PHYS1314 Electrodynamics
Problem Set 1 Solutions

1. (10 points) **The Levi-Civita symbol and vector identities**
In manipulating cross products, it is useful to use the Levi-Civita antisymmetric symbol:

$$\epsilon_{ijk} = \begin{cases} +1 \text{if} ijk = (123, 231, 312) \\ -1 \text{if} ijk = (213, 321, 132) \\ 0, \text{otherwise} \end{cases} . \tag{1}$$

That is, ϵ_{ijk} is nonzero only when all three indices are different; it is then equal to +1 if ijk is a cyclic permutation of 123, and -1 if ijk is an anti-cyclic permutation. ϵ_{ijk} is totally antisymmetric in the sense that it changes sign if any two indices are interchanged:

$$\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}. \tag{2}$$

With this definition, the i^{th} component of the cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as

$$\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk} A_j B_k, \tag{3}$$

where we have used the summation convention that repeated indices are summed over. For example, in the case of the first-rank tensors $\overleftrightarrow{C}$ and $\overleftrightarrow{D}$, $\epsilon_{ijk} C_{jl} D_{km} = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} C_{jl} D_{km}$. For the rest of this problem set, we will assume that this summation convention is implied, unless explicitly stated other wise.

(a) From the definition in Eq. 1, show that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{4}$$

Solution: The quantity $\epsilon_{ijk}\epsilon_{imn}$ can only be non-zero if $i \neq j \neq k$ and $i \neq m \neq n$. As we sum over i, ϵ_{ijk} will yield a non-zero value as long as $j \neq k$. However, the same value of i that, in this case, yields a non-zero ϵ_{ijk} will only yield a non-zero ϵ_{imn} if $m \neq i$ and $n \neq i$, which is to say that m and n must take on the same two values as j and k (or vice versa). There are then just two options for a non-zero value of $\epsilon_{ijk}\epsilon_{imn}$: $(m, n) = (j, k)$, for which $\epsilon_{imn} = \epsilon_{ijk}$ and $\epsilon_{ijk}\epsilon_{imn} = (\pm 1)^2$; or $(m, n) = (k, j)$, for which $\epsilon_{imn} = \epsilon_{ikj} = -\epsilon_{ijk}$ and $\epsilon_{ijk}\epsilon_{imn} = (\pm 1)(\mp 1) = -1$. Therefore, we arrive at

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{5}$$

(b) Show that for any vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, the following vector identities are true:

i. $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

Solution:

$$\begin{aligned}\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i &= \epsilon_{ijk} A_j \left(\vec{B} \times \vec{C}\right)_k = \epsilon_{ijk}\epsilon_{kmn} A_j B_m C_n \\ &= \epsilon_{kij}\epsilon_{kmn} A_j B_m C_n = \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j B_m C_n \\ &= B_i A_j C_j - C_i A_j B_j = B_i \left(\vec{A} \cdot \vec{C}\right) - C_i \left(\vec{A} \cdot \vec{B}\right). \end{aligned} \tag{6}$$

Since this is true for an arbitrary component i of $\vec{A} \times \left(\vec{B} \times \vec{C}\right)$, the identity is proved.

ii. $\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A}\left(\vec{\nabla} \times \vec{B}\right)$.

Solution:

$$\begin{aligned}\nabla \cdot \left(\vec{A} \times \vec{B}\right) &= \partial_i \left(\vec{A} \times \vec{B}\right)_i = \partial_i \left(\epsilon_{ijk} A_j B_k\right) \\ &= \epsilon_{ijk}\left[\left(\partial_i A_j\right) B_k + A_j \left(\partial_i B_k\right)\right] \\ &= B_k \left(\epsilon_{ijk}\partial_i A_j\right) + A_j \left(\epsilon_{ijk}\partial_i B_k\right) \end{aligned} \tag{7}$$

Making use of the antisymmetry of the Levi-Civita symbol, this becomes

$$\begin{aligned}\nabla \cdot \left(\vec{A} \times \vec{B}\right) &= B_k \left(\epsilon_{kij}\partial_i A_j\right) + A_j \left(-\epsilon_{jik}\partial_i B_k\right) \\ &= B_k \left(\nabla \times \vec{A}\right)_k - A_j \left(\nabla \times \vec{B}\right) \qquad = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A}\left(\vec{\nabla} \times \vec{B}\right). \end{aligned}$$

iii. $\vec{A} \times \left(\nabla \times \vec{B}\right) = \nabla\left(\vec{A} \cdot \vec{B}\right) - \vec{B} \times \left(\nabla \times \vec{A}\right) - \left(\vec{A} \cdot \nabla\right)\vec{B} - \left(\vec{B} \cdot \nabla\right)\vec{A}$.

Solution:

$$\begin{aligned}\left[\vec{A} \times \left(\nabla \times \vec{B}\right)\right]_i &= \epsilon_{ijk} A_j \left(\epsilon_{kmn}\partial_m B_n\right) = \epsilon_{kij}\epsilon_{kmn} A_j \partial_m B_n \\ &= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j \partial_m B_n \\ &= A_j \partial_i B_j - A_j \partial_j B_i. \end{aligned} \tag{8}$$

Noting that the first term of the last line can be written as

$$A_j \partial_i B_j = \partial_i \left(A_j B_j\right) - B_j \partial_i A_j = \partial_i \left(\vec{A} \cdot \vec{B}\right) - B_j \partial_i A_j, \tag{9}$$

we obtain

$$\begin{aligned}\left[\vec{A} \times \left(\nabla \times \vec{B}\right)\right]_i &= \partial_i \left(\vec{A} \cdot \vec{B}\right) - B_j \partial_i A_j - A_j \partial_j B_i \\ &= \partial_i \left(\vec{A} \cdot \vec{B}\right) - B_j \partial_i A_j - \left(\vec{A} \cdot \nabla\right) B_i \end{aligned} \tag{10}$$

We can switch the roles of $\vec{A}$ and $\vec{B}$, which yields

$$\left[\vec{B} \times \left(\nabla \times \vec{A}\right)\right]_i = \partial_i \left(\vec{A} \cdot \vec{B}\right) - A_j \partial_i B_j - \left(\vec{B} \cdot \nabla\right) A_i. \tag{11}$$

Adding the last two equations, we find

$$\left[\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)\right]_i=\partial_i\left(\vec{A}\cdot\vec{B}\right)-B_j\partial_iA_j-\left(\vec{A}\cdot\nabla\right)B_i$$
$$+\partial_i\left(\vec{A}\cdot\vec{B}\right)-A_j\partial_iB_j-\left(\vec{B}\cdot\nabla\right)A_i.$$

We note that $B_j\partial_iA_j+A_j\partial_iB_j=\partial_i\left(\vec{A}\cdot\vec{B}\right)$, so that

$$\left[\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)\right]_i=\partial_i\left(\vec{A}\cdot\vec{B}\right)-\left(\vec{A}\cdot\nabla\right)B_i-\left(\vec{B}\cdot\nabla\right)A_i,$$

and the identity is proved.

iv. $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right).$

Solution:

$$\begin{aligned}\left[\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)\right]_i&=\epsilon_{ijk}\partial_j\left(\vec{A}\times\vec{B}\right)_k=\epsilon_{ijk}\partial_j\left(\epsilon_{kmn}A_mB_n\right)\\&=\epsilon_{ijk}\epsilon_{kmn}\partial_j\left(A_mB_n\right)\\&=\epsilon_{kij}\epsilon_{kmn}\left[B_n\left(\partial_jA_m\right)+A_m\left(\partial_jB_n\right)\right]\\&=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\left[B_n\left(\partial_jA_m\right)+A_m\left(\partial_jB_n\right)\right]\\&=B_j\left(\partial_jA_i\right)-B_i\left(\partial_jA_j\right)+A_i\left(\partial_jB_j\right)-A_j\left(\partial_jB_i\right)\\&=\left(\vec{B}\cdot\nabla\right)A_i-B_i\left(\nabla\cdot\vec{A}\right)+A_i\left(\nabla\cdot\vec{B}\right)-\left(\vec{A}\cdot\nabla\right)B_i\end{aligned}$$

Since this is true for an arbitrary component i of $\nabla\times\left(\vec{A}\times\vec{B}\right)$, the identity is proved.

2. **Practice with δ-functions** (10 points)

(a) (4 points) Given an arbitrary, smooth function $\phi(x)$, evaluate $\int\limits_{-\infty}^{\infty}\phi(x)\delta\left(x^2-a^2\right)dx$.

Solution: There are various ways to evaluate this.
Wheeler has provided one approach based on the fact that the derivative of the Heaviside step function yields the delta function. First, note that $\theta(x^2-a^2)$ takes the values

$$\theta\left(x^2-a^2\right)=\begin{cases}1, & \text{for } x<-a\\0, & \text{for } -a<x<a\\1, & \text{for } x>a\end{cases}\tag{12}$$

and can be written as

$$\theta(x^2-a^2)=1-\left(\theta(x+|a|)-\theta(x-|a|)\right).\tag{13}$$

However, $\frac{d}{dx}\theta(x \pm a) = \delta(x \pm a)$, so

$$\frac{d}{dx}\theta(x^2 - a^2) = -\delta(x + |a|) + \delta(x - |a|). \tag{14}$$

Letting $u = x^2$ and taking the derivative of the left hand side of the last equation yields

$$\frac{d}{dx}\theta(x^2 - a^2) = \frac{du}{dx}\frac{d}{du}\theta(u - a^2) = 2x\delta(u - a^2) = 2x\delta(x^2 - a^2). \tag{15}$$

We see, then, that $\delta(x^2 - a^2) = \frac{1}{2x}\left[\delta(x - |a|) - \delta(x + |a|)\right]$, and therefore,

$$\int_{-\infty}^{\infty} \phi(x)\delta(x^2 - a^2)dx = \frac{1}{2|a|}\left[\phi(|a|) + \phi(-|a|)\right] = \frac{1}{2|a|}\left[\phi(a) + \phi(-a)\right]. \tag{16}$$

An alternative demonstration is as follows. We note that

$$\int_{-\infty}^{\infty} \phi(x)\delta(x^2 - a^2)dx = \int_{-\infty}^{\infty} \phi(x)\delta((x + a)(x - a))dx. \tag{17}$$

As we integrate, there are only two points where the delta function is non-zero: $x = \pm a$. In the vicinity of $x = a$, we have $(x + a) = 2a$, and in the vicinity of $x = -a$, we have $(x - a) = -2a$. We can rewrite the integral as

$$\int_{-\infty}^{\infty} \phi(x)\delta(x^2 - a^2)dx = \int_{-\infty}^{\infty} \phi(x)\left[\delta(2a(x - a)) + \delta(-2a(x + a))\right]dx. \tag{18}$$

Griffiths shows that $\delta(kx) = \frac{1}{|k|}\delta(x)$, so we find

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(x)\delta(x^2 - a^2)dx &= \int_{-\infty}^{\infty} \phi(x)\frac{1}{|2a|}\left[\delta(x - a) + \delta(x + a)\right]dx \\ &= \frac{1}{2|a|}\left[\phi(a) + \phi(-a)\right]. \end{aligned} \tag{19}$$

(b) (3 points) An cylinder of length L and radius R with its axis centered on the z axis and centered at the origin carries a total charge Q on its surface. Write a mathematical expression for the surface charge density in cylindrical coordinates, and confirm that this yields the proper total charge when integrated over all space.

Solution:

$$\rho(s,\phi,z) = \frac{Q}{2\pi RL}\delta(s-R)\left[\theta(z+L/2)-\theta(z-L/2)\right] \tag{20}$$

To confirm, we integrate the charge distribution over all space:

$$\begin{aligned}
\int_{z=-\infty}^{\infty}\int_{\phi=0}^{2\pi}\int_{s=0}^{\infty}\rho(s,\phi,z)s\,ds\,d\phi\,dz &= \frac{Q}{2\pi RL}\left[\int_{\phi=0}^{2\pi}d\phi\right]\left[\int_{s=0}^{\infty}\delta(s-R)s\,ds\right] \\
&\quad\times\left[\int_{z=-\infty}^{\infty}\left[\theta(z+L/2)-\theta(z-L/2)\right]dz\right] \\
&= \frac{Q}{2\pi RL}[2\pi][R]\left[\int_{z=-L/2}^{L/2}dz\right] = \frac{Q}{L}L = Q.
\end{aligned}$$

(c) (4 points) What is the charge distribution given by the form $\rho\left(\vec{x}\right) = A\,\delta\left(\vec{x}\cdot\vec{x}_0 - |\vec{x}_0|^2\right)$? What are the units of A (i.e., is it a point, line, surface, or volume charge density)? Explain.

3. (10 points) **Electric field from a charged circular disk**
Consider a flat circular disk of radius R that carries a uniform surface charge σ. The z axis coincides with the axis of continuous rotational symmetry of the disk.

(a) Find the electric field at all positions $\vec{x} = (0,0,z)$.

Solution: We first find the field for a loop of radius s and then integrate over $s=0$ to $s=R$. On the z axis, by symmetry, the x and y components of the field will be zero, so we just focus on the z component:

$$E_z(z) = k\int_{\phi=0}^{2\pi}\frac{\lambda\cos\theta}{z^2+s^2}\,s\,d\phi = 2\pi k\,s\lambda\frac{\cos\theta}{z^2+s^2} = 2\pi k\,s\lambda\frac{z}{(z^2+s^2)^{3/2}}. \tag{21}$$

We now need to turn the loop into an infinitesimally thin loop of charge density $\lambda = \sigma ds$ producing the field just calculated and integrate over s:

$$\begin{aligned}
\vec{E}(z) &= \hat{k}2\pi kz\sigma\int_0^R ds\,\frac{s}{(z^2+s^2)^{3/2}} = \hat{k}2\pi kz\sigma\left(-\frac{1}{\sqrt{z^2+s^2}}\bigg|_{s=0}^{R}\right) \\
&= \hat{k}2\pi kz\sigma\left(\frac{1}{z}-\frac{1}{z^2+R^2}\right) = \frac{\sigma z}{2\epsilon_0}\left(\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}\right)\hat{k}.
\end{aligned} \tag{22}$$

(b) What is the leading approximation to the field at $z >> R$?

Solution: We know the answer: $\vec{E}(z) = \frac{\pi R^2 \sigma}{4\pi\epsilon_0} \frac{1}{z^2}\hat{k}$. The key is to show that our exact solution yields that in the limit $z >> R$. We just need to expand the second term in brackets.

$$\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z}\frac{1}{\sqrt{1+\frac{R^2}{z^2}}} = \frac{1}{z}\left[1 - \frac{1}{2}\left(1+\delta\right)^{-3/2}\Big|_{\delta=0}\frac{R^2}{z^2}\right] = \frac{1}{z}\left[1-\frac{1}{2}\frac{R^2}{z^2}\right]. \tag{23}$$

The field is then approximately

$$\vec{E}(z) = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)\right]\hat{k} = \frac{\sigma}{4\epsilon_0}\frac{R^2}{z^2}\hat{k} = \frac{\pi R^2\sigma}{4\pi\epsilon_0}\frac{1}{z^2}\hat{k}. \tag{24}$$

(c) What is the first correction to the leading approximation to the field at $z << R$?

4. (20 points) **Electric field from a rectangular charged sheet**
Consider a uniformly charged, infinitesimally thin rectangular sheet with total charge Q extending from $x = -w$ to $x = -w$ and from $y = -L/2$ to $y = L/2$.

(a) Find the electric field at any position $x > 0$ on the positive x-axis.

(b) The solution in part (a) is not obvious. As in physics research, you must demonstrate that your solution exhibits the correct limiting behaviors.

i. Confirm that your result behaves exactly as expected at the origin.

ii. Find the leading behavior of your solution in the limit $x >> L, w$.

iii. Find the first correction to the leading behavior of your solution in the case that $x >> L >> w$; confirm that the first correction is the same as that for a line of charge of length L and the same total charge as the sheet.

Solution E field a distance x from a line of charge along the y axis is solved in Griffiths' (4th edition) Example 2.2:

$$\mathbf{E} = \frac{k\lambda L}{x\sqrt{x^2 + (L/2)^2}}\hat{x}. \tag{25}$$

Now we extend this:

$$\mathbf{E} = \hat{x}kL\sigma \int_{-w/2}^{w/2} \frac{dx\prime}{(x-x')\sqrt{(x-x')^2+(L/2)^2}}. \tag{26}$$

The solution of the last equation is somewhat involved. To simplify the notation, let $a = L/2$. Make the substitution $x - x' = a\tan\theta$, which yields $dx' = -d(x - x') =$

$-a\ \cos^{-2}\theta d\theta$. The field then becomes

$$\mathbf{E} = -\hat{x}kL\sigma \int\limits_{\theta=\theta(-w/2)}^{\theta(w/2)} d\theta \frac{a}{\cos^2\theta} \frac{1}{a\tan\theta\sqrt{a^2\tan^2\theta + a^2}} = -\hat{x}2k\sigma \int\limits_{\theta=\theta(-w/2)}^{\theta(w/2)} \frac{d\theta}{\sin\theta}, \tag{27}$$

where $\theta(\pm w/2) = \arctan\left(\frac{x \mp w/2}{a}\right) = \arctan\frac{(2x \mp w)}{L}$.

The integral of $1/\sin\theta$ can be evaluated by making use of the identities $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\sin^2(\theta/2) + \cos^2(\theta/2) = 1$:

$$\begin{aligned}\int \frac{d\theta}{\sin\theta} &= \int \frac{d\theta}{2\sin(\theta/2)\cos(\theta/2)} = \int \frac{\sin^2(\theta/2) + \cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} d\theta \\ &= \frac{1}{2}\int \left[\frac{\sin(\theta/2)}{\cos(\theta/2)} + \frac{\cos(\theta/2)}{\sin(\theta/2)}\right] d\theta = -\ln\left[\cos(\theta/2)\right] + \ln\left[\sin(\theta/2)\right] \\ &= \ln\left|\tan(\theta/2)\right|.\end{aligned} \tag{28}$$

The electric field is then

$$\begin{aligned}\mathbf{E} &= -\hat{x}\, 2k\sigma \left\{ \ln\left|\tan\left(\frac{1}{2}\arctan\frac{(2x-w)}{L}\right)\right| - \ln\left|\tan\left(\frac{1}{2}\arctan\frac{(2x+w)}{L}\right)\right|\right\} \\ &= -\hat{x}\, 2k\sigma \ln\left|\frac{\tan\left(\frac{1}{2}\arctan\frac{(2x-w)}{L}\right)}{\tan\left(\frac{1}{2}\arctan\frac{(2x+w)}{L}\right)}\right|\end{aligned} \tag{29}$$

As a check, what happens at distances $x >> L, w$? This is equivalent to setting $w/L << 2x/L$. In this case, we can eliminate the absolute values above and write

$$\mathbf{E} = -\hat{x}\, 2k\sigma \left\{ \ln\left[\tan\left(\frac{1}{2}\arctan[(2x/L) - \delta]\right)\right] - \ln\left[\tan\left(\frac{1}{2}\arctan[(2x/L) + \delta]\right)\right]\right\} \tag{30}$$

Taylor expanding about $\delta = 0$

$$\begin{aligned}\ln\left[\tan(\theta_\pm/2)\right] &= \ln\left[\tan\left(\frac{1}{2}\arctan[(2x/L) \mp \delta]\right)\right] \\ &= \ln\left[\tan(\theta_\pm(\delta)/2)\right]\Big|_{\delta=0} + \frac{d}{d\delta}\ln\left[\tan(\theta_\pm(\delta)/2)\right]\Big|_{\delta=0}\delta \\ &\quad + \frac{1}{2}\frac{d^2}{d\delta^2}\ln\left[\tan(\theta_\pm(\delta)/2)\right]\Big|_{\delta=0}\delta^2 + \cdots\end{aligned} \tag{31}$$

The first term, $\ln\left[\tan(\theta_\pm(\delta)/2)\right]$ is identical in both cases and so will cancel. The term linear in δ is given by

$$\frac{d}{d\delta}\ln\left[\tan(\theta_\pm(\delta)/2)\right]\bigg|_{\delta=0}\delta = \frac{1}{\tan\left(\theta_\pm(\delta)/2\right)}\frac{d}{d\delta}\tan\left[\theta_\pm(\delta)/2\right]\bigg|_{\delta=0}$$
$$= \frac{1}{\tan\left(\theta_\pm(\delta)/2\right)}\frac{1}{\cos^2\left[\theta_\pm(\delta)/2\right]}\frac{d}{d\delta}\left[\frac{1}{2}\theta_\pm(\delta)\right]\bigg|_{\delta=0}$$
$$= \frac{1}{2}\frac{1}{\sin\left(\theta_\pm(\delta)/2\right)\cos\left[\theta_\pm(\delta)/2\right]}\frac{d}{d\delta}\left[\theta_\pm(\delta)\right]\bigg|_{\delta=0}. \tag{32}$$

Noting that

$$\frac{d}{d\delta}\theta_\pm(\delta) = \frac{d}{d\delta}\left[\arctan\left(\frac{2x}{L}\mp\delta\right)\right] = \mp\frac{1}{1+\left(\frac{2x}{L}\mp\delta\right)^2}, \tag{33}$$

we find that

$$\frac{d}{d\delta}\ln\left[\tan(\theta_\pm(\delta)/2)\right]\bigg|_{\delta=0}\delta = \mp\frac{1}{2}\frac{1}{\sin\left[\theta_\pm(\delta)/2\right]\cos\left[\theta_\pm(\delta)/2\right]}\frac{1}{1+\left(\frac{2x}{L}\mp\delta\right)^2}\bigg|_{\delta=0}\delta \tag{34}$$
$$= \mp\frac{1}{2}\frac{1}{\sin\left[\frac{1}{2}\arctan\left(2x/L\right)\right]\cos\left[\frac{1}{2}\arctan\left(2x/L\right)\right]}\frac{1}{1+\left(\frac{2x}{L}\right)^2}\delta.$$

For $x >> L$, $\arctan(2x/L)\approx\pi/2$, so that $\sin\left[\frac{1}{2}\arctan(2x/L)\right] = \cos\left[\frac{1}{2}\arctan(2x/L)\right] = \sqrt{2}/2$ and

$$\frac{d}{d\delta}\ln\left[\tan(\theta_\pm(\delta)/2)\right]\bigg|_{\delta=0}\delta \approx \mp\frac{1}{1+\left(\frac{2x}{L}\right)^2}\frac{w}{L}\approx\frac{wL}{4x^2}. \tag{35}$$

To a first approximation, at $x >> w, L$, we then find

$$\mathbf{E}\approx -\hat{x}2k\sigma\left[-\frac{wL}{4x^2}-\left(\frac{wL}{4x^2}\right)\right] = \frac{1}{4\pi\epsilon_0}\frac{q}{x^2}, \tag{36}$$

where $q = \sigma wL$ is the total charge of the sheet. This is exactly what we expect at large distances, where the field should look like that from a point charge.

If we want to find the first correction to the point-charge approximation, in general we need to do a few things:

- We must come up with a better approximation of $\arctan\left(2x/L\right)$ than simply $\pi/2$;
- We must also write a better approximation for $\dfrac{1}{1+\left(2x/L\right)^2}\dfrac{w}{L}$ than $\dfrac{wL}{4x^2}$; and
- Most daunting of all, we must extend our Taylor expansion at least to the term of order δ^2.

If we stick with distances $x >> w, L$, we can dispense with the first item; our approximation for arctan remains good. Moreover, if we restrict ourselves to $w << L$, we can dispense with the next term in the Taylor expansion. The second item is straightforward:

$$\begin{aligned} \frac{1}{1+(2x/L)^2}\frac{w}{L} &= \left(\frac{L}{2x}\right)^2 \frac{1}{1+\left(\frac{L}{2x}\right)^2}\frac{w}{L} \\ &\approx \frac{wL}{4x^2}\left[1-\left(\frac{L}{2x}\right)^2\right] = \frac{wL}{4x^2} - \frac{wL^3}{16x^4}. \end{aligned} \tag{37}$$

We then find for the electric field

$$\mathbf{E} \approx -\hat{x}2k\sigma\left[-\left(\frac{wL}{4x^2}-\frac{wL^3}{16x^4}\right)-\left(\frac{wL}{4x^2}-\frac{wL^3}{16x^4}\right)\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{x^2}-\frac{qL^2}{8x^4}\right) \tag{38}$$

If we want to address the case of arbitrary ratio L/w at large distances ($x >> L, w$), we can still use our approximation that $\arctan(2x/L) \approx \pi/2$, but must consider at least the next term in the Taylor expansion. In fact, as we will see later in the course, this term will be zero, so we really need to go to the fourth term (the term of order δ^3) in the Taylor expansion.

Alternate approach to the approximate result: This was suggested by a student. Instead of doing a Taylor expansion of the electric field after integration, we can do so before integration.

$$\begin{aligned} \mathbf{E} &= \hat{x}kL\sigma \int_{-w/2}^{w/2} \frac{dx'}{(x-x')\sqrt{(x-x')^2+(L/2)^2}} \\ &= \hat{x}kL\sigma \int_{-w/2}^{w/2} \frac{dx'}{\sqrt{(x-x')^4+(L/2)^2(x-x')^2}} \\ &= \hat{x}kL\sigma \int_{-w/2}^{w/2} \frac{dx'}{\sqrt{x^4-4x^3x'+6x^2x'^2-4xx'^3+x'^4+(L/2)^2x^2-\frac{L^2}{2}xx'+(L/2)^2x'^2}} \\ &= \hat{x}\frac{kL\sigma}{x^2} \int_{-w/2}^{w/2} \frac{dx'}{\sqrt{1-4\frac{x'}{x}+6\frac{x'^2}{x^2}-4\frac{x'^3}{x^3}+\frac{x'^4}{x^4}+(L/2)^2\frac{1}{x^2}-\frac{L^2}{2}\frac{x'}{x^3}+(L/2)^2\frac{x'^2}{x^4}}}. \end{aligned} \tag{39}$$

We can rewrite the denominator of the integrand as $\sqrt{1+\delta}$, where δ is equal to everything from the $-4x'/x$ to $\left(\frac{L}{2}\right)^2\frac{x'^2}{x^4}$. We can then approximate the integrand as

$$\frac{1}{\sqrt{1+\delta}} \approx 1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2 \tag{40}$$

where we have gone to δ^2 to capture all contributions up to $\dfrac{1}{x^4}$ (when including the factor of x^{-2} before the integral. Although this would be very messy if we were to include all terms that make up δ, we only need to keep those that give terms up to x^{-2}. The electric field is then approximately

$$
\begin{aligned}
\mathbf{E}(x) &= \hat{x}\frac{kL\sigma}{x^2}\int\limits_{-w/2}^{w/2} dx' \left\{1 - \frac{1}{2}\left[-4\frac{x'}{x} + \left(6x'^2 + \frac{L^2}{4}\right)\frac{1}{x^2}\right] + \frac{3}{4}\left[-4\frac{x'}{x}\right]^2\right\} \qquad (41) \\
&= \hat{x}\frac{kL\sigma}{x^2}\int\limits_{-w/2}^{w/2} dx' \left\{1 + 2\frac{x'}{x} + \left(9x'^2 - \frac{L^2}{8}\right)\frac{1}{x^2}\right\} \\
&= \hat{x}\frac{kL\sigma}{x^2}\left\{w + \left[\left(\frac{w}{2}\right)^2 - \left(-\frac{w}{2}\right)^2\right]\frac{1}{x} + \left(\frac{3}{4}w^3 - \frac{L^2 w}{8}\right)\frac{1}{x^2}\right\} \\
&= \hat{x}\frac{kq}{x^2}\left[1 + \left(\frac{3}{4}w^2 - \frac{L^2}{8}\right)\frac{1}{x^2}\right].
\end{aligned}
$$

In the limit $w << L$, this is exactly the same result found above. Note, though, that this approach did not start at the exact solution. We have bypassed the exact solution entirely. That is good if we want just an approximate result, as this approach is relatively easy, but it doesn't lead us to the exact result found above.