Problem Set 9 董建宇 2019511017

1.(a) Since the free current density is 0 everywhere, we could get that $\nabla\times\vec{H}=\vec{0}$.

So that we could assume $\vec{H}=-\nabla U$, therefore, we get $\nabla^2 U=0$.

Using spherical coordinates, by the symmetry, we could easily get that U is independent with φ.

Assume the solution could be written as $U(r,\theta)=R(r)\Theta(\theta)$.

We could get that $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial U}{\partial\theta})=0$ which means $\frac{1}{R(r)}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})+\frac{1}{\Theta(\theta)\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})=0$

The general solution is $U(r,\theta)=\sum_{l=0}^{\infty}(A_l r^l+\frac{B_l}{r^{l+1}})P_l(\cos\theta)$.

For $r>R_{out}$, the solution satisfies when $r\to\infty$ $U(r,\theta)=\frac{B_0}{\mu_0}r\cos\theta$, So $U(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)+\frac{B_0}{\mu_0}r\cos\theta$

For $R_{in}<r<R_{out}$, the solution is $U(r,\theta)=\sum_{l=0}^{\infty}(A_l' r^l+\frac{B_l'}{r^{l+1}})P_l(\cos\theta)$

For $r<R_{in}$, the solution satisfies when $r=0$ $U(r,\theta)<\infty$, so $B_l=0$, $U(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$

So we have the total

$$\vec{H}(r,\theta)=-\nabla U=\begin{cases}-\sum_{l=0}^{\infty}\left[-(l+1)\frac{B_l}{r^{l+2}}P_l(\cos\theta)\hat{r}+\frac{B_l}{r^{l+2}}\frac{d}{d\theta}P_l(\cos\theta)\hat{\theta}\right]+\frac{B_0}{\mu_0}\hat{z} & r>R_{out}\\ -\sum_{l=0}^{\infty}\left\{\left[lA_l' r^{l-1}-(l+1)\frac{B_l'}{r^{l+2}}\right]P_l(\cos\theta)\hat{r}+(A_l' r^{l-1}+\frac{B_l'}{r^{l+2}})\frac{d}{d\theta}P_l(\cos\theta)\hat{\theta}\right\} & R_{in}<r<R_{out}\\ -\sum_{l=0}^{\infty}\left[lA_l r^{l-1}P_l(\cos\theta)\hat{r}+A_l r^{l-1}\frac{d}{d\theta}P_l(\cos\theta)\hat{\theta}\right] & r<R_{in}\end{cases}$$

For $r>R_{out}$ and $r<R_{in}$ $\vec{B}=\mu_0\vec{H}$. For $R_{in}<r<R_{out}$ $\vec{B}=\mu\vec{H}$

Using the boundary conditions $B_1^{\perp}=B_2^{\perp}$ and $H_1^{\parallel}=H_2^{\parallel}$ we could get that

$$\begin{cases}\mu_0\left[(l+1)\frac{B_l}{R_{out}^{l+2}}P_l(\cos\theta)+\frac{B_0}{\mu_0}\cos\theta\right]=-\mu\left[lA_l'R_{out}^{l-1}-(l+1)\frac{B_l'}{R_{out}^{l+2}}\right]P_l(\cos\theta) & ①\\ \frac{B_l}{R_{out}^{l+2}}\frac{d}{d\theta}P_l(\cos\theta)+\frac{B_0}{\mu_0}\sin\theta=(A_l'R_{out}^{l-1}+\frac{B_l'}{R_{out}^{l+2}})\frac{d}{d\theta}P_l(\cos\theta) & ②\\ -\mu_0 lA_l R_{in}^{l-1}P_l(\cos\theta)=-\mu\left[lA_l'R_{in}^{l-1}-(l+1)\frac{B_l'}{R_{in}^{l+2}}\right]P_l(\cos\theta) & ③\\ A_l R_{in}^{l-1}\frac{d}{d\theta}P_l(\cos\theta)=(A_l'R_{in}^{l-1}+\frac{B_l'}{R_{in}^{l+2}})\frac{d}{d\theta}P_l(\cos\theta) & ④\end{cases}$$

We could easily get that for all $l\neq 1$ $A_l=B_l=A_l'=B_l'=0$

We could get $A_1=\dfrac{-9\mu B_0}{(2\mu+\mu_0)(\mu+2\mu_0)-2(\mu-\mu_0)^2(\frac{R_{in}}{R_{out}})^3}$, which is that we care.

So for $r<R_{in}$, there is $\vec{B}=\mu_0\vec{H}$

$$\vec{B}(r,\theta)=-\mu_0A_1(\cos\theta\hat{r}-\sin\theta\hat{\theta})=-A_1\hat{z}=\frac{9\mu_0\mu B_0}{(2\mu+\mu_0)(\mu+2\mu_0)-2(\mu-\mu_0)^2(\frac{R_{in}}{R_{out}})^3}\hat{z}$$

1(b). If $\mu\gg\mu_0$, we could get that $\frac{\mu_0}{\mu}\to 0$. If we preserve small quantities of first order.

$$\vec{B}(r,\theta)=\frac{9\mu_0 B_0}{2\mu[1-(\frac{R_{in}}{R_{out}})^3]}\hat{z}\to 0$$

2. (a) Since $\mu >> \mu_0$, which means that there is no leakage of magnetic flux.

And we also assume that $\vec{B}$ and $\vec{H}$ inside the iron are uniform, smooth and continuous, so we could assume the magnitude of $\vec{B}$ inside the iron is B_1, and the direction is along $\hat{\theta}$

We could fill the gap and there is another circle current at the gap.

Using Ampere's Law, we have $\oint H \cdot dl = I_{f\text{-}in}$, which shows that $H 2\pi(R+r) = NI$ and we have $H = \frac{B}{\mu}$

So we get $\vec{B}$ and $\vec{H}$ inside is $\vec{B}_1 = \frac{\mu NI}{2\pi(R+r)}$ $\quad \vec{H}_1 = \frac{NI}{2\pi(R+r)}\hat{\theta}$ $\quad \vec{M}_1 = (\frac{\mu}{\mu_0} - 1)\frac{NI}{2\pi(R+r)}\hat{\theta}$

Then we could get that $I' = |\vec{M}| d = (\frac{\mu}{\mu_0} - 1)\frac{NId}{2\pi(R+r)}$. Using Biot-Savart Law, we could get the magnetic field at the circle current center is $B' = \frac{\mu_0 I'}{2}\frac{1}{r} = (\mu - \mu_0)\frac{NId}{4\pi r(R+r)}$

And the total magnetic field at the gap is $\vec{B}_2 = (|\vec{B}| - B')\hat{\theta} = \frac{\mu NI}{2\pi(R+r)}[1 - (1 - \frac{\mu_0}{\mu})\frac{d}{2r}]\hat{\theta}$

We also could get $\vec{H}$ in the gap is $\vec{H}_2 = \frac{\vec{B}_2}{\mu_0} = \frac{\mu NI}{2\pi\mu_0(R+r)}[1 - (1 - \frac{\mu_0}{\mu})\frac{d}{2r}]\hat{\theta}$

(b). Since we need to determine the position far from the toroid and in the plane that passes through the middle of the gap, we could seen the circle current as a magnetic diople $\vec{m} = I'\pi r^2 \hat{z}$

$\vec{r} = x\hat{x} + y\hat{y}$. So $\vec{m}\,\vec{r} = 0$. Then, $\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{(x^2+y^2)^{3/2}} \cdot (-I'\pi r^2)\hat{z}$

$$\vec{B} = \frac{\mu_0 - \mu}{4\pi}\,\frac{NIdr^2}{2(R+r)}\,\frac{1}{(x^2+y^2)^{3/2}}\,\hat{z}$$