第十四周 周一作业 6月1日.

Section 6.3

28. $\vec{v}=(a,b,c,d)$ $\quad \langle\vec{v},\vec{u}_1\rangle=0 \quad \langle\vec{v},\vec{u}_2\rangle=0$

We have $\vec{v}=s\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$

$\vec{v}_1=(1,0,1,0) \qquad \vec{v}_2=(2,-1,0,1)$

Then, assume $\vec{w}=\vec{w}_1+\vec{w}_2=\lambda_1\vec{u}_1+\lambda_2\vec{u}_2+\lambda_3\vec{v}_1+\lambda_4\vec{v}_2$

$$\begin{cases}-\lambda_1+\lambda_3+2\lambda_4=-1\\ \lambda_2-\lambda_4=2\\ \lambda_1+\lambda_3=6\\ 2\lambda_1+\lambda_2+\lambda_4=0\end{cases} \qquad \begin{bmatrix}-1&0&1&2\\0&1&0&-1\\1&0&1&0\\2&1&0&1\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix}=\begin{bmatrix}-1\\2\\6\\0\end{bmatrix}$$

We get $\lambda_1=\frac{5}{4} \quad \lambda_2=-\frac{1}{4} \quad \lambda_3=\frac{19}{4} \quad \lambda_4=-\frac{9}{4}$

$\vec{w}_1=\frac{5}{4}\vec{u}_1-\frac{1}{4}\vec{u}_2=(-\frac{5}{4},-\frac{1}{4},\frac{5}{4},\frac{9}{4}) \qquad \vec{w}_2=\frac{19}{4}\vec{v}_1-\frac{9}{4}\vec{v}_2=(\frac{1}{4},\frac{9}{4},\frac{19}{4},-\frac{9}{4})$

$\vec{w}=\vec{w}_1+\vec{w}_2=(-1,2,6,0)$

$\vec{w}_1$ is in the space W spanned by $\vec{u}_1$ and $\vec{u}_2$ and $\vec{w}_2$ is orthogonal to W.

33. $\vec{v}_1=\vec{u}_1=1$

$\vec{v}_2=\vec{u}_2-\text{proj}_{W_1}\vec{u}_2=x-\frac{1}{2} \qquad \|\vec{v}_2\|=\sqrt{\langle\vec{v}_2,\vec{v}_2\rangle}=\frac{1}{2\sqrt{3}}$

$\vec{v}_3=\vec{u}_3-\text{proj}_{W_2}\vec{u}_3=x^2-x+\frac{1}{6} \qquad \|\vec{v}_3\|=\sqrt{\langle\vec{v}_3,\vec{v}_3\rangle}=\frac{1}{6\sqrt{5}}$

$\vec{v}_{10}=\frac{\vec{v}_1}{\|\vec{v}_1\|}=1$

$\vec{v}_{20}=\frac{\vec{v}_2}{\|\vec{v}_2\|}=\sqrt{3}(2x-1)$

$\vec{v}_{30}=\frac{\vec{v}_3}{\|\vec{v}_3\|}=\sqrt{5}(6x^2-6x+1)$

Section 6.4

2 (b) $A=\begin{bmatrix}2&-2\\1&1\\3&1\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}2\\-1\\1\end{bmatrix}$

$A^T=\begin{bmatrix}2&1&3\\-2&1&1\end{bmatrix} \qquad A^TA=\begin{bmatrix}14&0\\0&6\end{bmatrix} \qquad (A^TA)^{-1}=\begin{bmatrix}\frac{1}{14}&0\\0&\frac{1}{6}\end{bmatrix} \qquad \vec{x}=(A^TA)^{-1}A^T\vec{b}=\begin{bmatrix}\frac{3}{7}\\-\frac{2}{3}\end{bmatrix}$

The least squares solution is $x_1=\frac{3}{7} \quad x_2=-\frac{2}{3}$

4 (a). $A=\begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix}$ $\vec{b}=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $A^TA=\begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix}$ $A^T\vec{b}=\begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 11 & 12 & -7 & 5 \\ 12 & 120 & -84 & 22 \\ -7 & -84 & 59 & -15 \end{array}\right]$ $A^TA\vec{x} = A^T\vec{b}$

$\vec{x}= t\begin{bmatrix} -\frac{1}{7} \\ \frac{5}{7} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{7} \\ \frac{13}{84} \\ 0 \end{bmatrix}$

9. (a). $A=\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ $\vec{u}=\begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix}$

$A^TA=\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$ $A^T\vec{u}=\begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$

$A^TA\vec{x}=A^T\vec{u}$ We have $\vec{x}=\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

$\text{proj}_W \vec{u} = A\vec{x} = \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}$

So the orthogonal projection of $\vec{u}$ is $(7, 2, 9, 5)$

11 (b) $A=\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \\ 4 & -5 & 3 \end{bmatrix}$ $A^T=\begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 1 & 0 & -5 \\ 3 & 1 & -2 & 3 \end{bmatrix}$

$\det(A^TA)=\begin{vmatrix} 21 & -22 & 20 \\ -22 & 27 & -17 \\ 20 & -17 & 23 \end{vmatrix} = 0$

So A doesn't have linearly independent column vectors.

第十四周周三作业　6月3日.

Prove Thm 7.1.2

证明：令 A. B 为正交矩阵

$\therefore A^{-1}=A^T \quad AA^T=A^TA=I \quad B^{-1}=B^T \quad BB^T=B^TB=I$

(a): $\because (A^{-1})^{-1}=(A^T)^{-1}=(A^{-1})^T$ $\therefore$ A^{-1} 为正交矩阵.

(b) $(AB)^{-1}=B^{-1}A^{-1}=B^TA^T=(AB)^T$ $\therefore$ AB 为正交矩阵

(c). $\det(AA^T)=\det(A)\det(A^T)=[\det(A)]^2=\det(I)=1$

$\therefore \det(A)=1$ 或 $\det(A)=-1$

Section 7.1

3 (d) $A=\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$\det A=1\neq 0$ So A is invertible.

$A^T=\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $A^{-1}=\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $\therefore A^{-1}=A^T$

So $A=\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is orthogonal and $A^{-1}=\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(e) $A=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

$\det(A)=1\neq 0$ So A is invertible.

$A^T=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ $A^{-1}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ $\therefore A^{-1}=A^T$

So $A=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ is orthogonal and $A^{-1}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

14. Assume $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and A is an orthogonal matrix so $\det A = a_{11}a_{22} - a_{12}a_{21}$ $\det A=1$ or $\det A=-1$

$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ $\quad A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

①. When $\det A=1$. $\quad A^T = A^{-1} \quad a_{11}=a_{22} \quad a_{21}=-a_{12}$ We have $a_{11}^2 + a_{12}^2 = 1$

So $\begin{cases} a_{11}=\cos\theta \\ a_{12}=\sin\theta \end{cases}$ or $\begin{cases} a_{11}=\cos\theta \\ a_{12}=-\sin\theta \end{cases}$ or $\begin{cases} a_{11}=\sin\theta \\ a_{12}=\cos\theta \end{cases}$ or $\begin{cases} a_{11}=\sin\theta \\ a_{12}=-\cos\theta \end{cases}$ or $\begin{cases} a_{11}=-\cos\theta \\ a_{12}=\sin\theta \end{cases}$ or $\begin{cases} a_{11}=-\cos\theta \\ a_{12}=-\sin\theta \end{cases}$ or $\begin{cases} a_{11}=-\sin\theta \\ a_{12}=\cos\theta \end{cases}$ or $\begin{cases} a_{11}=-\sin\theta \\ a_{12}=-\cos\theta \end{cases}$

So the form of A is $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

②. When $\det A=-1$ $\quad A^T = A^{-1} \quad a_{11}=-a_{22} \quad a_{21}=a_{12}$ We have $-a_{11}^2 - a_{12}^2 = -1$

So $\begin{cases} a_{11}=\cos\theta \\ a_{12}=\sin\theta \end{cases}$ or $\begin{cases} a_{11}=\cos\theta \\ a_{12}=-\sin\theta \end{cases}$ or $\begin{cases} a_{11}=\sin\theta \\ a_{12}=\cos\theta \end{cases}$ or $\begin{cases} a_{11}=\sin\theta \\ a_{12}=-\cos\theta \end{cases}$ or $\begin{cases} a_{11}=-\cos\theta \\ a_{12}=\sin\theta \end{cases}$ or $\begin{cases} a_{11}=-\cos\theta \\ a_{12}=-\sin\theta \end{cases}$ or $\begin{cases} a_{11}=-\sin\theta \\ a_{12}=\cos\theta \end{cases}$ or $\begin{cases} a_{11}=-\sin\theta \\ a_{12}=-\cos\theta \end{cases}$

So the form of A is $A=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

So A has only one of two possible forms: $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $A=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ $\quad \theta \in [0, 2\pi]$

Section 7.2

7. $A=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = \lambda(\lambda-3)^2$

We have $\lambda=0$ or $\lambda=3$

When $\lambda=0$ $\quad (\lambda I - A)\vec{x} = \vec{0}$ $\quad$ we have $\vec{x} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

When $\lambda=3$ $\quad (\lambda I - A)\vec{x} = 0$ $\quad$ we have $\vec{x} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \vec{v}_2 = \vec{u}_2 - \text{proj}_{W_1}\vec{u}_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$ $\quad \vec{v}_3 = \vec{u}_3 - \text{proj}_{W_2}\vec{u}_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$

$\vec{v}_1' = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ $\quad \vec{v}_2' = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$ $\quad \vec{v}_3' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

$$\therefore P=\begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt6} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & -\frac{2}{\sqrt6} & 0 \end{bmatrix} \qquad P^{-1}=\begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{2}{\sqrt6} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \end{bmatrix} \qquad P^{-1}AP=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

10. 令 $A=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ $(b\neq 0)$

$\det(\lambda I-A)=\begin{vmatrix} \lambda-a & -b \\ -b & \lambda-a \end{vmatrix}=\lambda^2-2a\lambda+a^2-b^2=0$ 得 $\lambda=a+b$ 或 $\lambda=a-b$

当 $\lambda=a+b$ $(\lambda I-A)\vec{x}=\begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\vec{x}=\vec{0}$ $\vec{x}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{u}_1=\begin{bmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{bmatrix}$

当 $\lambda=a-b$时 $(\lambda I-A)\vec{x}=\begin{bmatrix} -b & -b \\ -b & -b \end{bmatrix}\vec{x}=\vec{0}$ $\vec{x}=s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{u}_2=\begin{bmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \end{bmatrix}$

$P=\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}$ orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

12. (a) Assume $\vec{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{v}^T=[v_1 \ v_2 \cdots v_n]$ $\vec{v}\vec{v}^T=\begin{bmatrix} v_1v_1 & v_1v_2 & \cdots & v_1v_n \\ v_2v_1 & v_2v_2 & \cdots & v_2v_n \\ \vdots & \vdots & & \vdots \\ v_nv_1 & v_nv_2 & \cdots & v_nv_n \end{bmatrix}$

$I-\vec{v}\vec{v}^T=\begin{bmatrix} 1-v_1v_1 & -v_1v_2 & \cdots & -v_1v_n \\ -v_2v_1 & 1-v_2v_2 & \cdots & -v_2v_n \\ \vdots & \vdots & & \vdots \\ -v_nv_1 & -v_nv_2 & \cdots & 1-v_nv_n \end{bmatrix}$

$(I-\vec{v}\vec{v}^T)^T=I-\vec{v}\vec{v}^T$ $\therefore$ $I-\vec{v}\vec{v}^T$ is orthogonally diagonalizable.

(b) $\vec{v}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $I-\vec{v}\vec{v}^T=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ Assume $A=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\det(\lambda I-A)=\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 0 & \lambda \end{vmatrix}=(\lambda+1)(\lambda-1)^2=0$ We get $\lambda=-1$ or $\lambda=1$

When $\lambda=-1$: $(\lambda I-A)\vec{x}=\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}\vec{x}=\vec{0}$ $\vec{x}=t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda=1$: $(\lambda I-A)\vec{x}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\vec{x}=\vec{0}$ $\vec{x}=t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ orthogonally diagonalizes $I - \vec{v}\vec{v}^T$.