Electrodynamics
Problem Set 5

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According to Laplace function in cylindrical coordinate system, we could get

$$\frac{1}{\mathcal{S}}\frac{d}{ds}\left(s\frac{d\mathcal{S}(s)}{ds}\right)+\frac{1}{\Phi}\frac{1}{s^2}\frac{d^2\Phi}{d\varphi^2}+\frac{1}{\mathcal{Z}}\frac{d^2\mathcal{Z}}{dz^2}=0$$

Since the symmetry, we get the potential is independent of z. Thus we could let

$$\frac{1}{\mathcal{S}}\frac{d}{ds}\left(s\frac{d\mathcal{S}(s)}{ds}\right)=c_1,\ \ \frac{1}{\Phi}\frac{1}{s^2}\frac{d^2\Phi}{d\varphi^2}=c_2,$$

and let $c_2=-n^2$, so we could get the general solution is

$$V(s,\varphi)=(A\ln s+B)\,\Phi_{n=0}(\varphi)+\sum_{n=1}^{\infty}\left(A_ns^n+B_ns^{-n}\right)\left(C_n\cos(n\varphi)+D_n\sin(n\varphi)\right)$$

When n=0, we could have $\frac{d^2\Phi}{d\varphi^2}=0$, which means $\Phi_{n=0}(\varphi)=C\varphi+D$. When $\varphi=0$, we have

$$V(s,\varphi)=(A\ln s+B)\,D+\sum_{n=1}^{\infty}\left(A_ns^n+B_ns^{-n}\right)C_n=constant.$$

Which means that $V(s,\varphi)$ is indepnedent of s, so we could get the potential is

$$V(s,\varphi)=C_0\varphi+D_0.$$

We have $\varphi_0=2\arcsin\left(\frac{a}{L}\right)$, L is the length of of the plate. Then we have $V=C_0\varphi_0+D_0$. We could let the potential at negative electrode is 0, so the potential is

$$V(s,\varphi)=\frac{V}{2\arcsin\left(\frac{a}{L}\right)}\varphi.$$

We could calculate the electric field is

$$\vec{E}=-\nabla V(s,\varphi)=-\frac{V}{2\arcsin\left(\frac{a}{L}\right)}\frac{1}{s}\hat{e_\varphi},$$

then the surface charge density at the top plate is

$$\sigma(s)=\frac{\epsilon_0V}{2\arcsin\left(\frac{a}{L}\right)}\frac{1}{s}.$$

The total charge is

$$Q=\int_{L_1}^{L_1+L}\sigma(s)L'\,ds=\frac{\epsilon_0VL'}{2\arcsin\left(\frac{a}{L}\right)}\ln\left(\frac{L_1+L}{L_1}\right).$$

Using triangular similarity, we get

$$Q=\frac{\epsilon_0VL'}{2\arcsin\left(\frac{a}{L}\right)}\ln\left(\frac{d+a}{d-a}\right).$$

By the definition, we could determine the capacitance is

$$C=\frac{Q}{V}=\frac{\epsilon_0L'}{2\arcsin\left(\frac{a}{L}\right)}\ln\left(\frac{d+a}{d-a}\right).$$

Since a«d a«L and LL'=A, we could get the expression of capacitance is

$$C=\frac{\epsilon_0A}{d-a}\left(1-\frac{a}{d-a}\right)$$

So the lowest-order correction is

$$-\frac{\epsilon_0Aa}{(d-a)^2}$$

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Using Taylor expansion, we could get the form of the octopole potential is

$$\begin{aligned} V_{octo}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\iiint -\frac{\rho(\vec{x'})}{3!}\sum_{i,j,k=1}^{3} x'_i x'_j x'_k \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\frac{1}{R}\,d\tau' \\ &= -\frac{1}{4\pi\epsilon_0}\frac{1}{3!}\sum_{i,j,k=1}^{3}\left(\iiint x'_i x'_j x'_k \rho(\vec{x'})\,d\tau'\right)\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\frac{1}{R} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{1}{6}\sum_{i,j,k=1}^{3} O_{ijk}\frac{15x_i x_j x_k - 3r^2(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})}{r^7} \end{aligned}$$

$O_{ijk} = \iiint x'_i x'_j x'_k \rho(\vec{x'})\,d\tau'$ We could easily check that

$$\begin{aligned} \sum_{l=1}^{3} O_{ll1}\frac{15x_l^2 x - 3r^2(x_l\delta_{1l} + x_l\delta_{1l} + x)}{r^7} &= O_{111}\frac{15r^2 x - 3r^2 5x}{r^7} = 0 \\ \sum_{l=1}^{3} O_{ll2}\frac{15x_l^2 y - 3r^2(x_l\delta_{2l} + x_l\delta_{2l} + y)}{r^7} &= O_{222}\frac{15r^2 y - 3r^2 5y}{r^7} = 0 \\ \sum_{l=1}^{3} O_{ll3}\frac{15x_l^2 z - 3r^2(x_l\delta_{3l} + x_l\delta_{3l} + z)}{r^7} &= O_{333}\frac{15r^2 z - 3r^2 5z}{r^7} = 0 \end{aligned}$$

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3.1

We have the potential at $\vec{x} = (x, 0, 0)$ is

$$V(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x - \frac{3a}{2}} - \frac{1}{4\pi\epsilon_0}\frac{3q}{x - \frac{a}{2}} + \frac{1}{4\pi\epsilon_0}\frac{3q}{x + \frac{a}{2}} - \frac{1}{4\pi\epsilon_0}\frac{q}{x + \frac{3a}{2}} = \frac{3aq}{4\pi\epsilon_0 x^2}\left(\frac{1}{1 - \frac{9a^2}{4x^2}} - \frac{1}{1 - \frac{a^2}{4x^2}}\right)$$

When $x > \frac{3a}{2}$, we have $\frac{9a^2}{4x^2} < 1$ and $\frac{a^2}{4x^2} < 1$. By taking Taylor expansion, we could get

$$\frac{1}{1 - \frac{9a^2}{4x^2}} = \sum_{n=0}^{\infty}\left(\frac{9a^2}{4x^2}\right)^n, \quad \frac{1}{1 - \frac{a^2}{4x^2}} = \sum_{n=0}^{\infty}\left(\frac{a^2}{4x^2}\right)^n$$

Thus, we get the leading approximation is

$$V(x) = \frac{3qa^3}{2\pi\epsilon_0}\frac{1}{x^4}$$

3.2

According to last question, we could get the electric field at $|\vec{x}| > \frac{3a}{2}$ is

$$\vec{E}(x) = -\frac{dV(x)}{dx}\vec{i} = \sum_{n=0}^{\infty} 2n\left(9^n - 1\right)\left(\frac{a^2}{4}\right)^n \frac{1}{x^{2n+1}}\vec{i}$$

Thus, we could determine the energy of a dipole $\vec{p} = |p|\,\vec{k}$ is

$$U(x) = -\vec{p}\cdot\vec{E}(x) = 0$$

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4.1

According to the question, we could easily get the expression for the modification of the energy of the quadrupole due to the presence of this inhomogeneous electric field is

$$E_{mod} = \frac{1}{6} Q_{33} \frac{\partial E_z}{\partial z} = \frac{1}{6} eQ \frac{\partial E_z}{\partial z}$$

4.2

Since the nuclear has a uniform charge density, with total charge Ze, we could get the charge density is $\rho = \frac{Ze}{\frac{4}{3}\pi a^2 c}$. Then we could calculate

$$Q_{33} = \rho \iiint (3z^2 - (x^2 + y^2 + z^2))\, dV = \frac{2}{5} Ze(c^2 - a^2)$$

which is quadrupole moment.

For ^{153}Eu Z=63, we have

$$Q = \frac{Q_{33}}{e} = \frac{2}{5} Z(c + a)(c - a).$$

So we get $(c - a) = 7.086 \times 10^{-15} cm$, then fractional difference in radii is

$$\frac{c - a}{R} = 0.01$$