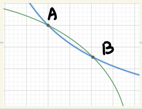
第八周周二作业 4月21日

习题 10.2



3.(1) $x^2+2y^2=3$ 与 $xy=1$ （不含原点部分）

如图第一象限部分 $A(1,1)$ $B(\sqrt{2},\frac{\sqrt{2}}{2})$ 总面积为2倍第一象限面积.

D: $x\in[1,\sqrt{2}]$ $y\in[\frac{1}{x},\sqrt{\frac{1}{2}(3-x^2)}]$

$$S=2\iint_D dxdy=2\int_1^{\sqrt{2}}dx\int_{\frac{1}{x}}^{\sqrt{\frac{1}{2}(3-x^2)}}dy=2\int_1^{\sqrt{2}}(\frac{\sqrt{2}}{2}\sqrt{3-x^2}-\frac{1}{x})dx=\sqrt{2}\int_1^{\sqrt{2}}\sqrt{3-x^2}dx-2\int_1^{\sqrt{2}}\frac{1}{x}dx=\sqrt{2}\left(\frac{1}{2}x\sqrt{3-x^2}+\frac{3}{2}\arcsin\frac{x}{\sqrt{3}}\right)\Big|_1^{\sqrt{2}}-2\ln x\Big|_1^{\sqrt{2}}$$

$$=\frac{3\sqrt{2}}{2}(\arcsin\frac{\sqrt{6}}{3}-\arcsin\frac{\sqrt{3}}{3})-\ln2=\frac{3\sqrt{2}}{2}\arcsin\frac{1}{3}-\ln2$$

(2). $(x-y)^2+x^2=a^2$ $(a>0)$

令 $u=x-y$ $v=x$ 则 $u^2+v^2=a^2$ $\begin{cases}x=v\\y=v-u\end{cases}$ $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=1$

$$\iint_D dxdy=\iint_{D'}1\,dudv=\int_{-a}^{a}du\int_{-\sqrt{a^2-u^2}}^{\sqrt{a^2-u^2}}dv=\int_{-a}^{a}2\sqrt{a^2-u^2}du=4\int_0^a\sqrt{a^2-u^2}du=4\cdot\left(\frac{1}{2}u\sqrt{a^2-u^2}+\frac{a^2}{2}\arcsin\frac{u}{a}\right)\Big|_0^a=\pi a^2$$

4. $$\iint_{x^2+y^2\le1}e^{x^2+y^2}dxdy\le\left[\int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}}e^{x^2}dx\right]^2$$

证明：对于左侧 $x=r\cos\theta$ $y=r\sin\theta$ D': $r\in[0,1]$ $\theta\in[0,2\pi]$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$

$$\iint_{x^2+y^2\le1}e^{x^2+y^2}dxdy=\iint_{D'}e^{r^2}r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1re^{r^2}dr=2\pi\ \frac{1}{2}(e-1)=\pi(e-1)<5.4$$

$e^{x^2}\ge1+x^2+\frac{x^4}{2}+\frac{x^6}{6}$ $\therefore\int_0^{\frac{\sqrt{\pi}}{2}}e^{x^2}dx\ge\int_0^{\frac{\sqrt{\pi}}{2}}(1+x^2+\frac{x^4}{2}+\frac{x^6}{6})dx=\frac{\sqrt{\pi}}{2}+\frac{1}{3}\cdot(\frac{\sqrt{\pi}}{2})^3+\frac{1}{10}(\frac{\sqrt{\pi}}{2})^5+\frac{1}{42}(\frac{\sqrt{\pi}}{2})^7$

$\therefore\left[\int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}}e^{x^2}dx\right]^2\ge4\cdot\left[\frac{\sqrt{\pi}}{2}+\frac{1}{3}(\frac{\sqrt{\pi}}{2})^3+\frac{1}{10}(\frac{\sqrt{\pi}}{2})^5+\frac{1}{42}(\frac{\sqrt{\pi}}{2})^7\right]^2>5.5$

$\therefore\left[\int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}}e^{x^2}dx\right]^2\ge\iint_{x^2+y^2\le1}e^{x^2+y^2}dxdy$

5. 证明: $\int_0^1 e^{f(x)}dx \cdot \int_0^1 e^{-f(y)}dy = \iint\limits_{[0,1]\times[0,1]} e^{f(x)-f(y)}dx\,dy \geq \iint_D e^{f(x)-f(y)}dx\,dy$ $\quad D:\ x\geq 0 \quad y\geq 0 \quad x^2+y^2\leq 1$

$\iint_D e^{f(x)-f(y)}dx\,dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{f(x)-f(y)}dx\,dy$

在 $[0,1]$ 上取分割 $T_1:\ 0=x_0<x_1<\cdots<x_n=1$

$\because$ $f(x)$在 $[0,1]$ 上连续、$\therefore$ $e^{f(x)}$ 在$[0,1]$上连续 $\therefore e^{f(x)}$在 $[0,1]$ 上可积

即 $\lim\limits_{\|T_1\|\to 0} \sum\limits_{i=1}^{n} e^{f(\xi_i)}\cdot(x_i-x_{i-1})$ $\quad \xi_i\in[x_{i-1},x_i]$ 存在且等于 $\int_0^1 e^{f(x)}dx$

$\therefore \int_0^1 e^{-f(y)}dy = \lim\limits_{\|T_2\|\to 0} \sum\limits_{i=1}^{m} e^{-f(y_i)}(y_i-y_{i-1}) \quad y_i\in[y_{i-1},y_i]$

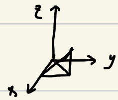
取 T_3 使得 $\|T_3\|<\|T_1\|$ 且 $\|T_3\|<\|T_2\|$

$$\int_0^1 e^{f(x)}dx\cdot\int_0^1 e^{-f(y)}dy = \lim_{\|T_3\|\to 0}\sum_{i=1}^{N} e^{f(\xi_i)}(x_i-x_{i-1})\cdot\lim_{\|T_3\|\to 0}\sum_{i=1}^{N} e^{-f(y_i)}(y_i-y_{i-1}) \geq \lim_{\|T_3\|\to 0}\sum_{i=1}^{N}(x_i-x_{i-1})(y_i-y_{i-1}) = 1$$

$\therefore \int_0^1 e^{f(x)}dx\cdot\int_0^1 e^{-f(y)}dy \geq 1$

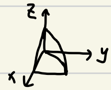
习题10.3

1(2). $\iiint_V xy^2z^3\,dx\,dy\,dz$ V: 由 $z=xy$, $y=x$, $x=1$, $z=0$ 围成.



$$\iiint_V xy^2z^3\,dx\,dy\,dz = \int_0^1 dx\int_0^x dy\int_0^{xy} xy^2z^3\,dz = \int_0^1 dx\int_0^x \frac{1}{4}x^5y^6\,dy = \int_0^1 \frac{1}{28}x^{12}\,dx = \frac{1}{364}$$

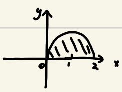
(3). $\iiint_V y\cos(x+z)\,dx\,dy\,dz$ V: 由 $y=\sqrt{x}$ $y=0$ $z=0$ $z+x=\frac{\pi}{2}$ 围成



$$\iiint_V y\cos(x+z)\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}dy\int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz = \int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}} y(1-\sin x)\,dy = \int_0^{\frac{\pi}{2}}\frac{1}{2}x(1-\sin x)\,dx = \frac{1}{2}\left(\frac{1}{2}x^2+x\cos x-\sin x\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{16}-\frac{1}{2}$$

2.(1). $\int_0^2 dx\int_0^{\sqrt{2x-x^2}}dy\int_0^a z\sqrt{x^2+y^2}\,dz$



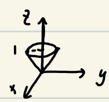
$$=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}\frac{1}{2}a^2\cdot\sqrt{x^2+y^2}\,dy$$

$$=\iint_D \frac{1}{2}a^2\cdot\sqrt{x^2+y^2}\,dx\,dy \qquad D:\ y\geq 0 \quad y^2+(x-1)^2\leq 1$$

令 $x=r\cos\theta$ $y=r\sin\theta$ $\theta\in[0,\frac{\pi}{2}]$ $r^2\sin^2\theta\leq 2r\cos\theta-r^2\sin^2\theta$ ∴ $r\in[0,2\cos\theta]$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$

$$\iint_D \frac{1}{2}a^2\sqrt{x^2+y^2}\,dx\,dy = \iint_{D'}\frac{1}{2}a^2\,r\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{2}a^2\,d\theta\int_0^{2\cos\theta}r^2\,dr = \int_0^{\frac{\pi}{2}}\frac{4}{3}a^2(1-\sin^2\theta)\,d(\sin\theta) = \frac{4}{3}a^2\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{8}{9}a^2$$

3(2). $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$ V: 由 $x^2+y^2=z^2$ $z=1$ 围成

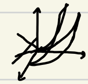


$\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_0^1 dz\iint_{x^2+y^2\leq z^2}\sqrt{x^2+y^2}\,dx\,dy$ 令 $x=r\cos\theta$ $y=r\sin\theta$ $\theta\in[0,2\pi]$ $r\in[0,z]$ $\frac{\partial(x,y)}{\partial(r,\theta)}=r$

$$\iint_{x^2+y^2\leq z^2}\sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}d\theta\int_0^z r^2\,dr = \frac{2\pi}{3}z^3$$

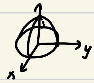
原积分 $=\int_0^1\frac{2\pi}{3}z^3\,dz = \frac{\pi}{6}$

(5). $\iiint_V x^2\,dx\,dy\,dz$ V是由曲面 $z=y^2$ $z=4y^2$ $(y>0)$ 及平面 $z=x$ $z=2x$ $z=1$ 围成区域



$$\iiint_V x^2\,dx\,dy\,dz = \int_0^1 dz \int_{\frac{z}{2}}^{z} dx \int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}} x^2\,dy = \int_0^1 dz \int_{\frac{z}{2}}^{z} \frac{\sqrt{z}}{2}\cdot x^2\,dx = \int_0^1 \frac{7}{48} z^{\frac{7}{2}}\,dz = \frac{7}{216}$$

4(1). $\iiint_V (x+y)\,dx\,dy\,dz$ V: 由 $z=1-x^2-y^2$ 和 $z=0$ 组成.



由对称性可知. 在Oxy平面内 第一. 四象限内积分互相抵消；在二. 三象限内 从$x+y=0$为对称轴, 两侧积分相互抵消。

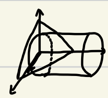
$\therefore \iiint_V (x+y)\,dx\,dy\,dz = 0$

$\int_0^1 dz \iint_D (x+y)\,dx\,dy$ D: $x^2+y^2 \le 1-z$ 令 $x=r\cos\theta$ $y=r\sin\theta$ $\theta\in[0,2\pi]$ $r\in[0,\sqrt{1-z}]$

$$\iint_D (x+y)\,dx\,dy = \iint_{D'} r(\cos\theta+\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}(\sin\theta+\cos\theta)\,d\theta\cdot\int_0^{\sqrt{1-z}} r^2\,dr = 0$$

$\therefore \int_0^1 0\cdot dz = 0$ $\therefore \iiint_V (x+y)\,dx\,dy\,dz = 0$

5(3) $z^2+x^2=1$ 和 $x+y+z=3$, $y=0$ 围成的体积.

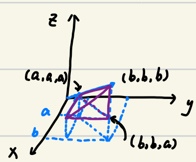


$\iiint_V dx\,dy\,dz = \iint_D dx\,dz \int_0^{3-x-z} dy = \iint_D (3-x-z)\,dx\,dz$ D: $x^2+z^2\le 1$

令 $x=r\cos\theta$ $z=r\sin\theta$ D': $r\in[0,1]$ $\theta\in[0,2\pi]$ $\left|\frac{\partial(x,z)}{\partial(r,\theta)}\right| = r$

$$\iint_D (3-x-z)\,dx\,dz = \iint_{D'} (3-r\cos\theta-r\sin\theta)\,r\,dr\,d\theta = \int_0^1 dr\int_0^{2\pi}[3r-r^2(\sin\theta+\cos\theta)]\,d\theta = \int_0^1 6\pi r = 3\pi$$

$\iiint_V dx\,dy\,dz = 3\pi$



9. 证明: $\int_a^b dx\int_a^x dy\int_a^y f(x,y,z)\,dz = \iiint_V f(x,y,z)\,dx\,dy\,dz$ V: $x\in[a,b]$, $y\in[a,x]$, $z\in[a,y]$

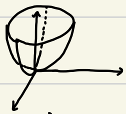
$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b dz\int_z^b dy\int_y^b f(x,y,z)\,dx$$

$$\therefore \int_a^b dx\int_a^x dy\int_a^y f(x,y,z)\,dz = \int_a^b dz\int_z^b dy\int_y^b f(x,y,z)\,dx$$

第八周周四作业 4月23日

习题 10.3

2.(4) $\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz$



$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{1}{3}\left[(2-x^2-y^2)^{\frac{3}{2}} - (x^2+y^2)^{\frac{3}{2}}\right] dy = \iint_D \frac{1}{3}\left[(2-x^2-y^2)^{\frac{3}{2}} - (x^2+y^2)^{\frac{3}{2}}\right] dxdy$ $D: x\in[0,1]$ $y\in[0,\sqrt{1-x^2}]$

令 $x=r\cos\theta$ $y=r\sin\theta$ $r\in[0,1]$ $\theta\in[0,\frac{\pi}{2}]$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

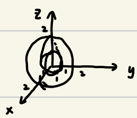
$\iint_{D'} \frac{1}{3}\left[(2-r^2)^{\frac{3}{2}} - r^3\right] r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \left[\frac{1}{3} r\cdot(2-r^2)^{\frac{3}{2}} - \frac{1}{3}r^4\right]dr = \frac{\pi}{2}\cdot\frac{1}{3}\cdot\left[-\frac{1}{5}(2-r^2)^{\frac{5}{2}} - \frac{1}{5}r^5\right]\Big|_0^1 = \frac{(2\sqrt{2}-1)\pi}{15}$

3(3). $\iiint_V z\,dxdydz$ V 由 $\sqrt{4-x^2-y^2}=z$ $x^2+y^2=3z$ 组成.

$\iiint_V z\,dxdydz = \iint_D dxdy \int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} z\,dz = \iint_D \frac{1}{2}\left[4-x^2-y^2-\left(\frac{x^2+y^2}{3}\right)^2\right] dxdy$ $D: x^2+y^2\le 3$

令 $x=r\cos\theta$ $y=r\sin\theta$ 则 D': $r\in[0,\sqrt{3}]$ $\theta\in[0,2\pi]$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

$\iint_D \frac{1}{2}\left[4-x^2-y^2-\left(\frac{x^2+y^2}{3}\right)^2\right]dxdy = \iint_{D'} \frac{1}{2}\left(4-r^2-\frac{r^4}{9}\right) r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\left(2r-\frac{1}{2}r^3-\frac{1}{18}r^5\right)dr = 2\pi\cdot\frac{13}{8} = \frac{13\pi}{4}$



(8) $\iiint_V (|x|+z)e^{-(x^2+y^2+z^2)}dxdydz$ $V: 1\le x^2+y^2+z^2\le 4$

$\iiint_V (|x|+z)e^{-(x^2+y^2+z^2)}dxdydz = \iiint_V |x|e^{-(x^2+y^2+z^2)}dxdydz + \iiint_V z e^{-(x^2+y^2+z^2)}dxdydz$

根据积分区间的对称性且 $z\cdot e^{-(x^2+y^2+z^2)}$ 关于z为奇函数. $|x|e^{-(x^2+y^2+z^2)}$ 关于x为偶

$\therefore \iiint_V z e^{-(x^2+y^2+z^2)}dxdydz = 0$ $\iiint_V |x|e^{-(x^2+y^2+z^2)}dxdydz = 2\iiint_{V'} x e^{-(x^2+y^2+z^2)}dxdydz$ V': $x\ge 0$ $1\le x^2+y^2+z^2\le 4$

$\iiint_{V'} x e^{-(x^2+y^2+z^2)}dxdydz = \iint_{D_1}dydz\int_0^{4-y^2-z^2} x e^{-(x^2+y^2+z^2)}dx - \iint_{D_2}dydz\int_0^{1-x^2-y^2} x\cdot e^{-(x^2+y^2+z^2)}dx$

$= \iint_{D_1}\frac{1}{2}\left[e^{-(y^2+z^2)} - e^{-4}\right]dydz - \iint_{D_2}\frac{1}{2}\left[e^{-(y^2+z^2)} - \frac{1}{e}\right]dydz$ D_1: $y^2+z^2\le 4$ D_2: $y^2+z^2\le 1$

对于 $\iint_{D_1}\frac{1}{2}\left[e^{-(y^2+z^2)} - \frac{1}{e^4}\right]dydz$ 令 $y=r\cos\theta$ $z=r\sin\theta$ D_1': $r\in[0,2]$ $\theta\in[0,2\pi]$ $\left|\frac{\partial(y,z)}{\partial(r,\theta)}\right| = r$

$\iint_{D_1}\frac{1}{2}\left[e^{-(y^2+z^2)} - \frac{1}{e^4}\right]dydz = \iint_{D_1'}\frac{1}{2}\left(e^{-r^2}-e^{-4}\right)r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}d\theta\int_0^2 r\left(e^{-r^2}-e^{-4}\right)dr = \pi\cdot\frac{1}{2}\left(1-\frac{5}{e^4}\right)$

同理可得 $\iint_{D_2} \frac{1}{2}[e^{-(y^2+z^2)} - \frac{1}{e}]dydz = \pi(\frac{1}{2}-\frac{1}{e})$

$\therefore \iiint_V (|x|+z)e^{-(x^2+y^2+z^2)}dxdydz = \pi(\frac{2}{e}-\frac{5}{e^4})$

5(8) $V: (x^2+y^2+z^2)^2 = a^3x$ 令 $x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$ $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\sin\theta$

$\iiint_V dxdydz = 4\iiint_{V'} r^2\sin\theta\, dr\,d\theta\,d\varphi$ $V': r\in[0, a\sqrt[3]{\sin\theta\cos\varphi}]$ $\theta\in[0,\frac{\pi}{2}]$ $\varphi\in[0,\frac{\pi}{2}]$

$\iiint_{V'} r^2\sin\theta\,drd\theta d\varphi = \int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^{a\sqrt[3]{\sin\theta\cos\varphi}} r^2dr = \int_0^{\frac{\pi}{2}}\frac{1}{3}a^3\cos\varphi\,d\varphi\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta = \frac{1}{3}a^3\cdot\frac{\pi}{4}$

$\therefore$ 体积为 $\frac{1}{3}\pi a^3$

6. $f(x,y,z) = x^2+y^2+z^2$ 在 $x^2+y^2+z^2 \le x+y+z$ 内平均值. $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2 \le \frac{3}{4}$

令 $x = \frac{1}{2}+r\sin\theta\cos\varphi$ $y = \frac{1}{2}+r\sin\theta\sin\varphi$ $z = \frac{1}{2}+r\cos\theta$ $V': r\in[0,\frac{\sqrt{3}}{2}]$ $\theta\in[0,\pi]$ $\varphi\in[0,2\pi]$ $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\sin\theta$

$\iiint_V (x^2+y^2+z^2)dxdydz = \iiint_{V'}[\frac{3}{4}+r(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)+r^2]\,r^2\sin\theta\,drd\theta d\varphi$

$\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\int_0^{\frac{\sqrt{3}}{2}}[\frac{3}{4}r^2+r^3(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)+r^4]\,dr$

$= \int_0^{2\pi}d\varphi\int_0^{\pi}\{\frac{3\sqrt{3}}{8}\sin\theta + \frac{9}{64}[\sin^2\theta(\cos\varphi+\sin\varphi)+\sin\theta\cos\theta]\}\,d\theta$

$= \int_0^{2\pi}[\frac{3\sqrt{3}}{4} + \frac{9}{64}\cdot\frac{\pi}{2}(\cos\varphi+\sin\varphi)]\,d\varphi$

$= \frac{3\sqrt{3}}{2}\pi$

平均值为 $\frac{\frac{3\sqrt{3}}{2}\pi}{\frac{4}{3}\pi(\frac{\sqrt{3}}{2})^3} = 3$

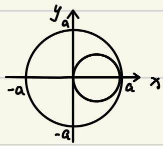
7. $F(t)=\iiint_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)\,dxdydz$

令 $x=r\sin\theta\cos\varphi$ $\quad y=r\sin\theta\sin\varphi$ $\quad z=r\cos\theta$ $\quad r\in[0,t]$ $\quad \theta\in[0,\pi]$ $\quad \varphi\in[0,2\pi]$ $\quad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right|=r^2\sin\theta$

则 $F(t)=\iiint_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)\,dxdydz=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\int_0^t f(r^2)\,r^2dr=4\pi\int_0^t f(r^2)\,r^2dr$

$\therefore F'(t)=4\pi t^2 f(t^2)$

13. $\rho=\sqrt{x^2+y^2}$ 建立如图坐标系 由对称性重心y坐标为0.



则 $x_G=\dfrac{\iint_D x\cdot\sqrt{x^2+y^2}\,dxdy}{\iint_D\sqrt{x^2+y^2}\,dxdy}$

$\iint_D\sqrt{x^2+y^2}\,dxdy=\iint_{x^2+y^2\le a^2}\sqrt{x^2+y^2}\,dxdy-\iint_{(x-\frac{a}{2})^2+y^2\le\frac{a^2}{4}}\sqrt{x^2+y^2}\,dxdy$ 令 $x=r\cos\theta$ $y=r\sin\theta$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$

$\iint_{x^2+y^2\le a^2}\sqrt{x^2+y^2}\,dxdy=\int_0^{2\pi}d\theta\int_0^a r^2dr=\frac{2}{3}\pi a^3$ $\qquad \iint_{(x-\frac{a}{2})^2+y^2\le\frac{a^2}{4}}\sqrt{x^2+y^2}\,dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}r^2dr=\frac{4}{9}a^3$

$\therefore \iint_D\sqrt{x^2+y^2}\,dxdy=\dfrac{6\pi-4}{9}a^3$

$\iint_D x\sqrt{x^2+y^2}\,dxdy=\iint_{x^2+y^2\le a^2}x\cdot\sqrt{x^2+y^2}\,dxdy-\iint_{x^2+y^2\le ax}x\cdot\sqrt{x^2+y^2}\,dxdy$ 令 $x=r\cos\theta$ $y=r\sin\theta$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$

$\iint_{x^2+y^2\le a^2}x\sqrt{x^2+y^2}\,dxdy=\int_0^{2\pi}\cos\theta\,d\theta\int_0^a r^3dr=0$ $\qquad \iint_{x^2+y^2\le ax}x\cdot\sqrt{x^2+y^2}\,dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta\int_0^{a\cos\theta}r^3dr=\frac{4}{15}a^4$

$\therefore\ x_G=\dfrac{-\frac{4}{15}a^4}{\frac{6\pi-4}{9}a^3}=-\dfrac{6}{5(3\pi-2)}a$

$\therefore$ 重心坐标为 $\left(-\dfrac{6}{5(3\pi-2)}a,\ 0\right)$

17. 要使重心恰好在球心，只需圆柱质量等于半球质量（密度均匀分布）

即 $\frac{1}{2}\cdot\frac{4}{3}\pi a^3=h\cdot\pi a^2$ 得 $h=\frac{2}{3}a$

习题 10.4.

1. (1). $\int\cdots\int_{[0,1]^n}(x_1^2+\cdots+x_n^2)\,dx_1\cdots dx_n$

令 $I_n=\int\cdots\int_{[0,1]^n}(x_1^2+\cdots+x_n^2)\,dx_1\cdots dx_n=\int\cdots\int_{[0,1]^{n-1}}\left(\frac{1}{3}+x_2^2+\cdots+x_n^2\right)dx_2\cdots dx_n$

$=\int\cdots\int_{[0,1]^{n-2}}\left(\frac{1}{3}+\frac{1}{3}+x_3^2+\cdots+x_n^2\right)dx_3\cdots dx_n=\cdots$

$=\frac{n}{3}$

(2) $\int\cdots\int_{[0,1]^n}(x_1+\cdots+x_n)^2\,dx_1\cdots dx_n$

$=\frac{1}{3}\int\cdots\int_{[0,1]^{n-1}}\left[(x_1+\cdots+x_{n-1}+1)^3-(x_1+\cdots+x_{n-1})^3\right]dx_1\cdots dx_{n-1}=\frac{1}{3}\int\cdots\int_{[0,1]^{n-1}}\left[3(x_1+\cdots+x_{n-1})^2+3(x_1+\cdots+x_{n-1})+1\right]dx_1\cdots dx_{n-1}$

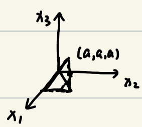
$=\int\cdots\int_{[0,1]^{n-1}}(x_1+\cdots+x_{n-1})^2dx_1\cdots dx_n+\int\cdots\int_{[0,1]^{n-1}}(x_1+\cdots+x_{n-1})\,dx_1\cdots dx_n+\frac{1}{3}$

$\int\cdots\int_{[0,1]^{n-1}}(x_1+\cdots+x_{n-1})\,dx_1\cdots dx_{n-1}=\frac{1}{2}\int\cdots\int_{[0,1]^{n-2}}\left[(x_1+\cdots+x_{n-2}+1)^2-(x_1+\cdots+x_{n-2})^2\right]dx_1\cdots dx_{n-2}=\frac{1}{2}+\int\cdots\int_{[0,1]^{n-2}}(x_1+\cdots+x_{n-2})\,dx_1\cdots dx_{n-2}$

$\therefore\int\cdots\int_{[0,1]^{n-1}}(x_1+\cdots+x_{n-1})\,dx_1\cdots dx_{n-1}=\frac{n-1}{2}$

$\therefore I_n=\frac{1}{6}+\frac{1}{2}\sum_{i=1}^{n}i=\frac{n}{6}+\frac{n(n+1)}{4}=\frac{n(n+3)}{12}$

3. $\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{n-1}}f(x_n)\,dx_n=\frac{1}{(n-1)!}\int_0^a f(t)\,(a-t)^{n-1}dt$



证明: 左边 $=\int_0^a dx_n\int_{x_n}^a dx_{n-2}\cdots\int_{x_2}^a f(x_n)\,dx_1$

$=\int_0^a dx_n\cdots\int_{x_3}^a\cdot f(x_n)\,(a-x_2)\cdot dx_2$

$=\int_0^a dx_n\cdots\int_0^{a-x_4}\frac{1}{2}f(x_n)\cdot(a-x_3)^2dx_3=\cdots$

$=\int_0^a\frac{1}{(n-1)!}f(x_n)\cdot(a-x_n)^{n-1}dx_n=\frac{1}{(n-1)!}\int_0^a f(t)\,(a-t)^{n-1}dt$

$\int_0^a dx_1\int_0^{x_1}dx_2\int_0^{x_2}f(x_3)\,dx_3$

$\int_0^a dx_3\cdot\int\ dx_2\cdot\int\ f(x_3)\,dx_1$

4. $\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1)\cdot f(x_2) \cdots f(x_n)\, dx_n = \frac{1}{n!} \left(\int_0^a f(t)\,dt\right)^n$

$\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1) \cdots f(x_n)\, dx_n = \left[\int_0^a dx_1 \cdots \int_0^{x_{n-1}} f(x_1) dx_n\right] \cdot \cdots \cdot \left[\int_0^a dx_1 \cdots \int_0^{x_{n-1}} f(x_n) dx_n\right]$

令 $I_k = \int_0^a dx_1 \cdots \int_0^{x_{n-1}} f(x_k)\, dx_n = \int_0^a dx_1 \cdots \int_0^{x_{k-1}} \frac{x_k^{n-k}}{(n-k)!} f(x_k)\, dx_k$

$I_k = \int_0^a dx_k \int_{x_k}^a dx_{k-1} \cdots \int_{x_2}^a \frac{x_k^{n-1}}{(n-k)!} f(x_k)\, dx_1 = \int_0^a \frac{x_k^{n-1}}{(n-k)!} f(x_k) \cdot (a-x_k)^{k-1} \cdot \frac{1}{(k-1)!}\, dx_k$

$\therefore\ I_1 \cdot I_2 \cdots I_n = \frac{1}{n!} \left(\int_0^a f(t)\,dt\right)^n$

即 $\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1) \cdots f(x_n)\, dx_n = \frac{1}{n!} \left(\int_0^a f(t)\,dt\right)^n$