PHYS1314 Electrodynamics Spring 2021
Problem Set 2 solutions

1. (5 points) **Divergence of the electric field**

   The electric field due to an infinite line with charge density λ on the z-axis is $\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{s}}{s}$. Find the divergence of the electric field at *all* points in space.

   Solution: The divergence at any given point is, by Gauss's Law, proportional to the charge density. Therefore, the divergence is everywhere zero except on the z-axis. Given the cylindrical symmetry of the problem, let's use cylindrical coordinates to calculate the divergence at $x = y = 0$. From the front cover of Griffiths, we find that in cylindrical coordinates, the divergence is

   $$\nabla \cdot \vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) + \frac{1}{s}\frac{\partial}{\partial \phi}E_\phi + \frac{\partial}{\partial z}E_z. \tag{1}$$

   However, this will be undefined at $s = 0$, where $1/s$ diverges. Instead, we note that the electric field is that due to a line of charge for which the charge density in cylindrical coordinates is $\nabla \cdot \vec{E} = f(s)\delta(s)$. The only question is what is the value of $f(s)$. Mathematically, we know that $\iiint_V \nabla \cdot \vec{E} d\tau = \oiint_S \vec{E} \cdot d\vec{a}$. Let's draw a cylindrical Gaussian cylinder centered on the line charge. We can rewrite this equation as

   $$\iiint_V f(s)\delta(s)s\, ds, d\phi\, dz = \oiint_S \frac{\lambda}{2\pi\epsilon_0 s} s\, d\phi\, dz, \tag{2}$$

   but the right hand side is just $\frac{\lambda}{2\pi\epsilon_0}\oiint_S d\phi\, dz$. Since this is true for any arbitrary cylinder centered on the line charge, we can just equate the s-dependent parts of the left and right hand sides:

   $$\int_0^\infty f(s)\delta(s)\, s\, ds = \frac{\lambda}{2\pi\epsilon_0} \tag{3}$$

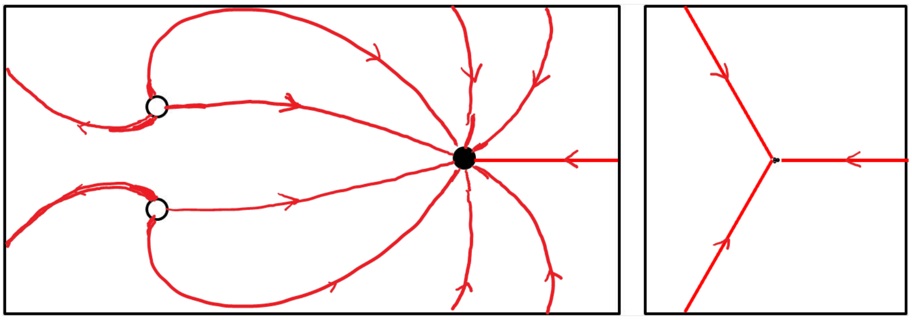
   We then see that $f(s) = \lambda/(2\pi\epsilon_0)$, so that

   $$\nabla \cdot \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\delta(s)}{s}. \tag{4}$$

2. (5 points) **Field lines**

   The following figure shows a pair of positive charges (open circles) with charge Q each and a negative charge (filled circle) with charge $-3Q$. The two positive charges are separated by a distance R, and they are each separated from the negative charge by a distance $3R$. Draw the electric field *lines* for these charges as viewed from close up (left box) and from far away (right box). The close-up view should be exactly what one would see if one were to zoom in from the far-away view.

   Solution: From our rules for drawing field lines, we know that the number of field lines attached to each charge must be in the ratio 1:1:3 given that $|q_A| : |q_B| : |q_C| :: 1 : 1 : 3$. We also know that the total charge is $q_A + q_B + q_C = -Q$, so at large distances, we should see as many field lines emanating from the group of charges as there are field lines emerging from charge A (since $q_A = Q$). One way to represent this, then is to have 4 lines originating on q_A, 4 lines originating on q_B, and 12 lines terminating on q_C (this yields the ratio of lines 1:1:3). Since the total charge is negative, at long distances, we should



see only field lines pointing towards the charges. This means that all of the lines ending on q_A and q_B must terminate on q_C.

In principle, we could draw as few as 1 line from each of A and B and 3 lines terminating on C, but this would give a grossly misleading (highly asymmetric) picture of the field. A minimum reasonable number of lines from A, B, and C is 3, 3, and 9.

3. (5 points) **Gauss' Law beyond three dimensions**
Suppose that there is a universe with four spatial dimensions, and suppose that Gauss's Law in its differential form holds true in that universe, i.e., $\nabla \cdot \vec{E} = \sigma/\epsilon_0$. Write the electric field associated with a point charge q at the origin in that universe?

Solution: We can draw an enclosing "3-sphere" (the surface of a "4-ball") centered on the charge. Applying Gauss's Law in its integral from to this 3-sphere:

$$\int\iiint \nabla \cdot \vec{E}\, d^4\mathbf{r} = \underset{\text{closed surface}}{\iiint} \vec{E} \cdot d^3\vec{a} = 2\pi^2 r^3 E, \tag{5}$$

where we have used the fact that the surface area of a 3-sphere is $2\pi^2$. We then see that

$$\vec{E} = \frac{q}{2\pi^2\epsilon_0}\frac{1}{r^3}\hat{r}. \tag{6}$$

4. (10 points) **Electrostatic potential from the electric field**
Suppose that Coulomb's law as we know it is slightly in error and that experiments of unprecedented sensitivity reveal the true force between two point charges q_1 and q_2 to be given by

$$\vec{F}_{\text{on2}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|r_{21}|^2}\left(1 + \frac{r_{21}}{\lambda}\right) e^{-r_{21}/\lambda}\,\hat{r}_{21} \tag{7}$$

where $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$ and λ is a new physical constant of magnitude of the order of half of the length of the universe. Find the potential of a point charge q in such a case.

Solution: We begin begin by simply noting that the potential is related to the electric field which is the force per unit test charge. To get the electric field, we just divide the force by q_1 and replace q_2 by q:

$$\vec{E}(\vec{r}) = k\frac{q}{r^2}\left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda}\hat{r}$$

Where we have chosen our origin to be at the location of q. The potential is then just

$$V(\vec{r}) = V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

Taking $\vec{dl} = \hat{r}dr$ and plugging in our electric field:

$$V(r) = -kq \int_{\infty}^{r} \frac{1}{r'^2}\left(1 + \frac{r'}{\lambda}\right) e^{-r'/\lambda} dr' = -kq \left[\int_{\infty}^{r} \frac{1}{r'^2} e^{-r'/\lambda} dr' + \int_{\infty}^{r} \frac{1}{\lambda r'} e^{-r'/\lambda} dr\right] \tag{8}$$

Let's solve the second integral first. We can do this by substitution. Let $u = \frac{1}{r}$ and $dv = e^{-r'/\lambda} dr'$ so that $du = -\frac{1}{r'^2} dr'$ and $v = -\lambda e^{-r'/\lambda}$:

$$\int_{\infty}^{r} \frac{1}{\lambda r'} e^{-r'/\lambda} dr = \frac{1}{\lambda}\left[uv\Big|_{r4'=\infty}^{r} - \int_{r'=\infty}^{r} vdu\right] = \frac{1}{\lambda}\left[-\frac{\lambda}{r'} e^{-r'/\lambda}\Big|_{r'=\infty}^{r} - \lambda \int_{r'=\infty}^{r} \frac{1}{r'^2} e^{-r'/\lambda} dr'\right] \tag{9}$$

We see that the second term in the preceding square brackets is equal in magnitude but opposite in sign to the first integral in our last equation for $V(r)$, so we are done:

$$V(r) = kq\frac{1}{r} e^{-r/\lambda} = \frac{q}{4\pi\epsilon_0} \frac{1}{r} e^{-r/\lambda}$$

.

5. (10 points) Assume an infinitely long cylinder of radius R that is uniformly charged with a charge density ρ. Calculate the electric potential at all points in space. Explain your choice of reference point.

   Solution: In principle, we can determine the electric potential by a volume integral over the entire charge distribution or by doing a path integral of the electric field. The electric field is easy to determine by Gauss's Law, so we will take the latter approach. By symmetry, the electric field is everywhere directed along $\hat{s}$, i.e., perpendicular and directly away from the z axis. Drawing a Gaussian cylinder of radius s and length L coaxial with the charged cylinder, the electric field is everywhere normal to the Gaussian surface except at the end caps, where the field is tangential to the surface. Only the curved surface then yields a non-zero flux. The electric flux is

$$\Phi_E = \oint\!\!\oint \vec{E} \cdot \vec{da} = \iint_{\text{curvedsurface}} E\, da = E \iint_{\text{curvedsurface}} da = 2\pi s L E. \tag{10}$$

   The enclosed charge is

$$q_{enc} = \begin{cases} \pi s^2 L\rho, & \text{for } s \leq R \\ \pi R^2 L\rho, & \text{for } s \geq R \end{cases}$$

   By Gauss's Law ($\Phi_E = q_{enc}/\epsilon_0$), the electric field is then

$$\vec{E}(\vec{s}) = \begin{cases} \dfrac{\rho}{2\epsilon_0} s\hat{s}, & \text{for } s \leq R \\ \dfrac{\rho}{2\epsilon_0} R^2 \dfrac{\hat{s}}{s}, & \text{for } s \geq R \end{cases}$$

   To calculate the electric potential $V = -\int \vec{E} \cdot \vec{dl}$, the natural path will be along the electric field direction (or antiparallel to it). Normally our reference point would be chosen at infinity. However, in this case, that leads to a divergence. If that is not obvious, we can see by considering the potential at $s > R$:

$$\begin{aligned} V(s) &= -\int_{s_0}^{s} \vec{E} \cdot \vec{dl} = -\frac{\rho}{2\epsilon_0} R^2 \int_{\infty}^{s} \left(\frac{1}{s'}\hat{s}\right) \cdot (\hat{s} ds') = -\frac{\rho}{2\epsilon_0} R^2 \int_{\infty}^{s} \frac{ds'}{s'} \\ &= -\frac{\rho}{2\epsilon_0} R^2 \ln s' \Big|_{\infty}^{s} = \frac{\rho}{2\epsilon_0} R^2 (\ln \infty - \ln s)\,. \end{aligned} \tag{11}$$

This blows up on account of the unphysical charge distribution. Note that our $\vec{dl} = \hat{s}ds$. It is easy to get the sign wrong by thinking at the direction we are going in the integration (towards the z axis) is in the $-\hat{s}$ direction, but the sign is naturally taken care of by the limits of integration, specifically that our limits are from farther to nearer.

Instead, natural choices of reference point would be either the z-axis or any point on the surface of the cylinder. Using the z axis as our reference,

$$
\begin{aligned}
V(s \leq R) &= -\frac{\rho}{2\epsilon_0}\int\limits_0^s (s'\hat{s})\cdot(\hat{s}ds') = -\frac{\rho}{2\epsilon_0}\int\limits_0^s s'ds' = -\frac{\rho}{4\epsilon_0}s'^2\Big|_0^s = -\frac{\rho}{4\epsilon_0}s^2 \\
V(s \geq R) &= V(s = R_-) - \int\limits_{s'=R}^{s} \vec{E}\cdot\vec{dl} = -\frac{\rho}{4\epsilon_0}R^2 - \frac{\rho}{2\epsilon_0}R^2\int\limits_{s'=R}^{s}\frac{\hat{s}}{s'}\cdot(\hat{s}ds') \\
&= -\frac{\rho}{4\epsilon_0}R^2\left[1 + \int\limits_{s'=R}^{s}\frac{1}{s'}ds'\right] = -\frac{\rho}{4\epsilon_0}R^2\left[1 + \ln s|_R^s\right] \\
&= -\frac{\rho}{4\epsilon_0}R^2\left[1 + 2\ln\frac{s}{R}\right]. \qquad (12)
\end{aligned}
$$

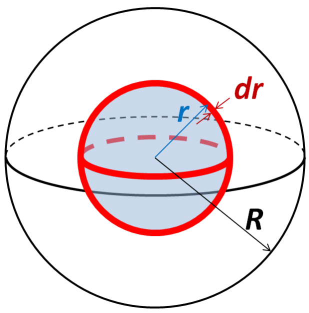
In summary,

$$
V(s) = \begin{cases} -\dfrac{\rho}{4\epsilon_0}s^2, \text{ for } s \leq R \\ -\dfrac{\rho}{4\epsilon_0}R^2\left[1 + 2\ln\dfrac{s}{R}\right], \text{ for } s \geq R \end{cases}
$$

Note that the sign of each result makes sense. For positive ρ it requires positive work by us (the field does negative work) to bring a positive charge closer to the z axis. The potential closer to the axis should then be greater than the potential further from the z axis, and our results show that the potential is indeed larger (less negative) for smaller s.

6. (10 points)

   (a) Find the electrostatic energy of a uniformly charged solid sphere, with total charge Q and radius R.



Solution: We have seen how to calculate the electrostatic potential energy in three related ways: by the work required to assemble the charge distribution piece by piece; by viewing each element of charge in the distribution as experiencing the electric potential due to all the other charges in the distribution and then accounting for double counting; or as being entirely associated with the electric field associated with the entire charge distribution.

Let's look at this from the first perspective. One way to calculate the electrostatic potential is to determine the work required to build up the shell of charge layer-by-layer. For each layer, the potential seen by that layer (the red layer in the figure, for example) is due to all the charge already brought interior to that layer (this interior charge is illustrated in blue. The potential due to the interior layers is just

$$V = k\frac{q}{r} = k\frac{\frac{4}{3}\pi r^3 \rho}{r} = \frac{\rho}{3\epsilon_0} r^2$$

This allows us to calculate the electrostatic potential energy as

$$\begin{aligned}
U = \int V(r) dq(r) &\xrightarrow{dq(r)=\rho d\tau=\rho r^2 dr d\Omega} U = \iint\limits_{\Omega=4\pi} \int\limits_{r=0}^{R} \frac{\rho}{3\epsilon_0} r^2 \rho r^2 dr d\Omega \\
&= \frac{\rho^2}{3\epsilon_0} 4\pi \int\limits_{r=0}^{R} r^4 dr = \frac{4\pi\rho^2}{15\epsilon_0} r^5\Big|_{r=0}^{R} \\
&= \frac{4\pi}{15\epsilon_0} \rho^2 R^5. \qquad (13)
\end{aligned}$$

Note that there is no factor of 1/2 in front of our integral. This is because here we are not double counting. We are only calculating the potential at a given point due to the charges *interior* to that point, not the potential at a given point due to all other charges.

Noting that the total charge is $Q = \frac{4\pi}{3} R^3 \rho$, so that $\rho^2 R^5 = \left(\frac{3Q}{4\pi}\right)^2 \frac{1}{R}$, we can write the electrostatic potential energy as

$$U = \frac{3Q^2}{20\pi\epsilon_0} \frac{1}{R} = \frac{3}{5} k \frac{Q^2}{R}. \qquad (14)$$

Instead, we could have just calculated the product of charge density and potential over the entire charge distribution making sure not to double count. Then, we need to know the electric potential of the charge distribution. For $r > R$, this is just the same as for a point charge of the same total charge at the origin:

$$V(r \geq R) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}. \qquad (15)$$

For $r < R$, though, we need to do a path integral of the electric field from the surface to the distance of interest. By Gauss's Law, the electric field inside the sphere is

$$\vec{E}(r) = \frac{1}{4\pi r^2} \frac{q_{\text{enc}}}{\epsilon_0} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{r^3}{R^3} \frac{Q}{r^2} \hat{r} = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3} \hat{r}. \qquad (16)$$

The potential difference between the surface of the sphere and a point inside the sphere is then

$$\begin{aligned}
V(r \leq R) - V(R) &= -\int\limits_{R}^{r} \vec{E} \cdot \vec{dl} = \frac{Q}{4\pi\epsilon_0 R^3} \int\limits_{r}^{R} r\, dr \\
&= \frac{Q}{4\pi\epsilon_0} \frac{1}{2} \left(\frac{1}{R} - \frac{r^2}{R^3}\right) \qquad (17)
\end{aligned}$$

or

$$V(r \leq R) = \frac{Q}{4\pi\epsilon_0} \frac{1}{2} \left(\frac{3}{R} - \frac{r^2}{R^3}\right) \qquad (18)$$

Since the charge density is zero outside the sphere, the electrostatic potential energy is just

$$\begin{aligned}
U &= \frac{1}{2}\frac{Q}{4\pi\varepsilon_0}\iint\limits_{\Omega=4\pi}\int\limits_{r=0}^{R}\frac{1}{2}\left(\frac{3}{R}-\frac{r^2}{R^3}\right)\rho r^2 dr d\Omega = \frac{1}{4}\frac{Q}{4\pi\varepsilon_0}4\pi\rho\int\limits_{r=0}^{R}\left(\frac{3}{R}r^2-\frac{1}{R^3}r^4\right)dr \\
&= \frac{1}{4}\frac{Q}{\varepsilon_0}\rho\left(\frac{1}{R}r^3-\frac{1}{R^3}\frac{1}{5}r^5\right)\Big|_{r=0}^{R} = \frac{1}{4}\frac{Q}{\varepsilon_0}\rho\left(R^2-\frac{1}{5}R^2\right) = \frac{1}{5}\frac{Q}{\epsilon_0}\rho R^2 \\
&= \frac{1}{5}\frac{Q}{\epsilon_0}\frac{3Q}{4\pi R^3}R^2 \\
&= \frac{3}{5}k\frac{Q^2}{R},
\end{aligned} \tag{19}$$

as before.

Alternatively, we could regard the energy stored in the electric field. The electric field for a uniformly charged solid sphere is, by application of Gauss's Law using a Gaussian sphere centered at the origin, Alternatively, we could regard the energy stored in the electric field. The electric field for a uniformly charged solid sphere is, by application of Gauss's Law using a Gaussian sphere centered at the origin,

$$\vec{E} = \frac{1}{A}\frac{q_{enc}}{\epsilon_0}\hat{r} = k\frac{q_{enc}}{r^2}\hat{r} = \begin{cases} k\dfrac{Q}{r^2}\hat{r}, \text{ for } r \geq R \\ k\dfrac{\frac{4}{3}\pi r^3\rho}{r^2}\hat{r} = \dfrac{4}{3}\pi\rho r\hat{r} = k\dfrac{Q}{R^3}r\hat{r}, \text{ for } r < R \end{cases}$$

The electrostatic potential energy can then be calculated as

$$\begin{aligned}
U &= \frac{1}{2}\epsilon_0\iiint E^2 d\tau = \frac{1}{2}\epsilon_0\left\{\iiint\limits_{r<R}\left(k\frac{Q}{R^3}r\right)^2 r^2 dr d\Omega + \iiint\limits_{r\geq R}\left(k\frac{Q}{r^2}\right)^2 r^2 dr d\Omega\right\} \\
&= \frac{1}{2}\epsilon_0 k^2 Q^2\left\{\frac{1}{R^6}\iiint\limits_{r<R} r^4 dr d\Omega + \iiint\limits_{r\geq R} r^{-2} dr d\Omega\right\} \\
&= \frac{1}{8\pi}kQ^2\left\{4\pi\frac{1}{5}\frac{r^5}{R^6}\Big|_{r=0}^{R} + 4\pi\left(-\frac{1}{r}\right)\Big|_{r=R}^{\infty}\right\} \\
&= \frac{1}{2}kQ^2\left\{\frac{1}{5R}+\frac{1}{R}\right\} \\
&= \frac{3}{5}k\frac{Q^2}{R}
\end{aligned} \tag{20}$$

Of course, the result is identical to that already calculated by the other approaches.

(b) Use (a) to compute the electrostatic energy of an atomic nucleus [charge = Ze, radius = $(1.2 \times 10^{-15}\,\mathrm{m})A^{1/3}$] in MeV times $Z^2/A^{1/3}$.

Solution: We just need to plug in the given numbers to the above equation:

$$U = \frac{3\left(1.6\times 10^{-19}Z\right)^2}{20\pi\left(8.854\times 10^{-12}\mathrm{CV^{-1}m^{-1}}\right)}\frac{1}{1.2\times 10^{-15}\mathrm{m}\ A^{1/3}} = 1.15\times 10^{-13}Z^2/A^{1/3}$$

$$U = 0.719 \text{ MeV } Z^2/A^{1/3}$$

(c) Calculate the change of electrostatic energy when a uranium nucleus ($Z = 92$, $A = 238$) fissions into two equal fragments.

Solution: The initial electrostatic potential energy is

$$U_i\left({}^{238}\mathrm{U}\right) = 0.719 \text{ MeV } 92^2/238^{1/3} = 982.0 \text{ MeV}.$$

The final electrostatic potential energy is

$$U_f = 2U\left({}^{119}\mathrm{Pd}\right) = 0.719 \text{ MeV } 46^2/119^{1/3} = 618.6 \text{ MeV}.$$

The change of electrostatic potential energy is then

$$\Delta U = U_i - U_f = -363 \text{ MeV}.$$

(d) The electron is a point charge. Suppose, though, that the energy of an electron at rest ($E = m_e c^2$) is entirely due to electrostatic potential energy of a uniform spherical charge distribution. Given that an electron has a charge $e = 1.6 \times 10^{-19}$ C, use the result of (a) to determine the classical radius that the electron would need to have to account for an energy of $m_e c^2$.

Solution: he radius is just given by inverting the answer to part (a): $R = \frac{3Q^2}{20\pi\epsilon_0}\frac{1}{U} = \frac{3Q^2}{20\pi\epsilon_0 m_e c^2}$. This yields

$$R = \frac{3\left(1.6 \times 10^{-19}\mathrm{C}\right)^2}{20\pi\left(8.85 \times 10^{-12}\mathrm{C^2m^{-2}N^{-1}}\right)\left(9.91 \times 10^{-31}\mathrm{kg}\right)\left(3.00 \times 10^8\mathrm{m/s}\right)^2}$$

$$R = 1.68 \times 10^{-15}\mathrm{m} = 1.68 \text{ fm}$$