董建宇　2019511017　大三

1. AD

2. 解. 令n表示该理想气体的物质的量, R为理想气体常数.

一定质量的理想气体内能只与温度有关, 因此在等温过程中内能不变。

① 考虑$A\to B$过程. 由热力学第一定律$\Delta U=\Delta Q-\Delta W=0$可知, 气体吸热等于气体对外做功.

理想气体状态方程为$PV=nRT_2$, 则气体对外做功为

$$W_{AB}=\int_{V_A}^{V_B}P\,dV=nRT_2\int_{V_A}^{V_B}\frac{1}{V}dV=nRT_2\ln\frac{V_B}{V_A}.$$

则该过程吸热$Q_{AB}=W_{AB}=nRT_2\ln\frac{V_B}{V_A}$.

② 考虑$C\to D$过程, 由热力学第一定律$\Delta U=\Delta Q-\Delta W=0$可知, 气体放热等于外界对气体做功.

理想气体状态方程为$PV=nRT_1$. 则外界对气体做功为

$$W_{DC}=\int_{V_D}^{V_C}P\,dV=nRT_1\int_{V_D}^{V_C}\frac{1}{V}dV=nRT_1\ln\frac{V_C}{V_D}.$$

则该过程放热$Q_{CD}=W_{DC}=nRT_1\ln\frac{V_C}{V_D}$.

③ 考虑绝热过程$B\to C$, $D\to A$. 有:

$$P_BV_B^{\gamma}=P_CV_C^{\gamma},\quad P_DV_D^{\gamma}=P_AV_A^{\gamma},\quad Q_{BC}=Q_{DA}=0$$

则有: $\frac{P_BV_B^{\gamma}}{P_AV_A^{\gamma}}=\frac{P_CV_C^{\gamma}}{P_DV_D^{\gamma}}$. 利用理想气体状态方程$PV=nRT$, 同时有$T_A=T_B=T_2$, $T_C=T_D=T_1$可得:

$\left(\frac{V_B}{V_A}\right)^{\gamma-1}=\left(\frac{V_C}{V_D}\right)^{\gamma-1}$. 两侧取对数, $(\gamma-1)\ln\frac{V_B}{V_A}=(\gamma-1)\ln\frac{V_C}{V_D}$. 由于$\gamma-1\neq0$, 有$\ln\frac{V_B}{V_A}=\ln\frac{V_C}{V_D}$.

综上所述、循环效率为$\eta=1-\frac{Q_{CD}}{Q_{AB}}=1-\frac{T_1}{T_2}$

注意到,论述过程中并无假设热容与温度的关系,因而适用多原子分子理想气体.

3. 解. 由于磁场存在,能级发生劈裂,可分别计算自旋向上电子$N_\uparrow$与自旋向下的电子$N_\downarrow$

$$N_\uparrow = V\int_0^{+\infty}\frac{D(\varepsilon)}{\exp(\beta(\varepsilon-\mu_eB-\mu))+1} = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^{+\infty}\sqrt{\varepsilon}\cdot\frac{d\varepsilon}{e^{\beta(\varepsilon-\mu_eB-\mu)}+1} = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^{+\infty}\frac{2}{3}\varepsilon^{\frac{3}{2}}\frac{e^{\beta(\varepsilon-\mu_eB-\mu)}}{\left[e^{\beta(\varepsilon-\mu_eB-\mu)}+1\right]^2}\cdot\beta d\varepsilon$$

令 $\xi = \beta(\varepsilon-\mu_eB-\mu)$ 则有 $\varepsilon = kT\xi+\mu_eB+\mu$ $\quad d\xi = \beta d\varepsilon$

则有 $$N_\uparrow = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_{-\beta(\mu_eB+\mu)}^{+\infty}d\xi\cdot\frac{2}{3}(\mu_eB+\mu+kT\xi)^{\frac{3}{2}}\frac{e^{\xi}}{(e^{\xi}+1)^2} \approx \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_{-\infty}^{+\infty}\frac{2}{3}(\mu+\mu_eB+kT\xi)^{3/2}\cdot\frac{e^{\xi}}{(e^{\xi}+1)^2}d\xi$$

令 $F(\mu+\mu_eB+kT\xi) = \frac{2}{3}(\mu+\mu_eB+kT\xi)^{\frac{3}{2}} \approx \frac{2}{3}(\mu+\mu_eB)^{\frac{3}{2}}+(\mu+\mu_eB)^{\frac{1}{2}}\cdot(kT\xi)+\frac{1}{4}(\mu+\mu_eB)^{-\frac{1}{2}}(kT\xi)^2+\cdots$

注意到 $\frac{e^{\xi}}{(e^{\xi}+1)^2}$ 为偶函数,上述展开式只有偶次幂贡献.

则可计算 $N_\uparrow = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left(\frac{2}{3}(\mu+\mu_eB)^{\frac{3}{2}}+\frac{\pi^2}{12}(kT)^2\cdot(\mu+\mu_eB)^{-\frac{1}{2}}\right)$

只需将 μ_eB 替换成 $-\mu_eB$. 可得 $N_\downarrow = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left(\frac{2}{3}(\mu-\mu_eB)^{\frac{3}{2}}+\frac{\pi^2}{12}(kT)^2(\mu-\mu_eB)^{-\frac{1}{2}}\right)$

则总磁矩保留至B领头阶有

$$M = \mu_e(N_\uparrow-N_\downarrow) = \frac{1}{2}\mu_e\cdot\frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\cdot\left\{\frac{2}{3}\left[(\mu+\mu_eB)^{\frac{3}{2}}-(\mu-\mu_eB)^{\frac{3}{2}}\right]+\frac{\pi^2}{12}(kT)^2\left[(\mu+\mu_eB)^{-\frac{1}{2}}-(\mu-\mu_eB)^{-\frac{1}{2}}\right]\right\}$$

$$= \mu_e\frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\cdot\left(2\mu^{\frac{1}{2}}\mu_eB-\frac{\pi^2}{12}(kT)^2\mu^{-\frac{3}{2}}\mu_eB\right)$$

$$= \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_e^2B\left(\frac{3}{2}\mu^{\frac{1}{2}}-\frac{\pi^2}{16}(kT)^2\mu^{-\frac{3}{2}}\right) \qquad \frac{3}{4}\cdot\frac{1}{\pi}$$

利用 $\mu = \varepsilon_F\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$ 可得:

$$M = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_e^2B\left(\frac{3}{2}\varepsilon_F^{\frac{1}{2}}\cdot\left(1-\frac{\pi^2}{24}\left(\frac{kT}{\varepsilon_F}\right)^2\right)-\frac{\pi^2}{16}(kT)^2\varepsilon_F^{-\frac{3}{2}}\cdot\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)\right)^{-\frac{3}{2}}\right)$$

$$= \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_e^2B\cdot\varepsilon_F^{\frac{1}{2}}\cdot\left(\frac{3}{2}-\frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$

$$= \frac{3}{2}\frac{V}{3\pi^2}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{\frac{3}{2}}\mu_e^2B\cdot\frac{1}{\varepsilon_F}\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$

利用 $\frac{N}{V} = n = \frac{1}{3\pi^2}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2}$ 替换可得 总磁化强度为

$$M = \frac{3nV\mu_e^2B}{2\varepsilon_F}\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$

磁化率为 $\chi = \mu_0\frac{\partial M}{\partial B}\Big|_{B=0} = \frac{3nV\mu_0\mu_e^2}{2\varepsilon_F}\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$

4. 解. (1) 对于气相，由 Mayer 第一定理，有：

$$\frac{P}{kT} = \sum_{l=1}^{\infty} b_l y^l = b_1 y + b_2 y^2 + \cdots$$

其中 $b_1 = \frac{1}{S}\int d^2\vec{r} = 1$， $b_2 = \frac{1}{2S}\int d^2\vec{r}_1 d^2\vec{r}_2 f_{12}$

由于最近邻相互作用为 u，则两体相互作用能量为

$$u_{ij} = \begin{cases} +\infty, & i=j \\ u, & i与j近邻 \\ 0, & 其它. \end{cases}$$

则有 $f_{ij} = e^{-\beta u_{ij}} - 1 = \begin{cases} -1, & i=j \\ e^{-\beta u}-1, & i,j近邻 \\ 0, & 其它. \end{cases}$

则 $b_2 = \frac{1}{2S}\int d^2\vec{r}_1 d^2\vec{r}_2 f_{ij} = \frac{1}{2}\cdot\Big[\underbrace{4(e^{-\beta u}-1)}_{\text{近邻相互作用}\\ \text{4为二维格气4个近邻}} - \underbrace{1}_{\text{自身与自身相互作用.}}\Big] = 2e^{-\beta u} - \frac{5}{2} = \frac{2}{x} - \frac{5}{2}$

则对于气相有 $\frac{P}{kT} = y + \left(\frac{2}{x} - \frac{5}{2}\right)y^2 + \cdots$

对于液相，有

$$\frac{P}{kT} = \ln y - \frac{nu}{2kT} + \sum_{l=1}^{\infty} b_l \left(\frac{e^{\beta u}}{y}\right)^l$$

对于该体系 $n=4$， $b_1 = 1$， $b_2 = \frac{2}{x} - \frac{5}{2}$，则有

$$\frac{P}{kT} = \ln y - 2\ln x + \frac{x^4}{y} + \left(\frac{2}{x} - \frac{5}{2}\right)\cdot\frac{x^8}{y^2} + \cdots$$

$$= \ln\frac{y}{x^2} + \frac{x^4}{y} + \left(2x^7 - \frac{5}{2}x^8\right)\cdot\frac{1}{y^2} + \cdots$$

综上所述：
$$\frac{P}{kT} = \begin{cases} y + \left(\frac{2}{x} - \frac{5}{2}\right)y^2 + \cdots & 气相 \\ \ln\frac{y}{x^2} + \frac{x^4}{y} + \left(2x^7 - \frac{5}{2}x^8\right)\cdot\frac{1}{y^2} + \cdots & 液相. \end{cases}$$

(2) 根据二维 Ising 模型 $k_c = 1$ 则有 $\tanh^2\frac{2|\varepsilon|}{kT_c} = \frac{1}{2}$

考虑到 $u = -4|\varepsilon|$，方程可化为 $\tanh\left(\frac{-u}{2kT_c}\right) = \frac{1}{\sqrt{2}}$，令 $z = e^{-\frac{u}{2kT_c}}$，则有 $\frac{z - \frac{1}{z}}{z + \frac{1}{z}} = \frac{z^2-1}{z^2+1} = \frac{1}{\sqrt{2}}$

即 $(\sqrt{2}-1)z^2 = \sqrt{2}+1$ 即 $e^{-\frac{u}{kT_c}} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{3-2\sqrt{2}}$. 则可解得邻界温度为 $kT_c = \frac{u}{\ln(3-2\sqrt{2})}$

两相密度差 $\rho_{A'} - \rho_A = \frac{1}{2}(1+M_{B'}) - \frac{1}{2}(1-M_{B'}) = M_{B'}$

其中 $M = \begin{cases} 0, & T > T_c \\ \left(1 - \frac{1}{\sinh^4\left(\frac{-u}{2kT}\right)}\right)^{1/8}, & T < T_c \end{cases}$

其中 $\sinh\left(-\frac{u}{kT}\right) = \sinh\left(-\frac{T_c\ln(3-2\sqrt{2})}{2T}\right) = \sinh\frac{T_c\ln(3+2\sqrt{2})}{2T} = \sinh\left(\frac{\ln(3+2\sqrt{2})}{2(1-z)}\right)$

令 $f(z) = 1 - \frac{1}{\sinh^4\left(\frac{\ln(3+2\sqrt{2})}{2(1-z)}\right)}$ 可以计算 $f(0) = 0$ 则 Taylor 展开保留一阶项有 $f(z) = f'(0)z$

则 $T<T_c$ 但 $T \to T_c$ 时有 $\rho_{A'} - \rho_A = M \to (f'(0)z)^{\frac{1}{8}}$.

(3). 适用于低密度区不适用于高密度区. 巨配分函数根分布在复 y-平面上为 (b).

5. 解. (1) 由定义可知, $\psi_E(\vec{r}', \vec{r}_2, \cdots, \vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \langle 0 | \psi(\vec{r}')\psi(\vec{r}_2)\cdots\psi(\vec{r}_N) | \Psi_E \rangle$

则有 $\psi_E^*(\vec{r}, \vec{r}_2, \cdots, \vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \langle \Psi_E | \psi^\dagger(\vec{r}_N) \cdots \psi^\dagger(\vec{r}_2)\psi^\dagger(\vec{r}) | 0 \rangle$

对于玻色子系统满足交换对称性, 即

$$\psi_E(\vec{r}_1, \cdots \vec{r}_i, \cdots \vec{r}_j, \cdots \vec{r}_N) = \psi_E(\vec{r}_1, \cdots, \vec{r}_j, \cdots \vec{r}_i, \cdots \vec{r}_N)$$

则可以计算关联函数

$$C_E(\vec{r}, \vec{r}') \equiv N \int \prod_{i=2}^{N} d^3\vec{r}_i \; \psi_E^*(\vec{r}, \vec{r}_2, \cdots, \vec{r}_N)\, \psi_E(\vec{r}', \vec{r}_2, \cdots, \vec{r}_N)$$

$$= \frac{1}{(N-1)!} \int \prod_{i=2}^{N} d^3\vec{r}_i \; \langle \Psi_E | \psi^\dagger(\vec{r}_N) \cdots \psi^\dagger(\vec{r}_2)\psi^\dagger(\vec{r}) | 0 \rangle \langle 0 | \psi(\vec{r}')\psi(\vec{r}_2)\cdots\psi(\vec{r}_N) | \Psi_E \rangle$$

$$= \frac{1}{(N-1)!} \int \prod_{i=2}^{N} d^3\vec{r}_i \; \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi^\dagger(\vec{r}_N) \cdots \psi^\dagger(\vec{r}_2)\; \psi(\vec{r}_2) \cdots \psi(\vec{r}_N)\, \psi(\vec{r}') | \Psi_E \rangle$$

$$= \frac{1}{(N-1)!} \int \prod_{i=3}^{N} d^3\vec{r}_i \; \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi^\dagger(\vec{r}_N) \cdots \psi^\dagger(\vec{r}_3)\; N_{op}\; \psi(\vec{r}_3) \cdots \psi(\vec{r}_N)\, \psi(\vec{r}') | \Psi_E \rangle$$

$$= \frac{1}{(N-1)!} \cdot 1 \int \prod_{i=4}^{N} d^3\vec{r}_i \; \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi^\dagger(\vec{r}_N) \cdots \psi^\dagger(\vec{r}_4)\; N_{op}\; \psi(\vec{r}_4) \cdots \psi(\vec{r}_N)\, \psi(\vec{r}') | \Psi_E \rangle$$

$$= \frac{1}{(N-1)!} \cdot 1 \cdot 2 \cdot \int \prod_{i=5}^{N} d\vec{r}_i \; \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi^\dagger(\vec{r}_N) \cdots \psi(\vec{r}_5)\; N_{op}\; \psi(\vec{r}_5) \cdots \psi(\vec{r}_N)\, \psi(\vec{r}') | \Psi_E \rangle$$

$$= \cdots = \frac{1}{(N-1)!} \cdot (N-1)! \; \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi(\vec{r}') | \Psi_E \rangle$$

$$= \langle \Psi_E | \psi^\dagger(\vec{r})\, \psi(\vec{r}') | \Psi_E \rangle$$

(2) 因为 Q 为正则配分函数, 则有

$$Q = \sum_E e^{-\frac{E}{kT}} = \mathrm{Tr}\; e^{-\frac{H}{kT}}$$

同时, 有

$$\sum_E e^{-\frac{E}{kT}} C_E(\vec{r}, \vec{r}') = \sum_E \langle \Psi_E | e^{-\frac{H}{kT}} \psi^\dagger(\vec{r})\psi(\vec{r}') | \Psi_E \rangle = \mathrm{Tr}\; e^{-\frac{H}{kT}} \psi^\dagger(\vec{r})\psi(\vec{r}')$$

则热力学平均为

$$G(\vec{r}, \vec{r}') \equiv \frac{1}{Q} \sum_E e^{-\frac{E}{kT}} C_E(\vec{r}, \vec{r}') = \frac{\mathrm{Tr}\; e^{-\frac{H}{kT}} \psi^\dagger(\vec{r})\psi(\vec{r}')}{\mathrm{Tr}\; e^{-\frac{H}{kT}}}$$