群论第四章作业

董建宇 202328000807038

1. 把下列置换化为无公共客体的轮换乘积:

(1)

$$\begin{aligned}(1\ 2)(2\ 3)(1\ 2) &= \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\\ &= \begin{pmatrix}3 & 1 & 2\\ 3 & 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = (1\ 3).\end{aligned}$$

(2)

$$\begin{aligned}(1\ 2\ 3)(1\ 3\ 4)(3\ 2\ 1) &= \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{pmatrix}\begin{pmatrix}3 & 2 & 1 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}\\ &= \begin{pmatrix}2 & 3 & 4 & 1\\ 3 & 1 & 4 & 2\end{pmatrix}\begin{pmatrix}2 & 1 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}\begin{pmatrix}3 & 2 & 1 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}\\ &= \begin{pmatrix}3 & 2 & 1 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} = (1\ 4\ 2).\end{aligned}$$

(3)

$$(1\ 2\ 3\ 4)^{-1} = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}^{-1} = \begin{pmatrix}2 & 3 & 4 & 1\\ 1 & 2 & 3 & 4\end{pmatrix} = (4\ 3\ 2\ 1).$$

(4)

$$\begin{aligned}(1\ 2\ 4\ 5)(4\ 3\ 2\ 6) =&(5\ 1\ 2)(2\ 4)(4\ 3\ 2)(2\ 6) = (5\ 1\ 2)(4\ 2)(2\ 4\ 3)(2\ 6)\\ =&(5\ 1\ 2)(4\ 3)(2\ 6) = (5\ 1\ 2\ 6)(4\ 3).\end{aligned}$$

(5)

$$\begin{aligned}(1\ 2\ 3)(4\ 2\ 6)(3\ 4\ 5\ 6) =&(1\ 2\ 3)(2\ 6\ 4)(6\ 3\ 4)(4\ 5) = (1\ 2\ 3)(2\ 6\ 4)(4\ 6\ 3)(4\ 5)\\ =&(1\ 2\ 3)(2\ 6\ 3)(4\ 5) = (1\ 2\ 3)(3\ 2\ 6)(4\ 5) = (1\ 2\ 6)(4\ 5).\end{aligned}$$

2. 写出对应下列杨表的杨算符

(1) $\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline 4 \\ \cline{1-1}\end{array}$

$$[E + (1\ 2) + (1\ 3) + (2\ 3) + (1\ 2\ 3) + (1\ 3\ 2)][E - (1\ 4)].$$

(2) $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$

$$[E+(1\ 2)][E+(3\ 4)][E-(1\ 3)][E-(2\ 4)].$$

(3) $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline \multicolumn{1}{|c|}{5}&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}$

$$\begin{aligned}&[E+(1\ 2)+(1\ 3)+(1\ 4)+(2\ 3)+(2\ 4)+(3\ 4)+(1\ 2\ 3)+(1\ 3\ 2)+(1\ 2\ 4)+(1\ 4\ 2)\\&+(1\ 3\ 4)+(1\ 4\ 3)+(2\ 3\ 4)+(2\ 4\ 3)+(1\ 2\ 3\ 4)+(1\ 2\ 4\ 3)+(1\ 3\ 2\ 4)+(1\ 3\ 4\ 2)\\&+(1\ 4\ 2\ 3)+(1\ 4\ 3\ 2)+(1\ 2)(3\ 4)+(1\ 3)(2\ 4)+(1\ 4)(2\ 3)][E-(1\ 5)].\end{aligned}$$

3. 具体写出 S_4 群恒元按杨算符的展开式。

Proof. 恒元可以写为

$$E=\frac{A}{24}+\frac{B}{12}+\frac{C}{8}.$$

其中:

$$\begin{aligned}A=&\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}=2[E+(1\ 2)(3\ 4)+(1\ 3)(2\ 4)+(1\ 4)(2\ 3)+(1\ 2\ 3)+(1\ 3\ 2)\\&+(1\ 2\ 4)+(1\ 4\ 2)+(1\ 3\ 4)+(1\ 4\ 3)+(2\ 3\ 4)+(2\ 4\ 3)];\end{aligned}$$

$$\begin{aligned}B=&\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}=E+(1\ 2)+(3\ 4)+(1\ 2)(3\ 4)-(1\ 3)-(2\ 1\ 3)-(4\ 3\ 1)\\&-(2\ 1\ 4\ 3)-(2\ 4)-(1\ 2\ 4)-(3\ 4\ 2)-(1\ 2\ 3\ 4)+(1\ 3)(2\ 4)+(1\ 3\ 2\ 4)\\&+(3\ 1\ 4\ 2)+(1\ 4)(3\ 2)+E+(1\ 3)+(2\ 4)+(1\ 3)(2\ 4)-(1\ 2)-(3\ 1\ 2)\\&-(4\ 2\ 1)-(3\ 1\ 4\ 2)-(3\ 4)-(1\ 3\ 4)-(2\ 4\ 3)-(1\ 3\ 2\ 4)+(1\ 2)(3\ 4)\\&+(1\ 2\ 3\ 4)+(2\ 1\ 4\ 3)+(1\ 4)(2\ 3);\end{aligned}$$

$$\begin{aligned}C=&\begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{|c|}{4}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{|c|}{2}&\multicolumn{1}{c}{}\\\cline{1-1}\multicolumn{1}{|c|}{3}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline 1&2&4\\\hline \multicolumn{1}{|c|}{3}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{2}&\multicolumn{1}{c}{}\\\cline{1-1}\multicolumn{1}{|c|}{4}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline 1&3&4\\\hline \multicolumn{1}{|c|}{2}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}&\multicolumn{1}{c}{}\\\cline{1-1}\multicolumn{1}{|c|}{4}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\\=&6E-2(2\ 3)(1\ 4)-2(2\ 4)(1\ 3)-2(4\ 3)(1\ 2).\end{aligned}$$

□

4. 下列两正则杨算符乘积 $\mathcal{Y}_1\mathcal{Y}_2$ 不为零，R 把正则杨表 $\mathcal{Y}_2$ 变成 $\mathcal{Y}_1$，试把 R 表示成属于杨表 $\mathcal{Y}_2$ 的横向置换 P_2 和纵向置换 Q_2 的乘积 P_2Q_2，再表示成属于杨表 $\mathcal{Y}_1$ 的横向置换 P_1 和纵向置

换 Q_1 的乘积 P_1Q_1.

$$\mathcal{Y}_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}, \quad \mathcal{Y}_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2}\end{array}.$$

Proof. 置换为:

$$\begin{aligned}
R =& \begin{pmatrix} 1 & 2 & 4 & 7 & 3 & 5 & 9 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = (4\ 3\ 5\ 6\ 8\ 9\ 7) \\
=&(7\ 4)(4\ 3\ 5\ 6\ 8\ 9) = (7\ 4)(3\ 5)(5\ 6\ 8\ 9\ 4) \\
=&(7\ 4)(3\ 5)(5\ 6\ 8\ 9)(9\ 4) = (7\ 4)(3\ 5)(9\ 5)(5\ 6\ 8)(9\ 4) \\
=&(7\ 4)(3\ 5\ 9)(6\ 8)(8\ 5)(9\ 4) = P_1Q_1.
\end{aligned}$$

其中

$$P_1 = (7\ 4)(3\ 5\ 9)(6\ 8); \quad Q_1 = (8\ 5)(9\ 4).$$

或者

$$\begin{aligned}
R =&(4\ 3\ 5\ 6\ 8\ 9\ 7) \\
=&(4\ 3)(3\ 5\ 6\ 8\ 9\ 7) = (4\ 3)(5\ 6)(6\ 8\ 9\ 7)(7\ 3) \\
=&(4\ 3)(5\ 6)(7\ 6)(6\ 8\ 9)(7\ 3) = (4\ 3)(5\ 6\ 7)(8\ 9)(9\ 6)(7\ 3).
\end{aligned}$$

其中

$$P_2 = (4\ 3)(5\ 6\ 7)(8\ 9); \quad Q_2 = (9\ 6)(7\ 3).$$

□

5. 用列表法计算 S_5 群生成元 $(1\ 2)$ 和 $(1\ 2\ 3\ 4\ 5)$ 在不可约表示 $[2,2,1]$ 中的表示矩阵。

Proof. 正交杨算符分别为 $\mathcal{Y}_1[E-(2\ 5)]$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, $\mathcal{Y}_4$, $\mathcal{Y}_5$，列表法计算表示矩阵如下:

(1 2)	$\begin{array}{\|c\|c\|}\hline 2&1\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&1\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 1&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 1&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&4\\\hline 1&5\\\hline 3\\\cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}-\begin{array}{\|c\|c\|}\hline 1&5\\\hline 3&4\\\hline 2\\\cline{1-1}\end{array}$	1-0	0-0	1-0	0-0	0+1
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	0	1	0	-1	1
$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$	0	0	-1	0	0
$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	0	0	0	-1	0
$\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	0	0	0	0	-1

□

6. 用等效方法计算 S_6 群各类在下列不可约表示中的特征标。

(1) 表示 $[3,2,1]$; (2) 表示 $[3,3]$; (3) 表示 $[2,2,2]$

Proof.

类	[3,2,1]	[3,3]	[2,2,2]
(1^6)	16	5	5
$(2,1^4)$	0	1	-1
$(2^2,1^2)$	0	1	1
(2^3)	0	-3	3
$(3,1^3)$	-2	-1	-1
$(3,2,1)$	0	1	-1
(3^2)	-2	2	2
$(4,1^2)$	0	-1	1
$(4,2)$	0	-1	-1
$(5,1)$	1	0	0
(6)	0	0	0

□

7. 分别写出 S_6 群相邻客体对换 P_a 在不可约表示 $[3,3]$ 和 $[2,2,2]$ 正交基中的实正交表示矩阵形式。因为下式两边的表示是等价的

$$[2,2,2] \simeq [1^6] \times [3,3].$$

试计算它们间的相似变换矩阵 X。

Proof. 对应杨图 $[2,2,2]$ 的正则杨表为:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array};$$

杨图 $[3,3]$ 的正则杨表为:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array}.$$

相邻客体对换 P_a 在表示 $[2,2,2]$ 和 $[1^6] \times [3,3]$ 中的表示矩阵分别为:

	$[2,2,2]$	$[1^6]\times[3,3]$
P_1	$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$
P_2	$\frac{1}{2}\begin{pmatrix}-1&0&\sqrt{3}&0&0\\0&-1&0&\sqrt{3}&0\\\sqrt{3}&0&1&0&0\\0&\sqrt{3}&0&1&0\\0&0&0&0&-2\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-2&0&0&0&0\\0&1&0&-\sqrt{3}&0\\0&0&1&0&-\sqrt{3}\\0&-\sqrt{3}&0&-1&0\\0&0&-\sqrt{3}&0&-1\end{pmatrix}$
P_3	$\frac{1}{3}\begin{pmatrix}3&0&0&0&0\\0&-3&0&0&0\\0&0&-3&0&0\\0&0&0&-1&\sqrt{8}\\0&0&0&\sqrt{8}&1\end{pmatrix}$	$\frac{1}{3}\begin{pmatrix}1&-\sqrt{8}&0&0&0\\-\sqrt{8}&-1&0&0&0\\0&0&-3&0&0\\0&0&0&-3&0\\0&0&0&0&3\end{pmatrix}$
P_4	$\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}&0&0&0\\\sqrt{3}&1&0&0&0\\0&0&-1&\sqrt{3}&0\\0&0&\sqrt{3}&1&0\\0&0&0&0&-2\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-2&0&0&0&0\\0&1&-\sqrt{3}&0&0\\0&-\sqrt{3}&-1&0&0\\0&0&0&1&-\sqrt{3}\\0&0&0&-\sqrt{3}&-1\end{pmatrix}$
P_5	$\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix}$

两组矩阵中，对应矩阵可以通过两次变换得到，先把非对角元改号，再关于反对角线做转置。

注意到 $[2,2,2]$ 非零对角元只在 $12,13,24,34,45$ 出现，只改变第一行、第四行符号就可以全部变号，则 X 矩阵可以写作 $X=YZ$，Y 对角元第 1、4 行为 -1，其余为 1，Z 的反对角元全为 1，则

$$X=\begin{pmatrix}0&0&0&0&-1\\0&0&0&1&0\\0&0&1&0&0\\0&-1&0&1&0\\1&0&0&0&0\end{pmatrix}$$

满足

$$X^{-1}[2,2,2]X = [1^6] \times [3,3].$$

$\square$

8. 用立特伍德-理查森规则计算下列置换群表示外积的约化

(1) $[3,2,1] \otimes [3]$,

Proof.

$$\begin{ytableau} \ & \ & \ \\ \ & \ \\ \ \end{ytableau} \times \begin{ytableau} 1 & 1 & 1 \end{ytableau} \simeq \begin{ytableau} \ & \ & \ & 1 & 1 & 1 \\ \ & \ \\ \ \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & 1 & 1 \\ \ & \ & 1 \\ \ \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & 1 & 1 \\ \ & \ \\ \ & 1 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & 1 & 1 \\ \ & \ \\ \ \\ 1 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & 1 \\ \ & \ & 1 \\ \ & 1 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & 1 \\ \ & \ & 1 \\ \ \\ 1 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & 1 \\ \ & \ \\ \ & 1 \\ 1 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ \\ \ & \ & 1 \\ \ & 1 \\ 1 \end{ytableau}.$$

$$\begin{aligned} [3,2,1] \times [3] \simeq & [6,2,1] \oplus [5,3,1] \oplus [5,2,2] \oplus [5,2,1,1] \oplus [4,3,2] \\ & \oplus [4,3,1,1] \oplus [4,2,2,1] \oplus [3,3,2,1]. \end{aligned}$$

维数验证

$$\frac{9!}{6!3!} \times 16 \times 1 = 1344 = 105 + 162 + 120 + 189 + 168 + 216 + 216 + 168.$$

$\square$

(2) $[3,2] \otimes [2,1]$,

Proof.

$$\begin{ytableau} \ & \ & \ \\ \ & \ \end{ytableau} \times \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \simeq \begin{ytableau} \ & \ & \ & 1 & 1 \\ \ & \ & 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & 1 & 1 \\ \ & \ \\ 2 \end{ytableau}$$

$$
\oplus \begin{ytableau} \; & \; & \; & 1 \\ \; & \; & 1 & 2 \end{ytableau}
\oplus \begin{ytableau} \; & \; & \; & 1 \\ \; & \; & 1 \\ 2 \end{ytableau}
$$

$$
\oplus \begin{ytableau} \; & \; & \; & 1 \\ \; & \; & 2 \\ 1 \end{ytableau}
\oplus \begin{ytableau} \; & \; & \; & 1 \\ \; & \; \\ 1 & 2 \end{ytableau}
$$

$$
\oplus \begin{ytableau} \; & \; & \; & 1 \\ \; & \; \\ 1 \\ 2 \end{ytableau}
\oplus \begin{ytableau} \; & \; & \; \\ \; & \; & 1 \\ 1 & 2 \end{ytableau}
$$

$$
\oplus \begin{ytableau} \; & \; & \; \\ \; & \; & 1 \\ 1 \\ 2 \end{ytableau}
\oplus \begin{ytableau} \; & \; & \; \\ \; & \; \\ 1 & 1 \\ 2 \end{ytableau}.
$$

$$
\begin{aligned}
[3,2]\times[2,1] \simeq & [5,3]\oplus[5,2,1]\oplus[4,4]\oplus 2[4,3,1]\oplus[4,2,2] \\
& \oplus[4,2,1,1]\oplus[3,3,2]\oplus[3,3,1,1]\oplus[3,2,2,1].
\end{aligned}
$$

维数验证

$$
\frac{8!}{5!3!}\times 5\times 2 = 560 = 28+64+14+2\times 70+56+90+42+56+70.
$$

$\square$

(3) $[2,1]\otimes[4,2^3]$.

Proof.

$$
\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}
\times
\begin{ytableau} \; & \; & \; & \; \\ \; & \; \\ \; & \; \\ \; & \; \end{ytableau}
\simeq
\begin{ytableau} \; & \; & \; & \; & 1 & 1 \\ \; & \; & 2 \\ \; & \; \\ \; & \; \end{ytableau}
\oplus
\begin{ytableau} \; & \; & \; & \; & 1 & 1 \\ \; & \; \\ \; & \; \\ \; & \; \\ 2 \end{ytableau}
$$

$$
\oplus
\begin{ytableau} \; & \; & \; & \; & 1 \\ \; & \; & 1 & 2 \\ \; & \; \\ \; & \; \end{ytableau}
\oplus
\begin{ytableau} \; & \; & \; & \; & 1 \\ \; & \; & 1 \\ \; & \; & 2 \\ \; & \; \end{ytableau}
$$

$$\oplus \begin{ytableau} \ & \ & \ & \ & 1 \\ \ & \ & 1 \\ \ & \ \\ \ & \ \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & \ & 1 \\ \ & \ & 2 \\ \ & \ \\ \ & \ \\ 1 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & \ & 1 \\ \ & \ \\ \ & \ \\ \ & \ \\ 1 & 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & \ & 1 \\ \ & \ \\ \ & \ \\ \ & \ \\ 1 \\ 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ & 1 & 1 \\ \ & \ & 2 \\ \ & \ \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ & 1 & 1 \\ \ & \ \\ \ & \ \\ 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ & 1 \\ \ & \ & 2 \\ \ & \ \\ 1 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ & 1 \\ \ & \ \\ \ & \ \\ 1 & 2 \end{ytableau}$$

$$\oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ & 1 \\ \ & \ \\ \ & \ \\ 1 \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & \ & \ \\ \ & \ \\ \ & \ \\ \ & \ \\ 1 & 1 \\ 2 \end{ytableau}.$$

$$\begin{aligned} [2,1] \times [4,2,2,2] \simeq & [6,3,2,2] \oplus [6,2,2,2,1] \oplus [5,4,2,2] \oplus [5,3,3,2] \\ & \oplus 2[5,3,2,2,1] \oplus [5,2,2,2,2] \oplus [5,2,2,2,1,1] \oplus [4,4,3,2] \\ & \oplus [4,4,2,2,1] \oplus [4,3,3,2,1] \oplus [4,3,2,2,2] \\ & \oplus [4,3,2,2,1,1] \oplus [4,2,2,2,2,1]. \end{aligned}$$

维数验证

$$\begin{aligned} \frac{13!}{3!10!} \times 2 \times 300 = & 171600 = 12012 + 9009 + 12870 + 11583 + 2 \times 21450 + 5005 \\ & + 10296 + 8580 + 12870 + 15015 + 8580 + 17160 + 5720. \end{aligned}$$

$\square$

9. 用立特伍德-查利森规则计算，S_6 群下列不可约表示关于子群 $S_3 \otimes S_3$ 的分导表示，按子群不可约表示的约化

(1) $[4,2]$,

Proof.

$$\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 1 \\ \cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 1 \\ \cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 2 \\ \cline{1-2}\end{array}.$$

$$[4,2] \simeq [3] \times [3] \oplus [3] \times [2,1] \oplus [2,1] \times [3] \oplus [2,1] \times [2,1].$$

维数验证:

$$9 = 1 \times 1 + 1 \times 2 + 2 \times 1 + 2 \times 2.$$

$\square$

(2) $[2,2,1,1]$,

Proof.

$$\begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}.$$

$$[2,2,1,1] \simeq [2,1] \times [2,1] \oplus [2,1] \times [1,1,1] \oplus [1,1,1] \times [2,1] \oplus [1,1,1] \times [1,1,1].$$

维数验证

$$9 = 2 \times 2 + 2 \times 1 + 1 \times 2 + 1 \times 1.$$

$\square$

(3) $[3,3]$.

Proof.

$$\begin{array}{|c|c|c|}\hline & & \\ \hline 1 & 1 & 1 \\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & 2 \\ \hline\end{array}.$$

$$[3,3] \simeq [3] \times [3] \oplus [2,1] \times [2,1]$$

维数验证

$$5 = 1 \times 1 + 2 \times 2.$$

$\square$