第十三周作业

习题 12.2

1. $f(x)=\begin{cases}1, & |x|<a\\ 0, & a\le |x|<\pi\end{cases}$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-a}^{a}1\cdot dx=\frac{2a}{\pi}$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx=\frac{1}{\pi}\int_{-a}^{a}\cos nx dx=\frac{2\sin na}{n\pi}$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx=\frac{1}{\pi}\int_{-a}^{a}\sin nx dx=0$

$\therefore \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{a}{\pi}+\sum_{n=1}^{\infty}\frac{2\sin na}{n\pi}\cos nx$

$\because$ $f(x)$ 在 $[-\pi,\pi]$ 上可积且平方可积

由 Parseval 等式： $\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx$

(1) 得 $\frac{2a^2}{\pi^2}+\sum_{k=1}^{\infty}\frac{4\sin^2 ka}{k^2\pi^2}=\frac{1}{\pi}2a$ $\quad\therefore \sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}=\frac{a}{2}(\pi-a)$

(2). $\frac{2a^2}{\pi^2}+\sum_{k=1}^{\infty}\frac{4(1-\cos^2 ka)}{k^2\pi^2}=\frac{2a}{\pi}$ $\quad\therefore \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{\cos^2 na}{n^2}\right)=\frac{2a}{\pi}-\frac{2a^2}{\pi^2}$

即 $\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2}=\frac{a}{2}(\pi-a)$

$\therefore \sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2}=\frac{\pi^2}{6}-\frac{a}{2}(\pi-a)$

2. 证明：要证 $\sum_{n=1}^{\infty}\frac{a_n}{n}$ 收敛. 令 $S_n=\sum_{k=1}^{n}\frac{a_k}{k}$

只需 $\forall\varepsilon>0$ $\forall p\in N_+$ $\exists N\in N_+$ $\forall n>N$ $|S_{n+p}-S_n|<\varepsilon$

$\because$ $f(x)$ 在 $[-\pi,\pi]$ 上可积且平方可积 且 a_n. b_n 是 Fourier 级数的系数

$\therefore \lim_{n\to\infty}a_n=0 \quad \lim_{n\to\infty}b_n=0$

即 $\forall\frac{\varepsilon}{p}>0$ $\exists N_1\in N_+$ $\forall n>N_1$ $|a_n|<\frac{\varepsilon}{p}$

令 $N_2=\max\{N,N_1\}$ 则 $|S_{n+p}-S_n|=\left|\frac{a_{n+1}}{n+1}+\frac{a_{n+2}}{n+2}+\cdots+\frac{a_{n+p}}{n+p}\right|<\frac{|a_{n+1}+\cdots+a_{n+p}|}{n}\le\frac{|a_{n+1}|+\cdots+|a_{n+p}|}{n}<\frac{p\cdot\frac{\varepsilon}{p}}{n}=\frac{\varepsilon}{n}\le\varepsilon$

$\therefore \sum_{n=1}^{\infty}\frac{a_n}{n}$ 收敛. 同理 $\sum_{n=1}^{\infty}\frac{b_n}{n}$ 收敛.

3. 解: $f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0\le x\le\pi.\end{cases}$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-1\,dx+\int_0^{\pi}1\,dx\right)=0$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-\cos nx\,dx+\int_0^{\pi}\cos nx\,dx\right)=0$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-\sin nx\,dx+\int_0^{\pi}\sin nx\,dx\right)=\frac{2}{n\pi}[1-(-1)^n]=\begin{cases}\frac{4}{n\pi}, & n=2k-1\\ 0, & n=2k\end{cases}\quad k\in N_+$

$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin(2k-1)x$

$\because f(x)\in L^2[-\pi,\pi]$

$\therefore \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx$

即 $\sum_{n=1}^{\infty}\frac{16}{(2n-1)^2\pi^2}=\frac{1}{\pi}2\pi$ $\quad\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

$x\in[0,\pi]$ 则 $\int_x^{\frac{\pi}{2}}f(t)\,dt=\sum_{n=1}^{\infty}\int_x^{\frac{\pi}{2}}\frac{4}{(2n-1)\pi}\sin(2n-1)t\,dt$

得 $\frac{\pi}{2}-x=\sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi}\cos(2n-1)x$

$\therefore \sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}=-\frac{\pi}{4}x+\frac{\pi^2}{8}$

即 $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$ $\quad\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}=-\frac{\pi}{4}x+\frac{\pi^2}{8}$

5. 解 $a_n=\int_0^{l}a\left(1-\frac{x}{l}\right)\sin\frac{n\pi}{l}x\,dx=\frac{al}{n\pi}$

$\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{l}x=\sum_{n=1}^{\infty}\frac{al}{n\pi}\sin\frac{n\pi}{l}x$

习题 12.3

2 (3) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\,dx=2$ $\quad a_1=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos x\,dx=-\frac{1}{2}$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x\cdot[\sin(n+1)x-\sin(n-1)x]\,dx=\frac{(-1)^{n-1}2}{n^2-1}$ $\quad(n\geq 2, n\in N_+)$

$b_1=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\sin x\,dx=0$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\sin nx\,dx=-\frac{1}{2\pi}\int_{-\pi}^{\pi}x\,[\cos(n+1)x-\cos(n-1)x]\,dx=0$

$\therefore \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}2}{n^2-1}$

7. 解. $x=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(nx)}{n}$

两侧对 x 积分 $\frac{1}{2}x^2=2\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2}+\frac{\pi^2}{6}$

$\therefore x^2$ 在 $(-\pi,\pi)$ 上 Fourier 展开式为 $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2}$ $\quad x=\pi$ 得 $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{6}\pi^2$

两侧对 x 积分 $\frac{1}{3}x^3=\frac{\pi^2}{3}x+4\sum_{n=1}^{\infty}(-1)^n\frac{\sin(nx)}{n^3}=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2\pi^2}{3n}-\frac{4}{n^3}\right)\sin(nx)$

$\therefore x^3$ 在 $(-\pi,\pi)$ 上 Fourier 展开式为 $x^3=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\left(\frac{2\pi^2}{n}-\frac{12}{n^3}\right)\sin(nx)$

两侧对 x 积分 $\frac{1}{4}x^4=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2\pi^2}{n}-\frac{12}{n^3}\right)\left(-\frac{\cos nx}{n}\right)+\frac{\pi^4}{20}$

$\therefore x^4$ 在 $(-\pi,\pi)$ 上 Fourier 展开式为 $x^4=\frac{\pi^4}{5}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{8\pi^2}{n^2}-\frac{48}{n^4}\right)\cos(nx)$ $\quad$ 令 $x=\pi$ 得 $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{90}\pi^4$

$\frac{1}{5}x^5=\frac{\pi^4}{5}x+\sum_{n=1}^{\infty}(-1)^n\left(\frac{8\pi^2}{n^3}-\frac{48}{n^5}\right)\sin(nx)$

得 $x^5=\sum_{n=1}^{\infty}(-1)^n\left(-\frac{2\pi^4}{n}+\frac{40\pi^2}{n^3}-\frac{240}{n^5}\right)\sin nx$

两侧对 x 积分 $\frac{1}{6}x^6=\sum_{n=1}^{\infty}(-1)^n\cdot\left(\frac{2\pi^4}{n^2}-\frac{40\pi^2}{n^4}+\frac{240}{n^6}\right)\cos nx+\frac{\pi^6}{42}$

得 $x^6=\frac{\pi^6}{7}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{12\pi^4}{n^2}-\frac{240\pi^2}{n^4}+\frac{1440}{n^6}\right)\cos nx$

令 $x=\pi$ 得 $\frac{6}{7}\pi^6=12\pi^4\sum_{n=1}^{\infty}\frac{1}{n^2}-240\pi^2\sum_{n=1}^{\infty}\frac{1}{n^4}+1440\sum_{n=1}^{\infty}\frac{1}{n^6}$ $\quad$ 得 $\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{1}{945}\pi^6$

两侧对x积分得 $\frac{1}{7}x^7 = \frac{\pi^6}{7}x + \sum_{n=1}^{\infty}(-1)^n\left(\frac{12\pi^4}{n^3} - \frac{240\pi^2}{n^5} + \frac{1440}{n^7}\right)\sin nx$

即 $x^7 = \sum_{n=1}^{\infty}(-1)^n\left(-\frac{2\pi^6}{n} + \frac{84\pi^4}{n^3} - \frac{1680\pi^2}{n^5} + \frac{10080}{n^7}\right)\sin(nx)$

两侧对x积分得 $\frac{1}{8}x^8 = \sum_{n=1}^{\infty}(-1)^n\left(\frac{2\pi^6}{n^2} - \frac{84\pi^4}{n^4} + \frac{1680\pi^2}{n^6} - \frac{10080}{n^8}\right)\cos nx + \frac{\pi^8}{72}$

即 $x^8 = \frac{1}{9}\pi^8 + \sum_{n=1}^{\infty}(-1)^n\left(\frac{16\pi^6}{n^2} - \frac{672\pi^4}{n^4} + \frac{13440\pi^2}{n^6} - \frac{80640}{n^8}\right)\cos nx$

令$x=\pi$得 $\frac{8}{9}\pi^8 = 16\pi^6\cdot\frac{\pi^2}{6} - 672\pi^4\cdot\frac{\pi^4}{90} + 13440\pi^2\cdot\frac{\pi^6}{945} - \sum_{n=1}^{\infty}80640\,\frac{1}{n^8}$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{1}{9450}\pi^8$

综上所述 $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2} \qquad x\in(-\pi,\pi)$

$x^3 = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\left(\frac{2\pi^2}{n} - \frac{12}{n^3}\right)\sin(nx) \qquad x\in(-\pi,\pi)$

$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{8\pi^2}{n^2} - \frac{48}{n^4}\right)\cos(nx) \qquad x\in(-\pi,\pi)$

$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945} \qquad \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{\pi^8}{9450}$

8. 解: 将$f(x)$延拓到$[-\pi,0)$ 其成为奇函数

$b_n = \frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx = \frac{2}{\pi}\left[\int_0^1\frac{\pi-1}{2}x\,d\left(-\frac{1}{n}\cos nx\right) + \int_1^{\pi}\frac{\pi-x}{2}\,d\left(-\frac{1}{n}\cos nx\right)\right]$

$= \frac{2}{\pi}\,\frac{\pi}{2n^2}\sin n = \frac{\sin n}{n^2}$

$\therefore f(x) = \sum_{n=1}^{\infty}b_n\sin nx = \sum_{n=1}^{\infty}\frac{\sin n}{n^2}\sin(nx) \quad x\in[0,\pi]$

9. 证明: (1) $\because f(x) = \sum_{n=1}^{\infty}\frac{\sin n}{n^2}\sin(nx)$

$\therefore \lim_{x\to0}f(x) = \lim_{x\to0}\sum_{n=1}^{\infty}\frac{\sin n}{n^2}\sin(nx) = \sum_{n=1}^{\infty}\lim_{x\to0}\frac{\sin n}{n^2}\sin nx = \sum_{n=1}^{\infty}\lim_{x\to0}\frac{\sin n}{n}x = \lim_{x\to0}x\cdot\sum_{n=1}^{\infty}\frac{\sin n}{n}$

$\therefore \sum_{n=1}^{\infty}\frac{\sin n}{n} = \lim_{n\to0}\frac{f(x)}{x} = \frac{\pi-1}{2}$

$f(1) = \frac{\pi-1}{2} = \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$

$\therefore \sum_{n=1}^{\infty}\frac{\sin n}{n} = \sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^2 = \frac{\pi-1}{2}$

(2) 将 $f(x)$ 延拓到 $[-\pi,0)$ 其成为奇函数

由 Parseval 等式 $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^4}=\frac{2}{\pi}\int_0^{\pi}f^2(x)dx=\frac{2}{\pi}\cdot\left(\int_0^1\left(\frac{\pi-1}{2}x\right)^2dx+\int_1^{\pi}\left(\frac{\pi-x}{2}\right)^2dx\right)$

$$=\frac{2}{\pi}\cdot\left[\frac{(\pi-1)^2}{12}+\frac{(\pi-1)^3}{12}\right]=\frac{2}{\pi}\cdot\frac{\pi(\pi-1)^2}{12}=\frac{(\pi-1)^2}{6}$$

$\therefore \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^4}=\frac{(\pi-1)^2}{6}$