Problem Set 1

1. (a) Assume: $\hat{e}_i, \hat{e}_j, \hat{e}_k$ are unit vectors in the direction of i, j and k, in Cartesian coordinates.

$\hat{e}_l, \hat{e}_m, \hat{e}_n$ are unit vectors in the direction of l, m and n, in Cartesian coordinates.

Then we have: $\hat{e}_k \varepsilon_{ijk} = \hat{e}_i \times \hat{e}_j$, $\hat{e}_k \cdot \hat{e}_k \varepsilon_{ijk} = \varepsilon_{ijk} = \hat{e}_k \cdot (\hat{e}_i \times \hat{e}_j)$

$\hat{e}_n \varepsilon_{lmn} = \hat{e}_l \times \hat{e}_m$, $\hat{e}_n \cdot \hat{e}_n \varepsilon_{lmn} = \varepsilon_{lmn} = \hat{e}_n \cdot (\hat{e}_l \times \hat{e}_m)$

$$\varepsilon_{ijk}\varepsilon_{lmn} = [\hat{e}_k \cdot (\hat{e}_i \times \hat{e}_j)][\hat{e}_n \cdot (\hat{e}_l \times \hat{e}_m)]$$

According to $\det A \det B = \det(AB)$

A and B are 3x3 matrix

$$= \begin{vmatrix} \hat{e}_k \cdot \hat{e}_n & \hat{e}_k \cdot \hat{e}_l & \hat{e}_k \cdot \hat{e}_m \\ \hat{e}_i \cdot \hat{e}_n & \hat{e}_i \cdot \hat{e}_l & \hat{e}_i \cdot \hat{e}_m \\ \hat{e}_j \cdot \hat{e}_n & \hat{e}_j \cdot \hat{e}_l & \hat{e}_j \cdot \hat{e}_m \end{vmatrix}$$

Assume $l \to i$, and m, n are not the same as j, k

then $\varepsilon_{ijk}\varepsilon_{imn} = \begin{vmatrix} \delta_{kn} & 0 & \delta_{km} \\ 0 & 1 & 0 \\ \delta_{jn} & 0 & \delta_{jm} \end{vmatrix} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

So $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

(b) (i) $[\vec{A} \times (\vec{B} \times \vec{C})]_i = \sum \varepsilon_{ijk} A_j (\vec{B} \times \vec{C})_k = \sum \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n = \sum \varepsilon_{kij} \varepsilon_{kmn} A_j B_m C_n$

$= \sum (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j B_m C_n$

$[\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})]_i = [(\sum \delta_{jn} A_j C_n)\vec{B} - (\sum \delta_{jm} A_j B_m)\vec{C}]_i = \sum \delta_{im}\delta_{jn} A_j C_n B_m - \sum \delta_{jm}\delta_{in} A_j B_m C_n$

$= \sum (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j B_m C_n$

So $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

(ii) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \partial_i \varepsilon_{ijk} A_j B_k = B_k \varepsilon_{ijk} \partial_i A_j + A_j \varepsilon_{ijk} \partial_i B_k = B_k \varepsilon_{kij} \partial_i A_j - A_j \varepsilon_{jik} \partial_i B_k$

$= \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

So $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

(iii) $[\vec{A} \times (\nabla \times \vec{B})]_i = \varepsilon_{ijk} A_j (\nabla \times \vec{B})_k = \varepsilon_{kij} A_j \varepsilon_{kmn} \partial_m B_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j \partial_m B_n = A_j \partial_i B_j - A_j \partial_j B_i$

$[\nabla(\vec{A} \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} - (\nabla \cdot \vec{B})\vec{A}]_i = \partial_i A_i B_i - \varepsilon_{ijk}\varepsilon_{kmn} B_j \partial_m A_n - B_i \partial_j A_j - A_i \partial_j B_j$

$= A_j \partial_i B_j - A_j \partial_j B_i$

So $\vec{A} \times (\nabla \times \vec{B}) = \nabla(\vec{A} \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} - (\nabla \cdot \vec{B})\vec{A}$

(iv). $[\nabla \times (\vec{A} \times \vec{B})]_i = \varepsilon_{ijk}\varepsilon_{imn} \partial_j (A_m B_n) = (\delta_{jm}\delta_{in} - \delta_{im}\delta_{jn})(B_n \partial_j A_m + A_m \partial_j B_n) \cdot (-1)$

$= B_j \partial_j A_i - B_i \partial_j A_j + A_i \partial_j B_j - A_j \partial_j B_i$

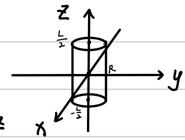
$= [(\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})]_i$

2. (a). $\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x+a)+\delta(x-a)\right]$

So $\int_{-\infty}^{+\infty} \phi(x)\,\delta(x^2-a^2)\,dx = \int_{-\infty}^{+\infty} \frac{1}{2|a|}\phi(x)\left[\delta(x+a)+\delta(x-a)\right]dx$

$= \frac{1}{2|a|}\left[\phi(-a)+\phi(a)\right]$

(b). The average surface charge density $\sigma_0 = \frac{Q}{2\pi R^2 + 2\pi RL}$



$\sigma = \sigma_0\, 2\pi r z\, \delta(r-R)\left[\delta(z-\tfrac{L}{2}) - \delta(z+\tfrac{L}{2})\right] + \sigma_0\, \pi r^2 \delta(r-R)\cdot\left[\delta(z-\tfrac{L}{2})+\delta(z+\tfrac{L}{2})\right]$

$q' = \iint \sigma\, dr\,dz = \sigma_0 \iint \left\{2\pi r z\, \delta(r-R)\left[\delta(z-\tfrac{L}{2})-\delta(z+\tfrac{L}{2})\right] + \pi r^2 \delta(r-R)\left[\delta(z-\tfrac{L}{2})+\delta(z+\tfrac{L}{2})\right]\right\} dr\,dz$

$= \sigma_0\left[2\pi R\left[\tfrac{L}{2}-(-\tfrac{L}{2})\right] + \pi R^2\cdot(1+1)\right] = \sigma_0\cdot(2\pi RL + 2\pi R^2) = Q$

Which shows this yields the proper total charge when integrated over all space.

(C). $\rho(\vec{x}) = A\,\delta(\vec{x}\cdot\vec{x}_0 - |\vec{x}_0|^2)$ The charge distribution given by the form is there is a point charge at position $\vec{x}_0$

The unit of A is a line charge density.

Because the unit of $\rho(\vec{x})$ is a volume charge density and the unit of $\delta(\vec{x}\cdot\vec{x}_0 - |\vec{x}_0|^2)$ is meter^{-2} so the unit of A is a line charge density.

3. (a) Since the rotational symmetry of the disk, the direction of the electric field at $(0,0,z)$ is along z axis

According to Coulumb's Law

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\cdot 2\pi r dr}{r^2+z^2}\cdot\cos\theta = \frac{\sigma z}{2\varepsilon_0}\frac{r}{(r^2+z^2)^{3/2}}dr$$

So $E = \int_0^R dE = \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r}{(r^2+z^2)^{3/2}}dr = \frac{\sigma}{2\varepsilon_0}(1-\frac{z}{\sqrt{R^2+z^2}})$

So the electric field at $(0,0,z)$ is $\vec{E} = \frac{\sigma}{2\varepsilon_0}(1-\frac{z}{\sqrt{R^2+z^2}})$

(b). If $z>>R$, then $\frac{R^2}{z^2}\to 0$ $\quad 1-\frac{z}{\sqrt{R^2+z^2}} = 1-(1+\frac{R^2}{z^2})^{-\frac{1}{2}} \doteq \frac{R^2}{2z^2}$

So $\vec{E} = \frac{\sigma R^2}{4\varepsilon_0}\cdot\frac{1}{z^2}\hat{e}_z$

The charged circular disk seems like a point charge.

(c) If $z<<R$ then $\frac{z^2}{R^2}\to 0$ $\quad 1-\frac{z}{\sqrt{R^2+z^2}}\doteq 1$

So $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{e}_z$

The charged circular disk seems like an infinite disk

4. (a) Since it is a uniformly charged, the surface charge $\sigma = \frac{Q}{2wL}$

According to symmetry, the electric field at any position $x>0$ on the positive x-axis.

$$E = \iint_S \frac{1}{4\pi\varepsilon_0}\frac{\sigma}{y^2+(x-x_0)^2}\cdot\cos\theta\, dx\,dy = \iint_S \frac{\sigma}{4\pi\varepsilon_0}\frac{x_0-x}{[y^2+(x-x_0)^2]^{3/2}}dx\,dy = \frac{\sigma}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}dy\int_{-w}^{w}\frac{x_0-x}{[y^2+(x-x_0)^2]^{3/2}}dx$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left[\frac{1}{\sqrt{y^2+(w-x_0)^2}}-\frac{1}{\sqrt{y^2+(w+x_0)^2}}\right]dy = \frac{\sigma}{4\pi\varepsilon_0}\left[2\operatorname{arcsinh}\frac{L}{2|w-x_0|}-2\operatorname{arcsinh}\frac{L}{2(w+x_0)}\right]$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left[\operatorname{arcsinh}\frac{L}{2|w-x_0|}-\operatorname{arcsinh}\frac{L}{2(w+x_0)}\right]$$

So the electric field at any position $x>0$ on the positive x-axis, for example, $(x_0,0)$ $\quad\vec{E}(x_0,0) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{wL}\left[\operatorname{arcsinh}\frac{L}{2|w-x_0|}-\operatorname{arcsinh}\frac{L}{2(w+x_0)}\right]\hat{i}$

(b). (i) At the origin, we have $x_0=0$.

Since the symmetry of the rectangular charged sheet, we can easily get $\vec{E}(0,0)=0$

My solution also says that $\vec{E}(0,0)=0$

So my result behaves exactly as expand at the origin.

(ii) If $x>>L,w$, we have $\frac{w}{x}\to 0$ $\quad\frac{L}{2(w+x)}\to 0$ $\quad\frac{L}{2(x-w)}\to 0$

$$\operatorname{arcsinh}\frac{L}{2(w+x)} \approx \frac{L}{2(w+x)} = \frac{L}{2x}\cdot\frac{1}{1+\frac{w}{x}} \approx \frac{L}{2x}(1-\frac{w}{x})$$

$\operatorname{arcsinh}\frac{L}{2(x-w)} \approx \frac{L}{2(x-w)} = \frac{L}{2x}\cdot\frac{1}{1-\frac{w}{x}} \approx \frac{L}{2x}\cdot\left(1+\frac{w}{x}\right)$

So $\operatorname{arcsinh}\frac{L}{2(x-w)} - \operatorname{arcsinh}\frac{L}{2(x+w)} \approx \frac{L}{2x}\cdot\left[\left(1+\frac{w}{x}\right)-\left(1-\frac{w}{x}\right)\right] = \frac{Lw}{x^2}$

So $\vec{E}(x,0) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}$ and this is the leading behavior, which shows that the rectangular charged sheet seems like a point charge.

(iii). If $x >> L >> w$, $\frac{L}{x+w} \to 0$ $\frac{L}{x-w} \to 0$ $\frac{w}{x} \to 0$

$\operatorname{arcsinh}\frac{L}{2(x+w)} \approx \frac{L}{2(x+w)} - \frac{1}{6}\frac{L^3}{8(x+w)^3} = \frac{L}{2x}\cdot\frac{1}{1+\frac{w}{x}} - \frac{L^3}{48x^3}\cdot\frac{1}{(1+\frac{w}{x})^3}$

$\approx \frac{L}{2x}\left(1-\frac{w}{x}+\frac{w^2}{x^2}-\frac{w^3}{x^3}\right) - \frac{L^3}{48x^3}\left(1-3\frac{w}{x}+6\frac{w^2}{x^2}-10\frac{w^3}{x^3}\right)$

$\approx \frac{L}{2x} - \frac{Lw}{2x^2} - \left(\frac{L^3}{48}-\frac{Lw^2}{2}\right)\cdot\frac{1}{x^3} + \left(\frac{L^3w}{16}-\frac{Lw^3}{2}\right)\frac{1}{x^4}$

$\operatorname{arcsinh}\frac{L}{2(x-w)} \approx \frac{L}{2(x-w)} - \frac{1}{6}\frac{L^3}{8(x-w)^3} = \frac{L}{2x}\cdot\frac{1}{1-\frac{w}{x}} - \frac{L^3}{48x^3}\cdot\frac{1}{(1-\frac{w}{x})^3}$

$\approx \frac{L}{2x}\left(1+\frac{w}{x}+\frac{w^2}{x^2}+\frac{w^3}{x^3}\right) - \frac{L^3}{48x^3}\left(1+3\frac{w}{x}+6\frac{w^2}{x^2}+10\frac{w^3}{x^3}\right)$

$\approx \frac{L}{2x} + \frac{Lw}{2x^2} - \left(\frac{L^3}{48}-\frac{Lw^2}{2}\right)\frac{1}{x^3} - \left(\frac{L^3w}{16}-\frac{Lw^3}{2}\right)\frac{1}{x^4}$

So $\operatorname{arcsinh}\frac{L}{2(x-w)} - \operatorname{arcsinh}\frac{L}{2(x+w)} \approx \frac{Lw}{x^2} - \left(\frac{L^3w}{8}-Lw^3\right)\frac{1}{x^4} \approx \frac{Lw}{x^2} - \frac{L^3w}{8x^4}$

So $\vec{E}(x,0) = \frac{Q}{4\pi\varepsilon_0}\cdot\left(\frac{1}{x^2}-\frac{L^2}{8x^4}\right)$

The first correction to the leading behavior is $-\frac{Q}{32\pi\varepsilon_0}\frac{L^2}{x^4}$

If there is a line of charge of length L and the same total charge as the sheet, since the symmetry, the electrical field on x-axis is along x-axis.

$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dy}{x^2+y^2}\cdot\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda x}{(x^2+y^2)^{3/2}}dy$ $\lambda = \frac{Q}{L}$

$E = \int_{-\frac{L}{2}}^{\frac{L}{2}} dE = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dy = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{1}{x}\cdot\frac{1}{(1+\frac{y^2}{x^2})^{3/2}}d\frac{y}{x} = \frac{\lambda}{4\pi\varepsilon_0}\cdot\frac{1}{x}\frac{2L}{\sqrt{4x^2+L^2}}$

If $x>>L$ $\frac{L}{x}\to 0$ $E = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{x^2}\cdot\frac{1}{\sqrt{1+(\frac{L}{2x})^2}} \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\cdot\left[1-\frac{1}{2!}\cdot\left(\frac{L}{2x}\right)^2\right] = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\left(1-\frac{L^2}{8x^2}\right)$

The first correction is also $-\frac{1}{32\pi\varepsilon_0}\frac{QL^2}{x^4}$ and this is the same as the first correction of my solution.