群论作业

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1. 找到相似变换矩阵 M 使

$$M^{-1}\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

由于矩阵 M 可逆，等式两侧左乘 M，并将矩阵 M 按列分块，即 $M = (\vec{m}_1\ \vec{m}_2\ \vec{m}_3)$。则题目等式化为:

$$\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} \vec{m}_1 & \vec{m}_2 & \vec{m}_3 \end{pmatrix} = \begin{pmatrix} \vec{m}_1 & \vec{m}_2 & \vec{m}_3 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \vec{m}_2 & -\vec{m}_1 & 0 \end{pmatrix}$$

对于每个列向量，记作: $\vec{m}_i = (a_i\ b_i\ c_i)^T$。考虑 $\vec{m}_3$ 有:

$$\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

可得:

$$a_3 = \tan\theta\cos\varphi\ c_3;$$
$$b_3 = \tan\theta\sin\varphi\ c_3.$$

考虑归一化 $|a_3|^2 + |b_3|^2 + |c_3|^2 = 1$，并选取 c_3 的相位因子为 0，可得:

$$\vec{m}_3 = (\sin\theta\cos\varphi \quad \sin\theta\sin\varphi \quad \cos\theta)^T.$$

考虑 $\vec{m}_1, \vec{m}_2$，有:

$$\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = -\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

即

$$\begin{pmatrix} 0 & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta & 0 & -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}^2 \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} = - \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix}$$

其中，$j = 1, 2$。则有:

$$\begin{aligned} \vec{m}_1 =& (\sin\varphi \quad -\cos\varphi \quad 0)^T; \\ \vec{m}_2 =& (\cos\theta\cos\varphi \quad \cos\theta\sin\varphi \quad -\sin\theta)^T. \end{aligned}$$

综上，矩阵 M 可为:

$$M = \begin{pmatrix} \sin\varphi & \cos\theta\cos\varphi & \sin\theta\cos\varphi \\ -\cos\varphi & \cos\theta\sin\varphi & \sin\theta\sin\varphi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

可以计算 M 的行列式为:

$$\mathbf{det}(M) = 1 \neq 0.$$

即矩阵 M 可逆，即矩阵 M 为题目所求相似变换矩阵。

2. 设 $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,$S = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$, 找到相似变换矩阵 X 使:

$$X^{-1}(R\otimes R)X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, X^{-1}(S\otimes S)X = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & -1 \end{pmatrix}$$

记 $A = R\otimes R$, $B = S\otimes S$，可以计算矩阵 A, B 分别为:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}.$$

将矩阵 X 按列向量分块，即:

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{pmatrix}.$$

则矩阵 X 满足:

$$A\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{pmatrix} = X\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \vec{x}_1 & -\vec{x}_2 & \vec{x}_3 & -\vec{x}_4 \end{pmatrix};$$

$$B\begin{pmatrix}\vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4\end{pmatrix} = X\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\vec{x}_1 & \vec{x}_2 & -\frac{1}{2}\vec{x}_3-\frac{\sqrt{3}}{2}\vec{x}_4 & \frac{\sqrt{3}}{2}\vec{x}_3-\frac{1}{2}\vec{x}_4\end{pmatrix}.$$

令 $\vec{x}_i = \begin{pmatrix}a_i & b_i & c_i & d_i\end{pmatrix}^T$，则根据 x_1, x_3 是 A 本征值为 1 的本征向量，x_2, x_4 是本征值为 -1 的本征向量，可以得知:

$$b_1 = c_1 = b_3 = c_3 = 0;\ a_2 = d_2 = a_4 = d_4 = 0.$$

则有:

$$\frac{1}{4}\begin{pmatrix}1 & \sqrt{3} & \sqrt{3} & 3\\ -\sqrt{3} & 1 & -3 & \sqrt{3}\\ -\sqrt{3} & -3 & 1 & \sqrt{3}\\ 3 & -\sqrt{3} & -\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}a_1\\0\\0\\d_1\end{pmatrix} = \begin{pmatrix}a_1\\0\\0\\d_1\end{pmatrix};$$

$$\frac{1}{4}\begin{pmatrix}1 & \sqrt{3} & \sqrt{3} & 3\\ -\sqrt{3} & 1 & -3 & \sqrt{3}\\ -\sqrt{3} & -3 & 1 & \sqrt{3}\\ 3 & -\sqrt{3} & -\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}0\\b_2\\c_2\\0\end{pmatrix} = \begin{pmatrix}0\\b_2\\c_2\\0\end{pmatrix};$$

$$\frac{1}{4}\begin{pmatrix}1 & \sqrt{3} & \sqrt{3} & 3\\ -\sqrt{3} & 1 & -3 & \sqrt{3}\\ -\sqrt{3} & -3 & 1 & \sqrt{3}\\ 3 & -\sqrt{3} & -\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}a_3\\0\\0\\d_3\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}a_3\\ -\frac{\sqrt{3}}{2}b_4\\ -\frac{\sqrt{3}}{2}c_4\\ -\frac{1}{2}d_4\end{pmatrix}.$$

可以解得:

$$\vec{x}_1 = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}^T;$$
$$\vec{x}_2 = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}^T;$$
$$\vec{x}_3 = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}^T;$$
$$\vec{x}_4 = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}^T.$$

综上所述，相似变换矩阵 X 为:

$$X = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}$$