高等量子力学第十次作业

董建宇 202328000807038

习题 5.3

证明方程

$$\sum_{i,j} |\phi_\alpha(i), \phi_\delta(j)\rangle \langle \phi_\beta(i), \phi_\tau(j)| = \hat{a}_\alpha^\dagger \hat{a}_\delta^\dagger \hat{a}_\tau \hat{a}_\beta.$$

Proof. 在 $\alpha \neq \delta \neq \beta \neq \tau$ 的情况下，不妨取 $\alpha < \delta < \beta < \tau$，可以计算:

$$\begin{aligned}
&\sum_{i,j} |\phi_\alpha(i), \phi_\delta(j)\rangle \langle \phi_\beta(i), \phi_\tau(j)| \, |n_1, n_2, \cdots, n_M\rangle \\
=&\frac{1}{\sqrt{N! n_1! n_2! \cdots n_M!}} \sum_\sigma \sigma \sum_{i,j} |\phi_\alpha(i), \phi_\delta(j)\rangle \langle \phi_\beta(i), \phi_\tau(j)| \, \psi^*(n_1, \cdots, n_M)\rangle \\
=&\frac{1}{\sqrt{N! n_1! n_2! \cdots n_M!}} \sum_\sigma \sigma \, |\psi^*(1, \cdots, n_\alpha + 1, \cdots, n_\delta + 1, \cdots, n_\beta - 1, \cdots, n_\tau - 1, \cdots, n_M)\rangle \\
=&\sqrt{(n_\alpha + 1)(n_\delta + 1) n_\beta n_\tau} \, |1, \cdots, n_\alpha + 1, \cdots, n_\delta + 1, \cdots, n_\beta - 1, \cdots, n_\tau - 1, \cdots, n_M\rangle \,.
\end{aligned}$$

即有:

$$\hat{a}_\alpha^\dagger \hat{a}_\delta^\dagger \hat{a}_\tau \hat{a}_\beta = \sum_{i,j} |\phi_\alpha(i), \phi_\delta(j)\rangle \langle \phi_\beta(i), \phi_\tau(j)| \,.$$

对于其他 $\alpha, \beta, \delta, \tau$ 的大小关系，可以得到同样的表达式。 □

习题 5.4

对全同费米子系统，用产生湮灭算符的对易关系证明:

(1) $\alpha \neq \beta$, $[\hat{n}_\alpha, \hat{n}_\beta] = 0$.

(2) $\hat{n}_\alpha$ 的本征值只能取 0 或者 1。

Proof. (1) 费米子产生湮灭算符满足反对易关系:

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}; \quad \{\hat{a}_i, \hat{a}_j\} = 0; \quad \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0.$$

则当 $i \neq j$ 时，有:

$$\hat{a}_i \hat{a}_j^\dagger = -\hat{a}_j^\dagger \hat{a}_i; \quad \hat{a}_i \hat{a}_j = -\hat{a}_j \hat{a}_i; \quad \hat{a}_i^\dagger \hat{a}_j^\dagger = -\hat{a}_j^\dagger \hat{a}_i^\dagger.$$

则可以计算，当 $\alpha \neq \beta$ 时:

$$[\hat{n}_\alpha, \hat{n}_\beta] = \hat{a}_\alpha^\dagger \hat{a}_\alpha \hat{a}_\beta^\dagger \hat{a}_\beta - \hat{a}_\beta^\dagger \hat{a}_\beta \hat{a}_\alpha^\dagger \hat{a}_\alpha = (-1)^4 \hat{a}_\beta^\dagger \hat{a}_\beta \hat{a}_\alpha^\dagger \hat{a}_\alpha - \hat{a}_\beta^\dagger \hat{a}_\beta \hat{a}_\alpha^\dagger \hat{a}_\alpha = 0.$$

(2) 注意到，对于费米子系统，有:

$$\{\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\} = 0.$$

则当 $\alpha=\beta$ 时，$2\hat{a}_\alpha^\dagger\hat{a}_\alpha^\dagger=0$，从而有:

$$\hat{a}_\alpha^\dagger\hat{a}_\alpha^\dagger\left|0\right\rangle=0.$$

对于量子态 $\left|n_1,n_2,\cdots,n_M\right\rangle$，可以由产生算符作用在真空态上得到，即

$$\left|n_1,n_2,\cdots,n_M\right\rangle\equiv {a_1^\dagger}^{n_1}{a_2^\dagger}^{n_2}\cdots{a_M^\dagger}^{n_M}\left|0\right\rangle.$$

但当 $n_i\geq 2$ 时，$\left|n_1,n_2,\cdots,n_M\right\rangle=0$，当 $n_i=0$ 或 1 时，$\hat{n}_i$ 的本征值为 0 或 1。综上所述，$\hat{n}_\alpha$ 的本征值只能取 0 或者 1。

□

习题 5.5

求黑体辐射中光子的化学势。

Proof. 由于黑体辐射中光子数不守恒，即产生或湮灭一个光子体系能量不发生变化，则黑体辐射中光子的化学势为 0。

对于黑体辐射，可以计算巨配分函数为:

$$\ln\varXi=\frac{\pi^2V}{45c^3\hbar^3}k_B^3T^3.$$

内能为:

$$U=-\frac{\partial\ln\varXi}{\partial\beta}=\frac{\pi^2V}{15c^3\hbar^3}k_B^4T^4.$$

压强为:

$$p=\frac{1}{3}\frac{U}{V}=\frac{\pi^2}{45c^3\hbar^3}k_B^4T^4.$$

自由能为:

$$F=-\frac{1}{3}U.$$

则 Gibbs 自由能为:

$$G=F+pV=0=N\mu.$$

则化学势为 0. □

习题 5.6

当光场的单个模式处在相干态 $\left|\alpha_{k\vec{e}_z,\vec{e}_x}\right\rangle$ 态时，求对应电场和磁场算符在该态上的平均值。

Proof. 电磁场算符分别为:

$$\vec{E}=\sum_{\vec{k},a}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0V}}\vec{e}_{\vec{k},a}i(\hat{a}_{\vec{k},a}e^{i\vec{k}\cdot\vec{r}}-\hat{a}_{\vec{k},a}^\dagger e^{-i\vec{k}\cdot\vec{r}})$$

$$\vec{B}=\sum_{\vec{k},a}\sqrt{\frac{\hbar}{2\epsilon_0V\omega_k}}(i\vec{k})\times\vec{e}_{\vec{k},a}(\hat{a}_{\vec{k},a}e^{i\vec{k}\cdot\vec{r}}-\hat{a}_{\vec{k},a}^\dagger e^{-i\vec{k}\cdot\vec{r}})$$

对于相干态 $\left|\alpha_{k\vec{e}_z,\vec{e}_x}\right\rangle$，满足本征值方程:

$$\hat{a}_{\vec{k},a}\left|\alpha_{k\vec{e}_z,\vec{e}_x}\right\rangle=\alpha_{k\vec{e}_z,\vec{e}_x}\left|\alpha_{k\vec{e}_z,\vec{e}_x}\right\rangle\delta_{a,\vec{e}_x}\delta_{\vec{k},k\vec{e}_z}.$$

则可以计算电场和磁场算符在该态上的平均值如下:

$$
\begin{aligned}
&\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{\vec{E}}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\\
=&\sum_{\vec{k},a}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\vec{e}_{\vec{k},a}i(\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{a}_{\vec{k},a}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\, e^{i\vec{k}\cdot\vec{r}}-\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{a}^{\dagger}_{\vec{k},a}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\, e^{-i\vec{k}\cdot\vec{r}})\\
=&\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\vec{e}_x i(\alpha_{k\vec{e}_z,\vec{e}_x}e^{ikz}-\alpha^*_{k\vec{e}_z,\vec{e}_x}e^{-ikz});\\
&\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{\vec{B}}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\\
=&\sum_{\vec{k},a}\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}(i\vec{k})\times\vec{e}_{\vec{k},a}(\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{a}_{\vec{k},a}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\, e^{i\vec{k}\cdot\vec{r}}-\langle\alpha_{k\vec{e}_z,\vec{e}_x}|\,\hat{a}^{\dagger}_{\vec{k},a}\,|\alpha_{k\vec{e}_z,\vec{e}_x}\rangle\, e^{-i\vec{k}\cdot\vec{r}})\\
=&\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}ik\vec{e}_y(\alpha_{k\vec{e}_z,\vec{e}_x}e^{ikz}-\alpha^*_{k\vec{e}_z,\vec{e}_x}e^{-ikz}).
\end{aligned}
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□