第十周周四作业.

习题 11.3

4 (1). $\oint_L (x+y)^2 dx + (x^2-y^2)dy$　　L: 是顶点为 $A(1,1)$　$B(3,3)$　$C(3,5)$ 的三角形周界沿逆时针.

解: $\oint_L (x+y)^2 dx + (x^2-y^2)dy = \iint_D [2x-2(x+y)]\,dxdy = \int_1^3 dx \int_x^{2x-1} -2y\,dy = \int_1^3 [x^2-(2x-1)^2]dx$

$= \int_1^3 (-3x^2+4x-1)dx = -12$

(2). $\oint_L (xy+x+y)dx + (xy+x-y)dy$　　L: 是椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 沿顺时针方向.

解: $\oint_L (xy+x+y)dx + (xy+x-y)dy = -\iint_D [(y+1)-(x+1)]\,dxdy = -\iint_D (y-x)\,dxdy$

令 $x=ka\cos\theta$　$y=kb\sin\theta$　$\left|\frac{\partial(x,y)}{\partial(k,\theta)}\right| = kab$　　$k\in[0,1]$　$\theta\in[0,2\pi]$

$\iint_D (y-x)dxdy = \iint_{D'} k(b\sin\theta - a\cos\theta)\cdot kab\,dkd\theta = ab\int_0^1 k^2dk \int_0^{2\pi}(b\sin\theta - a\cos\theta)\,d\theta = 0$

$\therefore \oint_L (xy+x+y)dx + (xy+x-y)dy = 0$

(3). $\oint_L (yx^3+e^y)dx + (xy^3+xe^y-2y)dy$　L是关于两坐标轴对称的闭曲线.

解: $\oint_L (yx^3+e^y)dx + (xy^3+xe^y-2y)dy = \iint_D [(y^3+e^y)-(x^3+e^y)]\,dxdy = \iint_D (y^3-x^3)\,dxdy$

D为关于两坐标轴对称的封闭区域. 则第一、三象限积分加和为0　第二、四象限积分加和为0.

$\therefore \oint_L (yx^3+e^y)dx + (xy^3+xe^y-2y)dy = 0$

(5). $\int_{AMB} (x^2+2xy-y^2)dx + (x^2-2xy+y^2)dy$

$\vec{v} = (x^2+2xy-y^2,\ x^2-2xy+y^2)$　$\exists \varphi = \frac{1}{3}x^3 + x^2y - xy^2 + \frac{1}{3}y^3$　满足 $\vec{v} = \nabla\varphi(x,y)$

$\therefore \int_{AMB} \vec{v}\cdot d\vec{r} = \varphi(B) - \varphi(A) = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

(6). $\int_{AMO} (e^x\sin y - my)dx + (e^x\cos y - m)dy$　　AMO 由 $A(a,0)$ 到 $O(0,0)$ 的上半圆周 $x^2+y^2=ax$ $(a>0)$.

解. 原式 $= \oint_{AMOA} (e^x\sin y - my)dx + (e^x\cos y - m)dy - \int_0^a (e^x\sin y - my)\,dx$

$= \iint_D [e^x\cos y - (e^x\cos y - m)]\,dxdy = \iint_D m\,dxdy$　令 $x=r\cos\theta$　$y=r\sin\theta$　$\theta\in[0,\frac{\pi}{2}]$　$r\in[0,\ a\cos\theta]$　$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

$\iint_D m\,dxdy = \int_0^{\frac{\pi}{2}} m\,d\theta \int_0^{a\cos\theta} r\,dr = \int_0^{\frac{\pi}{2}} m\,\frac{1}{2}a^2\cos^2\theta\,d\theta = \frac{1}{2}ma^2\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = \frac{\pi}{8}ma^2$

$\therefore \int_{AMO}(e^x\sin y - my)\,dx + (e^x\cos y - m)\,dy = \frac{\pi}{8}ma^2$

5. (1) $S = \oint_L x\,dy = \int_0^{2\pi} 3a\cos^3 t\cdot a\sin^2 t\cos t\,dt = 3a^2\int_0^{2\pi}\cos^4 t\,\sin^2 t\,dt = 3a^2\int_0^{2\pi}\frac{1}{16}(1+\cos 2t)(1-\cos 4t)\,dt$

$= \frac{3a^2}{16}\int_0^{2\pi}\left(1+\cos 2t-\cos 4t-\frac{1}{2}\cos 2t-\frac{1}{2}\cos 6t\right)dt$

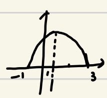
$= \frac{3\pi}{8}a^2$

(2) $S = \oint_L(-y\,dx) = -\int_0^{2\pi} -a(1-\cos t)\cdot a(1-\cos t)\,dt = a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)\,dt = 3\pi a^2$

6. (1). 令 $x=a\cos\theta$ $y=a\sin\theta$ $\theta\in[0,\pi]$

则 $\int_L \frac{-ydx+xdy}{x^2+y^2} = -\int_0^{\pi}\frac{a^2\sin^2\theta+a^2\cos^2\theta}{a^2}d\theta = -\pi$

(2). $\int_L \frac{-ydx+xdy}{x^2+y^2} = \oint_{L'}\frac{-ydx+xdy}{x^2+y^2}$



L'为L加上从$B(3,0)$沿x轴负方向到$A(-1,0)$

$P=\frac{-y}{x^2+y^2}$ $Q=\frac{x}{x^2+y^2}$ $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$

在原点附近挖去以原点为圆心 r 为半径的小圆 余下为 $\bar{D}$

$\int_{\partial\bar{D}} P\,dx+Q\,dy = \iint_{\bar{D}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = 0$

令 $x=r\cos t$ $y=r\sin t$ $t\in[0,\pi]$ $\int_0^{\pi}\frac{-ydx+xdy}{x^2+y^2} = \int_0^{\pi}dt = \pi$

$\therefore \int_L \frac{-ydx+xdy}{x^2+y^2} = -\pi$

7. (1). 证明: 令 $\vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j}$ $\vec{\tau} = -\cos\beta\,\vec{i} + \cos\alpha\,\vec{j}$ $\vec{\tau}\,ds = d\vec{r}$ ∴ $dx = -\cos\beta\,ds$ $dy = \cos\alpha\,ds$

∴ $\frac{\partial f}{\partial n} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$

∴ $\oint_L \frac{\partial f}{\partial n}ds = \oint_L \frac{\partial f}{\partial x}\cos\alpha\,ds + \frac{\partial f}{\partial y}\cos\beta\,ds = \oint_L -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy$

由格林公式 $\oint_L -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx\,dy = \iint_D \Delta f\,dx\,dy$

∴ $\oint_L \frac{\partial f}{\partial n}ds = \iint_D \Delta f\,dx\,dy$

(2). 令 $\vec{n} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$ ∵ $\vec{a}$ 为单位常值向量 ∴ 设 $\vec{a} = \cos\theta_0\,\vec{i} + \sin\theta_0\,\vec{j}$

$\cos(\vec{a},\vec{n}) = \frac{\vec{a}\cdot\vec{n}}{|\vec{a}|\cdot|\vec{n}|} = \cos\theta_0\cos\alpha + \sin\theta_0\sin\alpha$

$\vec{\tau} = -\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j}$ $\vec{\tau}\,ds = d\vec{r}$

∴ $\oint_L \cos(\vec{a},\vec{n})ds = \oint_L (-\sin\theta_0\,\vec{i} + \cos\theta_0\,\vec{j})\cdot\vec{\tau}\,ds = \oint_L (-\sin\theta_0\,\vec{i} + \cos\theta_0\,\vec{j})\,d\vec{r}$

$= \oint_L -\sin\theta_0\,dx + \cos\theta_0\,dy = 0$

∴ $\oint_L \cos(\vec{a},\vec{n})ds = 0$

(3). $\oint_L v\frac{\partial u}{\partial n}ds = \oint_L v\,\nabla u\cdot\vec{n}\,ds$

$= \oint_L v\left(\frac{\partial u}{\partial x}\sin\alpha - \frac{\partial u}{\partial y}\cos\alpha\right)ds$

$= \oint_L v\left(-\frac{\partial u}{\partial y}\vec{i} + \frac{\partial u}{\partial x}\vec{j}\right)\cdot\vec{\tau}\,d\vec{s}$

$= \oint_L v\left(-\frac{\partial u}{\partial y}\vec{i} + \frac{\partial u}{\partial x}\vec{j}\right)\cdot d\vec{r}$

$= \iint_D \left[\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-v\frac{\partial u}{\partial y}\right)\right]dx\,dy$

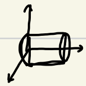
$= \iint_D (v\Delta u + \nabla u\cdot\nabla v)\,dx\,dy$

同理 $\oint_L u\frac{\partial v}{\partial n} = \iint_D (u\Delta v + \nabla u\cdot\nabla v)\,dx\,dy$

∴ $\oint_L \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) = \iint_D (v\Delta u - u\Delta v)\,dx\,dy$

第十周周六作业

习题 11.4



1.(2) $\iint_S xyz\, dxdy$

令 $x=R\cos\theta \quad z=R\sin\theta \quad \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\therefore \iint_S xyz\, dxdy = \iint_{S'} R^2\sin\theta\cos\theta\; y\; R\cdot(-\sin\theta)\, d\theta\, dy$

$= -R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d(\sin\theta)\int_0^h y\, dy$

$= -\frac{1}{3}h^2R^3$

(5) $\iint_S x^2dydz + y^2dxdz + z^2dxdy$

$\iint_S x^2dydz = \iint_S (1-y-z)^2dydz = \int_0^1 dy\int_0^{1-y}(1-y-z)^2dz = \int_0^1 \frac{1}{3}(1-y)^3dy$

$= \frac{1}{12}$

$\iint_S y^2dxdz = \iint_S (1-x-z)^2dxdz = \frac{1}{12}$

$\iint_S z^2dxdy = \iint_S (1-x-y)\, dxdy = \frac{1}{12}$

$\therefore \iint x^2dydz + y^2dxdz + z^2dxdy = \frac{1}{4}$

(6) $\iint_S (y-z)\, dydz + (z-x)\, dzdx + (x-y)\, dxdy$

令 $x = z\cos\theta \quad y = -z\sin\theta \quad z=z \quad \theta\in[0,2\pi] \quad z\in[0,1]$

$\therefore \iint_S (y-z)\, dydz = \iint_{S'} -z(1+\sin\theta)\frac{\partial(y,z)}{\partial(z,\theta)}\, dz\, d\theta$

$= \iint_{S'} z^2(1+\sin\theta)\cos\theta\, dz\, d\theta$

$= \int_0^1 z^2dz\int_0^{2\pi}(1+\sin\theta)\cos\theta\, d\theta = 0$ $\qquad -z\sin\theta$ -

$\iint_S (z-x)\, dzdx = \iint_{S'} z(1-\cos\theta)\frac{\partial(z,x)}{\partial(z,\theta)}\, dz d\theta = \iint_{S'} -z^2(1-\cos\theta)\sin\theta\, dz d\theta$

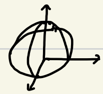
$= \int_0^1 z^2dz\int_0^{2\pi} -(1-\cos\theta)\sin\theta\, d\theta = 0$

$$\iint_S (x-y)\,dx\,dy = \iint_{S'} z(\cos\theta+\sin\theta)\,\frac{\partial(x,y)}{\partial(z,\theta)}\,dz\,d\theta = \iint_{S'} -z^2(\cos\theta+\sin\theta)\,dz\,d\theta$$

$$= \int_0^1 -z^2\,dz \int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta = 0$$

$\therefore \iint_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy = 0.$

(7). $\iint_S xz^2\,dy\,dz + x^2y\,dz\,dx + y^2z\,dx\,dy$



令 $x = a\sin\theta\cos\varphi \quad y = a\sin\theta\sin\varphi \quad z = a\cos\theta \quad \theta\in[0, \frac{\pi}{2}] \quad \varphi\in[0, 2\pi]$

$\vec{r} = (a\sin\theta\cos\varphi,\ a\sin\theta\sin\varphi,\ a\cos\theta) \qquad \vec{r}_\theta = (a\cos\theta\cos\varphi,\ a\cos\theta\sin\varphi,\ -a\sin\theta)$

$\vec{r}_\varphi = (-a\sin\theta\sin\varphi,\ a\sin\theta\cos\varphi,\ 0) \qquad \vec{r}_\theta\times\vec{r}_\varphi = (a^2\sin^2\theta\cos\varphi,\ a^2\sin^2\theta\sin\varphi,\ a^2\sin\theta\cos\theta)$

$\vec{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$ $\therefore$ (θ,φ) 为正向参数.

$$\iint_S xz^2\,dy\,dz = \iint_{S'} a\sin\theta\cos\varphi\cdot a^2\cos^2\theta\,\frac{\partial(y,z)}{\partial(\theta,\varphi)}\,d\theta\,d\varphi = \iint_{S'} a^3\sin\theta\cos^2\theta\cos\varphi\cdot a^2\sin^2\theta\cos\varphi\,d\theta\,d\varphi$$

$$= a^5\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta(1-\cos^2\theta)\,d\theta \int_0^{2\pi}\cos^2\varphi\,d\varphi$$

$$= a^5\cdot\frac{2}{15}\pi = \frac{2}{15}\pi a^5$$

$$\iint_S x^2y\,dz\,dx = \iint_{S'} a^3\sin^3\theta\sin\varphi\cos^2\varphi\,\frac{\partial(z,x)}{\partial(\theta,\varphi)}\,d\theta\,d\varphi = \iint_{S'} a^3\sin^3\theta\sin\varphi\cos^2\varphi\cdot a^2\sin^2\theta\sin\varphi\,d\theta\,d\varphi$$

$$= a^5\int_0^{\frac{\pi}{2}} \sin\theta(1-2\cos^2\theta+\cos^4\theta)\,d\theta \int_0^{2\pi}\frac{1}{4}\sin^2 2\varphi\,d\varphi$$

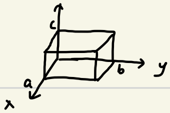
$$= a^5\,\frac{8}{15}\cdot\frac{1}{4}\cdot\pi = \frac{2\pi}{15}a^5$$

$$\iint_S y^2z\,dx\,dy = \iint_{S'} a^3\sin^2\theta\cos\theta\sin^2\varphi\,\frac{\partial(x,y)}{\partial(\theta,\varphi)}\,d\theta\,d\varphi = \iint_{S'} a^3\sin^2\theta\cos\theta\sin^2\varphi\cdot a^2\sin\theta\cos\theta\,d\theta\,d\varphi$$

$$= a^5\int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta \int_0^{2\pi}\sin^2\varphi\,d\varphi$$

$$= \frac{2\pi}{15}a^5$$

$\therefore \iint_S xz^2\,dy\,dz + x^2y\,dz\,dx + y^2z\,dx\,dy = \frac{2\pi}{5}a^5$



(8). $\iint_S f(x)\,dydz + g(y)\,dzdx + h(z)\,dxdy$

$\iint_S f(x)\,dydz = f(a)\int_0^b dy \cdot \int_0^c dz - f(0)\int_0^b dy \int_0^c dz = (f(a)-f(0))\,bc$

$\iint_S g(y)\,dzdx = g(b)\int_0^a dx \int_0^c dz - g(0)\int_0^a dx \int_0^c dz = (g(b)-g(0))\,ac$

$\iint_S h(z)\,dxdy = h(c)\int_0^a dx \int_0^b dy - h(0)\int_0^a dx \int_0^b dy = (h(c)-h(0))\,ab$

$\therefore \iint_S f(x)\,dydz + g(y)\,dzdx + h(z)\,dxdy = bc\,(f(a)-f(0)) + ac\,(g(b)-g(0)) + ab\,(h(c)-h(0))$

2. 解: $\vec{v} = (x^3 - yz,\ -2x^2y,\ z)$

$\Phi = \iint_S \vec{v}\cdot d\vec{S} = (\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6})\,\vec{v}\cdot d\vec{S}$

$\iint_{S_1} \vec{v}\cdot d\vec{S} = \iint_{S_1} yz\,dydz = \int_0^b y\,dy \int_0^c z\,dz = \frac{1}{4}b^2c^2$

$\iint_{S_2} \vec{v}\cdot d\vec{S} = \iint_{S_2} (a^3-yz)\,dydz = \int_0^c dz \int_0^b (a^3-yz)\,dy = a^3bc - \frac{1}{4}b^2c^2$

$\iint_{S_3} \vec{v}\cdot d\vec{S} = \iint_{S_3} 0\,dzdx = 0$

$\iint_{S_4} \vec{v}\cdot d\vec{S} = \iint_{S_4} -2bx^2\,dzdx = \int_0^a -2bx^2\,dx \int_0^c dz = -\frac{2}{3}a^3bc$

$\iint_{S_5} \vec{v}\cdot d\vec{S} = \iint_{S_5} 0\,dxdy = 0$

$\iint_{S_6} \vec{v}\cdot d\vec{S} = \iint_{S_6} c\,dxdy = abc$

$\therefore \Phi = \frac{1}{3}a^3bc + abc$

第十一章综合习题.

1. $I=\int_L z\,ds$ $\quad L:\ x^2+y^2=z^2 \quad y^2=ax \quad (a>0)$

令 $x=at^2$ $\quad y=at$ $\quad$ 则 $z=at\sqrt{t^2+1}$ $\quad t\in[0,1]$

$x'(t)=2at \quad y'(t)=a \quad z'(t)=a\sqrt{t^2+1}+\frac{at^2}{\sqrt{t^2+1}}=a\frac{2t^2+1}{\sqrt{t^2+1}}$

$I=\int_0^1 at\sqrt{t^2+1}\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,dt=\int_0^1 a^2t\cdot\sqrt{8t^4+9t^2+2}\,dt$ $\quad$ 令 $x=t^2$ $\quad dx=2tdt$ $\quad$ 则 $I=\frac{a^2}{2}\int_0^1\sqrt{8x^2+9x+2}\,dx$

$=\frac{a^2}{2}\int_0^1\sqrt{(2\sqrt2x+\frac{9}{4\sqrt2})^2-\frac{17}{32}}\,dx$ $\quad$ 令 $u=2\sqrt2x+\frac{9}{4\sqrt2}$ $\quad u\in[\frac{9}{4\sqrt2},\frac{25}{4\sqrt2}]$ $\quad du=2\sqrt2dx$

$I=\frac{a^2}{4\sqrt2}\int_{\frac{9}{4\sqrt2}}^{\frac{25}{4\sqrt2}}\sqrt{u^2-\frac{17}{32}}\,du=\frac{a^2}{4\sqrt2}\left\{\frac12u\sqrt{u^2-\frac{17}{32}}-\frac{17}{64}\left[\ln(u+\sqrt{u^2-\frac{17}{32}})-\frac12\ln\frac{17}{32}\right]\right\}\Big|_{\frac{9}{4\sqrt2}}^{\frac{25}{4\sqrt2}}$

$=\frac{a^2}{4\sqrt2}\left(\frac{25\sqrt{19}-9\sqrt2}{8\sqrt2}+\frac{17}{64}\ln(25-4\sqrt{38})\right)$

2. $(\frac{x}{a})^{2n+1}+(\frac{y}{b})^{2n+1}=c(\frac{x}{a})^n(\frac{y}{b})^n$

解：令 $x=ar(\cos\theta)^{\frac{2}{2n+1}}$ $\quad y=br(\sin\theta)^{\frac{2}{2n+1}}$

$r^{2n+1}=c\cdot r^{2n}\cdot(\frac12\sin2\theta)^{\frac{2n}{2n+1}}$ $\quad r=c\cdot(\frac12\sin2\theta)^{\frac{2n}{2n+1}}$ $\quad \theta\in[0,\frac{\pi}{2}]$ $\quad r\in[0,\ c(\frac12\sin2\theta)^{\frac{2n}{2n+1}}]$

$S=2\iint_S dxdy=2\iint_S\frac{\partial(x,y)}{\partial(r,\theta)}drd\theta=2\iint_S\frac{2}{2n+1}abr\cdot(\sin\theta\cos\theta)^{\frac{2}{2n+1}-1}drd\theta$

$=2\int_0^{\frac{\pi}{2}}\frac{1}{2n+1}ab(\sin\theta\cos\theta)^{\frac{2}{2n+1}-1}d\theta\int_0^{c(\frac12\sin2\theta)^{\frac{2n}{2n+1}}}2rdr$

$=2\int_0^{\frac{\pi}{2}}\frac{abc^2}{2n+1}(\sin\theta\cos\theta)d\theta$

$=\frac{abc^2}{2n+1}$

$\frac{1}{\sin\cos\theta}$