Problem Set 2 董建宇 2019511017

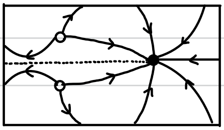
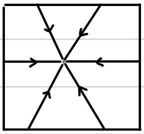
1. $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}$

① When $s>0$ $\nabla\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) = \frac{1}{s}\frac{\partial}{\partial s}(\frac{\lambda}{2\pi\varepsilon_0}) = 0$

② Whes $s=0$, according to Gauss's Law in differential form.

$\nabla\vec{E} = \frac{1}{\varepsilon_0}\lambda\,\delta(x)\delta(y)$ (Put the infinte line along the z-axis in Cartesian.

So the divergence of the electric field at all points in space is $\nabla\cdot\vec{E} = \frac{\lambda}{\varepsilon_0}\delta(x)\delta(y)$

2.  

3. Suppose that Gauss's Law in its differential form holds true in that universe.

We have $\oiint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$, if there is a point charge q at the origin.

Then we get $E\cdot 2\pi^2R^3 = \frac{q}{\varepsilon_0}$

So $\vec{E} = \frac{q}{2\pi^2\varepsilon_0}\frac{1}{R^3}\hat{R}$

4. According to the definition of electrostatic potential from the electric field, for a point charge q we have:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}(1+\frac{r}{\lambda})e^{-\frac{r}{\lambda}}\hat{r}$$

$$\varphi(r) = \int_\infty^r -\vec{E}(\vec{r})\cdot d\vec{r} = \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}(1+\frac{r}{\lambda})e^{-\frac{r}{\lambda}}dr$$

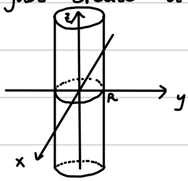
Let $t=\frac{r}{\lambda}$, $dt=\frac{1}{\lambda}dr$ then $\int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}(1+\frac{r}{\lambda})e^{-\frac{r}{\lambda}}dr = \frac{q}{4\pi\varepsilon_0}\frac{1}{\lambda}\int_{\frac{r}{\lambda}}^\infty \frac{1}{t^2}(1+t)e^{-t}dt$

$\int \frac{1}{t^2}(1+t)e^{-t}dt = \int(1+t)e^{-t}d(-\frac{1}{t}) = -\frac{1+t}{t}e^{-t} - \int e^{-t}dt = -\frac{1+t}{t}e^{-t} + e^{-t} = -\frac{1}{t}e^{-t} + C_0$ (C_0 is a constant)

So $\int_{\frac{r}{\lambda}}^\infty \frac{1}{t^2}(1+t)e^{-t}dt = -\frac{1}{t}e^{-t}\Big|_{\frac{r}{\lambda}}^\infty = 0 + \frac{\lambda}{r}e^{-\frac{r}{\lambda}} = \frac{\lambda}{r}e^{-\frac{r}{\lambda}}$

So $\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}e^{-\frac{r}{\lambda}}$, which shows the potential of a point charge q in such a case.

5. Since it is an infinitely long cylinder of radius R, we just create a cylindrical coordinate system as shown.



Since its infinite, the direction of electric field is always perpendicular to the z-axis.

We choose $|\vec{r}|=0$ as reference point which may not lead to infinite during the integral.

According to Gauss's Law we have $\oiint_S \vec{E}\cdot d\vec{s} = \iiint \frac{\rho}{\varepsilon_0} dV$

① When $r \geq R$, we have $E\, 2\pi r l = \frac{\rho}{\varepsilon_0}\cdot \pi R^2 \cdot l$, then $\vec{E} = \frac{\rho}{2\varepsilon_0}\frac{R^2}{r}\hat{r}$

② When $0<r<R$, we have $E\, 2\pi r l = \frac{\rho}{\varepsilon_0}\pi r^2 l$, then $\vec{E} = \frac{\rho r}{2\varepsilon_0}\hat{r}$

According to the definition of electric potential, $\varphi(r_0) = -\int_0^{r_0} \vec{E}(\vec{r})\cdot d\vec{r}$

① When $r_0 < R$ $\varphi(r_0) = -\int_0^{r_0} \frac{\rho r}{2\varepsilon_0} dr = -\frac{\rho}{4\varepsilon_0}\cdot r^2\Big|_0^{r_0} = -\frac{\rho r_0^2}{4\varepsilon_0}$

② When $r_0 \geq R$ $\varphi(r_0) = -\int_0^R \frac{\rho r}{2\varepsilon_0} dr - \int_R^{r_0} \frac{\rho R^2}{2\varepsilon_0}\cdot\frac{1}{r} dr = -\frac{\rho R^2}{4\varepsilon_0}(1+2\ln\frac{r_0}{R})$

So the electric potential at all points in space is:

$$\varphi(r) = \begin{cases} -\frac{\rho r^2}{4\varepsilon_0}, & r<R \\ -\frac{\rho r^2}{4\varepsilon_0}(1+2\ln\frac{r}{R}), & r\geq R \end{cases}$$

6. (a) The charge density $\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$

When there is a uniformly charged solid sphere, with charge density ρ and radius r, we can easily get the electric potential on its surface is $\varphi(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\frac{4}{3}\pi r^3\rho}{r} = \frac{\rho}{3\varepsilon_0}r^2$ while let the infinite potential is 0.

If we put a spherical shell with charge density ρ around, we need to do the work $dW = \varphi(r)\cdot\rho\, 4\pi r^2 dr$

Then the electrostatic energy is $E = \int_0^R dW = \int_0^R \frac{4\pi\rho^2}{3\varepsilon_0} r^4 dr = \frac{4\pi\rho^2}{15\varepsilon_0}R^5 = \frac{3Q^2}{20\varepsilon_0 R}$

(b) If $Q = Ze$ $r = (1.2\times10^{-15}m)A^{\frac{1}{3}}$ So $E = \frac{3e^2}{20\varepsilon_0\,(1.2\times10^{-15}m)}\cdot\frac{Z^2}{A^{1/3}} \approx 2.262\,MeV \times \frac{Z^2}{A^{1/3}}$

(c). When $Z=92$ $A=238$ $E_1 = 3089.3\,MeV$. When $Z=46$ $A=119$ $E_2 = 973.1\,MeV$

$\Delta E = E_1 - 2E_2 = 1143.1\,MeV$

(d). According to the question, we have $m_e c^2 = \frac{3e^2}{20\varepsilon_0 R_e}$, then $r_e = \frac{3e^2}{20\varepsilon_0 m_e c^2} \approx 5.312\times10^{-15}m$