$

Then we could determine that

$$C^{\frac{1}{2}\frac{1}{2}}_{11;\frac{1}{2}-\frac{1}{2}}=\sqrt{\frac{2}{3}};\quad C^{\frac{1}{2}\frac{1}{2}}_{10;\frac{1}{2}\frac{1}{2}}=-\sqrt{\frac{1}{3}};\quad C^{\frac{1}{2}-\frac{1}{2}}_{10;\frac{1}{2}-\frac{1}{2}}=\sqrt{\frac{1}{3}};\quad C^{\frac{1}{2}-\frac{1}{2}}_{1-1;\frac{1}{2}\frac{1}{2}}=-\sqrt{\frac{2}{3}}.$$

Then we could get the table of C-G coefficients for $j_1=1$ and $j_2=\frac{1}{2}$ as below

		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
		$+\frac{3}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
$+1$	$+\frac{1}{2}$	1	0	0	0	0	0
$+1$	$-\frac{1}{2}$	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	0	0	0
0	$+\frac{1}{2}$	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	0
0	$-\frac{1}{2}$	0	0	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0
-1	$+\frac{1}{2}$	0	0	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0
-1	$-\frac{1}{2}$	0	0	0	0	0	1

The top row of the table is $\{J,M\}$ and the left column of the table is $\{m_1,m_2\}$. $\square$

2. For $j = 1$, there is

$$d^j_{m'm}(\beta) = \langle 1, m'| \, e^{-\frac{i}{\hbar}\beta \hat{J}_y} \, |1, m\rangle \, .$$

Determine the matrix element

$$D^j_{m',m}(\alpha, \beta, \gamma).$$

Proof. In the $|1, m\rangle$ representation, we could get the matrix form of $\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$ is

$$J_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

Then, we could determine that

$$\begin{aligned}
J_y^2 =& \frac{\hbar^2}{-4}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix};\\
J_y^3 =& \frac{\hbar^3}{-8i}\begin{pmatrix} 0 & -4\sqrt{2} & 0 \\ 4\sqrt{2} & 0 & -4\sqrt{2} \\ 0 & 4\sqrt{2} & 0 \end{pmatrix} = \frac{\hbar^3}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \hbar^2 J_y;\\
J_y^4 =& \frac{\hbar^4}{16}\begin{pmatrix} 8 & 0 & -8 \\ 0 & 16 & 0 \\ -8 & 0 & 8 \end{pmatrix} = \frac{\hbar^4}{-4}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \hbar^2 J_y^2.
\end{aligned}$$

Which means

$$(J_y/\hbar)^{2n-1} = J_y/\hbar; \quad (J_y/\hbar)^{2n} = (J_y/\hbar)^2.$$

for $n = 1, 2, 3, \cdots$.

Then the matrix $d^j(\beta)$ is

$$\begin{aligned}
d^j(\beta) =& \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \beta^n J_y^n = 1 - i\sum_{k=0}^{\infty}(-1)^k\beta^{2k+1}(J_y/\hbar)^{2k+1} + \sum_{j=1}^{\infty}(-1)^j\beta^{2j}(J_y/\hbar)^{2j}\\
=& 1 - i\sum_{k=0}^{\infty}(-1)^k\beta^{2k+1}(J_y/\hbar) + \sum_{j=1}^{\infty}(-1)^j\beta^{2j}(J_y/\hbar)^2\\
=& 1 - i\sin\beta(J_y/\hbar) + (\cos\beta - 1)(J_y/\hbar)^2.
\end{aligned}$$

In matrix form, we get

$$d^1(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

The the matrix element satisfies $D^j_{m',m}(\alpha,\beta,\gamma) = e^{-im'\alpha - im\gamma} d^j_{m'm}(\beta)$, so we could write the matrix form of $D^1(\alpha,\beta,\gamma)$ as below

$$D^1(\alpha,\beta,\gamma) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta)e^{-i(\alpha+\gamma)} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i(\alpha+2\gamma)} & \frac{1}{2}(1-\cos\beta)e^{-i(\alpha+3\gamma)} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i(2\alpha+\gamma)} & \cos\beta e^{-i(2\alpha+2\gamma)} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i(2\alpha+3\gamma)} \\ \frac{1}{2}(1-\cos\beta)e^{-i(3\alpha+\gamma)} & \frac{1}{\sqrt{2}}\sin\beta e^{-i(3\alpha+2\gamma)} & \frac{1}{2}(1+\cos\beta)e^{-i(3\alpha+3\gamma)} \end{pmatrix}$$

□

3. The Hamiltonian of three spin-1/2 system is given by

$$H = J\hat{\sigma}_1 \cdot \hat{\sigma}_2 + K\hat{\sigma}_2 \cdot \hat{\sigma}_3,$$

where $\hat{\sigma}_i$ is the Pauli operator at the i-th site, and J and K are real. Solve the eigenstates and eigenvalues of the Hamiltonian. Discuss the degeneracy of eigenstates by analyzing the time-reverse symmetry of the system.

Proof. The Hamiltonian could be written as

$$\begin{aligned} H =& J\hat{\sigma}_1 \cdot \hat{\sigma}_2 + K\hat{\sigma}_2 \cdot \hat{\sigma}_3 \\ =& \frac{J}{2}(2\sigma_{1z}\sigma_{2z} + \sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}) + \frac{K}{2}(2\sigma_{2z}\sigma_{3z} + \sigma_{2+}\sigma_{3-} + \sigma_{2-}\sigma_{3+}). \end{aligned}$$

In $|s_1 m_1, s_2 m_2, s_3 m_3\rangle = |s_1 m_1\rangle \otimes |s_2 m_2\rangle \otimes |s_3 m_3\rangle$ represent, we could write the matrix form of the Hamiltonian

$$H = \begin{pmatrix} J+K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J-K & 2K & 0 & 0 & 0 & 0 & 0 \\ 0 & 2K & -(J+K) & 2J & 0 & 0 & 0 & 0 \\ 0 & 0 & 2J & K-J & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K-J & 2J & 0 & 0 \\ 0 & 0 & 0 & 0 & 2J & -(J+K) & 2K & 0 \\ 0 & 0 & 0 & 0 & 0 & 2K & J-K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J+K \end{pmatrix},$$

where $s_1 = s_2 = s_3 = 1/2$, $m_1, m_2, m_3 = \pm 1/2$.

We could consider the blocks of the Hamiltonian. It is clear that we get

$$H_1 = \begin{pmatrix} J+K \end{pmatrix}, \quad H_4 = \begin{pmatrix} J+K \end{pmatrix}$$

which are one-dimension matrix. So the eigenvalue of H_1 is $J+K$, corresponding eigenvector is $|1\rangle = |\frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2}\rangle$. The eigenvalue of H_4 is $J+K$, corresponding eigenvector is $|8\rangle = |\frac{1}{2} - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\rangle$.

Consider the matrix

$$H_2 = \begin{pmatrix} J-K & 2K & 0 \\ 2K & -(J+K) & 2J \\ 0 & 2J & K-J \end{pmatrix}.$$

The eigenvalues satisfy

$$\mathbf{Det}|H_2-\lambda\mathbf{I}| = -(\lambda-(J+K))(\lambda^2+2(J+K)\lambda-3(J-K)^2)=0.$$

So the eigenvalues are

$$\lambda_1 = J+K,\ \lambda_2 = 2\sqrt{J^2+K^2-JK}-(J+K),\ \lambda_3 = -2\sqrt{J^2+K^2-JK}-(J+K).$$

Corresponding eigenvectors are

$$\begin{aligned}
|2\rangle =& |\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle + |\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle + |\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\,;\\
|3\rangle =& -\frac{-\sqrt{J^2+K^2-JK}+J-K}{J}|\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\\
&-\frac{\sqrt{J^2+K^2-JK}+K}{J}|\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\\
&+|\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\,;\\
|4\rangle =& -\frac{-\sqrt{J^2+K^2-JK}+J-K}{J}|\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\\
&-\frac{K-\sqrt{J^2+K^2-JK}}{J}|\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\\
&+|\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\,.
\end{aligned}$$

Similarly, we could get the eigenvalues of H_3 are

$$\lambda_5 = J+K,\ \lambda_6 = 2\sqrt{J^2+K^2-JK}-(J+K),\ \lambda_7 = -2\sqrt{J^2+K^2-JK}-(J+K).$$

Corresponding eigenvectors are

$$\begin{aligned}
|5\rangle =& |\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle + |\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle + |\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\,;\\
|6\rangle =& -\frac{-\sqrt{J^2+K^2-JK}+J-K}{J}|\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\\
&-\frac{\sqrt{J^2+K^2-JK}+K}{J}|\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\\
&+|\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\,;\\
|7\rangle =& -\frac{-\sqrt{J^2+K^2-JK}+J-K}{J}|\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle\\
&-\frac{K-\sqrt{J^2+K^2-JK}}{J}|\frac{1}{2}-\frac{1}{2},\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\\
&+|\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2},\frac{1}{2}-\frac{1}{2}\rangle\,.
\end{aligned}$$

Then we could renormalize the six eigenvectors as

$$|i'\rangle = \frac{1}{\sqrt{\langle i|\,i\rangle}}\,|i\rangle\,,$$

where $i=1,2,3,4,5,6$.

Then we could get the table as below

eigenvalues	$J+K$	$2\sqrt{J^2+K^2-JK}-(J+K)$	$-2\sqrt{J^2+K^2-JK}-(J+K)$
eigenvectors	$\lvert 1\rangle, \lvert 8\rangle, \lvert 2\rangle, \lvert 5\rangle$	$\lvert 3\rangle, \lvert 6\rangle$	$\lvert 4\rangle, \lvert 7\rangle$
degeneracy	4	2	2

We could determine that

$$\begin{aligned}\hat{T}^{-1}\hat{H}\hat{T} =& iK\sigma_y(1)\sigma_y(2)\sigma_y(3)(J\hat{\sigma}_1\cdot\hat{\sigma}_2+\kappa\hat{\sigma}_2\cdot\hat{\sigma}_3)(-i)\sigma_y(1)\sigma_y(2)\sigma_y(3)K\\ =& JK\sigma_y(1)\sigma_y(2)\hat{\sigma}_1\cdot\hat{\sigma}_2\sigma_y(1)\sigma_y(2)K+\kappa K\sigma_y(2)\sigma_y(3)\hat{\sigma}_2\cdot\hat{\sigma}_3\sigma_y(2)\sigma_y(3)K\\ =& J\hat{\sigma}_1\cdot\hat{\sigma}_2+\kappa\hat{\sigma}_2\cdot\hat{\sigma}_3\\ =& \hat{H}.\end{aligned}$$

where κ is K in Hamiltonian.

Then we could determine that

$$\hat{T}\lvert 1\rangle = -\lvert 8\rangle;\ \hat{T}\lvert 2\rangle = \lvert 5\rangle;\ \hat{T}\lvert 3\rangle = \lvert 6\rangle;\ \hat{T}\lvert 4\rangle = \lvert 7\rangle.$$

□