Problem Set 11. 董建宇 2019511017

1. (a) According to Maxwell equations, when there are no free charges and no free currents, there is:

$$\nabla\cdot\vec{D}=0 \ \text{(i)} \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \ \text{(iii)} \qquad \vec{H}=\frac{1}{\mu_0}\vec{B} \qquad \vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon_0\vec{E}+\gamma\nabla\times\vec{E}$$

$$\nabla\cdot\vec{B}=0 \ \text{(ii)} \qquad \nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t} \ \text{(iv)}$$

Then, we could determine that $\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\nabla\times\vec{B})$. (*)

According to (i), $\nabla\cdot\vec{D}=\varepsilon_0\nabla\vec{E}+\gamma\nabla\cdot(\nabla\times\vec{E})=0$. We know that $\nabla(\nabla\times\vec{E})=0$, then $\nabla\cdot\vec{E}=0$.

According to (iv) $\nabla\times\vec{H}=\frac{1}{\mu_0}\nabla\times\vec{B}=\frac{\partial\vec{D}}{\partial t}=\varepsilon_0\frac{\partial}{\partial t}\vec{E}+\gamma\frac{\partial}{\partial t}(\nabla\times\vec{E})$.

Then, rewrite (*), there is that $\nabla^2\vec{E}=\frac{\partial^2}{\partial t^2}(\varepsilon_0\mu_0\vec{E}+\gamma\mu_0\nabla\times\vec{E})$

(b). Let $\vec{E}=\iiiint_{-\infty}^{+\infty}\vec{e}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}d^3k\,d\omega$, we could get that

$$-k^2\vec{e}=-\varepsilon_0\mu_0\omega^2\vec{e}-\mu_0\gamma\omega^2(i\vec{k}\times\vec{e})$$

$$i\vec{k}\cdot\vec{e}=0$$

Which shows that $i\vec{k}\times\vec{e}$ must be in the same direction of $\vec{e}$. Thus, $i\vec{k}\times\vec{e}=\pm k\vec{e}$

Then, there is $k^2\pm\mu_0\gamma\omega^2k-\mu_0\varepsilon_0\omega^2=0$.

Because of $k>0$, we could calculate that $k=\frac{1}{2}\left(\pm\mu_0\gamma\omega^2+\sqrt{\mu_0^2\gamma^2\omega^4+4\mu_0\varepsilon_0\omega^2}\right)$.

Since $n=\frac{c}{v}=\frac{ck}{\omega}$, we have

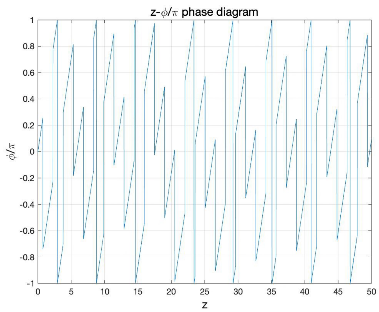
$$n_+=\frac{c}{2}\left(\mu_0\gamma\omega+\sqrt{\mu_0^2\gamma^2\omega^2+4\mu_0\varepsilon_0}\right) \qquad n_-=\frac{c}{2}\left(-\mu_0\gamma\omega+\sqrt{\mu_0^2\gamma^2\omega^2+4\mu_0\varepsilon_0}\right)$$

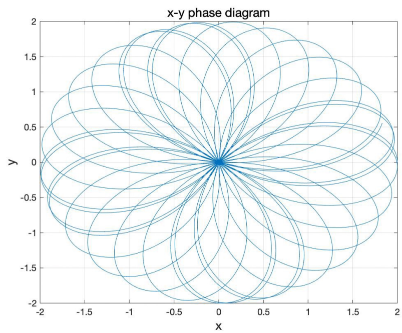
(c). We could assume the electric field goes along z-axis.

So $\vec{\tilde{E}}_L=E_0e^{i(k_+z-\omega t+\varphi_0)}(\hat{x}+i\hat{y})$ $\quad\vec{\tilde{E}}_R=E_0e^{i(k_-z-\omega t+\varphi_0)}(\hat{x}-i\hat{y})$ $\quad E_0,\varphi_0$ are constants.

Then, $\vec{E}_L=Re[\vec{\tilde{E}}_L]=E_0[\cos(k_+z-\omega t+\varphi_0)\hat{x}-\sin(k_+z-\omega t+\varphi_0)\hat{y}]$

$\vec{E}_R=Re[\vec{\tilde{E}}_R]=E_0[\cos(k_-z-\omega t+\varphi_0)\hat{x}+\sin(k_-z-\omega t+\varphi_0)\hat{y}]$





2\. (a). Assume the electrons density is n, the average velocity is $\vec{v}$. Then there are:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$$

$$m\frac{d\vec{v}}{dt} = m\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla v\right) = -e\vec{E} \Rightarrow \frac{\partial \vec{v}}{\partial t} = -\frac{e}{m}\vec{E}$$ (since v is a first order small)

Let n' be the electrons denstiy deviation value $n - n_e$.

Thus, there are

$$\frac{\partial n'}{\partial t} + n_e\nabla\cdot\vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{e}{m}(\vec{E}_i + \vec{E}_e)$$

$$\nabla\cdot\vec{E}_i = -\frac{e}{\varepsilon_0}n' \qquad \nabla\cdot\vec{E}_e = 0$$

So that we could get that $\frac{\partial \vec{v}}{\partial t} = -\frac{e}{m}\vec{E}_e$ (only consider the plasma influenced by the electromagnetic wave)

We know that the current density is $\vec{J} = -n_e e\vec{v}$, so $\frac{\partial \vec{J}}{\partial t} = \frac{n_e e^2}{m}\vec{E}_e$.

For the electromagnetic wave with frequence w. there are $\vec{E}_e(\vec{x},t) = \vec{E}_e(\vec{x})e^{-iwt} \Rightarrow \frac{\partial \vec{J}}{\partial t} = -iw\vec{J}$

So we get that $\vec{J} = i\frac{n_e e^2}{mw}\vec{E}_e$.

Then the conductivity is $\sigma(w) = i\frac{n_e e^2}{mw}$ (m is the mass of electron and w is the frequence).

(b). According to «电动力学» by Shuohong Guo, we have the complex capacitivity is

$$\varepsilon' = \varepsilon - \frac{n_e e^2}{mw^2} \approx \varepsilon_0 - \frac{n_e e^2}{mw^2}$$

So that $n(w) = \frac{c}{v} = c\cdot\sqrt{\mu_0\varepsilon'} = c\cdot\sqrt{\mu_0\varepsilon_0}\cdot\left(1 - \frac{n_e e^2}{mw^2\varepsilon_0}\right)^{\frac{1}{2}} = \left(1 - \frac{n_e e^2}{m\varepsilon_0 w^2}\right)^{\frac{1}{2}}$

(c) If $n^2(w) \le 0$, which means that $0 < w \le \sqrt{\frac{n_e e^2}{m\varepsilon_0}}$.

Such that the waves do not decay.

3. Suppose the waveguide is idealized, so that we have the boundary conditions are

$$E_{\parallel}=0\ ,\quad B_{\perp}=0.$$

We have the independent equations about E_z and B_z

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0,$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0.$$

We could assume that $E_z=0$ and $B_z\neq 0$.

Then we could use the method of separation of variables.

$$B_z(x,y)=X(x)\,Y(y)$$

So that there is $Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\left[\left(\frac{\omega}{c}\right)^2-k^2\right]XY=0.$

Then, divided by XY, we could get $\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\left[\left(\frac{\omega}{c}\right)^2-k^2\right]=0.$

So that there must be that $\frac{1}{X}\frac{d^2X}{dx^2}=k_x^2\ ,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}=k_y^2.$

So $X(x)=A_1\sin(k_x x)+A_2\cos(k_x x)\ ,\quad Y(y)=B_1\sin(k_y y)+B_2\cos(k_y y).$

Using the boundary conditions, there are $\frac{dX}{dx}\Big|_{x=0}=\frac{dX}{dx}\Big|_{x=a}=0\ ,\quad \frac{dY}{dy}\Big|_{y=0}=\frac{dY}{dy}\Big|_{y=b}=0.$

So $A_1=B_1=0\ ,\quad k_x=\frac{m\pi}{a}\ (m=0,1,2,\cdots)\ ,\quad k_y=\frac{n\pi}{b}\ (n=0,1,2,\cdots).$

Then, we could get $k=\sqrt{(\omega/c)^2-\pi^2[(m/a)^2+(n/b)^2]}.$

Thus, we could calculate the velocity is $v=\frac{\omega}{k}=\frac{c}{\sqrt{1-\frac{\pi^2c^2[(m/a)^2+(n/b)^2]}{\omega^2}}}$