第六周 周二作业 4月7日

习题9.5

4. $\vec{r}=\left(\frac{t}{1+t}, \frac{1+t}{t}, t^2\right)$ $t>0$

解：假设 $\vec{r}(t)$ 不是简单曲线，则 $\exists\, t_1, t_2>0$ 且 $t_1\neq t_2$ 使得 $\vec{r}(t_1)=\vec{r}(t_2)$

$\frac{t_1}{1+t_1}=\frac{t_2}{1+t_2}$ $\frac{1+t_1}{t_1}=\frac{1+t_2}{t_2}$ $t_1^2=t_2^2$ 得 $t_1=t_2$ 与 $t_1\neq t_2$ 矛盾 ∴假设不成立.

则 $\vec{r}(t)$ 为简单曲线

$x(t)=\frac{t}{1+t}$ $y(t)=\frac{1+t}{t}$ $z(t)=t^2$ $x'(t)=\frac{1}{(1+t)^2}$ $y'(t)=-\frac{1}{t^2}$ $z'(t)=2t$

∵ $x'(t), y'(t), z'(t)$ 连续 ∴ $\vec{r}(t)$ 为光滑曲线.

$t=1$ 时 $x(1)=\frac{1}{2}$ $y(1)=2$ $z(1)=1$ $x'(1)=\frac{1}{4}$ $y'(1)=-1$ $z'(1)=2$

∴ 切线方程为 $\frac{x-\frac{1}{2}}{\frac{1}{4}}=\frac{y-2}{-1}=\frac{z-1}{2}$ 即 $4x-2=-y+2=\frac{z-1}{2}$

法平面方程为 $\frac{1}{4}(x-\frac{1}{2})-(y-2)+2(z-1)=0$

6 (2). $x=a\sin\theta\cos\varphi$ $y=b\sin\theta\sin\varphi$ $z=c\cos\theta$ 在 (θ_0, φ_0)

解：$\vec{r}(\theta,\varphi)=(a\sin\theta\cos\varphi,\ b\sin\theta\sin\varphi,\ c\cos\theta)$

$\vec{r}_\theta'=(a\cos\theta\cos\varphi,\ b\cos\theta\sin\varphi,\ -c\sin\theta)$

$\vec{r}_\varphi'=(-a\sin\theta\sin\varphi,\ b\sin\theta\cos\varphi,\ 0)$

$\vec{r}_\theta'\times\vec{r}_\varphi'=(bc\sin^2\theta\cos\varphi,\ ac\sin^2\theta\sin\varphi,\ ab\sin\theta\cos\theta)$

$E=\vec{r}_\theta'\cdot\vec{r}_\theta'=(a^2\cos^2\varphi+b^2\sin^2\varphi)\cos^2\theta+c^2\sin^2\theta$

$F=\vec{r}_\theta'\cdot\vec{r}_\varphi'=\frac{1}{4}(b^2-a^2)\sin2\theta\sin2\varphi$

$G=\vec{r}_\varphi'\cdot\vec{r}_\varphi'=(a^2\sin^2\varphi+b^2\cos^2\varphi)\sin^2\theta$

$\vec{n}=\vec{r_\theta}\times\vec{r_\varphi}=(bc\sin^2\theta\cos\varphi,\ ac\sin^2\theta\sin\varphi,\ ab\sin\theta\cos\theta)$

则切平面为 $bc\sin\theta_0\cos\varphi_0(x-a\sin\theta_0\cos\varphi_0)+ac\sin^2\theta_0\sin\varphi_0(y-b\sin\theta_0\sin\varphi_0)+ab\sin\theta_0\cos\theta_0(z-c\cos\theta_0)=0$

法线方程为 $\frac{x-a\sin\theta_0\cos\varphi_0}{bc\sin^2\theta_0\cos\varphi_0}=\frac{y-b\sin\theta_0\sin\varphi_0}{ac\sin^2\theta_0\sin\varphi_0}=\frac{z-c\cos\theta_0}{ab\sin\theta_0\cos\theta_0}$ 即 $\frac{x-a\sin\theta_0\cos\theta_0}{bc\sin\theta_0\cos\theta_0}=\frac{y-b\sin\theta_0\sin\varphi_0}{ac\sin\theta_0\sin\varphi_0}=\frac{z-c\cos\theta_0}{ab\cos\theta_0}$

7. (3). $e^z-z+xy=3$ 在 $(2,1,0)$

解: 两侧对 x 求偏导数得 $e^z z_x'-z_x'+y=0$ 得 $z_x'=\frac{y}{1-e^z}$

两侧对 y 求偏导数得 $e^z z_y'-z_y'+x=0$ 得 $z_y'=\frac{x}{1-e^z}$

$\vec{r}(x,y)=(x,y,z)$ $\vec{r}_x'=(1,0,z_x')$ $\vec{r}_y'=(0,1,z_y')$

$\vec{n}=\frac{\vec{r}_x'\times\vec{r}_y'}{|\vec{r}_x'\times\vec{r}_y'|}=\left(\frac{y}{\sqrt{x^2+y^2+(1-e^z)^2}},\frac{x}{\sqrt{x^2+y^2+(1-e^z)^2}},\frac{1-e^z}{\sqrt{x^2+y^2+(1-e^z)^2}}\right)$ 在 $(2,1,0)$ 处 $\vec{n}=(\frac{\sqrt5}{5},\frac{2\sqrt5}{5},0)$

$\therefore$ 切平面为 $\frac{\sqrt5}{5}(x-2)+\frac{2\sqrt5}{5}(y-1)+0(z-0)=0$ 即 $(x-2)+2(y-1)=0$

法线方程为: $\vec{r}=(2,1,0)+t(\frac{\sqrt5}{5},\frac{2\sqrt5}{5},0)\quad t\in R$

(4) $4+\sqrt{x^2+y^2+z^2}=x+y+z$ 在 $(2,3,6)$

解: $x^2+y^2+z^2=(x+y+z-4)^2$ 得 $z(x,y)=\frac{4(x+y)-xy-8}{x+y-4}$

$\vec{r}(x,y)=(x,y,\frac{4(x+y)-xy-8}{x+y-4})$

$\vec{r}_x'=(1,0,\frac{4y-y^2-8}{(x+y-4)^2})$ $\vec{r}_y'=(0,1,\frac{4x-x^2-8}{(x+y-4)^2})$

$\vec{n}=\vec{r}_x'\times\vec{r}_y'=\left(\frac{y^2+8-4y}{(x+y-4)^2},\frac{x^2+8-4x}{(x+y-4)^2},1\right)$ 在 $(2,3,6)$ 处 $\vec{n}=(5,4,1)$

$\therefore$ 切平面为 $5(x-2)+4(y-3)+(z-6)=0$

法线为 $\vec{r}=(2,3,6)+t(5,4,1)\quad t\in R$

9. 解：设所求点为 (x_0, y_0, z_0)

则 $z_0 = x_0 y_0 \quad x_0 + 3y_0 + z_0 = 0$

$\vec{r} = (x, y, xy) \quad \vec{r}_x' = (1, 0, y) \quad \vec{r}_y' = (0, 1, x)$

$\vec{n} = \vec{r}_x' \times \vec{r}_y' = (-y, -x, 1) \quad \vec{n}_0 = (-y_0, -x_0, 1)$

平面 $x+3y+z=0$ 法向量 $\vec{n}_1 = (1, 3, 1)$ ∵法线垂直于平面 ∴ $\vec{n}_0 \parallel \vec{n}_1$

得 $\frac{-y_0}{1} = \frac{-x_0}{3} = \frac{1}{1}$ 即 $x_0 = -3 \quad y_0 = -1$

∴ $(x_0, y_0, z_0) = (-3, -1, 1)$

即法线方程为 $\frac{x+3}{-3} = \frac{y+1}{-1} = \frac{z-1}{1}$

10. 解：直线过 $P_1(6, 3, \frac{1}{2})$ 直线方向向量 $\vec{n}_1 = (2, 1, -1)$

设 $M(x_0, y_0, z_0)$ 平面 π 法向量为 $\vec{n}_0$ 则 $x_0^2 + 2y_0^2 + 3z_0^2 = 21$ ①

$F(x, y, z) = x^2 + 2y^2 + 3z^2 - 21 \quad F_x' = 2x \quad F_y' = 4y \quad F_z' = 6z$ ∴ $\vec{n}_0 = (2x_0, 4y_0, 6z_0)$

则平面 π 为 $2x_0(x-x_0) + 4y_0(y-y_0) + 6z_0(z-z_0) = 0$ 即 $x_0 x + 2y_0 y + 3z_0 z = 21$

$P_1(6, 3, \frac{1}{2}) \quad P_2(8, 4, -\frac{1}{2}) \in L$ ∴ $P_1, P_2 \in \pi$ $\quad 6x_0 + 6y_0 + \frac{3}{2}z_0 = 21$ ② $\quad 8x_0 + 8y_0 - \frac{3}{2}z_0 = 21$ ③.

联立 ① ② ③ 得 $(x_0, y_0, z_0) = (3, 0, 2)$ 或 $(x_0, y_0, z_0) = (1, 2, 2)$

π 的方程为 $x + 2z = 7$ 或 $x + 4y + 6z = 21$

12. 证明: 要证 π_1: $x^2+y^2+z^2=ax$ 与 π_2: $x^2+y^2+z^2=by$ 互相正交. 则证在交点处二者法向量垂直.

$F=x^2+y^2+z^2-ax$ $\quad G=x^2+y^2+z^2-by$ $\quad$ 令 $F=G$ 得 $ax=by$

$F'_x=2x-a$ $\quad F'_y=2y$ $\quad F'_z=2z$ $\quad \vec{n}_1=(2x-a,\ 2y,\ 2z)$

$G'_x=2x$ $\quad G'_y=2y-b$ $\quad G'_z=2z$ $\quad \vec{n}_2=(2x,\ 2y-b,\ 2z)$

$\vec{n}_1\cdot\vec{n}_2=(2x-a)\cdot 2x+2y(2y-b)+4z^2=4(x^2+y^2+z^2)-2(ax+by)=(2ax+2by)-2(ax+by)=0$

$\therefore$ $x^2+y^2+z^2=ax$ 与 $x^2+y^2+z^2=by$ 互相正交.

13. 解: $F=x+2y-\ln z+4$

$F'_x=1$ $\quad F'_y=2$ $\quad F'_z=-\frac{1}{z}$ $\quad$ 则在 $(2,-3,1)$ 处 $\vec{n}_1=(1,2,-1)$

π_1: $(x-2)+2(y+3)-(z-1)=0$ 即 π_1: $x+2y-z+5=0$

$G=x^2-xy-8x+z+5$

$G'_x=2x-y-8$ $\quad G'_y=-x$ $\quad G'_z=1$ $\quad$ 则在 $(2,-3,1)$ 处 $\vec{n}_2=(-1,-2,1)$

π_2: $-(x-2)-2(y+3)+(z-1)=0$ 即 π_2: $x+2y-z+5=0$

$\pi_1=\pi_2$ 即曲面 $x+2y-\ln z+4=0$ 与曲面 $x^2-xy-8x+z+5=0$ 在 $(2,-3,1)$ 处相切.

15(2) $\cos(xy)=x+2y$ 在 $(1,0)$

$\vec{r}(x,y)=x+2y-\cos(xy)$ $\quad \vec{r}'_x=1+y\sin(xy)$ $\quad \vec{r}'_y=2+x\cdot\sin(xy)$

在 $(1,0)$ 处 切向量 $\vec{\tau}=(1,2)$ $\quad \therefore$ 切线为 $x-1+2y=0$ 即 $x+2y-1=0$

法向量 $\vec{n}\cdot\vec{\tau}=0$ $\quad \therefore \vec{n}=(2,-1)$ $\quad \therefore$ 法线为 $2(x-1)-y=0$ 即 $2x-y-2=0$

16.(2). $\begin{cases} 2x^2+3y^2+z^2=47 \\ x^2+2y^2=z \end{cases}$ 在 $(-2, 1, 6)$

解: 令 $F=2x^2+3y^2+z^2-47$ $G=x^2+2y^2-z$

$F_x'=4x$ $F_y'=6y$ $F_z'=2z$ 则在 $(-2,1,6)$ 处 $\vec{n}_1=(-8,6,12)$

$G_x'=2x$ $G_y'=4y$ $G_z'=-1$ 则在 $(-2,1,6)$ 处 $\vec{n}_2=(-4,4,-1)$

$\vec{n}_0=\vec{n}_1\times\vec{n}_2=(-54,-56,-8)$

则切线方程为 $\frac{x+2}{-54}=\frac{y-1}{-56}=\frac{z-6}{-8}$ 即 $\frac{x+2}{27}=\frac{y-1}{28}=\frac{z-6}{4}$

法平面方程为 $-54(x+2)-56(y-1)-8(z-6)=0$ 即 $27(x+2)+28(y-1)+4(z-6)=0$

17. $\begin{cases} pu+qv-t^2=0 \\ qu+pv-s^2=0 \end{cases}$ $(p^2-q^2\neq0)$

证明: 两侧对 t 求导: $\begin{cases} pu_t'+qv_t'-2t=0 \\ qu_t'+pv_t'=0 \end{cases}$ 得 $u_t'=\frac{2tp}{p^2-q^2}$

两侧对 s 求导 $\begin{cases} pu_s'+qv_s'=0 \\ qu_s'+pv_s'-2s=0 \end{cases}$ 得 $v_s'=\frac{2sp}{p^2-q^2}$

两侧对 u 求导 $\begin{cases} p-2tt_u'=0 \\ q-2ss_u'=0 \end{cases}$ 得 $t_u'=\frac{p}{2t}$

两侧对 v 求导 $\begin{cases} q-2tt_v'=0 \\ p-2ss_v'=0 \end{cases}$ 得 $s_v'=\frac{p}{2s}$

则 $\frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial t}=\frac{p}{2t}\cdot\frac{2tp}{p^2-q^2}=\frac{p^2}{p^2-q^2}$ $\frac{\partial s}{\partial v}\cdot\frac{\partial v}{\partial s}=\frac{p}{2s}\cdot\frac{2sp}{p^2-q^2}=\frac{p^2}{p^2-q^2}$ 即 $\frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial t}=\frac{\partial s}{\partial v}\cdot\frac{\partial v}{\partial s}=\frac{p^2}{p^2-q^2}$

第六周周四作业 4月9日

习题 9.6

1. $\vec{e}=\frac{q}{r^3}\vec{r}$

解 $r=\sqrt{x^2+y^2+z^2}$ $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

散度: $\nabla\cdot\vec{e}=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\cdot\frac{q}{r^3}\vec{r}$

$$=q\left(\frac{y^2+z^2-2x^2}{r^5}+\frac{x^2+z^2-2y^2}{r^5}+\frac{x^2+y^2-2z^2}{r^5}\right)$$

当$|\vec{r}|=0$时 $\nabla\cdot\vec{e}=\infty$ 当$|\vec{r}|>0$ 时 $\nabla\cdot\vec{e}=0$

旋度: $\nabla\times\vec{e}=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times[\frac{q}{r^3}(x\vec{i}+y\vec{j}+z\vec{k})]$

$$=(\frac{\partial}{\partial y}\frac{qz}{r^3}-\frac{\partial}{\partial z}\frac{qy}{r^3})\vec{i}+(\frac{\partial}{\partial z}\frac{qx}{r^3}-\frac{\partial}{\partial x}\frac{qz}{r^3})\vec{j}+(\frac{\partial}{\partial x}\frac{qy}{r^3}-\frac{\partial}{\partial y}\frac{qx}{r^3})\vec{k}$$

$$=(-\frac{3qyz}{r^5}+\frac{3qyz}{r^5})\vec{i}+(-\frac{3qxz}{r^5}+\frac{3qxz}{r^5})\vec{j}+(-\frac{3qxy}{r^5}+\frac{3qxy}{r^5})\vec{k}=\vec{0}$$

2. $\vec{w}=w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$ $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

则 $\vec{v}=\vec{w}\times\vec{r}=(w_2z-w_3y)\vec{i}+(w_3x-w_1z)\vec{j}+(w_1y-w_2x)\vec{k}$

则 $\nabla\times\vec{v}=(w_1+w_1)\vec{i}+(w_2+w_2)\vec{j}+(w_3+w_3)\vec{k}=2\vec{w}$

物理意义: $\vec{w}$为刚体转动角速度 则 $\vec{v}=\vec{w}\times\vec{r}$为刚体$(x,y,z)$处的速度.

刚体运动速度的旋度等于2倍角速度.

3. (1). $\vec{v}=(3x^2-2yz,\ y^3+yz^2,\ xyz-3xz^2)$ 在 $M(1,-2,2)$

$\nabla\cdot\vec{v}=6x+3y^2+z^2+xy-6xz$ 当$(x,y,z)=(1,-2,2)$时

$\nabla\cdot\vec{v}=8$

(2). $\vec{v}=x^2\sin y\,\vec{i}+y^2\sin(xz)\,\vec{j}+xy\sin(\cos z)\,\vec{k}$ 在 $M(x,y,z)$ 处.

$\nabla\cdot\vec{v}=2x\sin y+2y\sin(xz)+xy\,[\cos(\cos z)]\cdot(-\sin z)$

$=2x\sin y+2y\sin(xz)-xy(\sin z)\cdot\cos(\cos z)$

4. $\vec{w}$是常向量 $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$

(1). $\mathrm{div}[(\vec{r}\cdot\vec{w})\vec{w}]=\nabla\cdot[(\vec{r}\cdot\vec{w})\vec{w}]=(\vec{r}\cdot\vec{w})\nabla\cdot\vec{w}+\vec{w}\cdot\nabla(\vec{r}\cdot\vec{w})$

$\because$ $\vec{w}$为常向量 $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ $\therefore\nabla\cdot\vec{w}=0$ $\nabla(\vec{r}\cdot\vec{w})=\vec{w}$

$\therefore\mathrm{div}[(\vec{r}\cdot\vec{w})\vec{w}]=|\vec{w}|^2$

(2). $\mathrm{div}(\frac{\vec{r}}{r})=\nabla\cdot(\frac{1}{r}\vec{r})=\frac{y^2+z^2}{r^3}+\frac{x^2+z^2}{r^3}+\frac{x^2+y^2}{r^3}=\frac{2}{r}$

(3). $\mathrm{div}(\vec{w}\times\vec{r})=\nabla\cdot(\vec{w}\times\vec{r})=\vec{r}\cdot\nabla\times\vec{w}-\vec{w}\cdot\nabla\times\vec{r}$

$\because$ $\vec{w}$为常向量 $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ $\nabla\times\vec{w}=\vec{0}$ $\nabla\times\vec{r}=\vec{0}$

$\therefore\mathrm{div}(\vec{w}\times\vec{r})=0$

(4). $\mathrm{div}(r^2\vec{w})=\nabla\cdot(r^2\vec{w})=2\vec{r}\cdot\vec{w}$

5. (1). $\vec{v}=y^2\vec{i}+z^2\vec{j}+x^2\vec{k}$

$\nabla\times\vec{v}=-2z\,\vec{i}-2x\,\vec{j}-2y\,\vec{k}$

(2). $\vec{v}=(xe^y+y)\vec{i}+(z+e^y)\vec{j}+(y+2ze^y)\vec{k}$

$\nabla\times\vec{v}=2ze^y\,\vec{i}+0\,\vec{j}-(xe^y+1)\,\vec{k}$

6. (1). $\mathrm{rot}(\vec{w}\times\vec{r}) = \nabla\times(\vec{w}\times\vec{r}) = 2\vec{w}$

(2). $\mathrm{rot}\,[f(r)\vec{r}] = \nabla\times[f(r)\vec{r}] = \nabla f(r)\times\vec{r} + f(r)\,\nabla\times\vec{r}$

$\nabla f(r) = f'(r)\cdot\frac{\partial r}{\partial x}\vec{i} + f'(r)\frac{\partial r}{\partial y}\vec{j} + f'(r)\frac{\partial r}{\partial z}\vec{k} = f'(r)\frac{x}{r}\vec{i} + f'(r)\frac{y}{r}\vec{j} + f'(r)\frac{z}{r}\vec{k}$

则 $\nabla f(r)\times\vec{r} = [f'(r)\frac{yz}{r} - f'(r)\frac{zy}{r}]\vec{i} + [f'(r)\frac{zx}{r} - f'(r)\frac{xz}{r}]\vec{j} + [f'(r)\frac{xy}{r} - f'(r)\frac{yx}{r}]\vec{k} = \vec{0}$

$\nabla\times\vec{r} = \vec{0}$

则 $\mathrm{rot}\,[f(r)\vec{r}] = \vec{0}$

(3). $\mathrm{rot}\,[f(r)\vec{w}] = \nabla\times[f(r)\vec{w}] = \nabla f(r)\times\vec{w} + f(r)\,\nabla\times\vec{w} = f'(r)\frac{1}{r}\vec{r}\times\vec{w} + \vec{0} = \frac{f'(r)}{r}\vec{r}\times\vec{w}$

(4). $\mathrm{div}\,[\vec{r}\times f(r)\vec{w}] = f(r)\vec{w}\cdot\nabla\times\vec{r} - \vec{r}\cdot\nabla\times[f(r)\vec{w}] = \vec{0} - \vec{r}\cdot[\frac{f'(r)}{r}\vec{r}\times\vec{w}] = -\frac{f'(r)}{r}\vec{r}\cdot(\vec{r}\times\vec{w})$

第九章综合习题

1. 证明：必要性 $\because f(x_1, x_2, \cdots, x_n)$ 在 R^n 上可微. $F(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$ 在 R 上可微　令 $s = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

$f(x_1, x_2, \cdots x_n) = F(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$

$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial s}\frac{\partial s}{\partial x_i} = a_i\frac{\partial F}{\partial s}$　　$\frac{\partial f}{\partial x_j} = a_j\frac{\partial F}{\partial s}$　　$\therefore a_j\frac{\partial f}{\partial x_i} = a_i\frac{\partial f}{\partial x_j}$　$(i, j = 1, 2, \cdots n)$

充分性 $\because a_j\frac{\partial f}{\partial x_i} = a_i\frac{\partial f}{\partial x_j}$　则 $\frac{\partial f}{\partial(a_ix_i)} = \frac{\partial f}{\partial(a_jx_j)}$

$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i = \sum_{i=1}^{n}\frac{\partial f}{\partial(a_ix_i)}\cdot d(a_ix_i) = \frac{\partial f}{\partial(a_ix_i)}\sum_{i=1}^{n}d(a_ix_i)$　　$\therefore \exists$ R 上一元可微函数 $F(s)$

$s = a_1x_1 + \cdots + a_nx_n$　使得　$F'(s) = \frac{\partial f}{\partial(a_ix_i)}$

则 $F(a_1x_1 + \cdots + a_nx_n) = f(x_1, x_2, \cdots x_n)$

2. 证明: 必要性: $\because$ 可微函数为k次齐次函数

$\therefore f(tx, ty, tz) = t^k f(x,y,z)$

两侧对t求导得 $x \cdot \frac{\partial f(tx,ty,tz)}{\partial (tx)} + y \frac{\partial f(tx,ty,tz)}{\partial (ty)} + z \frac{\partial f(tx,ty,tz)}{\partial (tz)} = k t^{k-1} \cdot f(x,y,z)$

令 $t=1$ 则 $x f'_x(x,y,z) + y f'_y(x,y,z) + z f'_z(x,y,z) = k f(x,y,z)$

充分性: $\because x f'_x(x,y,z) + y f'_y(x,y,z) + z f'_z(x,y,z) = k f(x,y,z)$

令 $g(x,y,z,t) = \frac{1}{t^k} F(tx, ty, tz)$

则 $\frac{\partial g}{\partial t} = \frac{1}{t^{k+1}} \left[x \frac{\partial F(tx,ty,tz)}{\partial (tx)} + y \frac{\partial F(tx,ty,tz)}{\partial (ty)} + z \frac{\partial F(tx,ty,tz)}{\partial (tz)} - k F(tx,ty,tz) \right]$

$\therefore \frac{\partial g}{\partial t} = 0$ 即 $g(x,y,z,t)$ 是只关于 x,y,z 的函数 记 $h(x,y,z) = g(x,y,z,t)$

则 $t^k h(x,y,z) = F(tx,ty,tz)$ 令 $t=1$ 得 $h(x,y,z) = F(x,y,z)$

$\therefore t^k F(x,y,z) = F(tx,ty,tz)$ 即 $F(x,y,z)$ 为k次齐次函数.

4. 证明: 不妨设 $\frac{\partial f}{\partial x}$ 连续.

$\Delta f = f(x+h, y+k) - f(x,y) = [f(x+h, y+k) - f(x+h, y)] + [f(x+h, y) - f(x,y)]$ $h \to 0,\ k \to 0$

且 $\frac{\partial f}{\partial x}$ 在连续.

由微分中值定理: $\Delta f = f'_y(x+h, y+\eta k) \cdot k + f'_x(x,y) h$ $\eta, \lambda \in [0,1]$

令 $I = f'_y(x+h, y+\eta k) k - f'_y(x,y) k$ 则 $\lim\limits_{(h,k)\to(0,0)} \left| \frac{I}{\sqrt{h^2+k^2}} \right| = \lim\limits_{(h,k)\to(0,0)} \left| \frac{k}{\sqrt{h^2+k^2}} [f'_y(x+h, y+\eta k) - f'_y(x,y)] \right| = 0$

$\therefore \Delta f = f'_y(x,y) k + f'_x(x,y) h + o(\sqrt{h^2+k^2})$

$\therefore (x,y) = (x_0, y_0)$ 时 $f(x,y)$ 可微.

5. 证明: $\because f'_x = f'_y = f'_z$ $f(x,y,z)$ 在 R^3 上有一阶连续偏导数.

$\therefore f'_x = f'_y = f'_z = C_1$ C_1 为常数.

$\therefore f(x,0,0) = C_1 x + C_2$ C_1, C_2 为常数.

$\because \forall x \in R$ $f(x,0,0) > 0$ $\therefore C_1 = 0$ $C_2 > 0$ $\therefore f(x,0,0) = C_2$ $f'_x = f'_y = f'_z = 0$

$\therefore f(x,y,z) = C_2 > 0$ $\forall (x,y,z) \in R^3$