高等量子力学第七次作业

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习题 4.1

试证明:

$$H\left|\psi_k^-\right\rangle = E_k\left|\psi_k^-\right\rangle.$$

Proof. 根据讲义内容可得:

$$\Omega(t+s) = e^{iH(t+s)}e^{-iH_0(t+s)} = e^{iHs}\Omega(t)e^{-iH_0 s}.$$

当 $t \to +\infty$ 时，有:

$$\Omega_- = e^{iHs}\Omega_- e^{-iH_0 s} = \Omega_- + is(H\Omega_- - \Omega_- H_0) + \mathcal{O}(s^2).$$

即有:

$$H\Omega_- = \Omega_- H_0.$$

从而可以计算:

$$H\left|\psi_k^-\right\rangle = H\Omega_-\left|k\right\rangle = \Omega_- H_0\left|k\right\rangle = E_k\Omega_-\left|k\right\rangle = E_k\left|\psi_k^-\right\rangle.$$

□

习题 4.2

证明散射矩阵 S 是幺正的。

Proof. 散射矩阵为;

$$S = U_I(+\infty,-\infty) = U_I(+\infty,0)U_I(0,-\infty) = \Omega_-^\dagger\Omega_+.$$

可以计算:

$$S^\dagger S = \Omega_+^\dagger\Omega_-\Omega_-^\dagger\Omega_+.$$

利用 $\left\langle\psi_{k'}^\pm\middle|\psi_k^\pm\right\rangle = \left\langle k'\right|\Omega_\pm^\dagger\Omega_\pm\left|k\right\rangle = \delta(k'-k)$，可得:

$$\Omega_\pm^\dagger\Omega_\pm = 1.$$

利用

$$\left|\psi_k^+\right\rangle = \Omega_+\left|k\right\rangle, \quad \left|\psi_k^-\right\rangle = \Omega_-\left|k\right\rangle.$$

可以将 $\Omega_\pm$ 形式上写作:

$$\Omega_+ = \int dk\left|\psi_k^+\right\rangle\left\langle k\right|; \quad \Omega_- = \int dk\left|\psi_k^-\right\rangle\left\langle k\right|.$$

则可以计算:

$$S^\dagger S = \Omega_+^\dagger\int dk\left|\psi_k^-\right\rangle\left\langle k\right|\int dk'\left|k'\right\rangle\left\langle\psi_{k'}^-\right|\Omega_+$$

$$
\begin{aligned}
&=\Omega_+^\dagger \int dk \left|\psi_k^-\right\rangle \left\langle\psi_k^-\right| \Omega_+ \\
&=\int dk_1 \left|k_1\right\rangle \left\langle\psi_{k_1}^+\right| (1-\sum_k \left|\phi_k\right\rangle \left\langle\phi_k\right|) \int dk_2 \left|\psi_{k_2}^+\right\rangle \left\langle k_2\right| \\
&=\int dk_1 dk_2 \left|k_1\right\rangle \left\langle k_2\right| \delta(k_1-k_2) \\
&=\int dk_1 \left|k_1\right\rangle \left\langle k_1\right| \\
&=1.
\end{aligned}
$$

其中，$|\phi_k\rangle$ 为 H 的束缚态，完备性关系为:

$$
\int dk \left|\psi_k^-\right\rangle \left\langle\psi_k^-\right| + \sum_k \left|\phi_k\right\rangle \left\langle\phi_k\right| = 1.
$$

此外，由于 $|\phi_k\rangle$ 为 H 的本征值为负数的本征态，$|\phi_k^+\rangle$ 为 H 的本征值为正数的本征态，则有正交关系:

$$
\left\langle\phi_k\middle|\psi_k^+\right\rangle = 0.
$$

类似地，可以计算:

$$
\begin{aligned}
SS^\dagger &=\Omega_-^\dagger \Omega_+ \Omega_+^\dagger \Omega_- \\
&=\Omega_-^\dagger \int dk \left|\psi_k^+\right\rangle \left\langle k\right| \int dk' \left|k'\right\rangle \left\langle\psi_{k'}^+\right| \Omega_- \\
&=\Omega_-^\dagger \int dk \left|\psi_k^+\right\rangle \left\langle\psi_k^+\right| \Omega_- \\
&=\int dk_1 \left|k_1\right\rangle \left\langle\psi_{k_1}^-\right| (1-\sum_k \left|\phi_k'\right\rangle \left\langle\phi_k'\right|) \int dk_2 \left|\psi_{k_2}^-\right\rangle \left\langle k_2\right| \\
&=\int dk_1 dk_2 \left|k_1\right\rangle \left\langle k_2\right| \delta(k_1-k_2) \\
&=\int dk_1 \left|k_1\right\rangle \left\langle k_1\right| \\
&=1.
\end{aligned}
$$

即有:

$$
SS^\dagger = S^\dagger S = 1.
$$

即散射矩阵 S 是幺正的。 □

习题 4.3

求解球面出射波 $\langle\vec{r}|\,\phi\rangle = \frac{1}{(2\pi)^{3/2}}\frac{e^{ikr}}{r}$ 的几率流分布。并讨论其当 $r\to\infty$ 时的极限。

Proof. 几率流密度算符为:

$$
\vec{J}(\vec{r}) = \frac{1}{2M}(\left|\vec{r}\right\rangle \left\langle\vec{r}\right| \vec{p} + \vec{p} \left|\vec{r}\right\rangle \left\langle\vec{r}\right|).
$$

则可以计算几率流为:

$$\begin{aligned}\langle\phi|\vec{J}|\phi\rangle =&\frac{1}{2M}(\langle\phi|\vec{r}\rangle\langle\vec{r}|\hat{\vec{p}}|\phi\rangle+\langle\phi|\hat{\vec{p}}|\vec{r}\rangle\langle\vec{r}|\phi\rangle)\\ =&\frac{1}{2M}\frac{1}{(2\pi)^3}\left(\frac{e^{-ikr}}{r}(-i\hbar\nabla)\frac{e^{ikr}}{r}+\left(-i\hbar\nabla\frac{e^{ikr}}{r}\right)^*\frac{e^{ikr}}{r}\right)\\ =&\frac{1}{2M}\frac{1}{(2\pi)^3}\left(\hbar\frac{2kr}{r^3}\vec{e}_r\right)\\ =&\frac{1}{(2\pi)^3}\frac{\hbar k}{Mr^2}\vec{e}_r.\end{aligned}$$

其中 $\nabla=\frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{e}_\phi$。当 $r\to\infty$ 时，几率流满足:

$$\langle\phi|\vec{J}|\phi\rangle\propto\frac{1}{r^2}\vec{e}_r.$$

$\square$