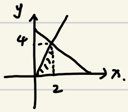
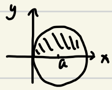
第七周周二作业 4月 14日

习题 10.1

1\. (2). $\int_0^2 dx \int_{2x}^{6-x} f(x,y)\,dy$

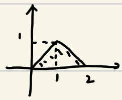


$= \int_0^4 dy \int_{\frac{y}{2}}^{2} f(x,y)\,dx$



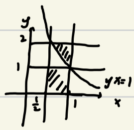
(3). $\int_0^a dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx$

$= \int_0^{2a} dx \int_0^{\sqrt{a^2-(x-a)^2}} f(x,y)\,dy$



(5) $\int_0^1 dx \int_0^x f(x,y)\,dy + \int_1^2 dx \int_0^{2-x} f(x,y)\,dy$

$= \int_0^1 dy \int_y^{2-y} f(x,y)\,dx$



(6). $\int_0^1 dy \int_{\frac{1}{2}}^{1} f(x,y)\,dx + \int_1^2 dy \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)\,dx$

$= \int_{\frac{1}{2}}^1 dx \int_0^1 f(x,y)\,dy + \int_{\frac{1}{2}}^1 dx \int_1^{\frac{2}{x}} f(x,y)\,dy$

2\. (1) $\iint_D \frac{y}{(1+x^2+y^2)^{3/2}}\,dx\,dy \qquad D=[0,1]\times[0,1]$

$= \int_0^1 dx \int_0^1 \frac{y}{(1+x^2+y^2)^{3/2}}\,dy = \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{2+x^2}}\right) dx = \left[\ln(\sqrt{1+x^2}+x) - \ln(\sqrt{2+x^2}+x)\right]\Big|_0^1 = \ln\left[\frac{(2+\sqrt{2})(\sqrt{3}-1)}{2}\right]$

(2). $\iint_D \sin(x+y)\,dx\,dy \qquad D=[0,\pi]^2$

$= \int_0^\pi dy \int_0^\pi \sin(x+y)\,dx = \int_0^\pi 2\cos y\,dy = 0$

(3). $\iint_D \cos(x+y)\,dx\,dy \qquad$ D：由 $y=\pi$ $y=x$ 与 $x=0$ 围成.

令 $y_1=x \quad y_2=\pi$ 　则原式 $= \int_0^\pi dx \int_{y_1}^{y_2} \cos(x+y)\,dy = \int_0^\pi (-\sin x - \sin 2x)\,dx = -2$

(4). $\iint_D (x+y)\,dx\,dy \qquad$ D：由 $x^2+y^2=a^2$ 围成的圆在第一象限

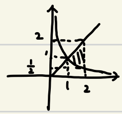
$y_1=0 \quad y_2=\sqrt{a^2-x^2}$ 　原式 $= \int_0^a dx \int_{y_1}^{y_2} (x+y)\,dy = \int_0^a \left[x\sqrt{a^2-x^2} + \frac{1}{2}(a^2-x^2)\right] dx = \int_0^a (x\sqrt{a^2-x^2})\,dx + \frac{1}{2}a^3 - \frac{1}{6}a^3$

$\int_0^a x\sqrt{a^2-x^2}\,dx$ 　令 $x = a\sin t \quad t\in[0,\frac{\pi}{2}]$ 　$\int_0^a x\sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a\sin t \cdot a\cos t \ \ a\cos t\,dt = \int_0^{\frac{\pi}{2}} -a^3\cos^2 t\,d(\cos t) = -\frac{1}{3}a^3\cos^3 t\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}a^3$

$dx = a\cos t\,dt$

$\therefore$ 原积分 $= \frac{2}{3}a^3$

(7). $\iint_D \frac{x^2}{y^2}\,dx\,dy$ D: 由 $x=2$ $y=x$ 及 $xy=1$ 围成



$y_1=\frac{1}{x}$ $y_2=x$ 则原积分 $=\int_1^2 dx\int_{y_1}^{y_2}\frac{x^2}{y^2}\,dy=\int_1^2(x^3-x)\,dx=(4-2)-(\frac{1}{4}-\frac{1}{2})=\frac{9}{4}$

4. 证明: $\because \varphi(x)$ 在 $[a,b]$ 上可积 $\psi(y)$ 在 $[c,d]$ 上可积.

在 $[a,b]$ 上取分割 T_1: $a\le x_0<x_1<\cdots<x_n=b$ 在 $[c,d]$ 上取分割 T_2: $c=y_0<y_1<\cdots<y_m=d$

则在 $[a,b]\times[c,d]$ 上有 $n\,m$ 个小矩形.

设 $N_i=\sup\varphi(x)$ $x\in[x_{i-1},x_i]$ $M_i=\sup\psi(y)$ $y\in[y_{i-1},y_i]$

$n_i=\inf\varphi(x)$ $x\in[x_{i-1},x_i]$ $m_i=\inf\psi(y)$ $y\in[y_{i-1},y_i]$

$\because \varphi(x)$ 和 $\psi(y)$ 可积 $\therefore \lim\limits_{|T_1|\to0}N_i=\lim\limits_{|T_1|\to0}n_i$ $\lim\limits_{|T_2|\to0}M_i=\lim\limits_{|T_2|\to0}m_i$ $\int_a^b\varphi(x)\,dx=\lim\limits_{|T_1|\to0}\sum\limits_{i=1}^{n}N_i\Delta x_i$ $\int_c^d\psi(y)\,dy=\lim\limits_{|T_2|\to0}\sum\limits_{j=1}^{m}M_j\Delta y_j$

S_{ij} 表示 $[x_{i-1},x_i]\cdot[y_{j-1},y_j]$ 的矩形区域. 则有 $\lim\limits_{\substack{|T_1|\to0\\|T_2|\to0}}N_iM_j=\lim\limits_{\substack{|T_1|\to0\\|T_2|\to0}}n_im_j$

$S^-=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}n_im_j\Delta x_i\Delta y_j\le\iint_D f(x,y)\le S^+=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}N_iM_j\Delta x_i\Delta y_j$

当 $|T_1|\,|T_2|\to0$ 即 $n\to\infty$ $m\to\infty$ 时 $S^-=S^+$ $\therefore f(x,y)=\varphi(x)\psi(y)$ 可积

$\iint_D f(x,y)=\lim\limits_{\substack{|T_1|\to0\\|T_2|\to0}}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}N_i\Delta x_i\,M_j\Delta y_j=\lim\limits_{|T_1|\to0}\sum\limits_{i=1}^{n}N_i\Delta x_i\ \lim\limits_{|T_2|\to0}\sum\limits_{j=1}^{m}M_j\Delta y_j=\int_a^b\varphi(x)\,dx\int_c^d\psi(y)\,dy$

6. 解: $\because f(x,y)$ 有二阶连续偏导数 $\therefore \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$

$\therefore \iint_D\frac{\partial^2 f}{\partial x\partial y}\,dx\,dy=\int_a^b dx\int_c^d\frac{\partial^2 f}{\partial x\partial y}\,dy=\int_a^b[f_x'(x,d)-f_x'(x,c)]\,dx=f(b,d)-f(a,d)-f(b,c)+f(a,c)$

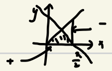
7. 解: $\lim\limits_{r\to0}\frac{1}{\pi r^2}\iint_{x^2+y^2\le r^2}f(x,y)\,dx\,dy$ $\because f(x,y)$ 连续 $\therefore \forall\varepsilon>0$ $\exists\delta=\delta(\varepsilon)$ $\forall|x|<\delta$ $|y|<\delta$ $|f(x,y)-f(0,0)|<\varepsilon$

$\therefore \lim\limits_{r\to0}\iint_{x^2+y^2\le r^2}f(x,y)\,dx\,dy=f(0,0)\pi r^2$ $\therefore \lim\limits_{r\to0}\frac{1}{\pi r^2}\iint_{x^2+y^2\le r^2}f(x,y)\,dx\,dy=f(0,0)$

第七周周四作业　4月16日

习题 10.1

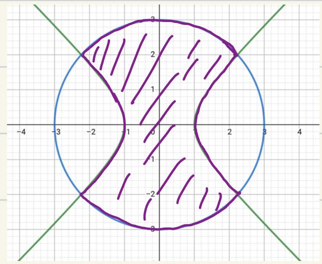
2. (8). $\iint_D |\cos(x+y)|\,dx\,dy$　由 $y=x$　$y=0$　$x=\frac{\pi}{2}$ 围成



$$=\int_0^{\frac{\pi}{2}}dx\int_0^{x}|\cos(x+y)|\,dy=\int_0^{\frac{\pi}{4}}dy\int_y^{-y+\frac{\pi}{2}}\cos(x+y)\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}dx\int_{-x+\frac{\pi}{2}}^{x}[-\cos(x+y)]\,dy$$

$$=\int_0^{\frac{\pi}{4}}(1-\sin 2y)\,dy+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\sin 2x)\,dx=\frac{\pi}{4}-\frac{1}{2}+\frac{\pi}{4}-\frac{1}{2}=\frac{\pi}{2}-1$$

3 (2). $\iint_D \sin x\sin y\,dx\,dy$　D $x^2-y^2=1$　$x^2+y^2=9$ 围成含圆点部分



由于 $f(x,y)=-f(-x,y)=-f(x,-y)=f(-x,-y)$

∴第一、二象限内积分相互抵消；第三、四象限内积分相互抵消

∴ $\iint_D \sin x\sin y\,dx\,dy=0$

5. 证明：∵ $f(x)$ 在 $[0,a]$ 上连续　令 $F(x)$ 为 $f(x)$ 的一个原函数　则 $F(a)-F(0)=\int_0^a f(x)\,dx$

$$\therefore\int_0^a dx\int_0^x f(x)f(y)\,dy=\int_0^a f(x)[F(x)-F(0)]\,dx=\int_{F(0)}^{F(a)}[F(x)-F(0)]\,d[F(x)]$$

$$=\frac{1}{2}[F(a)-F(0)]^2=\frac{1}{2}\left(\int_0^a f(x)\,dx\right)^2$$

$$\int_0^a dx\int_0^x f(y)\,dy=\int_0^a[F(x)-F(0)]\,dx=[F(x)-F(0)]x\Big|_0^a-\int_0^a x\cdot f(x)\,dx=a\int_0^a f(x)\,dx-\int_0^a xf(x)\,dx$$

$$=\int_0^a(a-x)f(x)\,dx$$

习题 10.2

1.(5) $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx \int_0^{Rx} (1+\frac{y^2}{x^2})\,dy + \int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx \int_0^{\sqrt{R^2-x^2}} (1+\frac{y^2}{x^2})\,dy$

$= \int_0^{\frac{R}{\sqrt{1+R^2}}} (R+\frac{R^3}{3})\,x\,dx + \int_{\frac{R}{\sqrt{1+R^2}}}^{R} \sqrt{R^2-x^2}\cdot(1+\frac{R^2-x^2}{3x^2})\,dx$

$= (R+\frac{R^3}{3})\,\frac{1}{2}\,\frac{R^2}{1+R^2} + (\frac{1}{3}R^2\arcsin\frac{x}{R}+\frac{1}{3}x\sqrt{R^2-x^2})\Big|_{\frac{R}{\sqrt{1+R^2}}}^{R} + \int_{\frac{R}{\sqrt{1+R^2}}}^{R} \frac{R^2\sqrt{R^2-x^2}}{3x^2}\,dx$

$\int \frac{R^2\sqrt{R^2-x^2}}{3x^2}\,dx$ 令 $t=\frac{R}{x}$ $x=\frac{R}{t}$ $dx=-\frac{R}{t^2}dt$ 原式 $=\int \frac{R}{3}\,t\cdot\sqrt{t^2-1}\cdot(-\frac{R}{t^2}dt) = -\frac{R^2}{3}\int\frac{\sqrt{t^2-1}}{t}\,dt = -\frac{R^2}{3}(\sqrt{t^2-1}-\arccos\frac{1}{t}+C)$ C属于常数.

则原式 $= \frac{R^2}{2(1+R^2)}(R+\frac{R^3}{3}) + (\frac{\pi}{6}R^2 - \frac{R^2}{3}\arcsin\frac{1}{\sqrt{1+R^2}} - \frac{1}{3}\frac{R^3}{1+R^2}) + \frac{R^2}{3}(R-\arccos\frac{1}{\sqrt{1+R^2}})$

$= \frac{R^3(R^2+3)}{6(1+R^2)} + \frac{\pi}{6}R^2 + \frac{R^5}{3(1+R^2)} - \frac{R^2}{3}(\arcsin\frac{1}{\sqrt{1+R^2}}+\arccos\frac{1}{\sqrt{1+R^2}})$

$= \frac{1}{2}R^3 + \frac{\pi}{6}R^2 - \frac{\pi}{6}R^2 = \frac{1}{2}R^3$

2.(1). $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ $D: x^2+y^2\le x+y$

D区域为 $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$ ∴令 $x=r\cos\theta$ $y=r\sin\theta$

则 $r^2\le r(\cos\theta+\sin\theta)$ 即 $r\le\cos\theta+\sin\theta$ $D': r\in[0,\cos\theta+\sin\theta]$ $\theta\in[-\frac{\pi}{4},\frac{3\pi}{4}]$

$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \iint_{D'} r\cdot\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \iint_{D'} r^2\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\cos\theta+\sin\theta} r^2 dr = \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{1}{3}(\cos\theta+\sin\theta)^3 d\theta$

$= \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^3(\theta+\frac{\pi}{4})\,d\theta = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} [1-\cos^2(\theta+\frac{\pi}{4})]\,d(-\cos(\theta+\frac{\pi}{4})) = \frac{2\sqrt{3}}{3}[\frac{1}{3}\cos^3(\theta+\frac{\pi}{4})-\cos(\theta+\frac{\pi}{4})]\Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{8\sqrt{2}}{9}$

(2). $\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,dx\,dy$ D: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ $y=0$ $y=x$ 围成第一象限部分

令 $x=ar\cos\theta$ $y=br\sin\theta$ D': $\theta\in[0,\arctan\frac{a}{b}]$ $r\in[0,2]$

$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,dx\,dy = \iint_{D'} r\cdot abr\,d\theta\,dr = \int_0^{\arctan\frac{a}{b}} d\theta\cdot\int_0^2 abr^2 dr = \frac{8ab}{3}\arctan\frac{a}{b}$

(5). $\iint_D xy\,dx\,dy$ D: $xy=a$ $xy=b$ $y^2=cx$ $y^2=dx$ 围成第一象限部分 $(0<a<b.\ 0<c<d)$

令 $xy=u$, $\frac{y^2}{x}=v$. 则 $x=\sqrt[3]{\frac{u^2}{v}}$ $y=\sqrt[3]{uv}$ $\frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{9}\frac{1}{v}+\frac{1}{9}\frac{1}{v} = \frac{1}{3v}$

D' : $u\in[a,b]$ $v\in[c,d]$

$\iint_D xy\,dxdy = \iint_{D'} u\cdot\frac{1}{3v}\,du\,dv = \int_a^b u\,du\cdot\int_c^d \frac{1}{3v}dv = \frac{1}{2}(b^2-a^2)\ \frac{1}{3}\ln\frac{d}{c} = \frac{1}{6}(b^2-a^2)\ln\frac{d}{c}$

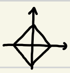
(6). $\iint_D 4xy\,dx\,dy$. D: $x^4+y^4\le 1$ $x\ge 0$ $y\ge 0$

$x^2=r\cos\theta$ $y^2=r\sin\theta$ $r\in[0,1]$ $\theta\in[0,\frac{\pi}{2}]$

$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \frac{1}{4}\ \frac{1}{\sqrt{\sin\theta\cos\theta}}$

$\iint_D 4xy\,dxdy = \iint_{D'} 4r\sqrt{\sin\theta\cos\theta}\ \frac{1}{4\sqrt{\sin\theta\cos\theta}}\ d\theta\,dr = \int_0^{\frac{\pi}{2}}d\theta\int_0^1 r\,dr = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$

(7). $\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}\,dx\,dy$ D: $|x|+|y|\le 1$



令 $u=x+y$ $v=x-y$ 则 $x=\frac{u+v}{2}$ $y=\frac{u-v}{2}$ $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$ D' $u\in[-1,1]$ $v\in[-1,1]$

$\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}\,dx\,dy = \iint_{D'} \frac{uv}{\sqrt{u+3}}\cdot\frac{1}{2}\,du\,dv = \int_{-1}^1 \frac{u}{2\sqrt{u+3}}\,du\cdot\int_{-1}^1 v\,dv = 0$

(9). $\iint_D |xy|\,dx\,dy$ D: $\{(x,y)\ |\ x^2+y^2\le a^2\}$

D' : $\{(x,y)\ |\ x^2+y^2\le a^2,\ y\ge 0,\ x\ge 0\}$

$\iint_D |xy|\,dxdy = 4\iint_{D'} xy\,dxdy = 4\int_0^a dx\int_0^{\sqrt{a^2-x^2}} xy\,dy = 4\int_0^a \frac{1}{2}x(a^2-x^2)\,dx = 2\int_0^a(a^2x-x^3)\,dx = \frac{1}{2}a^4$

6. 证明: 令 $u=x+y$ $\quad v=x-y$ $\quad$ 则 $x=\frac{u+v}{2}$ $\quad y=\frac{u-v}{2}$ $\quad$ $f(u)$为连续奇函数 $\quad \therefore f(0)=0$ $\quad f(-u)=-f(u)$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}$ $\qquad D'$: $u\in[-1,1]$ $\quad v\in[-1,1]$

$\iint_D e^{f(x+y)}dxdy=\iint_{D'}e^{f(u)}\cdot\frac{1}{2}\,du\,dv=\int_{-1}^{1}\frac{1}{2}dv\cdot\int_{-1}^{1}e^{f(u)}du=\int_{-1}^{1}e^{f(u)}du$

$\because f(u)=-f(-u)$ $\quad \therefore e^{f(u)}=e^{-f(u)}>0$ $\quad \therefore \int_{-1}^{1}e^{f(u)}du\geqslant 2\,e^{f(0)}=2e^0=2$

$\therefore \iint_D e^{f(x+y)}dxdy\geqslant 2$

7. 证明: 令 $u=x+y$ $\quad v=x-y$ $\quad$ 则 $x=\frac{u+v}{2}$ $\quad y=\frac{u-v}{2}$ $\quad D'$: $u\in[|v|-A,A-|v|]$ $v\in[-A,A]$ $\quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}$

$\iint_D f(x-y)\,dxdy=\iint_{D'}f(v)\frac{1}{2}du\,dv=\int_{-A}^{A}dv\int_{|v|-A}^{A-|v|}f(v)\frac{1}{2}du=\int_{-A}^{A}f(v)(A-|v|)\,dv=\int_{-A}^{A}f(t)(A-|t|)\,dt$

$\therefore \iint_D f(x-y)\,dxdy=\int_{-A}^{A}f(t)(A-|t|)\,dt$