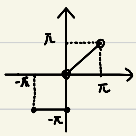
第十二周作业.

习题 12.1

1. (1).



$$f(x) = \begin{cases} -\pi, & -\pi \le x \le 0, \\ x, & 0 < x < \pi; \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right) = \frac{3}{2}\pi$$

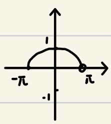
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\cos(nx)\,dx + \int_0^{\pi} x\cos(nx)\,dx\right) = \frac{1}{\pi}\left(\frac{\pi}{n}\sin n\pi + \frac{\pi}{n}\sin n\pi + \frac{(-1)^n-1}{n^2}\right) = \frac{(-1)^n-1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin(nx)\,dx + \int_0^{\pi} x\sin(nx)\,dx\right] = \frac{1}{n}\left(1-(-1)^n\right) + \frac{(-1)^{n+1}}{n} = \frac{1+2(-1)^{n+1}}{n}$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{3}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{\pi n^2}\cdot\cos(nx) + \frac{1+2(-1)^{n+1}}{n}\sin nx\right] = \begin{cases} -\frac{\pi}{2} & x = 2k\pi \\ 0 & x = (2k-1)\pi \\ f(x) & x \ne k\pi \end{cases} \quad k \in \mathbb{Z}$$

∴ Fourier 级数在 $x = 2k\pi$ 收敛到 $-\frac{\pi}{2}$，$x \ne 2k\pi$ 处收敛到 $f(x)$

(2).



$$f(x) = \cos\frac{x}{2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\frac{x}{2}\,dx = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\frac{x}{2}\cos(nx)\,dx = \frac{1}{\pi}\,\frac{1}{2}\int_{-\pi}^{\pi}\left(\cos\frac{2n+1}{2}x + \cos\frac{2n-1}{2}x\right)dx$$

$$= \frac{1}{2\pi}\quad \frac{4}{2n+1}(-1)^n + \frac{4}{2n-1}(-1)^{n-1} = \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n-1}}{2n-1}\right]$$

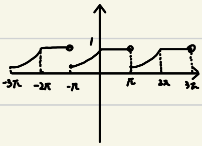
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\frac{x}{2}\sin(nx)\,dx = \frac{1}{\pi}\,\frac{1}{2}\int_{-\pi}^{\pi}\left(\sin\frac{2n+1}{2}x + \sin\frac{2n-1}{2}x\right)dx$$

$$= \frac{1}{2\pi}(0+0) = 0$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n-1}}{2n-1}\right]\cos(nx)$$

∵ $f(x)$ 在 $[-\pi, \pi)$ 上连续可微 ∴ $f(x)$ 的 Fourier 级数在整个数轴收敛到 $f(x)$

(3)



$$f(x)=\begin{cases} e^x, & -\pi\le x\le 0 \\ 1, & 0<x<\pi \end{cases}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}e^x dx+\int_0^{\pi}1\,dx\right]=\frac{1}{\pi}\left(1-e^{-\pi}+\pi\right)=\frac{1+\pi-e^{-\pi}}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}e^x\cos(nx)\,dx+\int_0^{\pi}\cos(nx)\,dx\right]=\frac{1}{\pi}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}+0\right]=\frac{1-(-1)^n e^{-\pi}}{(n^2+1)\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}e^x\sin(nx)\,dx+\int_0^{\pi}\sin nx\,dx\right]=\frac{1}{\pi}\cdot\left\{\frac{n[(-1)^n e^{-\pi}-1]}{n^2+1}+\frac{1-(-1)^n}{n}\right\}=\frac{(-1)^n(n^2e^{-\pi}-n^2-1)+1}{n(n^2+1)\pi}$$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{1+\pi-e^{-\pi}}{2\pi}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{(n^2+1)\pi}\cos(nx)+\frac{(-1)^n(n^2e^{-\pi}-n^2-1)+1}{n(n^2+1)\pi}\sin(nx)\right]$$

∵ $f(x)$ 分段可微 ∴ Fourier 级数收敛到 $\begin{cases}\frac{1+e^{-\pi}}{2} & x=(2k-1)\pi \\ f(x) & x\ne(2k-1)\pi\end{cases}$ $k\in\mathbb{Z}$

3.(1)、$f(x)=2x^2 \quad x\in[0,\pi]$

正弦级数：延拓 $f(x)$ 到 $[-\pi,0)$ 使其成为奇函数.

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx=\frac{2}{\pi}\int_0^{\pi}2x^2\sin(nx)\,dx=\frac{2}{\pi}\left\{\frac{2\pi^2}{n}(-1)^{n+1}+\frac{4}{n^3}[(-1)^n-1]\right\}=\frac{4\pi}{n}(-1)^{n+1}+\frac{8}{n^3\pi}[(-1)^n-1]$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\left\{\frac{4\pi}{n}(-1)^{n+1}+\frac{8}{n^3\pi}[(-1)^n+1]\right\}\sin(nx) \qquad x\in[0,\pi]$$

余弦级数. 延拓 $f(x)$ 到 $[-\pi,0)$ 使其成为偶函数.

$$a_0=\frac{2}{\pi}\int_0^{\pi}2x^2dx=\frac{2}{\pi}\cdot\frac{2}{3}x^3\Big|_0^{\pi}=\frac{4}{3}\pi^2$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}2x^2\cos(nx)\,dx=\frac{2}{\pi}\frac{4\pi}{n^2}(-1)^n=\frac{(-1)^n 8}{n^2}$$

$$\therefore f(x)=\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{8}{n^2}\cos(nx) \qquad x\in[0,\pi]$$

(2). $f(x)=\begin{cases} A, & 0\le x<\frac{l}{2}, \\ 0, & \frac{l}{2}\le x\le l; \end{cases}$

正弦级数：延拓 $f(x)$ 到 $[-l,0)$ 使之成为奇函数

$$b_n=\frac{2}{l}\int_0^{l}f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\cdot\int_0^{\frac{l}{2}}A\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l}0\,dx=\frac{2}{l}\frac{lA}{n\pi}\left(1-\cos\frac{n\pi}{2l}\right)=\frac{2A}{n\pi}\left(1-\cos\frac{n\pi}{2l}\right)$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{2A}{n\pi}\left(1-\cos\frac{n}{2l}\pi\right)\sin\frac{n\pi}{l}x$$

余弦级数：延拓 $f(x)$ 到 $[-l,0)$ 上 使其成为偶函数.

$$a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{l}\int_0^{\frac{l}{2}} A\,dx=\frac{A}{l}$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx=\frac{2}{l}\int_0^{\frac{l}{2}} A\cos\frac{n\pi}{l}x\,dx=\frac{2}{l}\frac{Al}{n\pi}\sin\frac{n\pi}{2l}=\frac{2A}{n\pi}\sin\frac{n}{2l}\pi$$

$$\therefore f(x)=\frac{A}{2l}+\sum_{n=1}^{\infty}\frac{2A}{n\pi}\sin\frac{n}{2l}\pi\cos\frac{n\pi x}{l}$$

(3). $f(x)=\begin{cases}1-\frac{x}{2h} & 0\le x\le 2h\\ 0 & 2h<x\le\pi\end{cases}$

正弦级数：延拓 $f(x)$ 到 $[-l,0)$ 使之成为奇函数

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx=\frac{2}{\pi}\int_0^{2h}(1-\frac{x}{2h})\sin(nx)dx=\frac{2}{\pi}(\frac{1}{n}-\frac{1}{2n^2h}\sin 2nh)$$

$$\therefore f(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\frac{2}{\pi}(\frac{1}{n}-\frac{1}{2n^2h}\sin 2nh)\sin nx$$

余弦级数：延拓 $f(x)$ 到 $[-l,0)$ 上 使其成为偶函数.

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{2h}(1-\frac{x}{2h})dx=\frac{2}{\pi}h$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx=\frac{2}{\pi}\int_0^{2h}(1-\frac{x}{2h})\cos(nx)dx=\frac{2}{\pi}\frac{1}{2n^2h}(1-\cos 2nh)$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{h}{\pi}+\sum_{n=1}^{\infty}\frac{1}{n^2\pi h}(1-\cos 2nh)\cdot\cos nx$$

5. (1). 解: $f(x)=\begin{cases}x, & 0\le x\le\frac{1}{2}.\\ 2-2x, & \frac{1}{2}<x<1.\end{cases}$ $S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ $a_n=2\int_0^1 f(x)\cos n\pi x\,dx$

$\therefore$ 延拓 $f(x)$ 到 $[-1,0)$ 上 使之成为偶函数.

$\therefore S(x)=\begin{cases}f(x) & x\in[0,\frac{1}{2})\cup(\frac{1}{2},1)\\ \frac{3}{4}, & x=\frac{1}{2}\end{cases}$ $S(x)=S(x+2)$ $S(-x)=S(x)$ $x\in[0,1)$

$\therefore S(\frac{9}{4})=S(\frac{1}{4})=f(\frac{1}{4})=\frac{1}{4}$ $S(\frac{5}{2})=S(-\frac{1}{2})=S(\frac{1}{2})=\frac{3}{4}$

(2). $S(x)=\begin{cases} 0, & x=0 \\ \frac{\pi^2}{2}, & x=\pi \\ f(x), & x\in(-\pi,0)\cup(0,\pi) \end{cases}$ $\quad S(x+2\pi)=S(x)$

$S(3\pi)=S(\pi)=\frac{\pi^2}{2}$ $\quad S(4\pi)=S(0)=0$

7. $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ $\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$\bar{a}_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\cos nx\,dx$

令 $t=x+h$ $\quad \bar{a}_n=\frac{1}{\pi}\int_{h-\pi}^{h+\pi} f(t)\cos(nt-nh)\,dt=\frac{1}{\pi}\int_{h-\pi}^{h+\pi} f(t)(\cos nt\cos nh+\sin nt\sin nh)\,dt$

$=\frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(t)\cos nt\cos nh\,dt+\int_{-\pi}^{\pi} f(t)\sin nt\sin nh\,dt\right]$

$=a_n\cos nh+b_n\sin nh.$

$\bar{b}_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\sin nx\,dx$

令 $u=x+h$ $\quad \bar{b}_n=\frac{1}{\pi}\int_{h-\pi}^{h+\pi} f(u)\sin(nu-nh)\,du=\frac{1}{\pi}\int_{h-\pi}^{h+\pi} f(u)(\sin nu\cos nh-\cos nu\sin nh)\,du$

$=\frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(u)\sin nu\cos nh\,du-\int_{-\pi}^{\pi} f(u)\cos nu\sin nh\,du\right]$

$=b_n\cos nh-a_n\sin nh$

8. 解: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(1-x^2)\,dx=2-\frac{2}{3}\pi^2$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(1-x^2)\cos nx\,dx=\frac{1}{\pi}\frac{4\pi}{n^2}(-1)^{n+1}=(-1)^{n+1}\frac{4}{n^2}$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(1-x^2)\sin nx\,dx$

$\because f(x)=(1-x^2)\sin nx$ 为奇函数 $\therefore b_n=0$

$\therefore y=1-x^2=1-\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}\cos nx$

(1) $x=0$ 时 $y=1=1-\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}$ $\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$

(2). $x^2=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx$ $x^2-\frac{1}{3}\pi^2=\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx\,dx$

$\therefore \int_{-\pi}^{\pi}(x^2-\frac{1}{3}\pi^2)^2dx=\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\frac{16}{n^4}\cos^2 nx\,dx$

即 $\frac{8}{45}\pi^5=\sum_{n=1}^{\infty}\frac{8}{n^4}2\pi$ $\therefore \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$

9.解: 将 $f(x)$ 延拓到 $[-\pi,0)$ 使其成为偶函数.

$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}(1+x)\,dx=2+\pi$

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}(1+x)\cos nx\,dx=\frac{2}{\pi}\cdot\frac{(-1)^n-1}{n^2}=\begin{cases}0 & n=2k\\ \frac{-4}{\pi n^2} & n=2k-1\end{cases}$ $k\in N_+$

$\therefore f(x)=1+x=1+\frac{\pi}{2}+\sum_{k=1}^{\infty}\frac{-4}{\pi(2k-1)^2}\cos(2k-1)x$

(1) 令 $x=1$ 得 $\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}=\frac{\pi^2}{8}-\frac{\pi}{4}$

(2). $f(4)=f(2\pi-4)$ 即 $\frac{3\pi}{2}-4=\sum_{n=1}^{\infty}\frac{-4\cos(2k-1)(4-2\pi)}{\pi(2n-1)^2}$

$\therefore \sum_{n=1}^{\infty}\frac{\cos(2n-1)4}{(2n-1)^2}=\pi-\frac{3\pi^2}{8}$