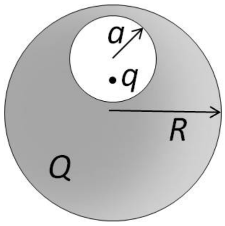
PHYS1314 Electrodynamics Spring 2021
Problem Set 3 solutions

1. (25 points) Dig a spherical cavity of radius a out of a larger solid conducting sphere of radius $R > 2a$ centered at the origin. The solid sphere is charged with a positive charge Q. The cavity is centered at a $\vec{x}_c = (0, 0, R/2)$. A negative point charge q is introduced at $\vec{x}_q = (0, 0, R/2 - a/2)$.



   (a) Calculate the electric field inside the cavity.
   (b) Calculate the surface charge density on the wall of the cavity.
   (c) Calculate the electric field outside the solid sphere.
   (d) Draw an accurate set of field lines representing the calculated electric field everywhere.
   (e) If you bring a new charge Q near the outside of the conductor, which (if any) of your preceding answers change? Explain.

Solution: (a) The calculation of the field was essentially laid out in class when we calculated the potential associated with a point charge outside a grounded conducting sphere. The idea here is the same. First, let us consider a grounded conducting sphere of radius a that is arbitrarily thin and centered on the origin. For a charge at position $z_Q\hat{z}$ (where $z_Q < a$), if we want to know the potential inside the sphere, we simply imagine that there is no conducting sphere and place an image charge, q', at point $\mathbf{x}' = (0, 0, z'$. The total potential of this charge configuration at a position $\mathbf{r}$ is

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\mathbf{r} - z_Q\hat{z}|} + \frac{q'}{|\mathbf{r} - z'\hat{z}|}\right]. \tag{1}$$

At the inner surface (at radial distance a from the center of the sphere) any position is described by the position $\mathbf{r} = a\hat{r}$, and the potential is zero:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|a\hat{r} - z_Q\hat{z}|} + \frac{q'}{|a\hat{r} - z'\hat{z}|}\right] = 0. \tag{2}$$

This can be rewritten by pulling a factor of a from the first denominator and a factor of z' from the second denominator:

$$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{a|\hat{r}-\frac{z_Q}{a}\hat{z}|}+\frac{q'}{z'|\frac{a}{z'}\hat{r}-\hat{z}|}\right]=\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{a|\hat{r}-\frac{z_Q}{a}\hat{z}|}+\frac{q'}{z'|\hat{z}-\frac{a}{z'}\hat{r}|}\right]=0. \qquad (3)$$

This will be satisfied if $\dfrac{Q}{a}=-\dfrac{q'}{z'}$ and $\dfrac{a}{z'}=\dfrac{z_Q}{a}$, so we must use an image charge $q'=-\dfrac{a}{z_Q}Q$ at a location $z'=\dfrac{a^2}{z_Q}$.

The previous is not necessary to the calculation, since we can get the previous result by just quoting the result given in the class notes for a charge q placed at a location $z\hat{z}$ outside a grounded conducting sphere. For that problem, we just need an image charge inside the sphere of charge $q'=-\dfrac{a}{z}q$ at location $z'\hat{z}=\dfrac{a^2}{z}\hat{z}$. In the current problem, we are instead given q' and z' inside the sphere with potential zero at radius a, so our image charge is $q=-\dfrac{z}{a}q'=\dfrac{a}{z'}q'=\dfrac{a}{z_Q}Q$ and $z=\dfrac{a^2}{z'}=\dfrac{a^2}{z_Q}$.

In the current assigned problem, if we place the origin at the center of the cavity, then he charge Q is located at $z_Q\hat{z}=-\dfrac{a}{2}\hat{z}$, so the image charge must be $q'=-2Q$ at $\mathbf{r}_{q'}=-2a\hat{z}$.

The potential inside the spherical cavity is then just that of the original charge and the image charge:

$$V(x,y,z)=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r}+\frac{a}{2}\hat{z}|}-\frac{2}{|\mathbf{r}+2a\hat{z}|}\right]. \qquad (4)$$

Likewise, the electric field is just the sum of the fields from each of these point charges:

$$\mathbf{E}(x,y,z)=\frac{Q}{4\pi\epsilon_0}\left[\frac{\mathbf{r}+\frac{a}{2}\hat{z}}{|\mathbf{r}+\frac{a}{2}\hat{z}|^3}-\frac{2\,(\mathbf{r}+2a\hat{z})}{|\mathbf{r}+2a\hat{z}|^3}\right]=\frac{Q}{4\pi\epsilon_0}\left[\frac{\left(x,y,z+\frac{a}{2}\right)}{|\left(x,y,z+\frac{a}{2}\right)|^3}-\frac{2\,(x,y,z+2a)}{|\,(x,y,z+2a)\,|^3}\right]$$

or

$$\boxed{\mathbf{E}(x,y,z)=\frac{Q}{4\pi\epsilon_0}\left[\frac{\left(x,y,z+\frac{a}{2}\right)}{\left[x^2+y^2+\left(z+\frac{a}{2}\right)^2\right]^{3/2}}-\frac{2\,(x,y,z+2a)}{\left[x^2+y^2+(z+2a)^2\right]^{3/2}}\right].} \qquad (5)$$

If we really want to write the electric field in terms of an origin centered at the center of the solid (other than the cavity) conducting sphere, we must displace the positions of the charges in the problem just solved by $\dfrac{1}{2}R\hat{z}$, so the electric field inside the cavity is then

$$\mathbf{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{\mathbf{r}-\left(\frac{R}{2}-\frac{a}{2}\right)\hat{z}}{|\mathbf{r}-\left(\frac{R}{2}-\frac{a}{2}\right)\hat{z}|^3} - \frac{2\left(\mathbf{r}-\left(\frac{R}{2}-2a\right)\hat{z}\right)}{|\mathbf{r}-\left(\frac{R}{2}-2a\right)\hat{z}|^3}\right]$$
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{(x,y,z-(R-a)/2)}{\left[x^2+y^2+(z-(R-a)/2)^2\right]^{3/2}} - \frac{2\,(x,y,z-(R-4a)/2)}{\left[x^2+y^2+(z-(R-4a)/2)^2\right]^{3/2}}\right]. \tag{6}$$

(b) The surface charge density at the surface of the cavity is just given by our standard boundary condition given to us by using Gauss's Law, the fact that the tangential part of the electric field at a surface is continuous (from the circulation of the field), and that the field inside a conductor is zero: $\vec{E}_{||} = 0$ and $\vec{E}_{\perp} = \sigma/\epsilon_0$. The latter tells us that the surface charge density is

$$\sigma = \epsilon_0 \mathbf{E}(\vec{r}_{\text{surface}}). \tag{7}$$

The latter is most easily solved by setting the origin at the center of the cavity. If we are willing to keep the origin at the center of the cavity, it is then easier to use spherical coordinates. The symmetry of the problem is such that there is no azimuthal (ϕ) dependence, so we can just as well calculate the charge density in the xz plane. We know that the field is normal to the surface of the cavity (note that here the surface normal is in the direction $-\hat{r}$, so $E_\perp = \mathbf{E}\cdot(-\hat{r})$, so

$$\sigma = -\epsilon_0\mathbf{E}\cdot\hat{r} = -\frac{Q}{4\pi}\left[\cdot\frac{\hat{r}\cdot\left(a\hat{r}+\frac{a}{2}\hat{z}\right)}{|a\hat{r}+\frac{a}{2}\hat{z}|^3} - \frac{2\hat{r}\cdot(a\hat{r}+2a\hat{z})}{|a\hat{r}+2a\hat{z}|^3}\right]$$
$$= -\frac{Qa}{4\pi}\left[\frac{1+\frac{1}{2}\cos\theta}{\left(\frac{5}{4}a^2+a^2\cos\theta\right)^{3/2}} - \frac{2+4\cos\theta}{(5a^2+4a^2\cos\theta)^{3/2}}\right]$$
$$= -\frac{1}{4\pi}\frac{Q}{a^2}\left[\frac{8+4\cos\theta}{(5+4\cos\theta)^{3/2}} - \frac{2+4\cos\theta}{(5+4\cos\theta)^{3/2}}\right]$$

$$\boxed{\sigma(\theta) = -\frac{1}{4\pi}\frac{Q}{a^2}\frac{6}{(5+4\cos\theta)^{3/2}}.} \tag{8}$$

This is enough; it tells us exactly what the surface charge distribution is. We could express this in terms of a coordinate system with the origin at the center of the conducting sphere, but there is no need.

Note that as a check, we could integrate the cavity surface charge to check that we get a total charge of $-Q$. That the total surface charge on the cavity must be $-Q$ can be seen by drawing a Gaussian surface around the cavity but inside the conductor. The field on the surface is everywhere zero (the static electric field inside a conductor must be zero), so the total charge inside must be zero.

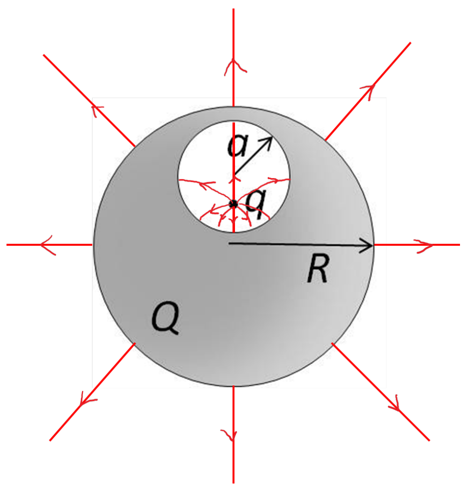
$$
\begin{aligned}
Q_s &= -\oiint \sigma(\theta)da = -\oiint \sigma(\theta)a^2 \sin\theta \, d\theta \, d\phi = \frac{3}{2\pi}Q2\pi \int_0^\pi d\theta \frac{\sin\theta}{(5+4\cos\theta)^{3/2}} \\
&= 3Q\frac{1}{2}\left.\frac{1}{(5+4\cos\theta)^{1/2}}\right|_0^\pi = \frac{3}{2}Q\left(\frac{1}{\sqrt{5-4}} - \frac{1}{\sqrt{5+4}}\right) \\
&= -Q. \qquad (9)
\end{aligned}
$$

At least that check is OK.

(c) Outside the sphere, the problem is simple. The charge on the surface of the cavity completely screens the field from the point charge inside the cavity. Together, these produce no field outside the cavity. However, the cavity surface charge came from the conductor and so must have left behind an opposite total charge distributed with spherical symmetry on the outer surface of the conductor (it must be spherical, or else this surface charge would produce a non-zero electric field inside the conductor). The field outside the conductor is then

$$
\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}, \qquad (10)
$$

where we are now writing the field in terms of an origin at the center of the conductor. Here q_s is the negative of the total surface charge on the cavity surface.



(d) The field lines are shown in the figure. The key points are that

(a) There are no field lines in the conducting (shaded) regions.

(b) The field lines inside the cavity have negative curvature (they bend downward as they go away from the charge).

(c) A corollary to the second point is that the number of field lines beginning or ending on the cavity wall increases as one goes from the top of the cavity to the bottom of the cavity.

(d) There are as many field lines outside the conductor as inside the cavity since the total charge in each region is the same.

(e) The field lines are normal to all conducting surfaces.

(e) If we bring a charge Q near the outside, there will be a non-uniform surface charge density σ_s induced on the outer surface of the conductor. At those points on the surface nearest the outer charge Q, σ_s will be of opposite sign to Q, but at the furthest points on the conductor surface, σ_s will be the same sign as Q. The total surface induced charge will be $Q_s = 0$. If not, there would be an opposite net charge left somewhere inside the conductor (on the cavity surface), but this extra charge on the cavity surface would produce a non-zero field in the conducting region, which is impossible. Therefore, the answers to (a) and (b) will be unchanged, but (c) will change.

2. (15 points) Consider an infinite (in the z direction) conducting pipe with sides at $x = \pm b$ and top and bottom at $y = 0$ and $y = a$. The potential on the upper wall (at $y = a$) is held at $V_0 > 0$, while the other three sides are grounded ($V = 0$).

(a) Determine the potential everywhere inside the pipe.

(b) Sketch the electric field lines and the equipotential surfaces.

(c) Determine the charge per unit length on the side at $y = 0$.

Solution:
(a) We will solve this by separation of variables in Cartesian coordinates. That is, we will look for solutions of the form $V(x, y, z) = X(x)Y(y)Z(z)$.

First, we note that the system is translationally invariant along z, so it must be the case that $Z(z) = \text{constant}$.

Therefore, we expect that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = (i\kappa)^2, \tag{11}$$

where κ is real. We might guess that the constant should have the form shown because the boundaries at $x = \pm b$ are identical and zero, which can be satisfied by sines and cosines, since these repeat themselves.

We have written the solutions to these second order differential equations before. For the choice of constant written above

$$X(x) = A_n \cos(\kappa_n x) + B_n \sin(\kappa_n x). \tag{12}$$

The fact that the boundary conditions are even in x means that the sine solutions will not satisfy our problem. We can, of course, also show this explicitly. At $x = b$, we have

$$A_n \cos(\kappa_n b) + B_n \sin(\kappa_n b) = 0, \tag{13}$$

which implies that

$$B_n = -A_n \frac{\cos(\kappa_n b)}{\sin(\kappa_n b)}. \tag{14}$$

Plugging this into our equation for $X(x)$ and making use of the boundary condition at $x = -b$,

$$A_n \left[\cos(-\kappa_n b) - \frac{\cos(\kappa_n b)}{\sin(\kappa_n b)} \sin(-\kappa_n b) \right] = 2A_n \cos(\kappa_n b) = 0. \tag{15}$$

This implies that $B_n = 0$, as we already knew, and that κ_n must satisfy the condition

$$\kappa_n = (2n - 1)\pi/(2b), \tag{16}$$

where n includes all positive integers.

Turning to $Y(y)$,

$$Y(y) = C_n e^{\kappa_n y} + D_n e^{-\kappa_n y}. \tag{17}$$

At $y = 0$, $V = 0$ so that we must have $Y(y) = 0$, which means that $D = -C$, so

$$Y(y) = C'_n \sinh(\kappa_n y). \tag{18}$$

Our solution to $V(x, y, z)$ is then

$$V(x, y) = \sum_{n=1,2,\ldots}^{\infty} \alpha_n \cos(\kappa_n x) \sinh(\kappa_n y). \tag{19}$$

To determine the α_n, we make use of the last boundary condition, $V(x, a) = V_0$, and the orthogonality of our basis functions in x: multiply $V(x, a)$ by $\cos(\kappa_m x)$ and integrate with respect to x:

$$\begin{aligned}
\int\limits_{x=-b}^{b} V_0 \cos(\kappa_m x) dx &= \sum_{n=1,2,\ldots}^{\infty} \alpha_n \sinh(\kappa_n a) \int\limits_{x=-b}^{b} \cos(\kappa_n x) \cos(\kappa_m x) dx \\
V_0 \left[\frac{\sin(\kappa_m b)}{\kappa_m} - \frac{\sin(-\kappa_m b)}{\kappa_m} \right] &= \sum_{n=1,2,\ldots}^{\infty} \alpha_n \sinh(\kappa_n a) \frac{1}{2} 2b\, \delta_{m,n} \\
2V_0 \frac{\sin(\kappa_m b)}{\kappa_m} &= \alpha_m b \sinh(\kappa_m a).
\end{aligned} \tag{20}$$

Since $\sin(\kappa_m b) = \sin\left[\dfrac{(2n-1)\pi}{2}\right] = -(-1)^n$, we find that

$$\boxed{\begin{aligned} V(x,y) &= -\frac{2V_0}{b}\sum_{n=1,2,\ldots}^{\infty}\frac{(-1)^n}{\kappa_n\sinh(\kappa_n a)}\cos(\kappa_n x)\sinh(\kappa_n y) \\ &= -4V_0\sum_{n=1,2,\ldots}^{\infty}\frac{(-1)^n\cos\left[\frac{(2n-1)\pi}{2b}x\right]\sinh\left[\frac{(2n-1)\pi}{2b}y\right]}{(2n-1)\pi\sinh\left[\frac{(2n-1)\pi}{2b}a\right]}. \end{aligned}}$$

Alternative answer:
There is a slightly different approach that we can take. $\kappa = 0$ is generally a legitimate value for our separation constant, but then the equation to be solved is

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = 0, \tag{21}$$

for which the solutions are

$$X(x) = Ax + B \tag{22}$$

and

$$Y(y) = Cy + D. \tag{23}$$

Since $V(x,0) = 0$, it must be that $D = 0$, and we have

$$X(x)Y(y) = BCy = \beta y. \tag{24}$$

This is just one contribution to the total potential, which we can now separate from the rest of the solution ($V_{\kappa\neq 0}(x,y)$):

$$V(x,y) = \beta y + V_{\kappa\neq 0}(x,y). \tag{25}$$

If we choose $\beta = V_0/a$, we find that $V_{\kappa\neq 0}(x,y)$ satisfies the B.C. $V_{\kappa\neq 0}(x,y) = 0$ at $y = 0$ and $y = a$ and $V_{\kappa\neq 0}(\pm b, y) = -V_0 y/a$. Now, the potential is zero at the boundaries in y, which suggests that we assign the sinusoidal solutions to $Y(y)$:

$$V_{\kappa\neq 0}(x,y) = \sum_{n=1,2,\ldots}^{\infty}\beta_n\cosh(n\pi x/a)\sin(n\pi y/a). \tag{26}$$

Going through the usual steps, one finds that

$$\beta_n = \frac{2V_0}{n\pi}\frac{(-1)^n}{\cosh(n\pi b/a)}, \tag{27}$$

and

$$\boxed{V(x,y) = V_0\left[\frac{y}{a} + \frac{2}{\pi}\sum_{n=1,2,\ldots}^{\infty}\frac{(-1)^n}{n}\frac{1}{\cosh(n\pi b/a)}\cosh(n\pi x/a)\sin(n\pi y/a)\right].}$$

(b) The equipotentials and field lines are shown in black and red, respectively. We have normalized the pipe dimensions so that the walls are at $x = \pm 1$ and $y = 0, 1$. The key points are

(a) There must be a reflection symmetry about $x = 0$.

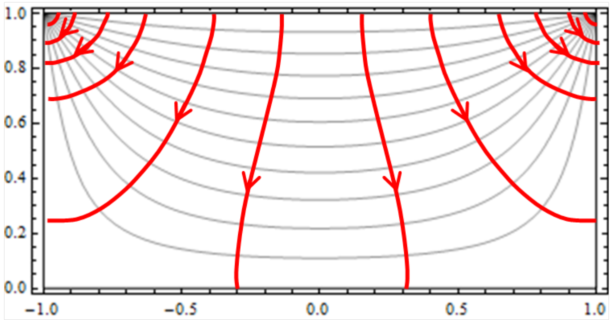
(b) The field lines must be most dense in the upper corners, since that is where the potential changes in the shortest distance.

(c) The field lines must be least dense at the lower corners.

(d) The field lines must be normal to the equipotentials.

(e) The field lines must be normal to the walls where they originate or terminate.

(f) At $y = a$, the field lines point away from the upper surface.



(c) The surface charge density is determined by Gauss's Law:

$$\sigma(x,0) = -\epsilon_0 \left.\frac{dV}{dy}\right|_{y=0}. \tag{28}$$

The surface charge density is then

$$\sigma(x,0) = \frac{2\epsilon_0 V_0}{b} \sum_{n=1,2,\ldots}^{\infty} \frac{(-1)^n}{\kappa_n \sinh(\kappa_n a)} \cos(\kappa_n x)\kappa_n \cosh(\kappa_n y)\Big|_{y=0}$$

or

$$\boxed{\sigma(x,0) = 4\epsilon_0 V_0 \sum_{n=1,2,\ldots}^{\infty} \frac{(-1)^n}{(2n-1)\pi \sinh[(2n-1)\pi a/(2b)]} \cos[(2n-1)\pi x/(2b)].}$$

This is even about $x = 0$ and goes to zero at $x = \pm b$. That the magnitude of the charge is maximum at $x = 0$ can be seen by noting that the cosine term in the sum is positive for all terms at $x = 0$ and that the factor multiplying this cosine term drops in magnitude as n increases. If you plot the charge density, you will see that the charge is negative at all x, as is to be expected, since the positive potential at $y = a$ will draw negative charges towards that upper surface.