相互作用表象: $|\psi_I(t_f)\rangle = U_I(t_f-t_i)|\psi_I(t_i)\rangle = e^{iH_0t}e^{-iHt_f}e^{iHt_i}e^{-iH_0t_i}|\psi_I(t_i)\rangle = \Omega^\dagger(t_f)\Omega(t_i)|\psi_I(t_i)\rangle$

其中 $\Omega(t)=U_I(0,t)=e^{iHt}e^{-iH_0t}$; $\lim_{t_i\to-\infty}\Omega(t_i)=U_I(0,-\infty)=\Omega_+$; $\lim_{t_f\to+\infty}\Omega(t_f)=U_I(0,+\infty)=\Omega_-$ 散射矩阵 $S=\Omega_-^\dagger\Omega_+$

$\Omega(t+s)=e^{iHs}\Omega(t)e^{-iH_0s} \Rightarrow \Omega_+=e^{iHs}\Omega_+e^{-iH_0s} \Rightarrow H\Omega_+=\Omega_+H_0 \Rightarrow H\Omega_+|k\rangle=\Omega_+H_0|k\rangle=E_k\Omega_+|k\rangle$ 记 $|\psi_k^+\rangle=\Omega_+|k\rangle$

$\begin{cases}\langle\psi_{k'}^+|\psi_k^+\rangle=\langle k'|k\rangle=\delta(k'-k)\\ \sum_k|\psi_k^+\rangle\langle\psi_k^+|+\sum_\alpha|\phi_\alpha\rangle\langle\phi_\alpha|=1\end{cases} \Rightarrow \begin{cases}\Omega_+^\dagger\Omega_+=1\\ \Omega_+\Omega_+^\dagger=1-\sum_\alpha|\phi_\alpha\rangle\langle\phi_\alpha|\end{cases}$ 注意到 $\frac{d\Omega(t)}{dt}=ie^{iHt}Ve^{-iH_0t}$ 则有 $\Omega(s)=1+i\int_0^s e^{iHt}Ve^{-\epsilon|t|}e^{-iH_0t}dt$

从而 $|\psi_k^+\rangle=\lim_{s\to-\infty}\Omega(s)|k\rangle=|k\rangle+\frac{1}{E_k-H+i\epsilon}V|k\rangle=|k\rangle+\frac{1}{E_k-H_0+i\epsilon}V|\psi_k^+\rangle$ (对$|\psi_k^-\rangle$,只须变成 $-\epsilon$)

散射矩阵在动量表象中的表示: $\langle k'|S|k\rangle=\langle\psi_{k'}^-|\psi_k^+\rangle=\langle k'|k\rangle-2\pi i\delta(E_k-E_{k'})\langle k'|T|k\rangle$

其中 $\langle k'|T|k\rangle=\langle k'|V|\psi_k^+\rangle=\langle k'|V\Omega_+|k\rangle$ 称作T矩阵. 微分散射截面为: $d\sigma=\frac{\langle\psi_k^+|\vec{j}|\psi_k^+\rangle\cdot d\vec{S}}{|\langle k|\vec{j}|k\rangle|}$

其中 $|\psi_k^+\rangle-|k\rangle$ (散射部分) $\vec{j}(\vec{r})=\frac{1}{2M}(|\vec{r}\rangle\langle\vec{r}|\vec{p}+\vec{p}|\vec{r}\rangle\langle\vec{r}|)$

4.2 散射势

入射态 $|\psi_{in}\rangle=|k\vec{e}_z\rangle$ $\langle\vec{r}|k\vec{e}_z\rangle=\frac{1}{(2\pi)^{3/2}}e^{ikz}$; 量子跃迁矩阵元为:

$\langle k'|U_I(s,-\infty)|k\rangle=\langle k'|\Omega^\dagger(s)|\psi_k^+\rangle=\langle k'|k\rangle+\langle k'|V|\psi_k^+\rangle\left(\frac{\exp(-i(E_k-E_{k'}-i\epsilon)s)}{E_k-E_{k'}-i\epsilon}-\frac{2i\epsilon}{(E_k-E_{k'})^2+\epsilon^2}\right)$

对任意有限时间s, $w(k\to k',s)=\frac{d}{dt}|\langle k'|U_I(s,-\infty)|k\rangle|^2=2\pi|\langle k'|T|k\rangle|^2\delta(E_k-E_{k'})$

微分散射截面: $d\sigma=\frac{V}{(2\pi)^3}\int k'^2dk'd\Omega\, w(k\to k')/|\langle k|\vec{j}|k\rangle|=(2\pi)^4M^2|\langle k'|T|k\rangle|^2d\Omega$

$T|k\rangle=V|k\rangle+V\frac{1}{E_k-H_0+i\epsilon}T|k\rangle \Rightarrow T|k\rangle=V\sum_{n=0}^\infty\left(\frac{1}{E_k-H_0+i\epsilon}V\right)^n|k\rangle$; $T=V\Omega_+$, $\Omega_+|k\rangle=|\psi_k^+\rangle$

总散射截面: $\sigma_t=-\frac{2M(2\pi)^3}{k}\text{Im}\langle k\vec{e}_z|T|k\vec{e}_z\rangle=\frac{4\pi}{k}\text{Im}f(k\vec{e}_z,k\vec{e}_z)$

$f(\vec{k}',\vec{k})=-(2\pi)^2M\langle\vec{k}'|T|\vec{k}\rangle$

4.3 旋转对称势

取 $H_0,\vec{L}^2,L_z$共同本征态 $|Elm\rangle$; $\langle Elm|E'l'm'\rangle=\delta(E-E')\delta_{ll'}\delta_{mm'}$ $\sum_{E,l,m}|Elm\rangle\langle Elm|=1$

$\langle Elm|S|E'l'm'\rangle=e^{i2\delta_l(E)}\delta(E-E')\delta_{ll'}\delta_{mm'}$ 归一化的动量本征态: $|E_p\vec{e}_p\rangle=\sqrt{Mp}|\vec{p}\rangle$; 满足: $\langle E_p\vec{e}_p|Elm\rangle=\delta(E_p-E)\langle\vec{e}_p|lm\rangle$

散射相移满足 $e^{i2\delta_l(E)}=1-2\pi i\langle Elm|T|Elm\rangle$; $f(k\vec{e}_r,k\vec{e}_z)=\sum_l(2l+1)f_l(E)P_l(\cos\theta)$; $f_l(E)=\frac{1}{k\cot\delta_l(E)-ik}$

球谐函数加法定理: $\sum_m\langle\vec{e}_r|lm\rangle\langle lm|\vec{e}_{r'}\rangle=\frac{2l+1}{4\pi}P_l(\cos\theta)$;

$\boxed{d\sigma=\left|\sum_l\frac{2l+1}{k\cot\delta_l-ik}P_l(\cos\theta)\right|^2d\Omega \qquad \sigma_t=\frac{4\pi}{k^2}\sum_l(2l+1)\sin^2\delta_l}$

相移计算: $H_0=\frac{\vec{p}^2}{2m}=\frac{1}{2m}(p_r^2+\frac{\vec{L}^2}{r^2})$; $p_r=\frac{1}{2}(\frac{\vec{r}}{r}\cdot\vec{p}+\vec{p}\cdot\frac{\vec{r}}{r})=\frac{\hbar}{i}(\frac{\partial}{\partial r}+\frac{1}{r})$

$\langle r\theta\phi|Elm\rangle=R_{kl}(r)\langle\theta\phi|lm\rangle$ 中径向波函数满足 $(p_r^2+\frac{l(l+1)}{r^2})R_{kl}(r)=k^2R_{kl}(r)$ 其中 $k=\sqrt{2mE}$; $\lim_{r\to0}rR_{kl}(r)=0$

令 $x=kr$, $y(x)=R_{kl}(r)$, 则有 $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+[1-\frac{l(l+1)}{x^2}]y=0$. 两个线性无关的解为 Bessel; Neumann.

$j_l(x)=(-1)^lx^l(\frac{1}{x}\frac{d}{dx})^l\frac{\sin x}{x}$; $n_l(x)=(-1)^{l+1}x^l(\frac{1}{x}\frac{d}{dx})^l\frac{\cos x}{x}$; $x\to0$时, $j_l(x)\propto\frac{1}{(2l+1)!!}x^l$; $n_l(x)\propto-(2l-1)!!\frac{1}{x^{l+1}}$

$x\to\infty$时, $j_l(x)\propto\frac{1}{x}\sin(x-\frac{l\pi}{2})$; $n_l(x)\propto-\frac{1}{x}\cos(x-\frac{l\pi}{2})$; 满足边界条件的解为 $R_{kl}(r)=N_{kl}j_l(kr)$; $N_{kl}=\sqrt{\frac{2Mk}{\pi}}$

其中$|Elm^+\rangle$满足径向方程 $(p_r^2+\frac{l(l+1)}{r^2}+2MV(r))R_{kl}^+(r)=k^2R_{kl}^+(r)$; $R_{kl}^+(r)=R_{kl}(r)+a_{kl}h_l(kr)$

则有 $ikf_l(E)=\frac{a_{kl}}{N_{kl}}$, 有 $e^{i2\delta_l(E)}=1+2\frac{a_{kl}}{N_{kl}}$; $R_{kl}^+(r)\to N_{kl}e^{i\delta_l(E)}\frac{\sin(kr-\frac{1}{2}l\pi+\delta_l(E))}{kr}$

低能散射, $k\ll1$, $\delta_l\ll1$, 有 $\delta_l(k)\simeq-\pi\int r^2dr|N_{kl}j_l(kr)|^2V(r)\propto k^{2l+1}$; $l=0$时 散射长度 $a_0=-\frac{\delta_0}{k}$

s波总散射截面 $\sigma_0=4\pi a_0^2$

4.4 球型方势散射

$V(r)=\begin{cases}V, & r\le a\\0, & 其它\end{cases}$ $V>0$ 势垒; $V<0$ 势阱

有: $R_{kl}^+(r)=\begin{cases}R_{kl}(r)+a_{kl}h_l(kr), & r\ge a;\\ b_{kl}j_l(k'r), & r\le a\end{cases}$ 其中 $k'=\sqrt{k^2-2MV}$; 令径向 $R_{kl}^+(r)=\frac{u_{kl}(r)}{r}$ 有:

$(-\frac{1}{2M}\frac{d^2}{dr^2}+\frac{l(l+1)}{2Mr^2}+V(r))u_{kl}(r)=E_ku_{kl}(r)$; $u_{kl}(0)=0$; $u_{kl}(r)$及其一阶导数在a点连续.

$N_{kl}j_l(ka)+a_{kl}h_l(ka)=b_{kl}j_l(k'a)$; $N_{kl}\frac{d}{dr}(rj_l(kr))\Big|_a+a_{kl}\frac{d}{dr}(rh_l(kr))\Big|_a=b_{kl}\frac{d}{dr}(rj_l(k'r))\Big|_a$

$$\boxed{\begin{aligned}a_{kl}&=\frac{j_l(ka)\frac{d}{dr}(rj_l(k'r))|_a-\frac{d}{dr}(rj_l(kr))|_a\,j_l(k'a)}{j_l(k'a)\frac{d}{dr}(rh_l(kr))|_a-\frac{d}{dr}(rj_l(k'r))|_a\,h_l(ka)}N_{kl}\\ b_{kl}&=\frac{j_l(ka)\frac{d}{dr}(rh_l(kr))|_a-\frac{d}{dr}(rj_l(kr))|_a\,h_l(ka)}{j_l(k'a)\frac{d}{dr}(rh_l(kr))|_a-\frac{d}{dr}(rj_l(k'r))|_a\,h_l(ka)}N_{kl}\end{aligned}}$$

$e^{i2\delta_l(E)}=-\frac{j_l(k'a)\,ka\,h_{l+1}^*(ka)-k'a\,j_{l+1}(k'a)h_l^*(ka)}{j_l(k'a)\,ka\,h_{l+1}(ka)-k'a\,j_{l+1}(k'a)h_l(ka)}$

利用了 $\frac{d}{dx}(x^{l+1}j_l(x))=x^{l+1}j_{l-1}(x)$

s波贡献散射截面为: $\boxed{s_0=k_0a,\ s=\frac{ka}{s_0}}$

当 $l=0$ 时 $e^{i2\delta_0(E)}=\frac{k'\cos k'a+ik\sin k'a}{k'\cos k'a-ik\sin k'a}e^{-i2ka}$; $\sigma_0=\frac{4\pi}{k^2}\sin^2\delta_0(E)=\frac{4\pi a^2}{(ss_0)^2}\sin^2\delta_0(s,s_0)$

考虑$V>0$时, 令 $k_0=\sqrt{2MV}$, $k<k_0$时, $k'=\sqrt{k^2-k_0^2}=i\varkappa$, 即 $\varkappa=\sqrt{k_0^2-k^2}$; $e^{i2\delta_0(k)}=\frac{\varkappa\cosh\varkappa a+ik\sinh\varkappa a}{\varkappa\cosh\varkappa a-ik\sinh\varkappa a}e^{-i2ka}$

于是 $\delta_0(k)=\arctan(\frac{k}{\varkappa}\tanh\varkappa a)-ka=\arctan(\frac{s}{\sqrt{1-s^2}}\tanh(\sqrt{1-s^2}\,s_0))-ss_0$; $s_0=k_0a$, $s=\frac{ka}{s_0}<1$

低能极限 $s\ll1$时, $\delta_0(s)=(\tanh(s_0)-s_0)s$; 散射长度 $a_0=a\frac{s_0-\tanh s_0}{s_0}$; $\sigma_0=\frac{4\pi a^2}{(ss_0)^2}\sin^2\delta_0(s)$

$s_0\to+\infty$时, $a_0=a$, $\sigma_0=4\pi a^2$.

当 $k>k_0$时, $\delta_0(k)=\arctan(\frac{k}{k'}\tan k'a)-ka$, $k'=\sqrt{k^2-k_0^2}$, 令 $s_0=k_0a$, $s=\frac{ka}{s_0}>1$, 有:

$\delta_0(s)=\arctan\left(\frac{s}{\sqrt{s^2-1}}\tan(\sqrt{s^2-1}\,s_0)\right)-ss_0$

考虑 $V<0$时, 令 $k_0=\sqrt{-2MV}$, 有 $k'=\sqrt{k^2+k_0^2}$, 令 $s_0=k_0a$, $s=\frac{ka}{s_0}$ 于是相移为

$\delta_0(s)=\arctan\frac{s}{\sqrt{s^2+1}}\tan\sqrt{s^2+1}\,s_0-ss_0$ 低能极限 $s\ll1$时, $\delta_0(s)=(\tan(s_0)-s_0)s$

散射长度 $a_0=a\frac{s_0-\tan s_0}{s_0}$, 当 $\delta_0(s)\ll1$时, s波散射截面为 $\sigma_0=\frac{4\pi a^2}{(ss_0)^2}\sin^2\delta_0(s)=\frac{4\pi a^2(\tan s_0-s_0)^2}{s_0^2}$

Example: 1. 散射势为 $V(r)=\mathcal{F}\delta(r-a)$ 下的s波散相移和散射截面, $(a>0)$

求解定态薛定谔方程 $(\frac{d^2}{dr^2}+k^2-2m\mathcal{F}\delta(r-a))u(r)=0$; $u(0)=0$, $u(\infty)\to\sin(kr+\delta_0)$

$r\neq a$时, $V(r)=0$ $u(r)=\begin{cases}C\sin(kr) & r<a\\ A\sin(kr)+B\cos(kr)=\sqrt{A^2+B^2}\sin(kr+\delta_0) & r>a\end{cases}$

连续性条件 $\begin{cases}u(a^+)=u(a^-)\\u'(a^+)-u'(a^-)=2m\mathcal{F}u(a)\end{cases}\Rightarrow\begin{cases}C\sin ka=A\sin ka+B\cos ka\\C(k\cos ka+2m\mathcal{F}\sin ka)=Ak\cos ka-Bk\sin ka\end{cases}$

$\Rightarrow\frac{k\cos ka+2m\mathcal{F}\sin ka}{k\sin ka}=\frac{A\cos ka-B\sin ka}{A\sin ka+B\cos ka}\Rightarrow\frac{B}{A}=-\frac{\frac{2m\mathcal{F}}{k}\sin^2(ka)}{1+\frac{m\mathcal{F}}{k}\sin(2ka)}$ $ka\ll1$时(s波)

$\frac{B}{A}\sim-\frac{2m\mathcal{F}ka^2}{1+2m\mathcal{F}a}$ 相移 $\delta_0=\arctan(\frac{B}{A})\sim-\arctan(\frac{2m\mathcal{F}ka^2}{1+2m\mathcal{F}a})$

散射截面 $\sigma_0=\frac{4\pi}{k^2}\sin^2\delta_0\sim4\pi a^2(\frac{2m\mathcal{F}a}{1+2m\mathcal{F}a})^2$

2. 刚球散射. $V=\infty$, $r<a$; $V=0$, $r>a$, $R_l(a)\to0$, 且 $R_l(r)$导数在a处有界, $\gamma_l\to\infty$

外部解 $R_l(r)=A_l(\cos\delta_lj_l(kr)-\sin\delta_ln_l(kr))$ 或 $\tan\delta_l=\frac{kj_l'(ka)-\gamma_lj_l(ka)}{kn_l'(ka)-\gamma_ln_l(ka)}\Rightarrow\tan\delta_l=\frac{j_l(ka)}{n_l(ka)}$

低能 $ka\ll l$ 有 $\tan\delta_l\sim\frac{(2l+1)(ka)^{2l+1}}{[(2l+1)!!]^2}$ 散射长度为a, 截面 $\sigma_0=4\pi a^2$

二次量子化 5.1 全同性及其量子态表示

N个自旋为S的全同粒子的量子态, 有如下公设: $\langle x_1,\cdots,x_i,\cdots,x_j,\cdots,x_N|P_{ij}|\psi\rangle=\langle x_1,\cdots,x_j,\cdots,x_i,\cdots,x_N|\psi\rangle$

定义3 相应置换算符满足 $P_{ij}|\psi\rangle=(-1)^{2S}|\psi\rangle$ $\quad=(-1)^{2S}\langle x_1,\cdots,x_i,\cdots,x_j,\cdots,x_N|\psi\rangle$

$|\psi^*(n_1,\ldots,n_M)\rangle=|\phi_1(1)\rangle\otimes\cdots\otimes|\phi_1(n_1)\rangle\otimes|\phi_2(n_1+1)\rangle\cdots\otimes|\phi_2(n_1+n_2)\rangle\otimes\cdots\otimes|\phi_M(n_1+\cdots+n_{M-1}+1)\rangle\otimes\cdots\otimes|\phi_M(N)\rangle$

对称化和归一化: $|n_1,n_2,\cdots,n_M\rangle=\frac{1}{\sqrt{N!n_1!n_2!\cdots n_M!}}\sum_\sigma\sigma|\psi^*(n_1,\ldots,n_M)\rangle$ Hilbert 维数 $\binom{N+M-1}{N}$

对于玻色子, $n_\alpha=\sum_{i=1}^N|\phi_\alpha(i)\rangle\langle\phi_\alpha(i)|$ $\quad\hat{a}_\alpha^\dagger\hat{a}_\beta=\sum_{i=1}^N|\phi_\alpha(i)\rangle\langle\phi_\beta(i)|$ $\quad[\hat{a}_\alpha,\hat{a}_\beta^\dagger]=\delta_{\alpha\beta}$

一般单体算符可写作 $A=\sum_{\alpha,\beta}\langle\phi_\alpha|A|\phi_\beta\rangle\hat{a}_\alpha^\dagger\hat{a}_\beta$; 两体算符: $B=\sum_{i<j}B(i,j)=\frac{1}{2!}\sum_{i\neq j}B(i,j)$

$=\frac{1}{2!}\sum_{\alpha\beta\delta\gamma}\sum_{i\neq j}|\phi_\alpha(i),\phi_\beta(j)\rangle\langle\phi_\alpha(i),\phi_\beta(j)|B(i,j)|\phi_\gamma(i),\phi_\delta(j)\rangle\langle\phi_\gamma(i),\phi_\delta(j)|$; $\sum_{i\neq j}|\phi_\alpha(i),\phi_\beta(j)\rangle\langle\phi_\gamma(i),\phi_\delta(j)|=a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma$

对于费米子: $\sum_{i=1}^N|\phi_\alpha(i)\rangle\langle\phi_\beta(i)|n_1,n_2,\cdots,n_M\rangle=\delta_{n_\beta,1}(-1)^{\sum_{i=1}^{\beta-1}n_i}\delta_{n_\alpha,0}(-1)^{\sum_{i=1}^{\alpha-1}n_i}|n_1,n_2,\cdots,n_\alpha+1,\cdots,n_\beta-1,\cdots,n_M\rangle$

5.3 电子场量子化: 光子

$A_{\vec{k}}(t)=A_{\vec{k}}(0)e^{-i\omega t}$; $\omega=ck$

矢势满足波动方程 $\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla^2\vec{A}=0$, 波动方程的解为 $\vec{A}=\sum_{\vec{k},\vec{e}_k}\vec{e}_k(A_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}}+A_{\vec{k}}^*(t)e^{-i\vec{k}\cdot\vec{r}})$

库仑规范: $\vec{k}\cdot\vec{e}_k=0$; 对同一传播方向,有两个独立电磁指向. $\vec{A}=\sum_{\vec{k},a}\vec{e}_{\vec{k},a}(A_{\vec{k},a}(t)e^{i\vec{k}\cdot\vec{r}}+A_{\vec{k},a}^*(t)e^{-i\vec{k}\cdot\vec{r}})$

$\frac{dA_{\vec{k},a}}{dt}=-i\omega_kA_{\vec{k},a}$ 电磁场能量为: $H=\frac{1}{2}\int d\vec{r}(\epsilon_0\vec{E}\cdot\vec{E}+\frac{1}{\mu_0}\vec{B}\cdot\vec{B})=\sum_{\vec{k},a}\epsilon_0V\omega_k^2(A_{\vec{k},a}A_{\vec{k},a}^*+A_{\vec{k},a}^*A_{\vec{k},a})$

广义坐标和广义动量: $X_{\vec{k},a}=\sqrt{\epsilon_0V}(A_{\vec{k},a}+A_{\vec{k},a}^*)$; $P_{\vec{k},a}=-i\omega_k\sqrt{\epsilon_0V}(A_{\vec{k},a}-A_{\vec{k},a}^*)$; $\frac{dX}{dt}=P$; $\frac{dP}{dt}=-\omega^2X$

从而电磁场能量 $H=\sum_{\vec{k},a}\frac{1}{2}(P_{\vec{k},a}^2+\omega_k^2X_{\vec{k},a}^2)$; 湮灭算符: $\hat{a}_{\vec{k},a}=\sqrt{\frac{2\epsilon_0V\omega_k}{\hbar}}A_{\vec{k},a}$; $H=\sum_{\vec{k},a}\frac{\hbar\omega_k}{2}(a_{\vec{k},a}^\dagger a_{\vec{k},a}+h.c.)$ $=\sum_{\vec{k},a}\hbar\omega_k(\hat{n}_{\vec{k},a}+\frac{1}{2})$

矢势算符: $\hat{A}=\sum_{\vec{k},a}\sqrt{\frac{\hbar}{2\epsilon_0V\omega_k}}\vec{e}_{\vec{k},a}(\hat{a}_{\vec{k},a}e^{i\vec{k}\cdot\vec{r}}+\hat{a}_{\vec{k},a}^\dagger e^{-i\vec{k}\cdot\vec{r}})$

电磁场算符: $\vec{E}=\sum_{\vec{k},a}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0V}}\vec{e}_{\vec{k},a}i(\hat{a}_{\vec{k},a}e^{i\vec{k}\cdot\vec{r}}-\hat{a}_{\vec{k},a}^\dagger e^{-i\vec{k}\cdot\vec{r}})$; $\vec{B}=\sum_{\vec{k},a}\sqrt{\frac{\hbar}{2\epsilon_0V\omega_k}}(i\vec{k})\times\vec{e}_{\vec{k},a}(\hat{a}_{\vec{k},a}e^{i\vec{k}\cdot\vec{r}}-\hat{a}_{\vec{k},a}^\dagger e^{-i\vec{k}\cdot\vec{r}})$

电磁场的动量算符. $\vec{P}=\int d\vec{r}\epsilon_0\vec{E}\times\vec{B}=\sum_{\vec{k},a}\hbar\vec{k}(\hat{n}_{\vec{k},a}+\frac{1}{2})$

单位体积腔体辐射能谱密度为 $u(\omega)=\frac{\hbar}{\pi^2c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$

5.4 相互作用电子气

无相互作用时, 箱归一化+周期性边界条件: $\vec{p}|\vec{k},\sigma\rangle=\hbar\vec{k}|\vec{k},\sigma\rangle$; $S_z|\vec{k},\sigma\rangle=\frac{\hbar}{2}\sigma|\vec{k},\sigma\rangle$

其中 $\vec{k}=k_L\vec{n}=\frac{2\pi}{L}\vec{n}$; $H_0=\sum_{\vec{k},\sigma}\frac{(\hbar\vec{k})^2}{2M}\hat{c}_{\vec{k},\sigma}^\dagger\hat{c}_{\vec{k},\sigma}$, 基态: $|G_0\rangle=\prod_{k\le k_F}\hat{C}_{\vec{k},+}^\dagger\hat{C}_{\vec{k},-}^\dagger|0\rangle$; $N=2\cdot\frac{\frac{4}{3}\pi k_F^3}{k_L^3}=\frac{Vk_F^3}{3\pi^2}$

基态总能量 $E_0=2\int_0^{k_F}\frac{\hbar^2k^2}{2M}\frac{4\pi k^2dk}{k_L^3}=\frac{3}{5}NE_F$

有相互作用: 简化处理 短程接触势: $V(\vec{r}_i,\vec{r}_j)=g\delta(\vec{r}_i-\vec{r}_j)$; 二次量子化: $V=\sum_{i<j}V(\vec{r}_i,\vec{r}_j)=\sum_{i<j}g\delta(\vec{r}_i-\vec{r}_j)$

$=\frac{g}{2}\int d\vec{r}_1d\vec{r}_1'd\vec{r}_2d\vec{r}_2'\sum_{\sigma_1\sigma_1'\sigma_2\sigma_2'}\langle\vec{r}_1,\sigma_1;\vec{r}_2,\sigma_2|\delta(\vec{r}_1-\vec{r}_2)|\vec{r}_1',\sigma_1';\vec{r}_2',\sigma_2'\rangle\psi^\dagger(\vec{r}_1,\sigma_1)\psi^\dagger(\vec{r}_2,\sigma_2)\psi(\vec{r}_2',\sigma_2')\psi(\vec{r}_1',\sigma_1')$

$=\frac{g}{2}\int d\vec{r}\sum_{\sigma_1\sigma_2}\psi^\dagger(\vec{r},\sigma_1)\psi^\dagger(\vec{r},\sigma_2)\psi(\vec{r},\sigma_2)\psi(\vec{r},\sigma_1)=\frac{g}{2}\int d\vec{r}\sum_{\vec{k}_1'\vec{k}_2'\vec{k}_1\vec{k}_2}\sum_{\sigma_1\sigma_2}\frac{1}{V^2}e^{i(\vec{k}_1+\vec{k}_2-\vec{k}_1'-\vec{k}_2')\cdot\vec{r}}c_{\vec{k}_1',\sigma_1}^\dagger c_{\vec{k}_2',\sigma_2}^\dagger c_{\vec{k}_2,\sigma_2}c_{\vec{k}_1,\sigma_1}$

$=\frac{g}{2V}\sum_{\vec{k}_1',\vec{k}_2',\vec{k}_1,\vec{k}_2}\sum_{\sigma_1\sigma_2}\delta_{\vec{k}_1'+\vec{k}_2',\vec{k}_1+\vec{k}_2}c_{\vec{k}_1',\sigma_1}^\dagger c_{\vec{k}_2',\sigma_2}^\dagger c_{\vec{k}_2,\sigma_2}c_{\vec{k}_1,\sigma_1}$; 引入磁化参数P: $\begin{cases}N_++N_-=N\\N_+-N_-=PN\end{cases}$

$M_z=-\mu_B\bar{\sigma}_z=-\mu_BNP$, $\mu_B=\frac{e\hbar}{2M}$. $|G\rangle=\prod_{k\le k_{F+}}c_{\vec{k},+}^\dagger\prod_{k\le k_{F-}}c_{\vec{k},-}^\dagger|0\rangle$; $k_{F\pm}=\sqrt[3]{\frac{6\pi^2N_\pm}{V}}$, 基态动能为

$E_k=\frac{3}{10}NE_F[(1+P)^{5/3}+(1-P)^{5/3}]=E_0\frac{(1+P)^{5/3}+(1-P)^{5/3}}{2}$.

基态势能: $E_V=\langle G|\hat{V}|G\rangle=\frac{g}{2V}\sum_{\vec{k}_1',\vec{k}_2',\vec{k}_1,\vec{k}_2}\sum_{\sigma_1\sigma_2}\delta_{\vec{k}_1'+\vec{k}_2',\vec{k}_1+\vec{k}_2}\langle G|c_{\vec{k}_1',\sigma_1}^\dagger c_{\vec{k}_2',\sigma_2}^\dagger c_{\vec{k}_2,\sigma_2}c_{\vec{k}_1,\sigma_1}|G\rangle$

由Wick定理, Hartree能: $E_{VH}=\frac{g}{2V}\sum_{\vec{k}_1',\vec{k}_2',\vec{k}_1,\vec{k}_2}\sum_{\sigma_1\sigma_2}\delta_{\vec{k}_1'+\vec{k}_2',\vec{k}_1+\vec{k}_2}\langle G|c_{\vec{k}_1',\sigma_1}^\dagger c_{\vec{k}_1,\sigma_1}|G\rangle\langle G|c_{\vec{k}_2',\sigma_2}^\dagger c_{\vec{k}_2,\sigma_2}|G\rangle$

$=\frac{g}{2V}\sum_{\vec{k}_1\vec{k}_2}\sum_{\sigma_1\sigma_2}n_{\vec{k}_1,\sigma_1}n_{\vec{k}_2,\sigma_2}=\frac{g}{2V}N^2$. 与P无关. Fock能(交换能) $E_{VF}=-\frac{g}{2V}\sum_{\vec{k}_1'\vec{k}_2'\vec{k}_1\vec{k}_2}\sum_{\sigma_1\sigma_2}\delta_{\vec{k}_1'+\vec{k}_2',\vec{k}_1+\vec{k}_2}$ (续)

$\langle G|c_{\vec{k}_1',\sigma_1}^\dagger c_{\vec{k}_2,\sigma_2}|G\rangle\langle G|c_{\vec{k}_2',\sigma_2}^\dagger c_{\vec{k}_1,\sigma_1}|G\rangle=-\frac{g}{2V}\sum_{\vec{k}_1\vec{k}_2}\sum_\sigma n_{\vec{k}_1,\sigma}n_{\vec{k}_2,\sigma}=-\frac{g}{2V}\sum_\sigma N_\sigma^2=-\frac{g}{4V}N^2(1+P^2)=-E_{VH}\frac{1+P^2}{2}$

总能量 $E(P)=E_0\frac{(1+P)^{5/3}+(1-P)^{5/3}}{2}+E_{VH}\frac{1-P^2}{2}$

5.5 相互作用玻色气体

$H=H_K+V=\sum_i\frac{\vec{p}_i^2}{2M}+\sum_{i<j}V(\vec{r}_i,\vec{r}_j)$; $V(\vec{r}_i,\vec{r}_j)=\frac{4\pi\hbar^2a}{M}\delta(\ )$

动量本征态满足 $\langle\vec{r}|\vec{m}\rangle=\frac{1}{L^3}e^{i\vec{m}\cdot\vec{r}k_L}$ $k_L=\frac{2\pi}{L}$, $m_i\in Z$; 二次量子化 $H_k=\sum_{\vec{m}}\frac{\hbar^2k_L^2}{2M}\vec{m}^2a_{\vec{m}}^\dagger a_{\vec{m}}$

相互作用二次量子化: $V=\frac{1}{2}\sum_{i\neq j}V(\vec{r}_i,\vec{r}_j)=\frac{1}{2}\sum_{\vec{m}_1\vec{m}_2\vec{m}_3\vec{m}_4}\langle\vec{m}_1,\vec{m}_2|V(\vec{r}_1,\vec{r}_2)|\vec{m}_3,\vec{m}_4\rangle a_{\vec{m}_1}^\dagger a_{\vec{m}_2}^\dagger a_{\vec{m}_4}a_{\vec{m}_3}$

$=\frac{2\pi\hbar^2a}{ML^3}\sum_{\vec{m}_1\vec{m}_2\vec{m}_3\vec{m}_4}\delta_{\vec{m}_1+\vec{m}_2,\vec{m}_3+\vec{m}_4}a_{\vec{m}_1}^\dagger a_{\vec{m}_2}^\dagger a_{\vec{m}_3}a_{\vec{m}_4}$ 记 $\epsilon_L=\frac{(\hbar k_L)^2}{2M}$, 则哈密顿量可写作:

$H=\epsilon_L\sum_{\vec{m}}\vec{m}^2a_m^\dagger a_m+\frac{a}{\pi L}\epsilon_L\sum_{\vec{m}_1\vec{m}_2\vec{m}_3\vec{m}_4}\delta_{\vec{m}_1+\vec{m}_2,\vec{m}_3+\vec{m}_4}a_{\vec{m}_1}^\dagger a_{\vec{m}_2}^\dagger a_{\vec{m}_3}a_{\vec{m}_4}$ $\qquad N=N_0+\sum_{m\neq0}a_m^\dagger a_m$

波戈留波夫变换, 对玻色气体基态为 $|G_0\rangle=\frac{1}{\sqrt{N!}}a_0^{\dagger N}|0\rangle$; 在哈密顿量H中, $a_0\simeq a_0^\dagger\simeq\sqrt{N_0}$

则常数项为 $H_0=\frac{a}{\pi L}\epsilon_La_0^\dagger a_0^\dagger a_0a_0=\frac{a}{\pi L}\epsilon_L(a_0^\dagger a_0a_0^\dagger a_0-a_0^\dagger a_0)=\frac{a}{\pi L}\epsilon_LN_0(N_0-1)=\frac{a}{\pi L}\epsilon_LN^2-2N\frac{a}{\pi L}\epsilon_L\sum_{m\neq0}a_m^\dagger a_m$

下一阶项为: $H_2=\sum_{m\neq0}\epsilon_L\vec{m}^2a_m^\dagger a_m+\frac{Na}{\pi L}\epsilon_L\sum_{m\neq0}(4a_m^\dagger a_m+a_m^\dagger a_{-m}^\dagger+a_ma_{-m})$ 则非0动量二次型为:

$H_e=\epsilon_L\sum_{m\neq0}\left((\vec{m}^2+\frac{2Na}{\pi L})a_m^\dagger a_m+\frac{Na}{\pi L}(a_m^\dagger a_{-m}^\dagger+a_ma_{-m})\right)$ 引入波戈留波夫变换对角化:

$b_{\vec{m}}=u_{\vec{m}}a_{\vec{m}}+v_{\vec{m}}a_{-\vec{m}}^\dagger$; $b_{\vec{m}}^\dagger=u_{\vec{m}}a_{\vec{m}}^\dagger+v_{\vec{m}}a_{-\vec{m}}$ 要求 $[b_{\vec{m}},b_{\vec{m}}^\dagger]=1\Rightarrow u_{\vec{m}}^2-v_{\vec{m}}^2=1$

令 $u_{-\vec{m}}=u_{\vec{m}}$, $v_{-\vec{m}}=v_{\vec{m}}$, 可得 $a_{\vec{m}}=u_{\vec{m}}b_{\vec{m}}-v_{\vec{m}}b_{-\vec{m}}^\dagger$; 则哈密顿量 H_e 变为

$H_e=\sum_{\vec{m}\neq0}(A_{\vec{m}}(u_m^2+v_m^2)-4Bu_mv_m)b_m^\dagger b_m+(A_mv_m^2-2Bu_mv_m)+\sum_{\vec{m}\neq0}(B(u_m^2+v_m^2)-A_mu_mv_m)(b_m^\dagger b_{-m}^\dagger+b_mb_{-m})$

其中 $A_m=\epsilon_L(\vec{m}^2+\frac{2Na}{\pi L})$, $B=\epsilon_L\frac{Na}{\pi L}$

考虑 $H=\lambda a^\dagger a+\mathcal{F}(a^\dagger a^\dagger+aa)$ 对角化. 取 $b=ua+va^\dagger$; $b^\dagger=ua^\dagger+va$

$[b,b^\dagger]=u^2-v^2=1$; 则 $H=\lambda(ub^\dagger-vb)(ub-vb^\dagger)+\mathcal{F}((ub^\dagger-vb)^2+(ub-vb^\dagger)^2)$ $\quad a=ub-vb^\dagger$; $a^\dagger=ub^\dagger-vb$

对角化有: $u^2+v^2=\frac{\lambda}{\mathcal{F}}uv$ $\quad=\lambda(u^2b^\dagger b-uvb^\dagger b^\dagger-uvbb+v^2\overset{(1+b^\dagger b)}{\ })+\mathcal{F}(u^2b^\dagger b^\dagger-uvb^\dagger b-uv\overset{(1+b^\dagger b)}{\ }+v^2bb)+(u^2bb+v^2b^\dagger b^\dagger-uv\overset{(1+b^\dagger b)}{bb^\dagger}-uvb^\dagger b)$

计 $u=\cosh x$, $v=\sinh x$.

$\quad=b^\dagger b(\lambda(u^2+v^2)-4uv\mathcal{F})+(\lambda v^2-2uv\mathcal{F})+(\mathcal{F}(u^2+v^2)-\lambda uv)(b^\dagger b^\dagger+bb)$

则 $\tanh2x=\frac{2\mathcal{F}}{\lambda}$

则 $u^2+v^2=\cosh(2x)=\frac{1}{\sqrt{1-\tanh^2(2x)}}=\frac{1}{\sqrt{1-(2\mathcal{F}/\lambda)^2}}$ $\qquad 2uv=\sinh(2x)=\frac{\tanh(2x)}{\sqrt{1-\tanh^2(2x)}}=\frac{2\mathcal{F}/\lambda}{\sqrt{1-(2\mathcal{F}/\lambda)^2}}$

基态性质：存在相互作用时，基态满足 对 $\vec{k}\neq 0$，$\hat{b}_{\vec{k}}|G\rangle=0$；关注基态中 $\vec{k}$ 和 $-\vec{k}$ 两个模式，$|G_{\vec{k}}\rangle=\sum_{i,j} C_{ij}|i,j\rangle$；$i,j$ 分别表示 $\vec{k}$，$-\vec{k}$ 粒子数. $b_{\vec{k}}=u_m a_m+v_m a^\dagger_{-m}$ 则有

$\sum_{i,j} u_m\sqrt{i}\,C_{ij}|i-1,j\rangle+v_m\sqrt{j+1}\,C_{ij}|i,j+1\rangle=0 \Rightarrow C_{i+1,j+1}=-\frac{v_m}{u_m}\sqrt{\frac{j+1}{i+1}}C_{i,j}=-\frac{v_m}{u_m}\sqrt{\frac{i+1}{j+1}}C_{i,j}$

则有 $C_{ij}=C_{ii}\delta_{ij}$；$C_{i+1,i+1}=-\frac{v_m}{u_m}C_{ii} \Rightarrow |G_{\vec{k}}\rangle=\sum_{i=0}^{\infty}\left(-\frac{v_m}{u_m}\right)^i C_{0,0}|i,i\rangle$，归一化 $C_{0,0}=\frac{1}{u_m}$

模式 $\vec{k}$ 粒子数. $n_{\vec{k}}=\langle G|n_{\vec{k}}|G\rangle=\sum_{i=0}^{\infty} i\,\frac{1}{u_m^2}\left(\frac{v_m^2}{u_m^2}\right)^i=v_m^2=\frac{\cosh(2\chi_m)-1}{2}=\frac{\frac{1}{\sqrt{1-(2B/A)^2}}-1}{2}$

相对论量子力学：6.1 KLEIN-GORDAN 方程

$E^2=c^2p^2+m^2c^4$，$E\to i\hbar\frac{\partial}{\partial t}$，$p\to -i\hbar\nabla$

$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi=\nabla^2\psi-\frac{m^2c^2}{\hbar^2}\psi \Rightarrow \frac{1}{c^2}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)=\nabla\cdot(\psi^*\nabla\psi-\psi\nabla\psi^*)$；$\vec{j}=\frac{\hbar}{2mi}(\psi^*\nabla\psi-\psi\nabla\psi^*)$

$\rho=\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)\Rightarrow\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0$；平面波解：$\psi(\vec{r},t)=Ce^{i\vec{k}\cdot\vec{r}-i\omega t}\Rightarrow E=\pm\sqrt{p^2c^2+m^2c^4}$

非相对论极限：$E=\sqrt{p^2c^2+m^2c^4}=mc^2\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}\approx mc^2+\frac{p^2}{2m}-\frac{1}{8}\frac{p^4}{m^3c^2}+\cdots$

令 $\psi(\vec{r},t)=e^{-imc^2t/\hbar}\varphi(\vec{r},t)$；$\frac{\partial^2}{\partial t^2}\psi(\vec{r},t)=\left(\frac{\partial^2}{\partial t^2}\varphi(\vec{r},t)-2\frac{imc^2}{\hbar}\frac{\partial}{\partial t}\varphi-\frac{m^2c^4}{\hbar^2}\varphi\right)e^{-imc^2t/\hbar}$

忽略第一项，代入K-G方程 $\Rightarrow -2\frac{im}{\hbar}\frac{\partial}{\partial t}\varphi-\frac{m^2c^2}{\hbar^2}\varphi=\nabla^2\varphi-\frac{m^2c^2}{\hbar^2}\varphi\Rightarrow i\hbar\frac{\partial}{\partial t}\varphi=-\frac{\hbar^2}{2m}\nabla^2\varphi$

在电磁场中，$E\to i\hbar\frac{\partial}{\partial t}-q\phi$，$\vec{p}\to -i\hbar\nabla-\frac{q}{c}\vec{A}$；$\left(i\hbar\frac{\partial}{\partial t}-q\phi\right)^2\psi=c^2\left(-i\hbar\nabla-\frac{q}{c}\vec{A}\right)^2\psi+m^2c^4\psi$

类似地，$\psi(\vec{r},t)=\varphi(\vec{r},t)e^{-imc^2t/\hbar}$ 得非相对论近似 $i\hbar\frac{\partial}{\partial t}\varphi(\vec{r},t)=\left[\frac{1}{2m}\left(-i\hbar\nabla-\frac{q}{c}\vec{A}\right)^2+q\phi\right]\varphi(\vec{r},t)$

规范不变性，$\vec{A}\to\vec{A}'=\vec{A}+\nabla f$；$\phi\to\phi'=\phi-\frac{1}{c}\frac{\partial}{\partial t}f$；则有 $\psi\to\psi'=e^{i\theta}\psi$；$\theta=i\frac{qf}{\hbar c}$

考虑负能解情形：$\left(i\hbar\frac{\partial}{\partial t}-q\phi\right)^2\psi^-=c^2\left(-i\hbar\nabla-\frac{q}{c}\vec{A}\right)^2\psi^-+m^2c^4\psi^-$；$\psi^-$ 表示负能波函数.

取共轭有：$\left(i\hbar\frac{\partial}{\partial t}+q\phi\right)^2\psi_c^-=c^2\left(-i\hbar\nabla+\frac{q}{c}\vec{A}\right)^2\psi_c^-+m^2c^4\psi_c^-$；有 $\psi_c^-=\psi^{-*}$；其中 ψ_c^- 为正能KG态，电荷为 $-q$，在相同势场下的波函数.

协变形式及Lorentz变换. def. $x^\mu=(x^0,x^1,x^2,x^3)=(ct,x,y,z)$；$x_\mu=(ct,-x,-y,-z)$

矩阵张量 $g=(g_{\mu\nu})=(g^{\mu\nu})=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$；$g_\mu^\nu=g_{\mu\sigma}g^{\sigma\nu}=\delta_\mu^\nu=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$ 逆变 x^μ，协变 x_μ $x^\mu=g^{\mu\nu}x_\nu$；$x_\mu=g_{\mu\nu}x^\nu$

$\partial_\mu=\frac{\partial}{\partial x^\nu}=(\partial_0,\nabla)$；$\partial^\mu=\frac{\partial}{\partial x_\nu}=(\partial_0,-\nabla)$；四动量：$p^\mu=\left(\frac{E}{c},\vec{p}\right)$，$p_\mu=\left(\frac{E}{c},-\vec{p}\right)$

$a_\mu b^\mu=g^{\mu\nu}a_\nu b_\mu=g_{\mu\nu}a^\nu b^\mu=a^0b^0-\vec{a}\cdot\vec{b}$；$\partial_\mu\partial^\mu=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2$；$p_\mu p^\mu=\frac{E^2}{c^2}-p^2$. $=g_{\mu\nu}\partial^\nu\partial^\mu$

KG方程 $\left[\frac{1}{c^2}\frac{\partial}{\partial t}-\nabla^2+\frac{m^2c^2}{\hbar}\right]\psi=0\Rightarrow\left[\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\right]\psi=0$. D'Almbert算子：$\Box=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2=\partial_\mu\partial^\mu$

Lorentz变换：$ct'=\frac{ct-\beta x}{\sqrt{1-\beta^2}}$，$x'=\frac{x-\beta ct}{\sqrt{1-\beta^2}}$；$ct=\frac{ct'+\beta x'}{\sqrt{1-\beta^2}}$，$x=\frac{x'+\beta ct'}{\sqrt{1-\beta^2}}$

逆变向量：
$x^{0'}=x^0\cosh\eta-x^1\sinh\eta$
$x^{1'}=-x^0\sinh\eta+x^1\cosh\eta$
$x^{2'}=x^2$
$x^{3'}=x^3$

协变张量
$x_0'=x_0\cosh\eta+x_1\sinh\eta$
$x_1'=x_0\sinh\eta+x_1\cosh\eta$
$x_2'=x_2$
$x_3'=x_3$

$x^\mu=L^\mu_\nu x^\nu$；$L^\lambda_\mu g^{\mu\nu}L^\rho_\nu=g^{\lambda\rho}$；$LgL^T=g$

6.2 Dirac方程

将能量写作 $E=c\alpha\cdot\hat{p}+\beta mc^2$；满足 $E^2=p^2c^2+m^2c^4$ 时有 $\boxed{\sigma_y\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$

$\{\alpha^k,\alpha^l\}=2\delta_{kl}$，$\{\alpha^k,\beta\}=0$ $\beta^2=I$；$i\hbar\frac{\partial}{\partial t}\psi=(c\alpha\cdot\hat{p}+\beta mc^2)\psi$；$\vec{\alpha},\beta$ 厄米. $\alpha^k=\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}$ $\beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}$

Dirac方程可重新写作 $\frac{1}{c}\frac{\partial\psi}{\partial t}+\sum_1^3\alpha^k\frac{\partial\psi}{\partial x_k}+i\frac{mc}{\hbar}\beta\psi=0\Rightarrow\frac{d}{dt}(\psi^\dagger\psi)+\sum_{k=1}^3\frac{\partial j^k}{\partial x_k}=0$

$j^k=c\psi^\dagger\alpha^k\psi$，令 $j^0=c\rho$. 则 $\partial_\mu j^\mu=0$

$H=-i\hbar c\vec{\alpha}\cdot\nabla+mc^2\beta$. $\frac{d}{dt}x_k=\frac{1}{i\hbar}[x_k,H]=\frac{1}{i\hbar}[x_k,c\sum_l\alpha^l p_l]=c\alpha^k$ 即 $v_k=\partial_t x_k=c\alpha^k$

$j^k=\psi^\dagger v_k\psi$. 角动量算符：$\vec{L}=\vec{r}\times\vec{p}=-i\hbar\vec{r}\times\nabla$；$[H,\vec{L}]=-\hbar^2c\,\vec{\alpha}\times\nabla$. 所以 $\vec{L}$ 不是守恒量.

自旋角动量 $\vec{S}=\frac{\hbar}{2}\vec{\Sigma}=\frac{\hbar}{2}\begin{pmatrix}\vec{\sigma}&0\\0&\vec{\sigma}\end{pmatrix}$ 利用等式：$[\vec{\alpha}\cdot\vec{A},\vec{\Sigma}]=\begin{pmatrix}0&[\vec{\sigma}\cdot\vec{A},\vec{\sigma}]\\ [\vec{\sigma}\cdot\vec{A},\vec{\sigma}]&0\end{pmatrix}$ 有 $[\vec{\alpha}\cdot\vec{A},\vec{\Sigma}]=2i\vec{\alpha}\times\vec{A}$

则有 $[H,\vec{\Sigma}]=-i\hbar c[\vec{\alpha}\cdot\nabla,\vec{\Sigma}]=2\hbar c\,\vec{\alpha}\times\nabla$. 则总角动量 $\vec{J}=\vec{L}+\vec{S}=\vec{L}+\frac{\hbar}{2}\vec{\Sigma}$ 满足 $[H,\vec{J}]=0$ 为守恒量.

螺旋度算符 $\hat{h}=\vec{\Sigma}\cdot\hat{p}$ $\hat{p}=\frac{\vec{p}}{|\vec{p}|}$ 也与 H 对易. $[H,\hat{h}]=0$

$[H,\vec{\Sigma}\cdot\vec{p}]=0$

Dirac方程协变形式 记 $\gamma^\mu=(\beta,\beta\alpha^i)\Rightarrow\left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\Psi=\left(\gamma^\mu\frac{p_\mu}{\hbar}-\frac{mc}{\hbar}\right)\Psi=0$

$p^\mu=(i\partial_0,\vec{p})$ $p_\mu=(i\partial_0,-\vec{p})$ $\partial_\mu=\frac{\partial}{\partial x^\mu}$ $\gamma_0=\begin{pmatrix}I&0\\0&-I\end{pmatrix}$ $\gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}$ $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

γ^0 厄米，γ^i 反厄米. 协变形式 $\gamma_\mu=g_{\mu\nu}\gamma^\nu$ $i\gamma\cdot\partial\Psi=m\Psi$

在Pauli度规下. $x_\mu=(x_1,x_2,x_3,x_4)=(x,y,z,ict)$；$\partial_\mu=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},-\frac{i}{c}\frac{\partial}{\partial t}\right)$；$\gamma_k=-i\beta\alpha^k$，$\gamma_4=\beta$

$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$，此时 γ 厄米. 且 $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}\Rightarrow$ Dirac方程 $\left(\gamma_\mu\partial_\mu+\frac{mc}{\hbar}\right)\Psi=0$

Dirac费米子哈密顿量：$H=c\vec{\alpha}\cdot\vec{p}+\beta mc^2$. 系统角动量算符为 L. 求 $[H,\vec{L}]=-\hbar^2c\,\vec{\alpha}\times\nabla$；$\vec{J}=\vec{L}+\vec{S}$ 守恒

两个无相互作用的全同玻色子处于一维谐振势中，基态及第一激发态波函数及本征能量

基态：$|2,0\rangle=\frac{1}{2!}(|\phi_1(1)\rangle\otimes|\phi_1(2)\rangle+|\phi_1(2)\rangle\otimes|\phi_1(1)\rangle)$ $E_0=\hbar\omega$；其中 $|\phi_i(j)\rangle$ 表示第 j 个粒子处于第 i 个量子态.

第一激发：$|1,1\rangle=\frac{1}{\sqrt{2!}}(|\phi_1(1)\rangle\otimes|\phi_2(2)\rangle+|\phi_2(1)\rangle\otimes|\phi_1(2)\rangle)$ $E_1=2\hbar\omega$.

若排斥作用无穷大，写出基态波函数和本征能量.

基态：$|1,1\rangle=\frac{1}{\sqrt{2!}}(|\phi_1(1)\rangle\otimes|\phi_2(2)\rangle+|\phi_2(1)\rangle\otimes|\phi_1(2)\rangle)$ $E_0'=2\hbar\omega$