Electrodynamics
Problem Set 4

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1.1

Since the system has axial symmetry, we choose spherical coordinates to describe the protential at all points in space and the protential has no connection with φ.

According to the Laplace function, we have that

$$\nabla^2 V = 0$$

Assume the protential could be written as

$$V(r, \theta) = S(r)\Theta(\theta)$$

So we could get that

$$\frac{1}{S}\frac{d}{dr}\left(r^2\frac{dS}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

So both of the items are constants.

$$\frac{1}{S}\frac{d}{dr}\left(r^2\frac{dS}{dr}\right) = l(l+1), \quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

So we could calculate that

$$S(r) = Ar^l + \frac{B}{r^{l+1}}, \quad \Theta(\theta) = P_l(\cos\theta)$$

Then the protential at all points could be written as

$$V_l(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

While $r \leqslant R$, to guarantee that when $r \to 0$, $V_l(r, \theta) \to V_0$, we need that B=0, so $V_l(r, \theta) = A_l r^l P_l(\cos\theta)$

While $r > R$, to guarantee that when $r \to \infty$, $V_l(r, \theta) \to 0$, we need that A=0, so $V_l(r, \theta) = \frac{B_l}{r^{l+1}}P_l(\cos\theta)$

So the protential function is

$$V(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & for\ r \leqslant R, \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & for\ r > R. \end{cases}$$

Since the function is continuous at r=R, so we have $B_l = A_l R^{2l+1}$.

Since we have the charge distribution on the disk has the form $1/\sqrt{R^2 - s^2}$, we could assume that the surface charge density is

$$\sigma = \frac{k}{\sqrt{R^2 - s^2}}$$

. Then we have that

$$V_0 = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{2\pi s\sigma}{s}\,ds = \frac{k\pi}{4\epsilon_0}, \quad k = \frac{4\epsilon_0 V_0}{\pi}$$

So the surface charge density is

$$\sigma = \frac{4\epsilon_0 V_0}{\pi}\frac{1}{\sqrt{R^2 - s^2}}$$

Then we could calculate the protential along z-axis

$$V(r, 0) = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{2\pi s\sigma}{\sqrt{r^2 + s^2}}\,ds = \frac{2V_0}{\pi}\int_0^R \frac{s}{\sqrt{R^2 - s^2}\sqrt{r^2 + s^2}}\,ds$$

With Taylor Expansion, we get

$$
\begin{aligned}
V(r,0) =& \frac{2V_0}{\pi}\int_0^R \frac{1}{\sqrt{R^2-s^2}}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{n!2^n}\left(\frac{r^2}{s^2}\right)^n\right]ds \\
=& V_0+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{n!2^n}\frac{2V_0}{\pi}\int_0^R\frac{1}{\sqrt{R^2-s^2}s^{2n}}\,ds\; r^{2n}
\end{aligned}
$$

So we could get that

$$
A_l = \begin{cases} V_0, & if\ l=0, \\ (-1)^{l/2}\frac{(l-1)!!}{(l/2)!2^{l/2}}\frac{2V_0}{\pi}\int_0^R\frac{1}{s^l\sqrt{R^2-s^2}}\,ds, & if\ l\ is\ even, \\ 0, & if\ l\ is\ odd. \end{cases}
$$

Then we could also get that

$$
B_l = \begin{cases} V_0R, & if\ l=0, \\ (-1)^{l/2}\frac{(l-1)!!}{(l/2)!2^{l/2}}\frac{2V_0}{\pi}R^{2l+1}\int_0^R\frac{1}{s^l\sqrt{R^2-s^2}}\,ds, & if\ l\ is\ even, \\ 0, & if\ l\ is\ odd. \end{cases}
$$

So the protential at all points in space is

$$
V(r,\theta) = \begin{cases} V_0+\sum_{k=1}^{\infty}\left[(-1)^k\frac{(2k-1)!!}{k!2^k}\frac{2V_0}{\pi}\int_0^R\frac{1}{s^l\sqrt{R^2-s^2}}\,ds\right]r^{2k}P_l(\cos\theta), \\ \qquad if\ r\leqslant R, \\ \frac{R}{r}V_0+\sum_{k=1}^{\infty}\left[(-1)^k\frac{(2k-1)!!}{k!2^{l/2}}\frac{2V_0}{\pi}R^{4k+1}\int_0^R\frac{1}{s^l\sqrt{R^2-s^2}}\,ds\right]r^{-(l+1)}P_l(\cos\theta), \\ \qquad if\ r>R. \end{cases}
$$

1.2

The capacitance of the disk is

$$
C=\frac{Q}{V_0}=\frac{1}{V_0}\int_0^R 2\pi s\sigma\,ds = 8\epsilon_0 R
$$

2

3

Firstly, we calculate the protential at all points in space. According to the Problem1, we have

$$
V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta), & for\ r\leqslant R, \\ \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta), & for\ r>R. \end{cases}
$$

where $B_l = A_l R^{2l+1}$ and we have the boundary at the e spherical shell

$$
\left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} = -\frac{\sigma(\theta)}{\varepsilon_0}
$$

So we get that

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

Then we have

$$\begin{aligned}A_l =&\frac{1}{2\varepsilon_0 R^{l-1}}\int_0^{\pi}\sigma_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\\ =&\frac{\sigma}{2\varepsilon_0 R^{l-1}}\left(\int_0^1 P_l(x)\,dx-\int_{-1}^0 P_l(x)\,dx\right)\end{aligned}$$

We know that

$$\int_{-1}^0 P_l(x)\,dx = (-1)^l\int_0^1 P_l(x)\,dx$$

So

$$A_l = \begin{cases}0, & if\ l\ is\ even,\\ \frac{\sigma}{\varepsilon_0 R^{l-1}}\int_0^1 P_l(x)\,dx, & if\ l\ is\ odd.\end{cases}$$

$$B_l = \begin{cases}0, & if\ l\ is\ even,\\ \frac{\sigma}{\varepsilon_0}R^{l+2}\int_0^1 P_l(x)\,dx, & if\ l\ is\ odd.\end{cases}$$

3.1

So we could calculate the protential at $\vec{r} = (6R, 0, 8R)$ is

$$V\left(10R,\arcsin\left(\frac{3}{5}\right)\right) = \frac{\sigma R^3}{2\varepsilon_0 r^2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k\left(\frac{R}{r}\right)^{2k}P_{2k+1}\left(\frac{4}{5}\right)$$

3.2

While $\vec{r} = \vec{0}$, we could easily get that

$$V(0) = 0$$