第十四周作业.

习题 12.4.

1.(1). $F(\lambda)=\int_{-\infty}^{+\infty} f(t)\,e^{-i\lambda t}\,dt=\int_0^T kt\,e^{-i\lambda t}\,dt=\frac{ikT}{\lambda}e^{-i\lambda T}+\frac{k}{\lambda^2}e^{-i\lambda T}-\frac{k}{\lambda^2}$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{ikT}{\lambda}e^{-i\lambda T}+\frac{k}{\lambda^2}e^{-i\lambda T}-\frac{k}{\lambda^2}\right)d\lambda=\begin{cases} f(x) & x\neq T \\ \frac{kT}{2} & x=T \end{cases}$$

(3). $F(\lambda)=\int_{-\infty}^{+\infty} f(t)\,e^{-i\lambda t}\,dt=\int_{-\infty}^{+\infty}\frac{1}{a^2+t^2}e^{-i\lambda t}\,dt$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x}\,d\lambda\int_{-\infty}^{+\infty}\frac{1}{a^2+t^2}e^{-i\lambda t}\,dt=f(x)$$

2.(3) $f(x)=\begin{cases}\cos x, & |x|\le\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$

$$a(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\cos(\lambda\xi)\,d\xi=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\xi\cos(\lambda\xi)\,d\xi=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\lambda+1)\xi+\cos(\lambda-1)\xi\,d\xi$$

$$=\frac{1}{2\pi}\left[\frac{2\sin(\lambda+1)\frac{\pi}{2}}{\lambda+1}+\frac{2\sin(\lambda-1)\frac{\pi}{2}}{\lambda-1}\right]=\frac{1}{\pi}\left[\frac{\sin(\lambda+1)\frac{\pi}{2}}{\lambda+1}+\frac{\sin(\lambda-1)\frac{\pi}{2}}{\lambda-1}\right]=\frac{2}{\pi(1-\lambda^2)}\cos\left(\lambda\frac{\pi}{2}\right)$$

$$b(\lambda)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\sin(\lambda\xi)\,d\xi=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\xi\sin(\lambda\xi)\,d\xi=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin(\lambda+1)\xi+\sin(\lambda-1)\xi\,d\xi$$

$$=\frac{1}{2\pi}\,0=0$$

$$\therefore f(x)=\int_0^{+\infty}\frac{2}{\pi}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\cos(\lambda x)\,d\lambda$$

3.(1). 偶延拓. $F(\lambda)=2\int_0^{+\infty} f(t)\cos(\lambda t)\,dt=2\lim_{A\to+\infty}\int_0^A e^{-t}\cos(\lambda t)\,dt$

$$=2\lim_{A\to+\infty}\frac{e^{-t}}{1+\lambda^2}(-\cos\lambda t+\lambda\sin\lambda t)\Big|_0^A$$

$$=\frac{2}{1+\lambda^2}$$

$$\frac{1}{\pi}\int_0^{+\infty}F(\lambda)\cos(\lambda x)\,d\lambda=f(x)$$

(2). 奇延拓. $F(\lambda) = 2\int_0^{+\infty} f(t)\sin\lambda t\,dt = 2\lim_{A\to+\infty}\int_0^A e^{-t}\sin(\lambda t)\,dt$

$$= 2\lim_{A\to+\infty} \frac{e^{-t}}{1+\lambda^2}(-\sin\lambda t - \lambda\cos\lambda t)\Big|_0^A$$

$$= \frac{2\lambda}{1+\lambda^2}$$

$$\frac{1}{\pi}\int_0^{+\infty}\frac{2\lambda}{1+\lambda^2}\sin\lambda x\,d\lambda = \begin{cases} 0, & x=0 \\ e^{-x}, & x\neq 0 \end{cases}$$

4. $f(x) = \begin{cases} 0, & |x|>1 \\ 1, & |x|<1 \end{cases}$ 为偶函数.

$$F(\lambda) = 2\int_0^{+\infty} f(t)\cos\lambda t\,dt = 2\int_0^1 \cos\lambda t\,dt = \frac{2}{\lambda}\sin\lambda$$

$$\frac{1}{\pi}\int_0^{+\infty}\frac{2}{\lambda}\sin\lambda\cos\lambda x\,d\lambda = \begin{cases} 0, & |x|>1 \\ \frac{1}{2}, & |x|=1 \\ 1, & |x|<1 \end{cases}$$

即 $\frac{2}{\pi}\int_0^{+\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha = \begin{cases} 0, & |x|>1 \\ \frac{1}{2}, & |x|=1 \\ 1, & |x|<1 \end{cases}$

$\therefore \int_0^{+\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha = \begin{cases} \frac{\pi}{2}, & |x|<1; \\ \frac{\pi}{4}, & |x|=1; \\ 0, & |x|>1. \end{cases}$

5. $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)e^{i\lambda x}\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lambda e^{-\beta|\lambda|+i\lambda x}\,d\lambda = \frac{1}{2\pi}\left[\int_0^{+\infty}\lambda e^{(-\beta+ix)\lambda}\,d\lambda + \int_{-\infty}^0 \lambda e^{(\beta+ix)\lambda}\,dx\right]$

$$= \frac{1}{2\pi}\left\{\lim_{A\to+\infty}\left[\frac{\lambda}{-\beta+ix} - \frac{1}{(-\beta+ix)^2}\right]e^{(-\beta+ix)\lambda}\Big|_0^A + \lim_{B\to-\infty}\left[\frac{\lambda}{\beta+ix} - \frac{1}{(\beta+ix)^2}\right]e^{(\beta+ix)\lambda}\Big|_B^0\right.$$

$$= \frac{1}{2\pi}\left[\frac{1}{(-\beta+ix)^2} - \frac{1}{(\beta+ix)^2}\right]$$

习题 13.1

1. (2). $\int_0^{+\infty} \sqrt{x}\, e^{-x} dx$

由 $e^{\frac{x}{2}} \geqslant \frac{x}{2}$ 可知 $\sqrt{x}\, e^{-\frac{x}{2}} \leq \frac{2\sqrt{x}}{x}$ 且 $\forall \varepsilon>0$ $\exists N=[\frac{4}{\varepsilon^2}]+1 \in N_+$ $\forall x>N$ $|\frac{2\sqrt{x}}{x}| < \varepsilon$

$\therefore \lim\limits_{x\to+\infty} \sqrt{x} e^{-\frac{x}{2}} = 0 < 1$

$\therefore$ 当 $x>N$ 时 $\sqrt{x} e^{-x} = \sqrt{x} e^{-\frac{x}{2}}\ e^{-\frac{x}{2}} \leq e^{-\frac{x}{2}}$

$\int_0^{+\infty} e^{-\frac{x}{2}} = \lim\limits_{A\to+\infty} \int_0^A e^{-\frac{x}{2}} dx = \lim\limits_{A\to+\infty} (2-2e^{-\frac{A}{2}}) = 2$

$\therefore \int_0^{+\infty} e^{-\frac{x}{2}}$ 收敛. 由比较判别法 $\int_0^{+\infty} \sqrt{x} e^{-x}$ 收敛.

(3). $\int_0^{+\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx$

$\int_0^{+\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx = \int_0^1 \frac{x\cdot \arctan x}{\sqrt[3]{1+x^4}} dx + \int_1^{+\infty} \frac{x\cdot \arctan x}{\sqrt[3]{1+x^4}} dx$

$\because \frac{x\cdot \arctan x}{\sqrt[3]{1+x^4}}$ 在 $[0,1]$ 上为连续函数. $\therefore \int_0^1 \frac{x\cdot\arctan x}{\sqrt[3]{1+x^4}} dx$ 有界.

当 $x\geqslant 1$ 时 $\frac{x \arctan x}{\sqrt[3]{1+x^4}} \geqslant \frac{\pi}{4} \frac{x}{\sqrt[3]{1+x^4}}$

$\forall \varepsilon>0$ $\exists N=N(\varepsilon)$ $\forall x>N$ $\left|\frac{x}{\sqrt[3]{1+x^4}} - \frac{x}{\sqrt[3]{x^4}}\right| < \varepsilon$

$\therefore \int_1^{+\infty} \frac{x}{\sqrt[3]{1+x^4}} = \int_1^N \frac{x}{\sqrt[3]{1+x^4}} dx + \lim\limits_{A\to+\infty} \int_N^A x^{-\frac{1}{3}} dx$

$\lim\limits_{A\to+\infty} \int_N^A x^{-\frac{1}{3}} dx$ 发散

$\therefore \int_0^{+\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}} dx$ 发散.

(5) $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,dx$

当 $x\to 0^+$ 时 $0\le\left|\frac{\ln x}{\sqrt{1-x^2}}\right|\le 2|\ln x|$

$$\int_0^1 2|\ln x|\,dx = -2\int_0^1 \ln x\,dx = -2\left(x\ln x\Big|_0^1 - \int_0^1 dx\right) = 2$$

$\therefore \int_0^1 2|\ln x|\,dx$ 收敛.

$\because \lim\limits_{x\to 1}\frac{\ln x}{\sqrt{1-x^2}} = \lim\limits_{x\to 1}\frac{\frac{1}{x}}{-\frac{x}{\sqrt{1-x^2}}} = 0$ $\quad\therefore \int_0^1\left|\frac{\ln x}{\sqrt{1-x^2}}\right|dx$ 收敛.

$\therefore \int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,dx$ 收敛.

(8). $\int_0^1 \frac{\sqrt{x}}{e^{\sin x}-1}\,dx$

$$\frac{\sqrt{x}}{e^{\sin x}-1} \le \frac{\sqrt{x}}{\sin x} \le \frac{\sqrt{x}}{x-\frac{1}{3}x^3} = \frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}}$$

$\int_0^1 \frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}} = \int_0^{\varepsilon}\frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}} + \int_{\varepsilon}^1 \frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}}$ $\quad \varepsilon>0$ 且 $\varepsilon\to 0^+$

$\int_0^{\varepsilon}\frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}} = \int_0^{\varepsilon}\frac{1}{\sqrt{x}} = 2\sqrt{\varepsilon}$ $\quad \int_{\varepsilon}^1 \frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}}$ 由一致连续可知收敛.

$\therefore \int_0^1 \frac{1}{\sqrt{x}-\frac{1}{3}x^{\frac{5}{2}}}$ 收敛.

$\therefore \int_0^1 \frac{\sqrt{x}}{e^{\sin x}-1}\,dx$ 收敛.

(10) $\int_0^{\frac{\pi}{2}} \frac{\ln\sin x}{\sqrt{x}}\,dx$

$$= 2\int_0^{\frac{\pi}{2}} \ln\sin x\, d(\sqrt{x}) = 2\cdot\left(\sqrt{x}\ln\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sqrt{x}\cdot\frac{\cos x}{\sin x}\,dx\right)$$

$\lim\limits_{x\to 0}\sqrt{x}\ln\sin x = 0$ $\quad\therefore \int_0^{\frac{\pi}{2}}\frac{\ln\sin x}{\sqrt{x}}\,dx = -2\int_0^{\frac{\pi}{2}}\frac{\sqrt{x}}{\tan x}\,dx$

$x\in(0,\frac{\pi}{2})$ 时 $\frac{\sqrt{x}}{\tan x} < \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$ $\quad \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{x}}\,dx = \sqrt{2\pi}$ $\quad\therefore \int_0^{\frac{\pi}{2}}\frac{\sqrt{x}}{\tan x}\,dx$ 收敛.

$\therefore \int_0^{\frac{\pi}{2}}\frac{\ln\sin x}{\sqrt{x}}\,dx$ 收敛.

(11). $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}\cos x}$

$x\in(0,\frac{\pi}{2})$ 时 $\frac{1}{\sqrt{\sin x}\cos x} \geqslant \frac{1}{\cos x}$

$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}dx = \ln\frac{1+\sin x}{\cos x}\Big|_0^{\frac{\pi}{2}}$

$\lim\limits_{x\to\frac{\pi}{2}^-} \ln\frac{1+\sin x}{\cos x} = +\infty$

$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}\cos x}$ 发散

2.(2). $\int_0^{+\infty} \frac{\sin x}{\sqrt[3]{x^2+x+1}}dx$

① $F(b)=\int_0^b \sin x\,dx = 1-\cos b \in[0,2]$ 即 $F(b)$有界

$g(x)=\frac{1}{\sqrt[3]{x^2+x+1}} \geqslant 0$ 且单调递减. $\lim\limits_{x\to+\infty} g(x)=0$

$\therefore \int_0^{+\infty} f(x)g(x)dx = \int_0^{+\infty}\frac{\sin x}{\sqrt[3]{x^2+x+1}}dx$ 收敛.

②. $\frac{|\sin x|}{\sqrt[3]{x^2+x+1}} \geqslant \frac{\sin^2 x}{\sqrt[3]{x^2+x+1}}$

$\exists\, x_0=3 \quad \forall x\geqslant x_0 \quad \sqrt[3]{x^2+x+1} < x$

$\therefore \int_0^{+\infty}\frac{|\sin x|}{\sqrt[3]{x^2+x+1}}dx \geqslant \int_0^3 \frac{\sin^2 x}{\sqrt[3]{x^2+x+1}}dx + \int_3^{+\infty}\frac{\sin^2 x}{x}dx$

$\int_0^3 \frac{\sin^2 x}{\sqrt[3]{x^2+x+1}}dx$收敛 $\qquad \int_3^{+\infty}\frac{\sin^2 x}{x} = \frac{1}{2}\int_3^{+\infty}\left(\frac{1}{x}-\frac{\cos 2x}{x}\right)dx$

$\int_3^{+\infty}\frac{1}{x}dx$ 发散到 $+\infty$. 由 Dirichlet 定理 $\int_3^{+\infty}\frac{\cos 2x}{x}$ 收敛.

$\therefore \int_0^{+\infty}\frac{|\sin x|}{\sqrt[3]{x^2+x+1}}dx$ 发散.

即 $\int_0^{+\infty}\frac{\sin x}{\sqrt[3]{x^2+x+1}}dx$ 条件收敛.

(3). $\int_2^{+\infty} \frac{\sin x}{x\ln x}\,dx$

① $F(b) = \int_2^b \sin x\,dx = \cos b - \cos 2$ 有界.

$g(x) = \frac{1}{x\ln x}$ 单调递减 $\lim_{x\to+\infty} g(x) = 0$

$\therefore \int_2^{+\infty} \frac{\sin x}{x\ln x}\,dx$ 收敛.

②. $\left|\frac{\sin x}{x\ln x}\right| \geqslant \frac{\sin^2 x}{x\ln x} = \frac{1}{2}\,\frac{1-\cos 2x}{x\ln x}$

由 Dirichlet 定理 $\int_2^{+\infty} \frac{\cos 2x}{x\ln x}$ 收敛.

$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \lim_{A\to+\infty} \ln(\ln A) - \ln(\ln 2) \to +\infty$

$\therefore \int_2^{+\infty} \left|\frac{\sin x}{x\ln x}\right| dx$ 发散.

$\therefore \int_2^{+\infty} \frac{\sin x}{x\ln x}\,dx$ 条件收敛.

(5). $\int_0^{+\infty} \frac{\sin x}{x(1+\sqrt{x})}\,dx$

① $F(b) = \int_0^b \sin x\,dx = 1-\cos b$ 有界.

$g(x) = \frac{1}{x(1+\sqrt{x})}$ 单调递减 $x\in(0,+\infty)$ 且 $\lim_{x\to+\infty} g(x) = 0$

$\therefore \int_0^{+\infty} \frac{\sin x}{x(1+\sqrt{x})}\,dx$ 收敛.

②. $\left|\frac{\sin x}{x(1+\sqrt{x})}\right| \leqslant \frac{1}{x(1+\sqrt{x})}$ $\quad \forall \varepsilon>0 \ \exists N=N(\varepsilon)>0 \ \forall x>N \ \left|\frac{1}{x(1+\sqrt{x})} - x^{-\frac{3}{2}}\right| < \varepsilon$

$\int_0^{+\infty} \frac{|\sin x|}{x(1+\sqrt{x})}\,dx \leqslant \int_0^N \frac{1}{1+\sqrt{x}}\,dx + \int_N^{+\infty} x^{-\frac{3}{2}}\,dx$

$\int_0^N \frac{1}{1+\sqrt{x}}\,dx$ 收敛. $\quad \int_N^{+\infty} x^{-\frac{3}{2}}\,dx = \lim_{A\to+\infty}\left(\frac{2}{\sqrt{N}} - \frac{1}{\sqrt{A}}\right) = \frac{2}{\sqrt{N}}$ 收敛 $\quad \therefore \int_0^{+\infty} \frac{|\sin x|}{x(1+\sqrt{x})}\,dx$ 收敛

$\therefore \int_0^{+\infty} \frac{\sin x}{x(1+\sqrt{x})}\,dx$ 绝对收敛.