第十六周周-作业 6月15日

Section 8.1

1. $\forall\ \vec{u},\vec{v}\in V$ $\quad T(\vec{u}+\vec{v})=\|\vec{u}+\vec{v}\|=\sqrt{\langle\vec{u}+\vec{v},\vec{u}+\vec{v}\rangle}=\sqrt{\|\vec{u}\|^2+\|\vec{v}\|^2+2\langle\vec{u},\vec{v}\rangle}$

$T(\vec{u})+T(\vec{v})=\|\vec{u}\|+\|\vec{v}\|=\sqrt{\langle\vec{u},\vec{u}\rangle}+\sqrt{\langle\vec{v},\vec{v}\rangle}$

即 $T(\vec{u}+\vec{v})\neq T(\vec{u})+T(\vec{v})$ 所以不是线性变换.

2. $\forall\ \vec{u},\vec{v}\in R^3$ $\lambda\in R$ $\quad T(\vec{u}+\vec{v})=(\vec{u}+\vec{v})\times\vec{v}_0=\vec{u}\times\vec{v}_0+\vec{v}\times\vec{v}_0=T(\vec{u})+T(\vec{v})$

$T(\lambda\vec{u})=(\lambda\vec{u})\times\vec{v}_0=\lambda(\vec{u}\times\vec{v}_0)=\lambda T(\vec{u})$

$\therefore$ 是线性变换.

4. $\forall\ \vec{u},\vec{v}\in M_{nn}$, $\lambda\in R$ $\quad T(\vec{u}+\vec{v})=tr(\vec{u}+\vec{v})=tr\,\vec{u}+tr\,\vec{v}=T(\vec{u})+T(\vec{v})$

$T(\lambda\vec{u})=tr(\lambda\vec{u})=\lambda\,tr(\vec{u})=\lambda T(\vec{u})$

$\therefore$ 是线性变换.

5. $\forall\ \vec{u},\vec{v}\in M_{mn}$ $\lambda\in R$ $\quad F(\vec{u}+\vec{v})=(\vec{u}+\vec{v})^T=(\vec{u})^T+(\vec{v})^T=F(\vec{u})+F(\vec{v})$

$F(\lambda\vec{u})=(\lambda\vec{u})^T=\lambda(\vec{u})^T=\lambda F(\vec{u})$

$\therefore$ 是线性变换.

7. $\forall\ \vec{u}=u_0+u_1x+u_2x^2$ $\quad \vec{v}=v_0+v_1x+v_2x^2$ $\quad \lambda\in R$

(a) $T(\vec{u}+\vec{v})=u_0+v_0+(u_1+v_1)(x+1)+(u_2+v_2)(x+1)^2=T(\vec{u})+T(\vec{v})$

$T(\lambda\vec{u})=\lambda u_0+\lambda u_1(x+1)+\lambda u_2(x+1)^2=\lambda T(\vec{u})$

$\therefore$ 是线性变换

(b) $T(\vec{u}+\vec{v})=u_0+v_0+1+(u_1+v_1+1)x+(u_2+v_2+1)x^2$

$T(\vec{u})+T(\vec{v})=u_0+v_0+2+(u_1+v_1+2)x+(u_2+v_2+2)x^2$

$\therefore T(\vec{u}+\vec{v})\neq T(\vec{u})+T(\vec{v})$ $\quad\therefore$不是线性变换.

10 $T(x_1,x_2)=T\left(\frac{x_2-3x_1}{7}\vec{v}_1+\frac{x_1+2x_2}{7}\vec{v}_2\right)=\frac{x_2-3x_1}{7}T(\vec{v}_1)+\frac{x_1+2x_2}{7}T(\vec{v}_2)=\left(\frac{3x_1-x_2}{7},-\frac{9x_1+4x_2}{7},\frac{5x_1+10x_2}{7}\right)$

$T(2,-3)=\left(\frac{9}{7},-\frac{6}{7},-\frac{20}{7}\right)$

23. 令 $\vec{x}=(x_1, x_2, x_3)$

$$T = A\vec{x} = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

(a) 由于 $\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} = \frac{14}{11}\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} - \frac{19}{11}\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ can be a basis for the range of T.

(b) 令 $\vec{v}_1, \vec{v}_2, \vec{v}_3$ 分别为A的三个列向量.

$\vec{v}_3 = \frac{14}{11}\vec{v}_1 - \frac{19}{11}\vec{v}_2$ 即 $14\vec{v}_1 - 19\vec{v}_2 - 11\vec{v}_3 = \vec{0}$

则 $\vec{x} = \begin{bmatrix} 14 \\ -19 \\ -11 \end{bmatrix}$ can be a basis for the kernel of T.

(c). Rank $(T) = 2$ and nullity $(T) = 1$

(d). Rank $(A) = 2$ and nullity $(A) = 1$

Section 8.2

3. (a). $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ 容易发现 $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $T = A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 即 $\ker(T) \neq \{\vec{0}\}$

所以A不是单射.

(b). $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & -1 & 2 & 4 \\ -1 & 3 & 0 & 0 \end{bmatrix}$ 当 $x = \begin{bmatrix} 3 \\ 1 \\ \frac{4}{6} \\ -\frac{13}{6} \end{bmatrix}$ 时 $T = A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 即 $\ker(T) \neq \{\vec{0}\}$

所以A不是单射.

(c). $A = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 5 & 3 \end{bmatrix}$ $T(\vec{x}) = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ 则 $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 即 $\ker(T) = \{\vec{0}\}$

所以A是单射.

8. 令 $\vec{u} = u_0 + u_1x + u_2x^2$

(a). $T(\vec{u}) = x\cdot(u_0 + u_1x + u_2x^2) = 0$ $\forall x \in R$ 得 $u_0 = u_1 = u_2 = 0$ 即 $\ker(T) = \{\vec{0}\}$

$\therefore$ T是单射.

(b). 令 $p(x) = v_0 + v_1x + v_2x^2$ 则 $T(p(x)) = v_0 + v_1(x+1) + v_2(x+1)^2 \overset{\forall x \in R}{=} 0$ 得 $v_0 = v_1 = v_2 = 0$ 即 $\ker(T) = \{\vec{0}\}$

$\therefore$ T是单射.

18. 证明: $\cos\theta_1 = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$ $\quad T(\vec{u}) = A\vec{u} \quad T(\vec{v}) = A\vec{v}$

$$\cos\theta_2 = \frac{\langle A\vec{u}, A\vec{v} \rangle}{\|A\vec{u}\| \cdot \|A\vec{v}\|} = \frac{\vec{u} \cdot A^T A \vec{v}}{A^T A \, \|\vec{u}\| \|\vec{v}\|} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

即 $\cos\theta_1 = \cos\theta_2$ 　又∵ $\theta_1, \theta_2 \in [0, \pi]$ 　∴ $\theta_1 = \theta_2$

即 $\vec{u}$与$\vec{v}$的夹角等于 $T(\vec{u})$与$T(\vec{v})$的夹角。

$$\|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle} = \sqrt{\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\langle \vec{u}, \vec{v} \rangle}$$

$$\|T(\vec{u}) - T(\vec{v})\| = \sqrt{\langle T(\vec{u})-T(\vec{v}), T(\vec{u})-T(\vec{v}) \rangle} = \sqrt{\|A\vec{u}\|^2 + \|A\vec{v}\|^2 - 2\langle A\vec{u}, A\vec{v} \rangle}$$

∵ 同积空间同构. ∴ $\|\vec{u} - \vec{v}\|_V = \|T(\vec{u}) - T(\vec{v})\|_W$

第十六周周三作业 6月7日

Section 8.3

2(a). $T_1(x, y) = (-2y, 3x, x-2y)$. $T_2(x,y,z) = (y, z, x)$ $T_3(x,y,z) = (x+z, y-z)$

$(T_3 \circ T_2 \circ T_1)(x,y) = (3x-2y, x)$

3.(a). $(T_1 \circ T_2)(A) = a+d$

(b). 不可以，因为 T_1 将A 映射到一个实数而非一个2x2矩阵.

13. (a) $a_1, a_2 \cdots a_n$ 均不为零.

(b). $T^{-1}(x_1, x_2, \cdots x_n) = (\frac{1}{a_1}x_1, \frac{1}{a_2}x_2, \cdots \frac{1}{a_n}x_n)$

18. 证明. $A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ $T(x,y) = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ 令 $T(x,y) = \vec{0}$ 则 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 即 $\ker(T) = \{\vec{0}\}$

$\therefore$ T是一对一的

$T^{-1} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} = T$

Section 8.4

2. (a). $T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}$

(b). $[\vec{x}]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $[T]_{B',B}[\vec{x}]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ -2a_2-3a_3 \end{bmatrix} = [T(\vec{x})]_{B'}$

6. (a). $T(\vec{v}_1) = (1,-1,0)$ $T(\vec{v}_2) = (-1,1,-1)$ $T(\vec{v}_3) = (0,0,1)$

$T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$[T]_B \cdot [\vec{x}]_B = [T(\vec{x})]_B$

(b). $[T]_B [\vec{x}]_B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1-x_2 \\ x_2-x_1 \\ x_3-x_1 \end{bmatrix} = [T(\vec{x})]_B$

(c) $\det([T]_B)=0$ $\therefore [T]_B$ 不可逆. $\therefore$ T不是一对一的.

Section 8.5

10. (a). $B=\{1, x, x^2, x^3, x^4\}$ $\quad p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$$T(p(x)) = p(2x+1) = a_0+a_1(2x+1)+a_2(2x+1)^2+a_3(2x+1)^3+a_4(2x+1)^4$$

$$= (a_0+a_1+a_2+a_3+a_4)+(2a_1+4a_2+6a_3+8a_4)x+(4a_2+12a_3+24a_4)x^2+(8a_3+32a_4)x^3+16a_4x^4$$

$$T=\begin{bmatrix}1&1&1&1&1\\0&2&4&6&8\\0&0&4&12&24\\0&0&0&8&32\\0&0&0&0&16\end{bmatrix}$$ $\quad \mathrm{rank}(T)=5 \qquad \mathrm{nullity}(T)=0$

(b). $\det(T)=1\times2\times4\times8\times16=1024$ $\therefore$ T可逆 $\therefore$ T是一对一的.

13. $$T=\begin{bmatrix}5&6&2\\0&-1&-8\\1&0&-2\end{bmatrix}$$

(a). $\det(\lambda I-T)=\begin{vmatrix}\lambda-5&-6&-2\\0&\lambda+1&8\\-1&0&\lambda+2\end{vmatrix}=(\lambda+4)(\lambda-3)^2=0$ 得 $\lambda=-4$ 或 $\lambda=3$

$\therefore$ T的特征值为 -4 或 3.

(b). 当 $\lambda=-4$ 时 $(\lambda I-T)\vec{u}=\begin{bmatrix}-9&-6&-2\\0&-3&8\\-1&0&-2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\vec{0}$ 得 $\vec{u}=t\begin{bmatrix}-6\\8\\3\end{bmatrix}$ 即特征空间基为 $-6+8x+3x^2$

当 $\lambda=3$ 时 $(\lambda I-T)\vec{u}=\begin{bmatrix}-2&-6&-2\\0&4&8\\-1&0&5\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\vec{0}$ 得 $\vec{u}=t\begin{bmatrix}5\\-2\\1\end{bmatrix}$ 即特征空间的基为 $5-2x+x^2$

14. $T(\vec{e}_{11})=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $\quad T(\vec{e}_{12})=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $\quad T(\vec{e}_{21})=\begin{bmatrix}2&1\\-2&0\end{bmatrix}$ $\quad T(\vec{e}_{22})=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$$T=\begin{bmatrix}0&0&2&0\\1&0&1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}$$

(a) $\det(\lambda I-T)=\begin{vmatrix}\lambda&0&-2&0\\-1&\lambda&-1&0\\0&-1&\lambda+2&0\\0&0&0&\lambda-1\end{vmatrix}=(\lambda-1)^2(\lambda+1)(\lambda+2)=0$ 得 $\lambda=1$ 或 $\lambda=-1$ 或 $\lambda=-2$

(6). 当$\lambda=1$时 $(\lambda I-T)\vec{x}=\begin{bmatrix}1&0&-2&0\\-1&1&-1&0\\0&-1&3&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\vec{0}$ $\vec{x}=t\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+s\begin{bmatrix}2\\3\\1\\0\end{bmatrix}$

特征空间的基为 $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ 和 $\begin{bmatrix}2\\3\\1\\0\end{bmatrix}$

当$\lambda=-1$时 $(\lambda I-T)\vec{x}=\begin{bmatrix}-1&0&-2&0\\-1&-1&-1&0\\0&-1&1&0\\0&0&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\vec{0}$ $\vec{x}=t\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}$

特征空间的基为 $\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}$

当$\lambda=-2$时 $(\lambda I-A)\vec{x}=\begin{bmatrix}-2&0&-2&0\\-1&-2&-1&0\\0&-1&0&0\\0&0&0&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\vec{0}$ $\vec{x}=t\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$

特征空间的基为 $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$