第四周作业　周二. 3月24日.

习题 9.1

16.(3) $f(x)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

解: 当 $(x,y)\neq(0,0)$ 时 $f(x,y)=\dfrac{x^2y}{x^2+y^2}$ 连续.

只需研究 $f(x,y)$ 在 $(0,0)$ 处的连续性.

$\forall\ \varepsilon>0\quad \exists\ \delta=2\varepsilon\quad \forall\ |x|<\delta\quad |y|<\delta$

$$\left|\frac{x^2y}{x^2+y^2}\right|\leqslant\left|\frac{x^2y}{2|x||y|}\right|=\frac{|x|}{2}<\varepsilon$$

$\therefore \lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{x^2+y^2}=0$ 即 $f(x,y)$ 在 $(0,0)$ 处连续.

即 $f(x,y)$ 在定义域内连续.

17. 证明: $f(t\cos\alpha, t\sin\alpha)=\dfrac{t^3\cos^2\alpha\sin\alpha}{t^4\cos^4\alpha+t^2\sin^2\alpha}=\dfrac{t\cdot\cos^2\alpha\sin\alpha}{t^2\cos^4\alpha+\sin^2\alpha}$

$\lim\limits_{t\to0}\dfrac{t\cos^2\alpha\sin\alpha}{t^2\cos^4\alpha+\sin^2\alpha}=0=f(0,0)$

即 $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ 在沿过 $(0,0)$ 的每一条射线连续.

当 $y=x^2$ 时 $\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{x^4+y^2}=\lim\limits_{x\to0}\dfrac{x^4}{x^4+x^4}=\dfrac{1}{2}\neq0$

即 $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ 在 $(0,0)$ 处不连续

18. 证明: $\forall \varepsilon>0 \quad \exists \delta=\varepsilon \quad (x_0, y_0) \in D \quad |x-x_0|<\delta \quad |y-y_0|<\delta$

$$|f(x,y)-f(x_0,y_0)| = |x-x_0| < \delta = \varepsilon$$

$\therefore$ 投影函数是连续函数.

若 $f(x,y)=\frac{1}{x}$ 则 $f(x,y)$ 的图象为闭集, 其映射为 $(-\infty,0)\cup(0,+\infty)$ 为开集

综上所述 投影函数为连续函数, 但它不一定将闭集映成闭集。

20. $\because x=x(u,v)\quad y=y(u,v)$ 在 (u_0,v_0) 处连续.

$\therefore \forall \varepsilon>0\quad \exists \delta_1=\delta_1(\varepsilon)\quad \forall\ |u-u_0|<\delta_1\quad |v-v_0|<\delta_1\qquad |x(u,v)-x_0(u_0,v_0)|<\varepsilon$

$\exists \delta_2=\delta_2(\varepsilon)\quad \forall\ |u-u_0|<\delta_2\quad |v-v_0|<\delta_2\qquad |y(u,v)-y_0(u_0,v_0)|<\varepsilon$

$\because f(x,y)$ 在 (x_0,y_0) 处连续.

$\therefore \forall \varepsilon>0\quad \exists \delta_3=\varepsilon\qquad \forall\ |x-x_0|<\delta_3\quad |y-y_0|<\delta_3\quad |f(x,y)-f(x_0,y_0)|<\varepsilon$

即 $|f(x_0,y)-f(x_0,y_0)|<\varepsilon\qquad |f(x,y)-f(x_0,y)|<\varepsilon$

令 $\delta'=\min\{\delta_1,\delta_2\}\quad \forall\ |u-u_0|<\delta'\quad |v-v_0|<\delta'$ 则有 $|x-x_0|<\delta_3\quad |y-y_0|<\delta_3$

$|f(x(u,v),y(u,v))-f(x(u_0,v_0),y(u_0,v_0))|\le|f(x(u,v),y(u,v))-f(x(u_0,v_0),y(u,v))|+$

$|f(x(u_0,v_0),y(u,v))-f(x(u_0,v_0),y(u_0,v_0))|<2\varepsilon$

即 $f(x(u,v),y(u,v))$ 在 (u_0,v_0) 处连续.

21. 证明: 设 $(x_0,y_0)\in[0,1]\times[0,1]$ 且 $(x_0,y_0)\neq(1,1)$

$\forall \varepsilon>0\quad \exists \delta=\frac{\varepsilon}{2}\qquad \forall\ |x-x_0|<\delta\quad |y-y_0|<\delta$

$|f(x,y)-f(x_0,y_0)|=\left|\frac{1}{1-xy}-\frac{1}{1-x_0y_0}\right|=\left|\frac{xy-x_0y_0}{(1-xy)(1-x_0y_0)}\right|<|xy-x_0y_0|=|xy-x_0y+x_0y-x_0y_0|$

$\le|y(x-x_0)|+|x_0(y-y_0)|<|x-x_0|+|y-y_0|<2\delta=\varepsilon$

$\therefore f(x,y)=\frac{1}{1-xy}\quad (x,y)\in[0,1]\times[0,1]$ 且 $(x,y)\neq(1,1)$ 连续.

$\exists \varepsilon_0=\frac{1}{3}$ 令 $x_n=y_n=1-\frac{1}{n}\quad n\in N_+$ 且 $x_n,y_n\in[0,1]$ 且 $(x_n,y_n)\neq(1,1)$

$f_n(x,y)=\frac{1}{1-x_ny_n}=\frac{1}{1-(1-\frac{1}{n})^2}\qquad |x_{n+1}-x_n|=|y_{n+1}-y_n|=\frac{1}{n}-\frac{1}{n+1}\to 0\quad (n\to+\infty)$

$$\left|f_{n+1}(x,y)-f_n(x,y)\right|=\left|\frac{1}{1-(1-\frac{1}{n+1})^2}-\frac{1}{1-(1-\frac{1}{n})^2}\right|=\left|\frac{2n^2-1}{4n^2-1}\right|$$

令 $a_n=\frac{2n^2-1}{4n^2-1}$ 则 $a_n>0$ 且 a_n 单调递增.

$\therefore a_n \geq a_1=\frac{1}{3}$

$\therefore \left|f_{n+1}(x,y)-f_n(x,y)\right| \geq \frac{1}{3}=\varepsilon_0$

$\therefore f(x,y)=\frac{1}{1-xy}$ $(x,y)\in[0,1]\times[0,1]$ 且 $(x,y)\neq(1,1)$ 不一致连续.

习题 9.2

1 (2). $f(x,y)=\sin x^2y$

$f'_x(x,y)=2xy\cos x^2y$ 则 $f'_x(1,\pi)=2\pi\cos\pi=-2\pi$

1 (3) $f(x,y)=\ln\left[xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}\right]$

$$f'_x(x,y)=\frac{y^2+2xy+\frac{(xy^2+yx^2)\cdot(y^2+2xy)}{\sqrt{1+(xy^2+yx^2)^2}}}{xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}}$$

$$f'_x(1,y)=\frac{y^2+2y+\frac{(y^2+y)(y^2+2y)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}}=(y^2+2y)\frac{(y^2+y)+\sqrt{1+(y^2+y)^2}}{(y^2+y)\left[\sqrt{1+(y^2+y)^2}+(y^2+y)\right]+1}$$

$$f'_y(x,y)=\frac{2xy+x^2+\frac{(xy^2+yx^2)\cdot(2xy+x^2)}{\sqrt{1+(xy^2+yx^2)^2}}}{xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}}$$

$$f'_y(1,y)=\frac{2y+1+\frac{(y^2+y)(2y+1)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}}$$

2 (2). $z = 3^{-\frac{y}{x}}$

$$\frac{\partial z}{\partial x} = 3^{-\frac{y}{x}} \cdot \frac{y}{x^2} \ln 3 = \frac{y}{x^2} 3^{-\frac{y}{x}} \cdot \ln 3 \qquad \frac{\partial}{\partial y} z = -3^{-\frac{y}{x}} \frac{1}{x} \ln 3 = -\frac{\ln 3}{x} e^{-\frac{y}{x}}$$

(4). $z = \ln(x + \sqrt{x^2+y^2})$

$$\frac{\partial}{\partial x} z = \frac{1 + \frac{x}{\sqrt{x^2+y^2}}}{x + \sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} \qquad \frac{\partial}{\partial y} z = \frac{\frac{y}{\sqrt{x^2+y^2}}}{x + \sqrt{x^2+y^2}} = \frac{y}{x^2+y^2+x\sqrt{x^2+y^2}}$$

(6) $u = e^{x(x^2+y^2+z^2)}$

$$\frac{\partial}{\partial x} u = (3x^2+y^2+z^2) e^{x(x^2+y^2+z^2)} \qquad \frac{\partial}{\partial y} u = 2xy\, e^{x(x^2+y^2+z^2)} \qquad \frac{\partial}{\partial z} u = 2xz \cdot e^{x(x^2+y^2+z^2)}$$

(8) $u = xe^{-z} + \ln(x + \ln y) + z$

$$\frac{\partial}{\partial x} u = e^{-z} + \frac{1}{x+\ln y} \qquad \frac{\partial}{\partial y} u = \frac{1}{y(x+\ln y)} \qquad \frac{\partial}{\partial z} u = -xe^{-z} + 1$$

3. $f(x,y) = \int_1^{x^2 y} \frac{\sin t}{t} dt$

$$\frac{\partial}{\partial x} f = 2xy \cdot \frac{\sin(x^2y)}{x^2y} = \frac{2\sin(x^2y)}{x} \qquad \frac{\partial}{\partial y} f = x^2 \cdot \frac{\sin(x^2y)}{x^2y} = \frac{\sin(x^2y)}{y}$$

4. $f(x,y) = \begin{cases} y \sin \frac{1}{x^2+y^2}, & x^2+y^2>0 \\ 0, & x^2+y^2=0 \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{y \cdot \sin\frac{1}{y^2} - 0}{y} = \lim_{y\to 0} \sin\frac{1}{y^2}$$

① 当 $\frac{1}{y^2}=\frac{\pi}{2}+2k\pi$ $k\in N_+$ 时 $\lim\limits_{k\to+\infty}\sin(\frac{\pi}{2}+2k\pi)=\lim\limits_{y\to0}\sin\frac{1}{y^2}=1$

②. 当 $\frac{1}{y^2}=k\pi$ $k\in N_+$ 时 $\lim\limits_{k\to+\infty}\sin(k\pi)=\lim\limits_{y\to0}\sin\frac{1}{y^2}=0$

$\therefore \lim\limits_{y\to0}\sin\frac{1}{y^2}$ 极限不存在.

综上所述 $f(x,y)$ 在 $(0,0)$ 处 关于 x 的偏导数为0

关于 y 的偏导数不存在.

5. $z=\sqrt{x^2+y^2}$

证明: $\forall\ \varepsilon>0$ $\exists\ \delta=\frac{\varepsilon}{\sqrt{2}}$ $|x|<\delta$ $|y|<\delta$

$$|z(x,y)-z(0,0)|=|\sqrt{x^2+y^2}-0|<\sqrt{\delta^2+\delta^2}=\varepsilon$$

$\therefore z=\sqrt{x^2+y^2}$ 在 $(0,0)$ 处连续.

$$\lim_{x\to0^+}\frac{z(x,0)-z(0,0)}{x}=\lim_{x\to0^+}\frac{\sqrt{x^2}-0}{x}=\lim_{x\to0^+}\frac{x}{x}=1$$

$$\lim_{x\to0^-}\frac{z(x,0)-z(0,0)}{x}=\lim_{x\to0^-}\frac{\sqrt{x^2}-0}{x}=\lim_{x\to0^-}\frac{-x}{x}=-1$$

$\therefore z(x,y)$ 在 $(0,0)$ 处关于 x 的偏导数不存在.

同理 $z(x,y)$ 在 $(0,0)$ 处关于 y 的偏导数不存在。

第四周 周四作业 3月26日

习题9.2

8. $u=\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}$

解 $\frac{\partial u}{\partial t}=-\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}+\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}\cdot\frac{x^2}{4t^2}=\frac{1}{4t^2\sqrt{t}}e^{-\frac{x^2}{4t}}(x^2-2t)$

$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}}(-\frac{x}{2t})=-\frac{x}{2t\sqrt{t}}e^{-\frac{x^2}{4t}}$ $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=-\frac{1}{2t\sqrt{t}}e^{-\frac{x^2}{4t}}+\frac{x^2}{4t^2\sqrt{t}}e^{-\frac{x^2}{4t}}$

$=\frac{1}{4t^2\sqrt{t}}e^{-\frac{x^2}{4t}}(x^2-2t)$

即 $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

16. $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

证明: $\forall\ \varepsilon>0\quad \exists\ \delta=2\varepsilon>0\quad \forall\ |x|<\delta\quad |y|<\delta$

$\left|\frac{x^2y}{x^2+y^2}-0\right|<\left|\frac{\delta^3}{2\delta^2}\right|=\varepsilon$

$\therefore$ $f(x,y)$ 在 $(0,0)$ 处连续.

$\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\frac{0-0}{x}=0$

$\lim\limits_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim\limits_{y\to 0}\frac{0-0}{y}=0$

$\therefore$ $f(x,y)$ 在 $(0,0)$ 处偏导数存在.

假设 $f(x,y)$ 在 $(0,0)$ 处可微. 则 $\exists\ a,b\in R$ $\frac{x^2y}{x^2+y^2}=ax+by+o\left(\frac{x^2y}{x^2+y^2}\right)$

即 $1=a\cdot\frac{x^2+y^2}{xy}+b\cdot\frac{x^2+y^2}{x^2}+o(1)$ 令 $y=kx$ $k\in R$ 则 $1=\frac{1+k^2}{k}a+(1+k^2)b$

不可能对任意 k成立.

即假设不成立 $f(x,y)$ 在 $(0,0)$ 处不可微.

17. $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

证明: $\forall \varepsilon>0 \quad \exists \delta=\sqrt{\frac{\varepsilon}{2}} \quad \forall |x|<\delta, |y|<\delta$

$$\left|(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}-0\right| \leq |x^2+y^2| < |\delta^2+\delta^2| = \varepsilon$$

$\therefore f(x,y)$ 在 $(0,0)$ 处连续.

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0} x\sin\frac{1}{x}=0$$

$$\frac{\partial f}{\partial x}=2x\cdot\sin\frac{1}{\sqrt{x^2+y^2}}+(x^2+y^2)\cos\frac{1}{\sqrt{x^2+y^2}}\cdot\left[-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right]=2x\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}\neq 0$$

$$\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0} y\cdot\sin\frac{1}{y}=0$$

$$\frac{\partial f}{\partial y}=2y\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{y}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial y}\neq 0$$

即 $f(x,y)$ 在 $(0,0)$ 处偏导数不连续

$$\lim_{(u,v)\to(0,0)}\frac{|f(u,v)-f(0,0)|}{\sqrt{u^2+v^2}}=\lim_{(u,v)\to(0,0)}\cdot\sqrt{u^2+v^2}\sin\frac{1}{\sqrt{u^2+v^2}}=0$$

$\therefore f(x,y)$ 在 $(0,0)$ 处可微.

18. (2). $z = u \arctan v \qquad u = \frac{xy}{x-y} \qquad v = x^2y + y - x$

解 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \arctan v \cdot \frac{\partial u}{\partial x} + \frac{u}{1+v^2} \frac{\partial v}{\partial x}$

$$= \arctan(x^2y+y-x) \cdot \frac{-y^2}{(x-y)^2} + \frac{xy}{x-y} \cdot \frac{1}{1+(x^2y+y-x)^2}(2xy-1)$$

$$= -\frac{y^2}{(x-y)^2} \arctan(x^2y+y-x) + \frac{xy(2xy-1)}{(x-y)[1+(x^2y+y-x)^2]}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\arctan v \cdot \frac{\partial u}{\partial x} + \frac{u}{1+v^2}\frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{1+v^2}\frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} + \arctan v \cdot \frac{\partial^2 u}{\partial x^2} + \frac{1}{1+v^2}\frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} + u\left[\frac{-2v}{(1+v^2)^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{1+v^2}\frac{\partial^2 v}{\partial x^2}\right]$$

$$= \arctan(x^2y+y-x)\frac{2y^2}{(x-y)^3} - \frac{4xy-2}{1+(x^2y+y-x)^2} \cdot \frac{y^2}{(x-y)^2} + \frac{xy}{x-y} \cdot \left\{\frac{2y}{1+(x^2y+y-x)^2} - \frac{2(x^2y+y-x)\cdot(2xy-1)^2}{[1+(x^2y+y-x)^2]^2}\right\}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \arctan v \frac{\partial u}{\partial y} + \frac{u}{1+v^2}\frac{\partial v}{\partial y}$$

$$= \arctan(x^2y+y-x) \cdot \frac{x^2}{(x-y)^2} + \frac{xy}{x-y} \cdot \frac{x^2+1}{1+(x^2y+y-x)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \arctan v \frac{\partial^2 u}{\partial y^2} + \frac{1}{1+v^2}\frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y} + \frac{1}{1+v^2}\frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y} + u \cdot \left[\frac{1}{1+v^2}\frac{\partial^2 v}{\partial y^2} - \frac{2v}{(1+v^2)^2}\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= \arctan(x^2y+y-x) \cdot \frac{2x^2}{(x-y)^3} + \frac{2x^2(x^2+1)}{[1+(x^2y+y-x)^2] \cdot (x-y)^2} - \frac{xy}{x-y} \cdot \frac{2(x^2y+y-x)}{[1+(x^2y+y-x)^2]^2} \cdot (x^2+1)^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{2xy-1}{1+(x^2y+y-x)^2} \cdot \frac{x^2}{(x-y)^2} + \arctan(x^2y+y-x) \cdot \frac{-2xy}{(x-y)^3} - \frac{y^2}{(x-y)^2} \cdot \frac{x^2+1}{1+(x^2y+y-x)^2} + \frac{xy}{x-y} \cdot \left\{\frac{2x}{1+(x^2y+y-x)^2} - \frac{2(x^2+1)(2xy-1)(x^2y+y-x)}{[1+(x^2y+y-x)^2]^2}\right\}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{-\frac{y^2}{(x-y)^2}\arctan(x^2y+y-x) + \frac{xy(2xy-1)}{(x-y)[1+(x^2y+y-x)^2]}\right\}$$

$$= -\frac{2xy}{(x-y)^3} \cdot \arctan(x^2y+y-x) - \frac{y^2}{(x-y)^2} \cdot \frac{x^2+1}{1+(x^2y+y-x)^2} + \frac{x^2}{(x-y)^2} \cdot \frac{2xy-1}{1+(x^2y+y-x)^2} + \frac{xy}{x-y}\left\{\frac{2x}{1+(x^2y+y-x)^2} - \frac{2(x^2+1)(2xy-1)(x^2y+y-x)}{[1+(x^2y+y-x)^2]^2}\right\}$$

19.(3) $u=\ln(x^2+y^2)$ $\quad x=e^{t+s+r}$ $\quad y=4(s^2+t^2)$

解: $\dfrac{\partial u}{\partial r}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial r}=\dfrac{2x}{x^2+y^2}\cdot e^{t+s+r}=\dfrac{2e^{2(t+s+r)}}{e^{2(t+s+r)}+16(s^2+t^2)^2}$

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}=\frac{2x}{x^2+y^2}\cdot e^{t+s+r}+\frac{2y}{x^2+y^2}\cdot 8s$$

$$=\frac{2e^{2(t+s+r)}+64s(s^2+t^2)}{e^{2(s+t+r)}+16(s^2+t^2)^2}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}=\frac{2x}{x^2+y^2}\cdot e^{t+s+r}+\frac{2y}{x^2+y^2}\cdot 8t$$

$$=\frac{2e^{2(t+s+r)}+64t(s^2+t^2)}{e^{2(s+t+r)}+16(s^2+t^2)^2}$$

19 (4) $u=\frac{e^{ax}(y-z)}{a^2+1}$ $\quad y=a\sin x \quad z=\cos x$

$$\frac{du}{dx}=\frac{d}{dx}\frac{e^{ax}(a\sin x-\cos x)}{a^2+1}=\frac{1}{a^2+1}\left[ae^{ax}(a\sin x-\cos x)+e^{ax}\cdot(a\cos x+\sin x)\right]$$

$$=e^{ax}\cdot\sin x$$

20.(4) $u=f(x+y+z,\ x^2+y^2+z^2)$

解. $$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial(x+y+z)}\cdot\frac{\partial(x+y+z)}{\partial x}+\frac{\partial f}{\partial(x^2+y^2+z^2)}\cdot\frac{\partial(x^2+y^2+z^2)}{\partial x}$$

$$=\frac{\partial f}{\partial(x+y+z)}+2x\cdot\frac{\partial f}{\partial(x^2+y^2+z^2)}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 f}{\partial(x+y+z)^2}+\frac{2\partial f}{\partial(x^2+y^2+z^2)}+4x^2\cdot\frac{\partial^2 f}{\partial(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\cdot\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 f}{\partial(x+y+z)^2}+4xy\cdot\frac{\partial^2 f}{\partial(x^2+y^2+z^2)^2}$$

28. $u=\varphi(x-at)+\psi(x+at)$

证明: $$\frac{\partial u}{\partial t}=\frac{\partial\varphi}{\partial(x-at)}\cdot\frac{\partial(x-at)}{\partial t}+\frac{\partial\psi}{\partial(x+at)}\cdot\frac{\partial(x+at)}{\partial t}=-a\cdot\frac{\partial\varphi}{\partial(x-at)}+a\cdot\frac{\partial\psi}{\partial(x+at)}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=-a\frac{\partial^2\varphi}{\partial(x-at)^2}\cdot\frac{\partial(x-at)}{\partial t}+a\cdot\frac{\partial^2\psi}{\partial(x+at)^2}\cdot\frac{\partial(x+at)}{\partial t}$$

$$=a^2\frac{\partial^2\varphi}{\partial(x-at)^2}+a^2\cdot\frac{\partial^2\psi}{\partial(x+at)^2}$$

$$\frac{\partial u}{\partial x}=\frac{\partial\varphi}{\partial(x-at)}\cdot\frac{\partial(x-at)}{\partial x}+\frac{\partial\psi}{\partial(x+at)}\cdot\frac{\partial(x+at)}{\partial x}=\frac{\partial\varphi}{\partial(x-at)}+\frac{\partial\varphi}{\partial(x+at)}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2\varphi}{\partial(x-at)^2}\cdot\frac{\partial(x-at)}{\partial x}+\frac{\partial^2\psi}{\partial(x+at)^2}\cdot\frac{\partial(x+at)}{\partial x}$$

$$=\frac{\partial^2\varphi}{\partial(x-at)^2}+\frac{\partial^2\psi}{\partial(x+at)^2}$$

即 $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$

31. 证明: 由 $\xi = x - \sin x + y \quad \eta = x + \sin x - y \qquad x = \frac{\xi+\eta}{2} \quad y = \frac{\xi-\eta}{2} + \sin\frac{\xi+\eta}{2}$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\cos x - \frac{\partial^2 u}{\partial y^2}\sin^2 x - \frac{\partial u}{\partial y}\sin x = 0$$

可知 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}\cdot(1-\cos x) + \frac{\partial u}{\partial \eta}(1+\cos x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \eta}{\partial x}$$

$$= \left[\frac{\partial^2 u}{\partial \xi^2}(1-\cos x) + \frac{\partial u}{\partial \xi}\cdot\frac{\sin x}{2} + \frac{\partial^2 u}{\partial \xi \partial \eta}(1+\cos x) - \frac{\partial u}{\partial \eta}\frac{\sin x}{2}\right]\cdot(1-\cos x)$$

$$+\left[\frac{\partial^2 u}{\partial \xi \partial \eta}(1-\cos x) + \frac{\partial u}{\partial \xi}\frac{\sin x}{2} + \frac{\partial^2 u}{\partial \eta^2}(1+\cos x) - \frac{\partial u}{\partial \eta}\cdot\frac{\sin x}{2}\right](1+\cos x)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \eta}{\partial y}$$

$$= \left[\frac{\partial^2 u}{\partial \xi^2}(1-\cos x) + \frac{\partial^2 u}{\partial \xi \partial \eta}(1+\cos x)\right] - \left[\frac{\partial^2 u}{\partial \xi \partial \eta}(1-\cos x) + \frac{\partial^2 u}{\partial \eta^2}(1+\cos x)\right]$$

$$= \frac{\partial^2 u}{\partial \xi^2}(1-\cos x) - \frac{\partial^2 u}{\partial \eta^2}(1+\cos x) + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\cos x$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

则 $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\cos x - \frac{\partial^2 u}{\partial y^2}\sin^2 x - \frac{\partial u}{\partial y}\sin x$

$$= \frac{\partial^2 u}{\partial \xi^2}(1-\cos x)^2 + \frac{\partial^2 u}{\partial \eta^2}(1+\cos x)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\sin^2 x + \frac{\partial u}{\partial \xi}\cdot\sin\theta - \frac{\partial u}{\partial \eta}\sin\theta$$

$$+ 2\frac{\partial^2 u}{\partial \xi^2}(1-\cos x)\cdot\cos x - 2\frac{\partial^2 u}{\partial \eta^2}(1+\cos x)\cdot\cos x$$

$$+ 4\frac{\partial^2 u}{\partial \xi \partial \eta}\cos^2\theta - \frac{\partial^2 u}{\partial \xi^2}\sin^2 x - \frac{\partial^2 u}{\partial \eta^2}\sin^2 x + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\sin^2 x - \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\sin x$$

$$= \frac{\partial^2 u}{\partial \xi^2}\left[(1-\cos x)^2 + 2\cos x(1-\cos x) - \sin^2 x\right] + \frac{\partial^2 u}{\partial \eta^2}\left[(1+\cos x)^2 - 2\cos x(1+\cos x) - \sin^2 x\right)$$

$$+ 4\frac{\partial^2 u}{\partial \xi \partial \eta} = 4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \qquad 即 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

32 $\begin{cases} u = x-2y \\ v = x+ay \end{cases}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial y}$$

$$= -2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v}\right) + a\left(\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\cdot\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\cdot\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial v}{\partial y}$$

$$= -2\cdot\left(-2\frac{\partial^2 z}{\partial u^2} + a\frac{\partial^2 z}{\partial u\partial v}\right) + a\cdot\left(-2\frac{\partial^2 z}{\partial u\partial v} + a\frac{\partial^2 z}{\partial v^2}\right)$$

则 $6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y^2} = 6\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u\partial v}\right) - 2\frac{\partial^2 z}{\partial u^2} + a\frac{\partial^2 z}{\partial v^2} + (a-2)\frac{\partial^2 z}{\partial u\partial v}$

$$-4\frac{\partial^2 z}{\partial u^2} - a^2\frac{\partial^2 z}{\partial v^2} + 4a\frac{\partial^2 z}{\partial u\partial v}$$

$= (6+a-a^2)\frac{\partial^2 z}{\partial v^2} + (10+5a)\frac{\partial^2 z}{\partial u\partial v} = 0$　　可以得到 $\frac{\partial^2 z}{\partial u\partial v} = 0$

即 $\begin{cases} 6+a-a^2=0 \\ 10+5a\neq 0 \end{cases}$　　得 $a=3$

33. $\frac{\partial z}{\partial y} = x^2+2y$　　得 $z(x,y) = x^2y + y^2 + C$　　C 为与 y 无关的量

$z(x, x^2) = x^4 + x^4 + C = 1$　　即 $C = -2x^4+1$

$z(x,y) = x^2y + y^2 - 2x^4 + 1$

35. 解：$\because u(x,2x)=x$ 对 x 求偏导数 得 $u_x'(x,2x)+2u_y'(x,2x)=1$ $\because u_x'(x,2x)=x^2$ ①

则 $u_y'(x,2x)=\frac{1-x^2}{2}$ ②.

①.② 对 x 求偏导 得

$u_{xx}''(x,2x)+2u_{xy}''(x,2x)=2x$

$u_{xy}'(x,2x)+2u_{yy}''(x,2x)=-x$

由于 $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$

得 $u_{xx}''(x,2x)=-\frac{4}{3}x$ $u_{xy}''(x,2x)=\frac{5}{3}x$ $u_{yy}''(x,2x)=-\frac{4}{3}x$

37 解：设球坐标 (r,θ,φ) 对应直角坐标 (x,y,z)

则 $x=r\sin\theta\cos\varphi$ $y=r\cdot\sin\theta\sin\varphi$ $z=r\cos\theta$ $u=u(x,y,z)$ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

$dr=\sin\theta\cos\varphi\,dx+\sin\theta\sin\varphi\,dy+\cos\theta\,dz$

$d\theta=\frac{\cos\theta\cos\varphi}{r}dx+\frac{\cos\theta\sin\varphi}{r}dy-\frac{\sin\theta}{r}dz$

$d\varphi=-\frac{\sin\varphi}{r\sin\theta}dx+\frac{\cos\varphi}{r\sin\theta}dy$

则 $\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\cdot\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\cdot\frac{\partial}{\partial\varphi}$

$=\sin\theta\cos\varphi\cdot\frac{\partial}{\partial r}+\frac{\cos\theta\cos\varphi}{r}\cdot\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\varphi}\cdot\frac{\partial}{\partial\varphi}$

$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\cdot\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\cdot\frac{\partial}{\partial\varphi}$

$=\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$

$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\cdot\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\cdot\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\cdot\frac{\partial}{\partial\varphi}$

$=\cos\theta\cdot\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial r}{\partial x}\left(\frac{\partial}{\partial x}\right)\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\left(\frac{\partial}{\partial x}\right)\cdot\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\left(\frac{\partial}{\partial x}\right)\cdot\frac{\partial}{\partial \varphi}$$

$$= \sin^2\theta\cos^2\varphi\,\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\cos^2\varphi}{r^2}\,\frac{\partial^2}{\partial \theta^2} + \frac{\sin^2\varphi}{r^2\sin^2\theta}\,\frac{\partial^2}{\partial \varphi^2}$$

$$+ \frac{2\sin\theta\cos\theta\cos^2\varphi}{r}\,\frac{\partial^2}{\partial r\partial\theta} - \frac{2\sin\varphi\cos\varphi}{r}\,\frac{\partial^2}{\partial r\partial\varphi} - \frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\,\frac{\partial^2}{\partial\theta\partial\varphi}$$

$$+ \frac{\cos^2\theta\cos^2\varphi + \sin^2\theta}{r}\cdot\frac{\partial}{\partial r} + \frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}\,\frac{\partial}{\partial\varphi} + \frac{-2\sin^2\theta\cos\theta\cos^2\varphi + \cos\theta\sin^2\varphi}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial r}{\partial y}\cdot\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial\varphi}$$

$$= \sin^2\theta\sin^2\varphi\,\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\sin^2\varphi}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}$$

$$+ \frac{2\sin\theta\cos\theta\sin^2\varphi}{r}\,\frac{\partial^2}{\partial r\partial\theta} + \frac{2\sin\varphi\cos\varphi}{r}\,\frac{\partial^2}{\partial r\partial\varphi} + \frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\,\frac{\partial^2}{\partial\theta\partial\varphi}$$

$$+ \frac{\cos^2\theta\sin^2\varphi + \cos^2\varphi}{r}\,\frac{\partial}{\partial r} - \frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}\,\frac{\partial}{\partial\varphi} + \frac{-2\sin^2\theta\cos\theta\sin^2\varphi + \cos\theta\cos^2\varphi}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) = \frac{\partial r}{\partial z}\left(\frac{\partial}{\partial z}\right)\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\left(\frac{\partial}{\partial z}\right)\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial z}\left(\frac{\partial}{\partial z}\right)\frac{\partial}{\partial\varphi}$$

$$= \cos^2\theta\,\frac{\partial^2}{\partial z^2} + \frac{\sin^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\,\frac{\partial}{\partial r}$$

$$\because\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\therefore\ \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0$$