Problem Set 10 董建宇 2019511017

1\. (a) After the system returns to steady state, the current in the secondary winds is 0.

According to Ampere's Law, we have $\oint \vec{H}\, d\vec{l} = N_p I_p$ $\qquad |\vec{H}| = \frac{N_p I_p}{2\pi R_0}$, since $R \gg r$

Thus, we could get that the incremental change in ΔH is $\frac{N_p \Delta I_p}{2\pi R_0}$.

(b). According to Maxwell equation and Faraday's Law, we have that

$$2\pi R_0 H = N_p I_p - N_s I_s$$

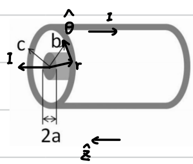
$$N_s \pi r^2 \frac{dB}{dt} = I_s R_g$$

Then, there is $P = \frac{dW}{dt} = N_p \pi r^2 \frac{dB}{dt} I_p = \pi r^2 \frac{dB}{dt}(N_s I_s + 2\pi R_0 H) = I_s^2 R_g + 2\pi^2 R_0 r^2 H \frac{dB}{dt}$

So the work done is $W = \int_{t_0}^{t} \frac{dW}{dt} dt = \int_{t_0}^{t} I_s^2 R_s\, dt + \int_{t_0}^{t} 2\pi^2 R_0 r^2 H \frac{dB}{dt} dt$

We could calculate that $\int 2\pi R_0 \pi r^2 H\, dB = 2\pi^2 R_0 r^2 S$, where S is the enclosed area in the figure "$B-\mu_0 H$".

So that the work is $W = \int_{t_0}^{t} I_s^2 R_s\, dt + 2\pi^2 R_0 r^2 S$.



2\. (a) For $r \le a$, the current density is $\vec{J}_1 = \frac{I_0}{\pi a^2}\cos(\omega t)\,\hat{z}$.

For $b \le r \le c$, the current density is $\vec{J}_2 = \frac{-I_0}{\pi(c^2-b^2)}\cos(\omega t)\,\hat{z}$.

Using Ampere's Law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{in}$.

When $r>c$, $I_{in}=0$, so $\vec{B}=0$. When $r<a$, $B\, 2\pi r = \pi r^2 |\vec{J}_1| \mu_0$ so $\vec{B} = \frac{\mu_0 I_0 r}{2\pi a^2}\cos(\omega t)\,\hat{\theta}$

When $a<r<b$, $I_{in} = I(t)$ so $\vec{B} = \frac{\mu_0 I_0}{2\pi r}\cos(\omega t)\,\hat{\theta}$. Since $c-b \ll b$, we could ignore the $\vec{B}$ in $b<r<c$

According to Faraday's Law, there is $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S}$.

When $r<a$, there is $-E(\vec{r},t)\cdot s = -\frac{d}{dt}\int_0^r \frac{\mu_0 I_0 r}{2\pi a^2}\cos(\omega t)\, s\, dr = \frac{\mu_0 I_0 r^2 s \omega}{4\pi a^2}\sin(\omega t)$

so $\vec{E}(\vec{r},t) = -\frac{\mu_0 I_0 \omega r^2}{4\pi a^2}\sin(\omega t)\,\hat{z} \qquad 0<r<a$

When $a<r<b$, there is $-E(\vec{r},t)\cdot s = -\frac{d}{dt}\left[\int_0^a \frac{\mu_0 I_0 r}{2\pi a^2}\cos(\omega t)\, s\, dr + \int_a^r \frac{\mu_0 I_0}{2\pi r}\cos(\omega t)\, s\, dr\right] = \frac{\mu_0 I_0 \omega s}{2\pi}\sin(\omega t)\left(\frac{1}{2} + \ln\frac{r}{a}\right)$

so $\vec{E}(\vec{r},t) = -\frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\cdot\left(\frac{1}{2}+\ln\frac{r}{a}\right)\hat{z} \qquad a<r<b$

When $r>c$, $\vec{E}(\vec{r},t) = 0$

(b) According to question (a), we have the electric field as a function of $\vec{r}$ and t.

Then, according to Maxwell Equation, we could get the displacement current density is

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \begin{cases} -\frac{\varepsilon_0 \mu_0 I_0 \omega^2 r^2}{4\pi a^2}\cos(\omega t)\,\hat{z}, & 0<r<a; \\ -\frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi}\cos(\omega t)\left(\frac{1}{2}+\ln\frac{r}{a}\right)\hat{z}, & a<r<b \\ 0 & , r>c \end{cases}$$

So the total displacement current is

$$I_d = \oint \vec{J}_d \cdot d\vec{a} = -\frac{\varepsilon_0 \mu_0 I_0 \omega^2}{4\pi a^2} \cos(\omega t) \int_0^a r^2 \cdot 2\pi r\, dr - \frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos(\omega t) \int_a^b \left(\frac{1}{2} + \ln\frac{r}{a}\right) 2\pi r\, dr$$

$$= -\frac{1}{8} \varepsilon_0 \mu_0 I_0 \omega^2 a^2 \cos(\omega t) - -\frac{1}{4} \varepsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t)(b^2 - a^2) - \varepsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t) \cdot \left[\frac{1}{2} b^2 \ln\frac{b}{a} - \frac{1}{4}(b^2 - a^2)\right]$$

$$= -\frac{1}{8} \varepsilon_0 \mu_0 I_0 \omega^2 a^2 \cos(\omega t) - \varepsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t) \cdot \frac{1}{2} b^2 \ln\frac{b}{a}$$

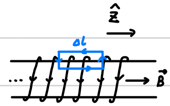
$$I_{d,max} = \varepsilon_0 \mu_0 I_0 \omega^2 \left(\frac{1}{8} a^2 + \frac{1}{2} b^2 \ln\frac{b}{a}\right)$$

So $I_{d\,max} = 0.01\, I_0$, the frequency is

$$f = \frac{1}{2\pi} \omega = \frac{0.1}{2\pi} \sqrt{\frac{1}{\varepsilon_0 \mu_0 \left(\frac{1}{8} a^2 + \frac{1}{2} b^2 \ln\frac{b}{a}\right)}} = 603\ \text{MHz}$$

Which is smaller than 860 MHz

3. (a) Since the solenoid is long, using Ampere's Law, we could get $\oint \vec{B}_0 \cdot d\vec{l} = B\Delta l = \mu_0 n \Delta l I_0$.

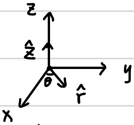


So the magnetic field is $\vec{B}_0 = \mu_0 n I_0 \hat{z}$.

Then, the pressure of the electromagnetic field per unit length on the solenoid is

$$\vec{F} = n\oint I_0 d\vec{l} \times \vec{B} = 2\pi R \mu_0 n^2 I_0^2 \hat{r}$$

, $\hat{r}$ is the unit vector in cylindrical coordinate.



(b) If the current is slowly ramped up from zero to its final value of I_0, we could assume the current is a function of t, and $I(t) = I_0 f(t)$. $f(t)$ increases from 0 to 1 as t increases from 0 to infinity. So we could get the magnetic field is $\vec{B}(t) = \mu_0 n I_0 f(t)\hat{z}$.

Then, using Ampere's Law, we have $\oint \vec{E}\cdot d\vec{l} = -\iint \frac{d\vec{B}}{dt}\cdot d\vec{a}$. So we could determine that the electric field is $\vec{E} = -\frac{1}{2}\mu_0 n I_0 f'(t) r\hat{\theta}$ where $f'(t) = \frac{df(t)}{dt}$

Then, by the definition, we could determine the Poynting vector at $r = R-\delta$ is

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = -\frac{1}{2}\mu_0 n^2 I_0^2 (R-\delta) f(t) f'(t)\hat{r}$$

The total energy that passes through the walls of the solenoid is

$$E_1 = \int_0^{\infty}\left(\oiint \vec{S}\cdot d\vec{a}\right)dt = \int_0^{\infty} \frac{1}{2}\mu_0 n^2 I_0^2 (R-\delta)\cdot 2\pi(R-\delta)\cdot L\ f(t)f'(t)dt = \mu_0\pi n^2 I_0^2 (R-\delta)^2 L\ \frac{1}{2}f^2(t)\Big|_0^{+\infty} = \frac{1}{2}\mu_0\pi n^2 I_0^2 (R-\delta)^2 L$$

The total magnetic field energy stored in the solenoid is

$$E_2 = \frac{1}{2\mu_0}\iiint B_0^2\, d\tau = \frac{1}{2\mu_0}(\mu_0 n I_0)^2 \pi(R-\delta)^2 L = \frac{1}{2}\mu_0\pi n^2 I_0^2 (R-\delta)^2 L$$

where L is the length of the solenoid.

We can see that $E_1 = E_2$ which means that the total energy that passes through the walls equals to the energy stored in the solenoid.

4. (a). Since it is an uniform magnetized conducting sphere of radius R and $\vec{M} = M\vec{k}$, we could determine that the magnetic field is

$$\vec{B} = \begin{cases} \frac{2}{3}\mu_0 M\hat{k}, & r < R; \\ \frac{1}{3}\mu_0 M\frac{R^3}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}), & r > R. \end{cases}$$

Since it carries net charge Q, and it is a conductor, the charge only exist on the surface of the sphere. Then we could determine the electric field is

$$\vec{E} = \begin{cases} 0, & r < R; \\ \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}, & r \geq R. \end{cases}$$

Thus, the momentum density is

$$\vec{g} = \varepsilon_0(\vec{E}\times\vec{B}) = \begin{cases} 0, & r<R; \\ \frac{\mu_0}{12\pi}\cdot MQR^3\frac{1}{r^5}\sin\theta\,\hat{\varphi}, & r>R. \end{cases}$$

So the angular momentum density is

$$\vec{l} = \vec{r}\times\vec{g} = \begin{cases} 0, & r<R; \\ -\frac{\mu_0}{12\pi}MQR^3\frac{1}{r^4}\sin\theta\,\hat{\theta}, & r>R. \end{cases}$$

By the symmetry, we could easily get that the total angular momentum is along z-axis.
Thus, the total angular momentum is

$$\vec{L} = \iiint \frac{\mu_0}{12\pi\varepsilon_0} MQR^3 \frac{1}{r^4} \sin^2\theta \cdot r^2\sin\theta \, d\theta \, d\varphi \, \hat{z} = \frac{\mu_0 MQR^3}{12\pi} \int_R^{+\infty} \frac{1}{r^2} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi \, \hat{z}$$

$$= \frac{2}{9} \mu_0 MQR^2 \hat{z}$$

(b). Since the charge only exists on the surface, we only need to calculate the electric field on the surface.
According to Faraday's Law, there is $\oint \vec{E}_1 \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\pi r_1^2 \cdot \frac{2}{3}\mu_0 \frac{dM}{dt}$ where $r_1 = R\sin\theta$.
So we could get $\vec{E}_1 = -\frac{1}{3}\mu_0 R\sin\theta \frac{dM}{dt} \hat{\varphi}$.
Then the torque exerted is $\vec{N} = \iint R\hat{r} \times \sigma da \vec{E}_1 = \iint \frac{1}{3}\mu_0 R^2 \frac{dM}{dt} \sin\theta \cdot \sigma R^2 \sin\theta d\theta d\varphi \, \hat{\theta}$
By the symmetry, we could easily get the total torque is along z-axis.
Thus, the total torque is $\vec{N} = -\frac{1}{3}\mu_0 R^4 \sigma \frac{dM}{dt} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi = -\frac{2}{9}\mu_0 R^2 Q \frac{dM}{dt} \hat{z}$
Then the angular momentum produce by $\vec{E}_1$ is $\vec{L}_1 = \int \vec{N} dt = -\frac{2}{9}\mu_0 R^2 Q \int_M^0 dM \, \hat{z} = \frac{2}{9}\mu_0 MQR^2 \hat{z}$
So the final angular momentum is $\vec{L}_1 = \frac{2}{9}\mu_0 MQR^2$.

(c). According to Maxwell equation, we have the displacement current density is

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \begin{cases} 0 & , \quad r<0 \\ \frac{1}{4\pi} \frac{1}{r^2} \frac{dQ}{dt} \hat{r}, & \quad r>0 \end{cases}$$

Then the torque exerted is $\vec{N} = \iiint \vec{r} \times (\vec{J}_d \times \vec{B}) \, d\tau = \iiint -\frac{1}{12\pi} \mu_0 MR^3 \frac{dQ}{dt} \sin\theta \frac{1}{r^4} d\tau \, \hat{\theta}$
By the symmetry, we could see that the total torque is along z-axis.
So that we could calculate $\vec{N} = -\frac{1}{12\pi}\mu_0 MR^3 \frac{dQ}{dt} \iiint \frac{1}{r^4} \sin^2\theta \cdot r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$$= -\frac{1}{12\pi}\mu_0 MR^3 \frac{dQ}{dt} \int_R^{+\infty} \frac{1}{r^2} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= -\frac{2}{9}\mu_0 MR^2 \frac{dQ}{dt}$$

So the final angular momentum is $\vec{L}_2 = \int \vec{N} dt = \int_Q^0 -\frac{2}{9}\mu_0 MR^2 dQ = \frac{2}{9}\mu_0 MQR^2$