高等热力学与统计物理第四次作业

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1 计算下列系统内电子气体的 Fermi 能，Fermi 温度和 Fermi 速度

(1) 室温下的金属钠：密度为 $0.97g/cm^3$，每个原子贡献一个传导电子，假设他们的能量和动量关系为 $E=\frac{p^2}{2m}$，即忽略晶格场对电子运动的影响。

在 0K 下，费米狄拉克分布为：

$$f=\begin{cases}1, & E<\epsilon_F,\\ 0, & E>\epsilon_F.\end{cases}$$

则粒子数 N 为：

$$\begin{aligned}N&=2\left(\frac{L}{2\pi}\right)^3\iiint dk_x\,dk_y\,dk_z\\ &=2\frac{V}{(2\pi)^3}\int 4\pi k^2\,dk\\ &=\frac{V}{2\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\int_0^{\epsilon_F}\sqrt{E}\,dE\\ &=\frac{V}{3\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\epsilon_F^{3/2}.\end{aligned}$$

其中粒子数 $N=nN_A$，体积 $V=\frac{nM}{\rho}$，n 为物质的量，N_A 为阿伏伽德罗常数，M 为钠的摩尔质量。则有：

$$\epsilon_F=\frac{1}{2m}\left(\frac{3\pi^2\hbar^3N_A\rho}{M}\right)^{2/3}.$$

代入数据可得费米能量为：

$$\epsilon_F=5.048\times10^{-19}J=3.15eV.$$

费米温度为：

$$T_F=\frac{\epsilon_F}{k_B}=36562K.$$

费米速度为：

$$v_F=\sqrt{\frac{2\epsilon_F}{m}}=1.05\times10^6m/s.$$

(2) 天狼星的伴星（白矮星）：其质量约为太阳质量的 0.98 倍，半径约为太阳半径的 0.0084 倍。假设星体全部由氦构成。

对于白矮星而言，密度为:

$$\rho = \frac{0.98 m_{sun}}{\frac{4}{3}\pi(0.0084 r_{sun})^3} = 2.315 \times 10^9 kg/m^3.$$

每一个氦原子贡献 2 个电子，考虑相对论条件粒子数满足:

$$N = \frac{2V}{\pi^2}\int_0^{k_F} k^2\, dk = \frac{2V}{3\pi^2}k_F^3.$$

则有:

$$k_F = \sqrt[3]{\frac{6\pi^2 N_A \rho}{M}} = 2.743 \times 10^{12} m^{-1}.$$

费米能量为:

$$\epsilon_F = \sqrt{p^2c^2 + m^2c^4} - mc^2 = 3.739 \times 10^{-14} J = 2.33 \times 10^5 eV.$$

费米温度为:

$$T_F = \frac{\epsilon_F}{k_B} = 2.708 \times 10^9 K.$$

费米速度为:

$$v_F = \sqrt{\frac{p^2c^2}{p^2 + m^2c^2}} = 2.180 \times 10^8 m/s.$$

2 证明非相对论简并电子气体的热力学函数是

$$G = N\mu = N\epsilon_F\left[1 - \frac{1}{12}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2 - \frac{1}{80}\pi^4\left(\frac{kT}{\epsilon_F}\right)^4 \dots\right]$$

$$E = \frac{3}{5}N\epsilon_F\left[1 + \frac{5}{12}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2 - \frac{1}{16}\pi^4\left(\frac{kT}{\epsilon_F}\right)^4 \dots\right]$$

$$C_V = \frac{1}{2}N\pi^2\frac{k^2T}{\epsilon_F}\left[1 - \frac{3}{10}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2 \dots\right]$$

$$S = \frac{1}{2}N\pi^2\frac{k^2T}{\epsilon_F}\left[1 - \frac{1}{10}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2 \dots\right]$$

其中 $N = \frac{V}{3\pi^2}\left(\frac{2m\varepsilon_F}{\hbar}\right)^{3/2}$ 为电子总数，$\dots$ 代表 $\frac{kT}{\varepsilon_F}$ 的更高阶项。

粒子数 N 满足:

$$\begin{aligned} N &= \frac{V(2m)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1}\, d\epsilon \\ &= \frac{V(2m)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{2\beta}{3}\epsilon^{3/2}\frac{e^{\beta(\epsilon-\mu)}}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2}\, d\epsilon. \end{aligned}$$

内能为:

$$E=\frac{V(2m)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1}\,d\epsilon$$
$$=\frac{V(2m)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{2\beta}{5}\epsilon^{5/2}\frac{e^{\beta(\epsilon-\mu)}}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2}\,d\epsilon.$$

令 $\xi=\beta(\epsilon-\mu)$，则积分化为:

$$N=\frac{V(2m)^{3/2}}{3\pi^2\hbar^3}\int_{-\beta\mu}^\infty (kT\xi+\mu)^{3/2}\frac{e^\xi}{(e^\xi+1)^2}\,d\xi.$$
$$E=\frac{V(2m)^{3/2}}{5\pi^2\hbar^3}\int_{-\beta\mu}^\infty (kT\xi+\mu)^{5/2}\frac{e^\xi}{(e^\xi+1)^2}\,d\xi.$$

令

$$I=\int_{-\beta\mu}^\infty (kT\xi+\mu)^{3/2}\frac{e^\xi}{(e^\xi+1)^2}\,d\xi\approx\int_{-\infty}^\infty (kT\xi+\mu)^{3/2}\frac{e^\xi}{(e^\xi+1)^2}\,d\xi.$$

对 $(kT\xi+\mu)^{3/2}$ 做泰勒展开得:

$$(kT\xi+\mu)^{3/2}=\mu^{3/2}+\frac{3}{2}kT\mu^{1/2}\xi+\frac{3}{8}(kT)^2\mu^{-1/2}\xi^2-\frac{1}{16}(kT)^3\mu^{-3/2}\xi^3+\frac{3}{128}(kT)^4\mu^{-5/2}\xi^4+\mathcal{O}(\xi^4).$$

对 $(kT\xi+\mu)^{5/2}$ 做泰勒展开可得:

$$(kT\xi+\mu)^{5/2}=\mu^{5/2}+\frac{5}{2}kT\mu^{3/2}\xi+\frac{15}{8}(kT)^2\mu^{1/2}\xi^2+\frac{5}{16}(kT)^3\mu^{-1/2}\xi^3-\frac{5}{128}(kT)^4\mu^{-3/2}\xi^4+\mathcal{O}(\xi^4).$$

由于 $\frac{e^\xi}{(e^\xi+1)^2}$ 是关于 ξ 的偶函数,则泰勒展开只有偶次幂的积分不为零。令 $c_n=\int_0^\infty\frac{\xi^{2n}e^\xi}{(e^\xi+1)^2}\,d\xi$。当 $n=0$ 时，$c_0=\frac{1}{2}$。当 $n>0$ 时，有:

$$\frac{e^\xi}{(e^\xi+1)^2}=\frac{e^{-\xi}}{(e^{-\xi}+1)^2}=\sum_{l=1}^\infty(-1)^{l-1}le^{-l\xi}.$$

则有:

$$c_n=\sum_{l=1}^\infty(-1)^{l-1}l\int_0^\infty \xi^{2n}e^{-l\xi}\,d\xi=(2n)!\sum_{l=1}^\infty\frac{(-1)^{l-1}}{l^{2n}}.$$

则有:

$$c_1=2\left(\sum_{l=odd}\frac{1}{l^2}-\sum_{l=even}\frac{1}{l^2}\right)=2\left(\sum_{l=1}^\infty\frac{1}{l^2}-\frac{2}{2^2}\sum_{l=1}^\infty\frac{1}{l^2}\right)=\frac{\pi^2}{6}.$$
$$c_2=24\left(\sum_{l=odd}\frac{1}{l^4}-\sum_{l=even}\frac{1}{l^4}\right)=24\left(\sum_{l=1}^\infty\frac{1}{l^4}-\frac{2}{16}\sum_{l=1}^\infty\frac{1}{l^4}\right)=\frac{7\pi^4}{30}.$$

则有:

$$N=\frac{V(2m)^{3/2}}{3\pi^2\hbar^3}\mu^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\dots\right].$$
$$E=\frac{V(2m)^{3/2}}{5\pi^2\hbar^3}\mu^{5/2}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2-\frac{7\pi^4}{384}\left(\frac{kT}{\mu}\right)^4\dots\right].$$

则有:

$$E=\frac{3}{5}N\mu\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\mu}\right)^2-\frac{11\pi^4}{120}\left(\frac{kT}{\mu}\right)^4\right].$$

对比

$$N=\frac{V(2m)^{3/2}}{3\pi^2\hbar^3}\epsilon_F^{3/2}.$$

有:

$$\epsilon_F=\mu\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\dots\right]^{2/3}$$

泰勒展开可得:

$$\epsilon_F=\mu\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2+\frac{\pi^4}{180}\left(\frac{kT}{\mu}\right)^4\right]$$

利用级数反演可得:

$$\mu=\epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2-\frac{\pi^4}{80}\left(\frac{kT}{\epsilon_F}\right)^4\cdots\right]$$

则 Gibbs 势为:

$$G=N\mu=N\epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2-\frac{\pi^4}{80}\left(\frac{kT}{\epsilon_F}\right)^4\cdots\right].$$

内能为:

$$E=\frac{3}{5}N\epsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2-\frac{\pi^4}{16}\left(\frac{kT}{\epsilon_F}\right)^4\cdots\right]$$

则等容热熔为:

$$C_V=\left(\frac{\partial E}{\partial T}\right)_V=\frac{1}{2}N\pi^2\frac{k^2T}{\epsilon_F}\left[1-\frac{3}{10}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]$$

在非相对论条件下有:

$$PV=\frac{2}{3}E.$$

则气体自由能为:

$$F=G-PV=G-\frac{2}{3}E=N\epsilon_F\left[\frac{3}{5}-\frac{\pi^2}{4}\left(\frac{kT}{\epsilon_F}\right)^2+\frac{\pi^4}{80}\left(\frac{kT}{\epsilon_F}\right)^4+\dots\right]$$

则熵为:

$$S=-\left(\frac{\partial F}{\partial T}\right)_V=N\epsilon_F\left(\frac{\pi^2}{2}\frac{k^2T}{\epsilon_F^2}-\frac{\pi^4}{20}\frac{k^4T^3}{\epsilon_F^4}+\cdots\right).$$