Problem Set 8 董建宇 2019511017.

1. Assume the scalar potential of the spherical could be written as $U(r,\theta) = R(r)\Theta(\theta)$.

So the magnetic field could be written as $\vec{B} = -\nabla U(r,\theta) = -\Theta(\theta)\frac{\partial R}{\partial r}\hat{r} - \frac{1}{r}R(r)\frac{\partial \Theta(\theta)}{\partial \theta}\hat{\theta}$

We have that $\nabla^2 U = 0$, so the general solution is $U(r,\theta) = \sum_{l=0}^{\infty}(C_l r^l + \frac{D_l}{r^{l+1}})P_l(\cos\theta)$

When $r<R$, we need $U(r,\theta)$ is finite for $r=0$, so $D_l = 0$.

When $r>R$, we need $U(r,\theta)$ in finite for $r\to\infty$, so $C_l = 0$

So $U(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta), & \text{when } r<R. \\ \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta), & \text{when } r>R \end{cases}$

Using the boundary conditions, we have $\lim_{r\to R^+} B_\perp = \lim_{r\to R^-} B_\perp$ and $\lim_{r\to R^+} B_\parallel - \lim_{r\to R^-} B_\parallel = \mu_0\sigma\omega R\sin\theta$

So we get that $\begin{cases} lC_l R^{l-1} = -(l+1)\frac{D_l}{R^{l+2}} \\ \sum_{l=0}^{\infty}(C_l R^{l-1} - \frac{D_l}{R^{l+2}})\frac{d}{d\theta}P_l(\cos\theta) = \mu_0\sigma\omega R\sin\theta \end{cases}$

So only for $l=1$ C_l and D_l are unequal to 0, $C_1 = -\frac{2\mu_0\sigma\omega R}{3}$ $D_1 = \frac{\mu_0\sigma\omega R^4}{3}$

$C_l = D_l = 0$ for $l\neq 1$.

So the scalar potential is $U(r,\theta) = \begin{cases} -\frac{2}{3}\mu_0\sigma\omega R r\cos\theta, & r<R \\ \frac{\mu_0\sigma\omega R^4}{3r^2}\cos\theta, & r>R \end{cases}$

So we could get $\vec{B} = -\nabla U = \begin{cases} \frac{2}{3}\mu_0\sigma\omega R\cos\theta\,\hat{r} - \frac{2}{3}\mu_0\sigma\omega R\sin\theta\,\hat{\theta} = \frac{2}{3}\mu_0\sigma\omega R\,\hat{z}, & r<R \\ \frac{\mu_0\sigma\omega R^4}{3r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}), & r>R \end{cases}$

2. (a). By the definition, we have the magnetic dipole moment of the electron is

$$\vec{m} = \frac{1}{2}\iiint \vec{x}'\times\vec{J}(\vec{x}')\,d\tau'$$

Assume the angular velocity of the electron is ω. So using the spherical coordinates, we could get the current density is $\vec{J}(\vec{x}') = \rho_e\omega r\sin\theta\,\hat{\varphi}$ $\vec{x}' = r\hat{r}$

So $\vec{x}'\times\vec{J}(\vec{x}') = -\rho_e\omega r^2\sin\theta\,\hat{\theta}$. then according to the symmetry, we have that the total diople moment only has $\hat{z}$ component. So we could get that

$$\vec{m} = \frac{1}{2}\iiint \rho_e\omega r^2\sin\theta\cdot\sin\theta\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi\,\hat{z} = \frac{1}{2}\rho_e\omega\int_0^{r_e}r^4dr\int_0^{\pi}\sin^3\theta\,d\theta\cdot\int_0^{2\pi}d\varphi\,\hat{z} = \frac{4}{15}\pi\rho_e\omega r_e^5\,\hat{z}$$

By the definition, we have that its angular momentum is $\vec{L} = I\vec{\omega} = \frac{2}{5}m_e r_e^2\omega\,\hat{z}$

Thus we could get that $\gamma = \frac{|\vec{m}|}{|\vec{L}|} = \frac{2}{3}\frac{\pi\rho_e r_e^3}{m_e}$

(b). $|\vec{m}| = \gamma|\vec{L}| = \frac{1}{3}\frac{\pi\hbar\rho_e r_e^3}{m_e} = \frac{e\hbar}{4m_e} = 4.61\times10^{-24}\ A\cdot m^2$

3. Using the cylindrical coordinates (r, θ, z) and put the origin at the center of the solenoid.

By the symmetry, we have that the vector potential is independent with θ

By the definition, we have that the vector potential $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d\tau'$.

According to the question, we have that $\vec{J}(\vec{x}') = \frac{NI}{L} \delta(r-R) \cdot u(\frac{L}{2} - |z|)\, \hat{\theta}$ for $u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$.

In order to find the leading approximation, we only need to determine the approximation of $\vec{A}$.

Using multipole expansion, we could get that the approximation of $\vec{A}$ is diople.

So that we could determine the vector potential is

$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}$ Now, we should determine the diople moment $\vec{m} = \frac{1}{2} \iiint \vec{x}' \times \vec{J}(\vec{x}')\, d\tau'$

$\vec{x}' \times \vec{J}(\vec{x}') = \frac{NI}{L} r \delta(r-R)\, u(\frac{L}{2} - |z|)\, \hat{z} - \frac{NI}{L} \delta(r-R)\, z\, u(\frac{L}{2} - |z|)\, \hat{r}$

So $m_z = \frac{1}{2} \iiint \frac{NI}{L} r \delta(r-R)\, u(\frac{L}{2} - |z|) \cdot d\tau' = \frac{1}{2} \frac{NI}{L} \int_0^{+\infty} r \delta(r-R) \cdot r\, dr \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \cdot \int_0^{2\pi} d\varphi = \pi N I R^2$

Because of the symmetry, $m_r = 0$. So $\vec{m} = \pi N I R^2\, \hat{z}$

Then, the vector potential is $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0}{4\pi} \frac{\pi N I R^2 (-y\, \vec{i} + x\, \vec{j})}{(x^2+y^2+z^2)^{3/2}}$

By the theorem, we have the leading approximation for the magnetic field is

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}) = \frac{\mu_0}{4} N I R \quad \frac{3xz\, \vec{i} + 3yz\, \vec{j} + (2z^2 - x^2 - y^2)\, \vec{k}}{(x^2+y^2+z^2)^{5/2}}$$

According to Biot-Savart Law, $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \vec{r}}{r^3}$

4. We know that a circle current produce the magnetic field at z is

$$B_0(z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}.$$ and the direction of the magnetic field is up

Then, we could get the z component at sphere is

$$B(z) = \int_z^{z+L} \frac{\mu_0 I n}{2} \frac{a^2}{(a^2+s^2)^{3/2}} ds = \frac{\mu_0 I n}{2} \frac{s}{\sqrt{s^2+a^2}} \Big|_z^{z+L} = \frac{\mu_0 I n}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z}{\sqrt{z^2+a^2}}\right]$$

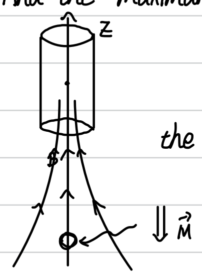
We assert that the force is max if and only if $\vec{M}$ is parallel with

So the energy is $U = \frac{4}{3}\pi R^3 \vec{M} \cdot \vec{B} = \frac{\mu_0 I M n}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z}{\sqrt{z^2+a^2}}\right]\cos\theta$ $\quad \theta = 0$ or π

So the force is

$$F_z = \frac{\partial U}{\partial z} = \frac{4}{3}\pi R^3 \frac{\mu_0 I M n}{2} \cos\theta \cdot \left[\frac{a^2}{((z+L)^2+a^2)^{3/2}} - \frac{a^2}{(z^2+a^2)^{3/2}}\right] \qquad \theta = \pi \quad \cos\pi = -1$$

And the maximum force is $\vec{F}_z = \frac{2}{3}\pi R^3 M \mu_0 I n \left[\frac{a^2}{(z^2+a^2)^{3/2}} - \frac{a^2}{((z+L)^2+a^2)^{3/2}}\right] \hat{z}$



the direction of magnetic field produced by the solenoid is the opposite direction of magnetization of the small sphere.