第七周周一　4月13日作业

Section 4.1

7. $k(x,y,z)=(k^2x, k^2y, k^2z)$

$(\lambda+\mu)(x,y,z)=((\lambda+\mu)^2x, (\lambda+\mu)^2y + (\lambda+\mu)^2z)$

$\lambda(x,y,z)+\mu(x,y,z)=((\lambda^2+\mu^2)x, (\lambda^2+\mu^2)y, (\lambda^2+y^2)z)$

When $\lambda, \mu \neq 0$ $(\lambda+\mu)(x,y,z)\neq \lambda(x,y,z)+\mu(x,y,z)$.

Such that this is not a vector space.

11. This is a vector space with the given operations.

Prove: ① $\vec{u}=(1,y)$, $\vec{v}=(1,y')\in V$ 则 $\vec{u}+\vec{v}=(1,y+y')\in V$

② $\vec{u}+\vec{v}=(1,y+y')=(1,y'+y')=\vec{v}+\vec{u}$

③ $\vec{w}=(1,y'')$ $(\vec{u}+\vec{v})+\vec{w}=(1,y+y'+y'')=\vec{u}+(\vec{v}+\vec{w})$

④ $\vec{0}=(1,0)$ 则 $\vec{0}+\vec{u}=(1,y)=\vec{u}+\vec{0}=\vec{u}$

⑤ $-\vec{u}=(1,-y)$ $\vec{u}+(-\vec{u})=(1,0)=(-\vec{u})+\vec{u}=\vec{0}$

⑥ $\lambda\vec{u}=(1,\lambda y)\in V$

⑦ $\lambda(\vec{u}+\vec{v})=(1,\lambda(y+y'))$ $\lambda\vec{u}+\lambda\vec{v}=(1,\lambda y)+(1,\lambda y')=(1,\lambda(y+y'))$

⑧ $(\lambda+\mu)\vec{u}=(1,(\lambda+\mu)y)=\lambda\vec{u}+\mu\vec{u}$

⑨ $(\lambda\mu)\vec{u}=(1,\lambda\mu y)=\lambda(\mu\vec{u})$

⑩ $1\cdot\vec{u}=(1,y)=\vec{u}$

So the set is a vector space with the given operations.

21. Prove. Assume V = the set of M_{mn} of all $m\times n$ matrices with the usual operations of addition and scalar multiplication.

$\vec{u},\vec{v},\vec{w}\in V$ λ is a scalar(纯量)

① $\vec{u}+\vec{v}=(u_{ij}+v_{ij})_{m\times n}\in V$

② $\vec{u}+\vec{v}=(u_{ij}+v_{ij})_{m\times n}=(v_{ij}+u_{ij})_{m\times n}=\vec{v}+\vec{u}$

③ $(\vec{u}+\vec{v})+\vec{w}=(u_{ij}+v_{ij}+w_{ij})_{m\times n}=\vec{u}+(\vec{v}+\vec{w})$

④ $\vec{0}=0_{m\times n}$ $\vec{0}+\vec{u}=(u_{ij})_{m\times n}=\vec{u}+\vec{0}=\vec{u}$

⑤ $-\vec{u}=(-u_{ij})_{m\times n}$ $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=(0)_{m\times n}=\vec{0}$

⑥. $\lambda\vec{u} = (\lambda u_{ij})_{m\times n} \in V$

⑦. $\lambda(\vec{u}+\vec{v}) = [\lambda(u_{ij}+v_{ij})]_{m\times n} = (\lambda u_{ij}+\lambda v_{ij})_{m\times n} = \lambda\vec{u}+\lambda\vec{v}$

⑧. $(\lambda+\mu)\vec{u} = ((\lambda+\mu)u_{ij})_{m\times n} = (\lambda u_{ij}+\mu u_{ij})_{m\times n} = \lambda\vec{u}+\mu\vec{u}$

⑨. $(\lambda\mu)\vec{u} = ((\lambda\mu)u_{ij})_{m\times n} = (\lambda(\mu u_{ij}))_{m\times n} = \lambda(\mu\vec{u})$

⑩. $1\vec{u} = (1\cdot u_{ij})_{m\times n} = (u_{ij})_{m\times n} = \vec{u}$

So the set is a vector space with the given operations.

Section 4.2

2. (a). W is the set of all diagonal $n\times n$ matrices.

Suppose $A.B \in W$ λ is a scalar (纯量).

Then $A+B = (a_{ij}+b_{ij})_{n\times n} \in W$ $\quad \lambda A = (\lambda a_{ij})_{n\times n} \in W$

So W_a is a subspace of $M_{n\times n}$

(b). W is the set of all $n\times n$ matrices such that $\det(A)=0$.

Suppose $A.B \in W$ $\det(A)=\det(B)=0$ $\quad\lambda$ is a scalar (纯量)

$\det(\lambda B) = \lambda^n \det(B) = 0 \quad \lambda B \in W$

But as usual, $\det(A+B) \neq \det(A)+\det(B) = 0$ $\quad A=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $B=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $A+B=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $\det(A)=\det(B)=0$ but $\det(A+B)=1\neq 0$

$A+B \notin W$

So W_b is not a subspace of $M_{n\times n}$

(C) W is the set of all $n\times n$ matrices such that $tr(A)=0$

Assume $A=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $\quad B=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $\quad$ 则 $A+B=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

$tr(A)=tr(B)=0 \quad \therefore A.B\in W$ but $tr(A+B)=1\neq 0$ then $A+B\notin W$

So W_c is not a subspace of $M_{n\times n}$

(d) W is the set of all symmetric $n\times n$ matrices.

Assume $A.B \in W$ λ is a scalar(纯量) $\forall\ i\neq j$ $a_{ij}=a_{ji}$ $b_{ij}=b_{ji}$ $\therefore a_{ij}+b_{ij}=a_{ji}+b_{ji}$ $\lambda a_{ij}=\lambda a_{ji}$

$A+B=(a_{ij}+b_{ij})_{n\times n}\in W.$

$\lambda A=(\lambda a_{ij})_{n\times n}\in W$

So W_d is a subspace of $M_{n\times n}$

(e) W is the set of all $n\times n$ matrices such that $A^T=-A$

Assume $A.B\in W$ then $\forall\ i=j$ $a_{ij}=b_{ij}=0$ $\forall\ i\neq j$ $a_{ij}=-a_{ji}$ $b_{ij}=-b_{ji}$ $\therefore a_{ij}+b_{ij}=-(a_{ji}+b_{ji})$

$C=A+B=(a_{ij}+b_{ij})_{n\times n}=(C_{ij})_{n\times n}$ $\forall\ i=j$ $C_{ij}=0$ $\forall\ i\neq j$ $C_{ij}=-C_{ji}$ $\therefore C=(A+B)\in W$

$D=(d_{ij})_{n\times n}=\lambda A=(\lambda a_{ij})_{n\times n}$ $\forall\ i=j$ $d_{ij}=0$ $\forall\ i\neq j$ $d_{ij}=-d_{ji}$ $\therefore D=(\lambda A)\in W.$

So W_e is a subspaces of $M_{n\times n}$

(f). W is the set of all $n\times n$ matrices A for which $A\vec{x}=0$ has only the trivial solution

Assume $A.B\in W$ λ is a scalar(纯量). Then $A\vec{x}=0$ and $B\vec{x}=0$ has only the trivial solution

$C=A+B$ $D=\lambda A$ Then $C\vec{x}=(A+B)\vec{x}=A\vec{x}+B\vec{x}=0$ and $D\vec{x}=\lambda(A\vec{x})=0$ has only the trivial solution.

$C=A+B\in W$ $D=\lambda A\in W$

So W_f is a subspaces of $M_{n\times n}$

(g). W is the set of all $n\times n$ matrices A such that $AB=BA$ for some fixed $n\times n$ matrix B.

Assume A_1 $A_2\in W$ λ is a scalar(纯量). Then $A_1B=BA_1$ $A_2B=BA_2$

$C=A_1+A_2$ $D=\lambda A_1$ Then $CB=(A_1+A_2)B=A_1B+A_2B=BA_1+BA_2=B(A_1+A_2)=BC$

$DB=(\lambda A_1)B=\lambda(A_1B)=\lambda(BA_1)=B(\lambda A_1)=BD$ $C.D\in W$ Then $A_1+A_2\in W$ $\lambda A_1\in W$

So W_g is a subspaces of $M_{n\times n}$

So the answer is (a) (d) (e) (f) (g)

3. (a) W is all polynomial $a_0+a_1x+a_2x^2+a_3x^3$ for which $a_0=0$

Assume $\alpha_1, \alpha_2 \in W$ λ is a scalar (标量)

$\alpha_1: a_0'+a_1'x+a_2'x^2+a_3x^3$ $\alpha_2: a_0''+a_1''x+a_2''x^2+a_3''x^3$ $a_0'=a_0''=0$ So $a_0'+a_0''=0$ $\lambda a_0'=0$

$\alpha_1+\alpha_2=(a_0'+a_0'')+(a_1'+a_1'')x+(a_2'+a_2'')x^2+(a_3'+a_3'')x^3 \in W$

$\lambda\alpha_1=(\lambda a_0')+(\lambda a_1')x+(\lambda a_2')x^2+(\lambda a_3')x^3 \in W$

So W_a is a subspace of P_3

(b) W is all polynomial $a_0+a_1x+a_2x^2+a_3x^3$ for which $a_0+a_1+a_2+a_3=0$

Assume $\alpha_1: a_0'+a_1'x+a_2'x^2+a_3x^3$ $\alpha_2: a_0''+a_1''x+a_2''x^2+a_3''x^3 \in W$ λ is a scalar

Then $a_0'+a_1'+a_2'+a_3'=0$ $a_0''+a_1''+a_2''+a_3''=0$ $\therefore (a_0'+a_0'')+(a_1'+a_1'')+(a_2'+a_2'')+(a_3'+a_3'')=0$ $\lambda(a_1'+a_2'+a_3'+a_4')=0$

$\alpha_1+\alpha_2=(a_0'+a_0'')+(a_1'+a_1'')x+(a_2'+a_2'')x^2+(a_3'+a_3'')x^3 \in W$

$\lambda\alpha_1=\lambda a_0'+\lambda a_1'x+\lambda a_2'x^2+\lambda a_3'x^3 \in W$

So W_b is a subspace of P_3

(c) W is all polynomial of the form $a_0+a_1x+a_2x^2+a_3x^3$ in which a_0 a_1 a_2 a_3 are integers.

$\exists\ \alpha_1=1+x+x^2+x^3 \in W$ $\lambda_0=\frac{1}{2}$

$\lambda_0\alpha_1=\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3$ $a_1'=a_2'=a_3'=a_4'=\frac{1}{2}$ $\therefore \lambda_0\alpha_1 \notin W$

So W_c is not a subspace of P_3.

(d) W is all polynomial of the form a_0+a_1x where a_0 a_1 are real number.

Assume $\alpha_1, \alpha_2 \in W$ $\lambda \in R$ $\alpha_1=a_0'+a_1'x$ $\alpha_2=a_0''+a_1''x$

$\alpha_1+\alpha_2=(a_0'+a_0'')+(a_1'+a_1'')x \in W$ $\lambda\alpha_1=\lambda a_0'+\lambda a_1'x \in W$

So W_d is a subspace of P_3.

So the answer is (a) (b) (d).

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Section 4.2

5. (a). W is the set of all sequences $\vec{v}$ in R^∞ of the form $\vec{v} = (v, 0, v, 0, v, 0 \cdots)$

Assume $\forall \vec{v}_1, \vec{v}_2 \in W \quad \lambda \in R$

$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = (v_1+v_2, 0, v_1+v_2, 0, v_1+v_2, 0 \cdots) \in R^\infty \qquad \vec{v}_4 = \lambda \vec{v}_1 = (\lambda v_1, 0, \lambda v_1, 0, \lambda v_1, 0, \cdots) \in W$

So W_a is a subspace of R^∞

(b). W is the set of all sequences $\vec{v}$ in R^∞ of the form $\vec{v} = (v, 1, v, 1, \cdots)$

Assume $\forall \vec{v}_1, \vec{v}_2 \in W \quad \lambda \in R$

$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = (v_1+v_2, 2, v_1+v_2, 2, v_1+v_2, 2 \cdots) \notin W$

$\therefore W_b$ is not a subspace of R^∞

(c). W is the set of all sequences $\vec{v}$ in R^∞ of the form $\vec{v} = (v, 2v, 4v, 8v, 16v \cdots)$

Assume $\forall \vec{v}_1, \vec{v}_2 \in W \quad \lambda \in R$

$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = (v_1+v_2, 2(v_1+v_2), 4(v_1+v_2), 8(v_1+v_2), 16(v_1+v_2) \cdots) \in W$

$\vec{v}_4 = \lambda \vec{v}_1 = (\lambda v_1, 2(\lambda v_1), 4(\lambda v_1), 8(\lambda v_1), 16(\lambda v_1) \cdots) \in W$

$\therefore W_c$ is a subspace of R^∞

(d). W is the set of all sequences in R^∞ whose components are 0 from some point on.

Assume $\vec{v}_1, \vec{v}_2 \in W \quad \lambda \in R$.

Then $\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \in W \qquad \vec{v}_4 = \lambda \vec{v}_1 \in W$

$\therefore W_d$ is a subspace of R^∞

So the answer is (a) (c) (d)

9. $A=\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$ $B=\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ $C=\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

(a). Suppose $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = xA + yB + zC$

Then $\begin{cases} 4x + y = 6 \\ -y + 2z = -8 \\ -2x + 2y + z = -1 \\ -2x + 3y + 4z = -8 \end{cases}$ $\left(\begin{array}{ccc:c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right)$

According to Gaussian Elimination:

$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right)$ So (a) is a linear combination of A. B.C.

(b). Suppose $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = xA + yB + zC$

Then $\begin{matrix} 4x + y = 0 \\ -y + 2z = 0 \\ -2x + 2y + z = 0 \\ -2x + 3y + 4z = 0 \end{matrix}$ $\left(\begin{array}{ccc:c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array}\right)$

It's easy to find that $x = y = z = 0$ is one solution.

So (b) is a linear combination of A. B. C

(c). Suppose $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = xA + yB + zC$

Then $\begin{cases} 4x + y = 6 \\ -y + 2z = 0 \\ -2x + 2y + z = 3 \\ -2x + 3y + 4z = 8 \end{cases}$ $\left(\begin{array}{ccc:c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right)$

According to Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

So (c) is a linear combination of A. B.C.

(d) Suppose $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix} = xA + yB + zC$

Then
$$\begin{aligned} 4x + y &= -1 \\ -y + 2z &= 5 \\ -2x + 2y + z &= 7 \\ -2x + 3y + 4z &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 0 & -1 \\ 0 & -1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ -2 & 3 & 4 & 1 \end{array}\right)$$

According to Gaussian Elimination we have:

$$\left(\begin{array}{ccc|c} 4 & 1 & 0 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & \frac{19}{6} \\ 0 & 0 & 1 & \frac{18}{11} \end{array}\right)$$

$\frac{19}{6} \neq \frac{18}{11}$

So (d) is not a linear combination of A B C

So the answer is (a) (b) (c)

Section 4.3

3. (c). Suppose $x(0, 3, -3, -6) + y(-2, 0, 0, -6) + z(0, -4, -2, -2) + t(0, -8, 4, -4) = (0, 0, 0, 0)$

$$\begin{cases} -2y = 0 \\ 3x - 4z - 8t = 0 \\ -3x - 2z + 4t = 0 \\ -6x - 6y - 2z - 4t = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 0 & -2 & 0 & 0 & 0 \\ 3 & 0 & -4 & -8 & 0 \\ -3 & 0 & -2 & 4 & 0 \\ -6 & -6 & -2 & -4 & 0 \end{array}\right)$$

According to Gaussian Elimination we have:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

So $x=y=z=t=0$

The set is not linearly dependent

4 (d). Suppose $a(1+3x+3x^2)+b(x+4x^2)+c(5+6x+3x^2)+d(7+2x-x^2)=0$

Then:
$$\begin{aligned} a \quad\quad +5c+7d &= 0 \\ 3a+b+6c+2d &= 0 \\ 3a+4b+3c-d &= 0 \end{aligned} \qquad \left(\begin{array}{cccc|c} 1 & 0 & 5 & 7 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 3 & 4 & 3 & -1 & 0 \end{array}\right)$$

According to Gaussion Elimination we have:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{17}{4} & 0 \\ 0 & 1 & 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 & \frac{9}{4} & 0 \end{array}\right)$$

So the set of vectors in P_2 is linearly dependent.

19. (a). They are linearly independent. Because if put the three vectors' initial points at the same, they don't lie in the same plane. So that $\forall\ \vec{v} \in R^3$ $\exists\ \lambda_1, \lambda_2, \lambda_3 \in R$ and $\lambda_1, \lambda_2, \lambda_3$ don't equal to zero at the same time. $\vec{v} = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3$

(b). They are linearly dependent. Because if put the three vectors' initial point at same point, all of them lie in the plane. So that $\exists\ \lambda_1, \lambda_2, \lambda_3 \in R$ and $\lambda_1, \lambda_2, \lambda_3$ don't equal to zero at same time $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3 = \vec{0}$

24. $f_1(x) = e^x \quad f_2(x) = xe^x \quad f_3(x) = x^2e^x$

$$W(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & (x^2+2x)e^x \\ e^x & (x+2)e^x & (x^2+4x+2)e^x \end{vmatrix} = \left[(x+1)(x^2+4x+2) + x(x^2+2x) + x^2(x+2) - x^2(x+1) - x(x^2+4x+2) - (x+2)(x^2+2x)\right]e^{3x}$$

$$= 2e^{3x} > 0 \qquad \forall x \in R$$

$\therefore f_1(x) = e^x \quad f_2(x) = xe^x \quad f_3(x) = x^2e^x$ are linearly independent.