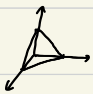
5月12日 第十一周作业

习题 11.5.

1. (1). $\iint_S (x+1)\,dydz + y\,dzdx + (xy+z)\,dxdy$



$\iint_S (x+1)\,dydz + y\,dzdx + (xy+z)\,dxdy = \iiint_V (1+1+1)\,dxdydz$

$= \int_0^1 3dz \int_0^{1-z} dy \int_0^{1-y-z} dx = \frac{1}{2}$

(2). $\iint_S xy\,dydz + yz\,dzdx + zx\,dxdy$

$= \iiint_V (y+z+x)\,dxdydz$

$= \int_0^1 dz \int_0^{1-z} dy \int_0^{1-z-y} (x+y+z)\,dx = \int_0^1 dz \int_0^{1-z} (1-y-z)\cdot\frac{1+y+z}{2}\,dy$

$= \int_0^1 \frac{1}{2}\left(\frac{1}{3}z^3 - z + \frac{2}{3}\right)dz = \frac{1}{8}$

(3). $\iint_S x^2\,dydz + y^2\,dzdx + z^2\,dxdy$

$= \iiint_V (2x+2y+2z)\,dxdydz$

令 $x = a + r\sin\theta\cos\varphi$ $\quad y = b + r\sin\theta\sin\varphi$ $\quad z = c + r\cos\theta$

$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\sin\theta$ $\quad \theta\in[0,\pi]$ $\quad \varphi\in[0,2\pi]$ $\quad r\in[0,R]$

原式 $= \iiint_{V'} 2\{a+b+c+r[\sin\theta(\sin\varphi+\cos\varphi)+\cos\theta]\}\cdot r^2\sin\theta\,drd\theta d\varphi$

$= 2\left[\iiint_{V'}(a+b+c)\cdot r^2\sin\theta\,drd\theta d\varphi + \iiint_{V'} r^3(\sin^2\theta(\sin\varphi+\cos\varphi)+\sin\theta\cos\theta)\,drd\theta d\varphi\right]$

$= 2\cdot\left[\frac{4\pi}{3}R^3(a+b+c) + 0\right]$

$= \frac{8\pi}{3}R^3(a+b+c)$

(4). $\iint_S xy^2\,dydz + yz^2\,dzdx + zx^2\,dxdy$

$= \iiint_V (x^2+y^2+z^2)\,dxdydz$ $\quad x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$

令 $x = r\sin\theta\cos\varphi$ $\quad y = r\sin\theta\sin\varphi$ $\quad z = \frac{1}{2}+r\cos\theta$ $\quad r\in[0,\frac{1}{2}]$ $\quad \theta\in[0,\pi]$ $\quad \varphi\in[0,2\pi]$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\sin\theta$$

原式 $= \iiint_{V'} \left[r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + (\frac{1}{2}+r\cos\theta)^2\right] r^2\sin\theta \; dr\,d\theta\,d\varphi$

$= \iiint_{V'} (r^4\sin\theta + r^3\sin\theta\cos\theta + \frac{1}{4}r^2\sin\theta)\,dr\,d\theta\,d\varphi$

$= \frac{\pi}{15}$

(5). $\iint_S (x-z)\,dy\,dz + (y-x)\,dz\,dx + (z-y)\,dx\,dy$

$= \iiint_V (1+1+1)\,dx\,dy\,dz - \iint_{S'}(1-y)\,dx\,dy$ $\quad S': x^2+y^2=1$

$= \int_0^1 3\,dz \iint_{S''} dx\,dy - \iint_{S'}(1-y)\,dx\,dy$

其中 $\iint_{S''} dx\,dy = \pi(\sqrt{z})^2 = \pi z$ $\quad \int_0^1 3\pi z = \frac{3}{2}\pi$

对于 $\iint_{S'}(1-y)\,dx\,dy$ 令 $x=r\cos\theta$ $y=r\sin\theta$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

$\iint_{S'}(1-y)\,dx\,dy = \iint (1-r\sin\theta)\cdot r\,dr\,d\theta = \int_0^1 r\,dr\int_0^{2\pi}(1-r\sin\theta)\,d\theta = \pi$

$\therefore \iint_S (x-z)\,dy\,dz + (y-x)\,dz\,dx + (z-y)\,dx\,dy = \frac{\pi}{2}$

(6). $\iint_S (y^2+z^2)\,dy\,dz + (z^2+x^2)\,dz\,dx + (x^2+y^2)\,dx\,dy$

$= \iiint_V (0+0+0)\,dx\,dy\,dz - 0$

$= 0$

3. 解: $\iint_S \vec{V}\cdot d\vec{S} = \iint_{S_1}\vec{V}\cdot d\vec{S} + \iint_{S_2}\vec{V}\cdot d\vec{S}$ $\quad S_1$ 为S与以原点为原心 r为半径的球面(内为正向) S_2为原点为原心 r为半径球面外向.

$\iint_{S_1}\vec{V}\cdot d\vec{S} = \iiint_{V_1}\left[\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}\right] dx\,dy\,dz$

$\iint_{S_2}\vec{V}\cdot d\vec{S} = \iint_{S_2}\frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{\sqrt{(x^2+y^2+z^2)^3}}$ 令 $x=r\sin\theta\cos\varphi$ $y=r\sin\theta\sin\varphi$ $z=r\cos\theta$ $\frac{\partial(y,z)}{\partial(\theta,\varphi)} = r^2\sin^2\theta\cos\varphi$ $\frac{\partial(z,x)}{\partial(\theta,\varphi)} = r^2\sin^2\theta\sin\varphi$

$\frac{\partial(x,y)}{\partial(\theta,\varphi)} = r^2\sin\theta\cos\theta$ $\quad \therefore \iint_{S_2}\vec{V}\cdot d\vec{S} = \iint_{S_2'}\frac{1}{r^3}\cdot\left(r\sin\theta\cos\varphi\cdot r^2\sin^2\theta\cos\varphi + r\sin\theta\sin\varphi\cdot r^2\sin^2\theta\sin\varphi + r\cos\theta\; r^2\sin\theta\cos\theta\right)d\theta\,d\varphi$

$= \iint_{S_2'}(\sin^3\theta\cos^2\varphi + \sin^3\theta\sin^2\varphi + \sin\theta\cos^2\theta)\,d\theta\,d\varphi = \int_0^{\pi}(\sin\theta+\cos\theta)\,d\theta\int_0^{2\pi}d\varphi = 4\pi$

$$\therefore \iint_S \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{\sqrt{(x^2+y^2+z^2)^3}} = 4\pi$$

4. $\oiint_S xf(x)\,dydz + (-xyf(x))\,dzdx + (-ze^{2x})\,dxdy = 0$

$\therefore \iiint_V (f(x) + xf'(x) - xf(x) - e^{2x})\,dxdydz = 0$

即 $f(x) + x\cdot f'(x) - xf(x) - e^{2x} = 0$

$[x\cdot e^{-x}\cdot f(x)]' = e^{x}$ 两侧不定积分得 $xe^{-x}f(x) = e^{x} + C \quad C \in R$

$\because \lim_{x\to 0^+} f(x) = 1 \quad \therefore C = -1$

即 $f(x) = \dfrac{e^{2x} - e^{x}}{x}$

6. 证明：设液体密度为ρ 取外侧为正向.

$F = \oiint_S \rho g z\,dxdy$

$= \iiint_V \rho g\,dxdydz = \rho g V$

即 浮力等于排开液体的重力.

10. 解：(1). S 取 Oxy 平面上的圆面 $x^2+y^2 \le R^2$：

$\oint_L x^2y^3dx + dy + zdz = \iint_S 0\,dydz + 0\,dzdx + (-3x^2y^2)\,dxdy = -3\iint_S x^2y^2\,dxdy$

令 $x = r\cos\theta \quad y = r\sin\theta \qquad r\in[0,R] \quad \theta\in[0,2\pi] \qquad \left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

$-3\iint_S x^2y^2\,dxdy = -3\iint_{D'} r^5\sin^2\theta\cos^2\theta\,dr\,d\theta = -3\int_0^R r^5dr\int_0^{2\pi}(\frac{1}{8} - \frac{1}{8}\cos 4\theta)\,d\theta = -\frac{1}{2}R^6\cdot\frac{\pi}{4} = -\frac{\pi}{8}R^6$

(2). S 取半球面 $z = \sqrt{R^2 - x^2 - y^2}$

$\oint_L x^2y^3dx + dy + zdz = \iint_S 0\,dydz + 0\,dzdx + (-3x^2y^2)\,dxdy = -3\iint_S x^2y^2\,dxdy$

令 $x = R\sin\theta\cos\varphi \qquad y = R\sin\theta\sin\varphi \qquad \left|\dfrac{\partial(x,y)}{\partial(\theta,\varphi)}\right| = R^2\sin\theta\cos\theta \qquad \theta\in[0,\frac{\pi}{2}] \quad \varphi\in[0,2\pi]$

$$-3\iint_S x^2y^2\,dxdy = -3R^6\iint_{S'} \sin^5\theta\cos\theta\sin^2\varphi\cos^2\varphi\,d\theta d\varphi = -3R^6\int_0^{\frac{\pi}{2}}\sin^5\theta\cos\theta\,d\theta\int_0^{2\pi}\sin^2\varphi\cos^2\varphi\,d\varphi$$

$$= -\frac{R^6}{2}\cdot\frac{\pi}{4} = -\frac{\pi R^6}{8}$$

即 (1) (2) 两种取法结果相同.

11. 证明：$\because$ $\vec{c}$ 为常向量场. $\therefore$ $\vec{c}$ 为光滑向量场. 令 $\vec{c} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\therefore \oint_L \vec{c}\cdot d\vec{r} = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = 0$$

5月14日.

习题 11.7.

1\. (1). L_1: $W_1 = \int_{L_1} \vec{F}\cdot d\vec{r} = \int_0^1 0dx + \int_0^1 1dy = 1$

L_2: $W_2 = \int_{L_2} \vec{F}\cdot d\vec{r} = \int_0^1 -x^2dx + 2x^2dx = \frac{1}{3}$

L_3: $W_3 = \int_{L_3} \vec{F}\cdot d\vec{r} = \int_0^1 -xdx + xdx = 0$

L_4: $W_4 = \int_{L_4} \vec{F}\cdot dr = \int_0^1 0dy + \int_0^1 -1dx = -1$

$\vec{F} = -y\vec{i} + x\vec{j}$ 不是一个保守场. 即不存在标量函数 φ 使得 $F = \nabla\varphi$

(2). L_1: $W_1 = \int_{L_1} \vec{F}\cdot d\vec{r} = \int_0^1 0dx + \int_0^1 1dy = 1$

L_2: $W_2 = \int_{L_2} \vec{F}\cdot d\vec{r} = \int_0^1 (2x^3 + 2x^3)dx = 1$

L_3: $W_3 = \int_{L_3} \vec{F}\cdot d\vec{r} = \int_0^1 (2x^2 + x^2)dx = 1$

L_4: $W_4 = \int_{L_4} \vec{F}\cdot d\vec{r} = \int_0^1 0dy + \int_0^1 2xdx = 1$

$W_1 = W_2 = W_3 = W_4$ 因为 $\vec{F} = 2xy\vec{i} + x^2\vec{j} = \nabla\varphi$ 其中 $\varphi = x^2y$

3\. (1) $\nabla\times\vec{v} = [\frac{\partial}{\partial x}(2y\cos x - x^2\sin y) - \frac{\partial}{\partial y}(2x\cos y - y^2\sin x)]\vec{k}$

$= (-2y\sin x - 2x\sin y + 2x\sin y + 2y\sin x)\vec{k}$

$= \vec{0}$

$\therefore \vec{v} = (2x\cos y - y^2\sin x)\vec{i} + (2y\cos x - x^2\sin y)\vec{j}$ 是有势场.

$\frac{\partial\varphi}{\partial x} = 2x\cos y - y^2\sin x$ $\quad \frac{\partial\varphi}{\partial y} = 2y\cos x - x^2\sin y$

即势函数 $\varphi = x^2\cos y + y^2\cos x$

(2). $\vec{v} = yz(2x+y+z)\vec{i} + xz(2y+x+z)\vec{j} + xy(2z+x+y)\vec{k}$

$\nabla \times \vec{v} = \left\{\frac{\partial}{\partial y}[xy(2z+x+y)] - \frac{\partial}{\partial z}[xz(2y+x+z)]\right\}\vec{i}$

$+\left\{\frac{\partial}{\partial z}[yz(2x+y+z)] - \frac{\partial}{\partial x}[xy(2z+x+y)]\right\}\vec{j}$

$+\left\{\frac{\partial}{\partial x}[xz(2y+x+z)] - \frac{\partial}{\partial y}[yz(2x+y+z)]\right\}\vec{j}$

$= (2xz + x^2 + 2xy - 2xy - x^2 - 2xz)\vec{i} + (2xy + y^2 + 2yz - 2yz - 2xy - y^2)\vec{j} + (2yz + 2xz + z^2 - 2xz - 2yz - z^2)\vec{k}$

$= \vec{0}$

$\therefore \vec{v}$为有势场. $\frac{\partial \varphi}{\partial x} = 2xyz + y^2z + yz^2$ $\frac{\partial \varphi}{\partial y} = 2xyz + xz^2 + x^2z$ $\frac{\partial \varphi}{\partial z} = 2xyz + x^2y + xy^2$

$\therefore \varphi = xyz(x+y+z)$

(3) $\vec{v} = r^2\vec{r} = x(x^2+y^2+z^2)\vec{i} + y(x^2+y^2+z^2)\vec{j} + z(x^2+y^2+z^2)\vec{k}$

$\nabla \times \vec{v} = \left[\frac{\partial}{\partial y} z(x^2+y^2+z^2) - \frac{\partial}{\partial z} y(x^2+y^2+z^2)\right]\vec{i} + \left[\frac{\partial}{\partial z} x(x^2+y^2+z^2) - \frac{\partial}{\partial x} z(x^2+y^2+z^2)\right]\vec{j} + \left[\frac{\partial}{\partial x} y(x^2+y^2+z^2) - \frac{\partial}{\partial y} x(x^2+y^2+z^2)\right]\vec{k}$

$= (2yz - 2yz)\vec{i} + (2xz - 2xz)\vec{j} + (2xy - 2xy)\vec{k} = \vec{0}$

$\therefore \vec{v}$为有势场. $\frac{\partial \varphi}{\partial x} = x^3 + xy^2 + xz^2$ $\frac{\partial \varphi}{\partial y} = x^2y + y^3 + yz^2$ $\frac{\partial \varphi}{\partial z} = x^2z + y^2z + z^3$

$\therefore \varphi = \frac{1}{4}(x^4 + y^4 + z^4) + \frac{1}{2}(x^2y^2 + x^2z^2 + y^2z^2)$

7. 证明: (1) 令 $u = x^2 + y^2$ $du = 2(xdx + ydy)$

$\therefore \oint_L f(x^2+y^2)(xdx+ydy) = \frac{1}{2}\oint_L f(u)du$

$\because f(u)$是连续函数 $\therefore$存在 $F(u)$ 满足 $F'(u) = f(u)$ $dF(u) = f(u)du$

$\therefore \oint_L f(x^2+y^2)(xdx+ydy) = \frac{1}{2}\oint_L dF(u) = 0$

$\therefore \oint_L f(x^2+y^2)(xdx+ydy) = 0$

(2). 令 $u=\sqrt{x^2+y^2+z^2}$ $du=\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}$

则 $\oint_L f(\sqrt{x^2+y^2+z^2})(xdx+ydy+zdz)=\oint_L f(u)\cdot udu = \oint_L uf(u)\,du$

$\because$ $f(u)$为连续函数 $\therefore$ $uf(u)$为连续函数 $\therefore \exists\, G(u)$ 满足 $G'(u)=uf(u)$ 即 $dG(u)=uf(u)du$

$\therefore \oint_L f(\sqrt{x^2+y^2+z^2})(xdx+ydy+zdz) = \oint_L dG(u)=0$

$\therefore \oint_L f(\sqrt{x^2+y^2+z^2})(xdx+ydy+zdz)=0$

8. 解: $\vec{B}=\frac{2I}{x^2+y^2}(-y\vec{i}+x\vec{j})$ $(x^2+y^2\neq 0)$

①. 光滑闭曲线不包含 $(0,0)$ S为以L为边界的曲面.

$\oint_L \vec{B}\cdot d\vec{r} = \iint_S 2I\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dxdy$

$= \iint_S 2I\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}+\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]dxdy = \iint_S 2I\ 0\,dxdy=0$

②. 光滑闭曲线包含 $(0,0)$ 取以原点为圆心 $R>0$ 为半径的且全部位于闭曲线内部的圆弧.

$\oint_L \vec{B}\cdot d\vec{r} = \oint_{L_1}\vec{B}\cdot d\vec{r}+\oint_R \vec{B}\cdot d\vec{r}$

$=\iint_{S_1} 2I\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dxdy+\oint_R \vec{B}\cdot d\vec{r}$

$=\oint_R \vec{B}\cdot d\vec{r} = \oint_R 2I\,\frac{-ydx+xdy}{x^2+y^2}$

令 $x=R\cos\theta$ $y=R\sin\theta$ $\theta\in[0,2\pi]$

$\therefore \oint_R 2I\frac{-ydx+xdy}{x^2+y^2} = \int_0^{2\pi} 2I\,\frac{R^2\sin^2\theta+R^2\cos^2\theta}{R^2\cos^2\theta+R^2\sin^2\theta}d\theta = 4\pi I$

11. 解：$\because \int_L 2xy\,dx + Q(x,y)\,dy$ 与路径无关.

$\therefore$ 令 $\vec{v} = (2xy,\ Q(x,y))$ $\quad \nabla\times\vec{v} = \vec{0}$ 即 $\frac{\partial}{\partial x} Q(x,y) - \frac{\partial}{\partial y} 2xy = 0$ $\quad \therefore Q(x,y) = x^2 + C(y)$

$\int_{(0,0)}^{(t,1)} 2xy\,dx + Q(x,y)\,dy$ 令 $x = ty$ $\quad dx = t\,dy$

则原式 $= \int_0^1 2t^2y^2 dy + [t^2y^2 + C(y)]\,dy = t^2 + \int_0^1 C(y)\,dy$

$\int_{(0,0)}^{(1,t)} 2xy\,dx + Q(x,y)\,dy$ 令 $y = tx$ $\quad dy = t\,dx$

则原式 $= \int_0^1 2tx^2 dx + tx^2 dx + \int_0^t C(y)\,dy = t + \int_0^t C(y)\,dy$

$\therefore\ t^2 + \int_0^1 C(y)\,dy = t + \int_0^t C(y)\,dy$

$t^2 - t = \int_1^t C(y)\,dy$

得 $C(y) = 2y - 1$

$\therefore\ Q(x,y) = x^2 + 2y - 1$

12.(1) $(xy^2 + 2y - 2y\cos x - y\sin x)\,dx + (x^2y + 2x + \cos x - 2\sin x)\,dy = 0$

令 $\vec{r} = (xy^2 + 2y - 2y\cos x - y\sin x,\ x^2y + 2x + \cos x - 2\sin x)$

$\nabla\times\vec{r} = [(2xy + 2 - \sin x - 2\cos x) - (2xy + 2 - 2\cos x - \sin x)]\,\vec{k} = \vec{0}$ $\quad \therefore \vec{r}$ 为保守场.

$\varphi = \frac{1}{2}x^2y^2 + 2xy - 2y\sin x + y\cos x$ 满足 $\vec{r} = \nabla\varphi$

$\nabla\varphi = 0$ 只需 $\varphi = C$

即原方程解为 $\frac{1}{2}x^2y^2 + 2xy - 2y\sin x + y\cos x$

(2). $2xy\,dx+(y^2-x^2)\,dy=0$

$$y'=\frac{2xy}{x^2-y^2}$$

令 $y=ux$ $\quad y'=u'x+u$

$$u'x+u=\frac{2u}{1-u^2}$$

$$u'x=\frac{u+u^3}{1-u^2}$$ 则 $$\frac{u+u^3}{1-u^2}\,du=\frac{1}{x}\,dx$$

$\frac{1}{2}\frac{1+u^2}{1-u^2}\,d(u^2)=\frac{1}{x}\,dx$ 两侧不定积分得

$$-\ln(1-u^2)-\frac{1}{2}u^2=\ln x+C \quad C\in R$$

即原方程解为 $-\ln(1-\frac{y^2}{x^2})-\frac{1}{2}\frac{y^2}{x^2}=\ln x+C$