第八周周一作业 4月20日.

Section 4.4

4. (a). Assume $a(1-3x+2x^2)+b(1+x+4x^2)+c(1-7x)=0$

Then
$$\begin{aligned} a+b+c&=0 \\ -3a+b-7c&=0 \\ 2a+4b&=0 \end{aligned} \qquad \left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & 0 \\ 2 & 4 & 0 & 0 \end{array}\right)$$

According to Gaussion Elimination : $\left(\begin{array}{ccc:c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Then $(1-3x+2x^2)$, $(1+x+4x^2)$, $(1-7x)$ are linear dependent.

So (a) is not a basis for P_2

(b). Assume $a(4+6x+x^2)+b(-1+4x+2x^2)+c(5+2x-x^2)=0$

Then
$$\begin{aligned} 4a-b+5c&=0 \\ 6a+4b+2c&=0 \\ a+2b-c&=0 \end{aligned} \qquad \left(\begin{array}{ccc:c} 4 & -1 & 5 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

According to Gaussion Elimination: $\left(\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Then $(4+6x+x^2)$, $(-1+4x+2x^2)$, $(5+2x-x^2)$ are linear dependent

So (b) is not a basis for P_2

(c.) Assume $a(1+x+x^2)+b(x+x^2)+cx^2=0$

Then
$$\begin{aligned} a&=0 \\ a+b&=0 \\ a+b+c&=0 \end{aligned} \qquad \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

According to Gaussion Elimination $\left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

So (c) is a basis for P_2

(d). Assume $a(-4+x+3x^2) + b(6+5x+2x^2) + c(8+4x+x^2) = 0$

Then
$$\begin{aligned} -4a + 6b + 8c &= 0 \\ a + 5b + 4c &= 0 \\ 3a + 2b + c &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} -4 & 6 & 8 & 0 \\ 1 & 5 & 4 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

According to Gaussion Elimination: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

So (d) is a basis for P_2

So the anwsers are (c) (d).

6. (a) $\exists\ \lambda_1 = 1\ \ \lambda_2 = -1\ \ \lambda_3 = -1$ such that $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3 = \cos^2 x - \sin^2 x - \cos 2x = 0$

$\therefore\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3$ are linear dependent.

So $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not a basis for V.

(b). $\vec{u}_1 = \cos^2 x \quad \vec{u}_2 = \sin^2 x$

$S = \{\vec{u}_1, \vec{u}_2\}$ is a basis for V.

18. Yes. $\vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\det(\vec{v}_1) = 1 \neq 0 \quad \det(\vec{v}_2) = -1 \neq 0 \quad \det(\vec{v}_3) = -1 \neq 0 \quad \det(\vec{v}_4) = 1 \neq 0 \quad \therefore\ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are invertible matrices.

$\dim M_{2\times2} = 4$ So we only need to justify $M_{2\times2} = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

$\forall\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{22}$

$$A = \frac{1}{2}\left[a_{11}\cdot(\vec{v}_1 + \vec{v}_2) + a_{12}(\vec{v}_3 + \vec{v}_4) + a_{21}(\vec{v}_3 - \vec{v}_4) + a_{22}(\vec{v}_1 - \vec{v}_2)\right] = \frac{a_{11}+a_{22}}{2}\vec{v}_1 + \frac{a_{11}-a_{22}}{2}\vec{v}_2 + \frac{a_{12}+a_{21}}{2}\vec{v}_3 + \frac{a_{12}-a_{21}}{2}\vec{v}_4$$

Which shows $M_{2\times2} = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

So there is a basis for M_{22} consisting of invertible matrices.

Section 4.5

4. $x_1 - 3x_2 + x_3 = 0$

$2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

Basis : $(3, 1, 0)$ $(-1, 0, 1)$ dimension $= 2$

8. (a). dimension $= 3$

(b) dimension $= 2$

(c). dimension $= 1$

9. (a). dimension $= n$

(b). dimension $= \frac{n(n+1)}{2}$

(c). dimension $= \frac{n(n+1)}{2}$

第八周周三作业 4月22日.

Section 4.5

12 (a) $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 3 \\ 1 & -2 & -2 \end{vmatrix} = 2\vec{i} + 1\vec{j} + 0\vec{k} = (2,1,0)$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for R^3

(b). $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 3 & 1 & -2 \end{vmatrix} = 2\vec{i} + 2\vec{j} + 4\vec{k} = (2,2,4)$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ can be a basis for R^3

17. (a). Proof:

$\forall n \in N_+$ Assume $F_0 = 1$ $F_1 = x$ $\cdots$ $F_n = x^n$

$W(x) = \begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & n \cdot x^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & n! \end{vmatrix} = (n!) \cdot [(n-1)!] \cdot \cdots \cdot 2 \cdot 1 > 0$

$\therefore F_0, F_1, \cdots F_n$ are linearly independent.

(b). Assume $F(-\infty, +\infty)$ is finite-dimensional and $\dim(F) = m$.

Then if $n > m+1$, we can find $n+1$ linearly independent vectors in $F(-\infty, +\infty)$

So the hypothesis can't establish.

So $F(-\infty, +\infty)$ is infinite-dimensional.

(c). ① Assume $C(-\infty, +\infty)$ is finite-dimensional vector space and $\dim(C) = N_1$

$\therefore \exists\, n > N_1$ and $n \in N_+$ x^n can't be a linear combination of $C(-\infty, +\infty)$ $\therefore$ The assumption is not right.

So $C(-\infty, +\infty)$ is infinite-dimensional vector space

② Assume $C^m(-\infty, +\infty)$ in finite-dimensional vector space and $\dim(C^m) = N_2$

And a basis for $C^m = \{f_1, \cdots f_{N_2}\}$

We can easily find $\exists n > N_2$ $f_{N_2+1} = x^n$ can't be a linear combination of C^m

So the assumption is not right.

So $C^m(-\infty, +\infty)$ is infinite-dimsional vector space.

③. Assume $C^\infty(-\infty, +\infty)$ in finite-dimensional vector space and $\dim(C^\infty) = N_3$

And a basis for $C^\infty = \{f_1, \cdots f_{N_3}\}$

We also can easily find $\exists n > N_3$ $\overset{n \in N_+}{}$ $f_{N_3+1} = x^n$ $A = \{f_1, f_2 \cdots f_{N_3}, f_{N_3+1}\}$ $W(A) \neq 0$ for any $x \in R$.

So the assumption is not right.

So $C^\infty(-\infty, +\infty)$ is infinite-dimsional vector space.

So $C(-\infty, +\infty)$, $C^m(-\infty, +\infty)$. $C^\infty(-\infty, +\infty)$ are infinite-dimsional vector space.

Section 4.6

4. Assume $A = aA_1 + bA_2 + cA_3 + dA_4$

Then
$$\begin{aligned} -a + b &= 2 \\ a + b &= 0 \\ c &= -1 \\ d &= 3 \end{aligned}$$
We have $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$

So $[A]_S = (-1, 1, -1, 3)$

6. (a). $\vec{u}_1' = 2\vec{u}_1 + \vec{u}_2$ $\quad [\vec{u}_1']_B = (2, 1)$

$\vec{u}_2' = -3\vec{u}_1 + 4\vec{u}_2$ $\quad [\vec{u}_2']_B = (-3, 4)$

$P_{B' \to B} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$

(b). $\vec{u}_1 = \frac{4}{11}\vec{u}_1' + (-\frac{1}{11})\vec{u}_2'$ $\quad [\vec{u}_1]_{B'} = (\frac{4}{11}, -\frac{1}{11})$

$\vec{u}_2 = \frac{3}{11}\vec{u}_1' + \frac{2}{11}\vec{u}_2'$ $\quad [\vec{u}_2]_{B'} = (\frac{3}{11}, \frac{2}{11})$

$P_{B \to B'} = \begin{pmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{pmatrix}$

(c). $[\vec{w}]_B = (3, -5)$

$[\vec{w}]_{B'} = P_{B \to B'} [w]_B = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$

(d). $\vec{w} = 3\vec{u}_1 - 5\vec{u}_2 = 3(\frac{4}{11}\vec{u}_1' - \frac{1}{11}\vec{u}_2') - 5(\frac{3}{11}\vec{u}_1' + \frac{2}{11}\vec{u}_2')$

$= -\frac{3}{11}\vec{u}_1' - \frac{13}{11}\vec{u}_2'$

$\therefore [\vec{w}]_{B'} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$ The same as (c).

12. (a). $S = \{\vec{u}_1, \vec{u}_2\}$ $\quad \vec{u}_1 = (1, 0)$ $\quad \vec{u}_2 = (0, 1)$

$[\vec{v}_1]_S = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad [\vec{v}_2]_S = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$P_{B \to S} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$

(b) $\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$

$(-\frac{1}{2})$ row① $\xrightarrow{+}$ row② $\quad \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & \frac{11}{2} & -\frac{1}{2} & 1 \end{array}\right]$

$(\frac{6}{11})$ row② $\xrightarrow{+}$ row①
row② $\times \frac{2}{11}$ $\quad \left[\begin{array}{cc|cc} 2 & 0 & \frac{8}{11} & \frac{6}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$

row① $\times \frac{1}{2}$ $\quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$ $\quad \therefore P_{S \to B} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$

(c) $P_{B\to S} P_{S\to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P_{S\to B} P_{B\to S} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $P_{S\to B}$ and $P_{B\to S}$ are inverses of one another.

(d). Assume $\vec{w} = a\vec{v}_1 + b\vec{v}_2$ Then $\begin{matrix} 2a - 3b = 5 \\ a + 4b = -3 \end{matrix}$ we have $a=1$ $b=-1$

$[\vec{w}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$[\vec{w}]_S = P_{B\to S} [\vec{w}]_B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

(e) $[\vec{w}]_S = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$[\vec{w}]_B = P_{S\to B} [\vec{w}]_S = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$