第五周周二作业 3月31日.

习题9.3

1. (2). $F(x,y) = x\cos xy$

解: $F(1,\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$ $\qquad \frac{\partial F}{\partial x} = \cos xy - xy\sin xy \qquad \frac{\partial F}{\partial y} = -x^2\sin xy$

$\because$ $\frac{\partial F}{\partial x}$ 与 $\frac{\partial F}{\partial y}$ 在 $(1,\frac{\pi}{2})$ 的邻域内连续 且 $\frac{\partial F}{\partial y}(1,\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$\therefore$ $F(x,y)$ 在 $(1,\frac{\pi}{2})$ 附近对 y 有唯一解

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{\cos xy - xy\sin xy}{x^2\sin xy} = \frac{\cos xy}{x^2\sin xy} - \frac{y}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\cos xy}{x^2\sin xy} - \frac{y}{x}\right) = \frac{\left[-\sin(xy)\cdot y - \sin(xy)\cdot x\cdot\frac{dy}{dx}\right]x^2\sin xy - \cos(xy)\cdot\left[2x\sin(xy) + x^2\cos(xy)\cdot y + x^2\cos(xy)\cdot x\cdot\frac{dy}{dx}\right]}{x^4\sin^2(xy)} + \frac{y}{x^2} - \frac{1}{x}\frac{dy}{dx}$$

$$= -\frac{y}{x^2} - \frac{1}{x}\left(\frac{\cos xy}{x^2\sin xy} - \frac{y}{x}\right) - \frac{2\cos(xy)}{x^3\sin(xy)} - \frac{y\cos(xy)}{x^2\sin xy} - \frac{\cos^2(xy)}{x\sin^2(xy)}\cdot\left(\frac{\cos xy}{x^2\sin xy} - \frac{y}{x}\right) + \frac{y}{x^2} - \frac{1}{x}\left(\frac{\cos xy}{x^2\sin xy} - \frac{y}{x}\right)$$

$$= -\frac{4\cos(xy)}{x^3\sin(xy)} - \frac{y\cos(xy)}{x^2\sin(xy)} - \frac{\cos^3(xy)}{x^3\sin^3(xy)} + \frac{y\cos^2(xy)}{x^2\sin^2(xy)} + 2\frac{y}{x^2}$$

2. (1) $F(x,y) = \sin(xy) - e^{xy} - x^2y$

$$\frac{\partial F}{\partial x} = y\cos(xy) - ye^{xy} - 2xy \qquad \frac{\partial F}{\partial y} = x\cos(xy) - xe^{xy} - x^2$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y\left[\cos(xy) - e^{xy} - 2x\right]}{x\left[\cos(xy) - e^{xy} - x\right]}$$

(6) $F(x,\ x+y,\ x+y+z) = 0 \qquad F(x,\ u,\ v(x,u)) = 0$

两侧对 x 求偏导: $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial(x+y)}\cdot\frac{\partial(x+y)}{\partial x} + \frac{\partial F}{\partial(x+y+z)}\cdot\frac{\partial(x+y+z)}{\partial x} = 0 \qquad \frac{\partial y}{\partial x} = 0$

两侧对 y 求偏导: $\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial F}{\partial(x+y)}\cdot\frac{\partial(x+y)}{\partial y} + \frac{\partial F}{\partial(x+y+z)}\cdot\frac{\partial(x+y+z)}{\partial y} = 0 \qquad \frac{\partial x}{\partial y} = 0$

则 $\frac{\partial z}{\partial x} = -\frac{F'_x + F'_{(x+y)}}{F'_{(x+y+z)}} - 1 \qquad \frac{\partial z}{\partial y} = -\frac{F'_{(x+y)}}{F'_{(x+y+z)}} - 1$

(7). $F(xz, yz)=0$

两侧对 x 求导. $F'_{(xz)}\cdot\left(z+x\cdot\frac{\partial z}{\partial x}\right)+F'_{(yz)}\cdot y\frac{\partial z}{\partial x}=0$

两侧对 y 求导: $F'_{(xz)}\cdot x\frac{\partial z}{\partial y}+F'_{(yz)}\left(z+y\frac{\partial z}{\partial y}\right)=0$

得 $\frac{\partial z}{\partial x}=-\frac{zF'_{(xz)}}{xF'_{(xz)}+yF'_{(yz)}}$ $\qquad\frac{\partial z}{\partial y}=-\frac{zF'_{(yz)}}{xF'_{(xz)}+yF'_{(yz)}}$

3. 解 令 $F(x,y)=x^2+xy+y^2-27$ $\qquad F(x,y)=0$

$\frac{\partial F}{\partial x}=2x+y$ $\qquad\frac{\partial F}{\partial y}=2y+x$

$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{2x+y}{2y+x}$ $(2y+x\neq0)$ 若求 $y=y(x)$ 极值. 则 $\frac{dy}{dx}=0$ 得 $y=-2x$

$3x^2-27=0$ 得 $x=3$ 或 $x=-3$

则 $y=y(x)$ 极大值为 6 极小值为 -6

4. (3) $u^3-3(x+y)u^2+z^3=0$

解: 两侧对 x 求偏导: $3u^2u'_x-[3u^2+6(x+y)u\cdot u'_x]=0$ 得 $u'_x=\frac{u}{u-2(x+y)}$

两侧对 y 求偏导: $3u^2u'_y-[3u^2+6(x+y)u\,u'_y]=0$ 得 $u'_y=\frac{u}{u-2(x+y)}$

两侧对 z 求偏导: $3u^2u'_z-6(x+y)u\,u'_z+3z^2=0$ 得 $u'_z=\frac{-z^2}{u^2-2(x+y)u}$

则 $du=u'_x\,dx+u'_y\,dy+u'_z\,dz=\frac{u}{u-2(x+y)}dx+\frac{u}{u-2(x+y)}dy+\frac{-z^2}{u^2-2(x+y)u}dz$

(4). $F(x-y,\ y-z,\ z-x)=0$

两侧对x求偏导: $F'_{(x-y)} - F'_{(y-z)}\cdot\frac{\partial z}{\partial x} + F'_{(z-x)}(\frac{\partial z}{\partial x}-1)=0$ 得 $\frac{\partial z}{\partial x}=\frac{F'_{(z-x)}-F'_{(x-y)}}{F'_{(z-x)}-F'_{(y-z)}}$

两侧对y求偏导 $-F'_{(x-y)}+F'_{(y-z)}(1-\frac{\partial z}{\partial y})+F'_{(z-x)}\frac{\partial z}{\partial y}=0$ 得 $\frac{\partial z}{\partial y}=\frac{F'_{(x-y)}-F'_{(y-z)}}{F'_{(z-x)}-F'_{(y-z)}}$

则 $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{F'_{(z-x)}-F'_{(x-y)}}{F'_{(z-x)}-F'_{(y-z)}}dx+\frac{F'_{(x-y)}-F'_{(y-z)}}{F'_{(z-x)}-F'_{(y-z)}}dy$

6. 证明: $2\sin(x+2y-3z)=x+2y-3z$

两侧对x求偏导得 $2\cos(x+2y-3z)(1-3\frac{\partial z}{\partial x})=1-3\frac{\partial z}{\partial x}$ $\forall (x,y)\in R^2$都成立 $\therefore\ 1-3\frac{\partial z}{\partial x}=0$

两侧对y求偏导得 $4\cos(x+2y-3z)(2-3\frac{\partial z}{\partial y})=2-3\frac{\partial z}{\partial y}$ $\forall (x,y)\in R^2$都成立 $\therefore\ 2-3\frac{\partial z}{\partial y}=0$

$\therefore \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{1}{3}+\frac{2}{3}=1$

9. $y=f(x+t)$ $y+g(x,t)=0$

解: $t=f^{-1}(y)-x$ $y=-g(x,t)$

两侧对x求偏导: $\frac{dy}{dx}=-g'_x-g'_t\cdot\frac{\partial t}{\partial x}$ $=-g'_x-g'_t\cdot(\frac{dy}{dx}\cdot\frac{1}{f'_{(x+t)}}-1)$

$\frac{dy}{dx}=f'_{(x+t)}\cdot(1+\frac{\partial t}{\partial x})$ $\frac{dy}{dx}\cdot\frac{1}{f'_{(x+t)}}-1$

得 $[1+\frac{g'_t}{f'_{(x+t)}}]\frac{dy}{dx}=g'_t-g'_x$

即 $\frac{dy}{dx}=\frac{f'_{(x+t)}}{g'_t+f'_{(x+t)}}(g'_t-g'_x)$

11. (1). $\begin{cases} u^2+v^2+x^2+y^2=1 \\ u+v+x+y=0 \end{cases}$

解：两侧对 x , y 分别求偏导数.

$2uu_x'+2vv_x'+1=0$ $\qquad$ $2uu_y'+2vv_y'+1=0$

$u_x'+v_x'+1=0$ $\qquad$ $u_y'+v_y'+1=0$

得 $u_x'=\frac{2v-1}{2u-2v}$ $\qquad$ $u_y'=\frac{2v-1}{2u-2v}$

$v_x'=\frac{1-2u}{2u-2v}$ $\qquad$ $v_y'=\frac{1-2u}{2u-2v}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{2v-1}{2u-2v}\cdot\frac{1-2u}{2u-2v}-\frac{2v-1}{2u-2v}\frac{1-2u}{2u-2v}=0$$

(2). $\begin{cases} xu-yv=0 \\ yu+xv=1 \end{cases}$ $\qquad$ $v_x'=\frac{1}{y}(u+xu_x')$ $\qquad$ $\frac{(x^2+y^2)u-x^2u-xyv}{x^2+y^2}$

解：两侧分别对 x , y 求偏导数.

得 $\begin{cases} u+xu_x'-yv_x'=0 \\ yu_x'+v+xv_x'=0 \end{cases}$ $\qquad$ $\begin{cases} xu_y'-v-yv_y'=0 \\ u+yu_y'+xv_y'=0 \end{cases}$

$u_x'=-\frac{xu+yv}{x^2+y^2}$ $\qquad$ $u_y'=\frac{xv-yu}{x^2+y^2}$

$v_x'=\frac{yu-xv}{x^2+y^2}$ $\qquad$ $v_y'=-\frac{xu+yv}{x^2+y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=u_x'v_y'-u_y'v_x'=\frac{(xu+yv)^2}{(x^2+y^2)^2}-\frac{(xv-yu)(yu-xv)}{(x^2+y^2)^2}$$

$$=\frac{(xu+yv)^2+(xv-yu)^2}{(x^2+y^2)^2}$$

(3) $\begin{cases} u = f(ux,\ v+y) \\ v = g(u-x,\ v^2y) \end{cases}$

解：两侧分别对 x, y 求偏导数.

$$u'_x = f'_{(ux)}(u + xu'_x) + f'_{(v+y)}\, v'_x \qquad u'_y = f'_{(ux)}\, x u'_y + f'_{(v+y)}(v'_y + 1)$$

$$v'_x = g'_{(u-x)}(u'_x - 1) + g'_{(v^2y)}\, 2yv v'_x \qquad v'_y = g'_{(u-x)}\, u'_y + g'_{(v^2y)}(v^2 + 2yvv'_y)$$

得 $$u'_x = \frac{uf'_{(ux)}[1-2yvg'_{(v^2y)}] - g'_{(u-x)}}{[1-xf'_{(ux)}][1-2yvg'_{(v^2y)}] - g'_{(u-x)}} \qquad u'_y = \frac{f'_{(v+y)}\cdot[v^2g'_{(v^2y)} - 2yvg'_{(v^2y)} + 1]}{[1-2yvg'_{(v^2y)}][1-xf'_{(ux)}] - f'_{(v+y)}g'_{(u-x)}}$$

$$v'_x = \frac{g'_{(u-x)}[uf'_{(ux)} + xf'_{(ux)} - 1]}{[1-xf'_{(ux)}][1-2yvg'_{(v^2y)}] - g'_{(u-x)}} \qquad v'_y = \frac{f'_{(v+y)}g'_{(u-x)} + v^2g'_{(v^2y)}[1-xf'_{(ux)}]}{[1-2yvg'_{(v^2y)}][1-xf'_{(ux)}] - f'_{(v+y)}g'_{(u-x)}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = u'_x v'_y - u'_y v'_x$$

$$= \frac{[1-xf'_{(ux)}]\left\{[1-2yvg'_{(v^2y)}][1+uv^2f'_{(ux)}g'_{(v^2y)}] + v^2g'_{(u-x)}g'_{(v^2y)}[f'_{(v+y)}-1]\right\} - f'_{(x+y)}[g'_{(u-x)} + uv^2f'_{(ux)}g'_{(v^2y)}]}{\{[1-xf'_{(ux)}][1-2yvg'_{(v^2y)}] - g'_{(u-x)}\}\{[1-2yvg'_{(v^2y)}][1-xf'_{(ux)}] - f'_{(v+y)}g'_{(u-x)}\}}$$

$uf'_{(ux)}[1-2yvg'_{(v^2y)}]\{f'_{(v+y)}g'_{(u-x)} + v^2g'_{(v^2y)}[1-xf'_{(ux)}]\} - (g'_{(u-x)})^2 f'_{(v+y)} - v^2 g'_{(u-x)} g'_{(v^2y)}[1-xf'_{(ux)}]$

$- v^2 f'_{(v+y)} g'_{(v^2y)} g'_{(u-x)}[uf'_{(ux)} + xf'_{(ux)} - 1] - f'_{(v+y)}[1-2yvg'_{(v^2y)}]\, g'_{(u-x)}[uf'_{(ux)} + xf'_{(ux)} - 1]$

$+[1-xf'_{(ux)}]\cdot f'_{(v+y)}g'_{(u-x)}$

$[1-2yvg'_{(v^2y)}]\quad uf'_{(ux)}\cancel{f'_{(v+y)}}g'_{(u-x)} + uv^2f'_{(ux)}g'_{(v^2y)}[1-xf'_{(ux)}] - uf'_{(ux)}\cancel{f'_{(v+y)}}g'_{(u-x)} - xf'_{(ux)}f'_{(v+y)}g'_{(u-x)} + f'_{(v+y)}g'_{(u-x)}$

$[1-2yvg'_{(v^2y)}]\cdot[1-xf'_{(ux)}]\cdot[1+uv^2f'_{(ux)}g'_{(v^2y)}] - g'_{(u-x)}\cdot[f'_{(v+y)}g'_{(u-x)} + \boxed{v^2g'_{(v^2y)}} - xv^2f'_{(ux)}g'_{(v^2y)} + uv^2f'_{(v+y)}f'_{(ux)}g'_{(v^2y)} + xv^2f'_{(ux)}f'_{(v+y)}g'_{(v^2y)} - \boxed{v^2f'_{(v+y)}g'_{(v^2y)}}]$

$xv^2f'_{(ux)}g'_{(v^2y)}[f'_{(v+y)}-1] - v^2g'_{(v^2y)}[f'_{(v+y)}-1]$

$g'_{(u-x)}[xf'_{(ux)}+1]\quad v^2g'_{(v^2y)}[f'_{(v+y)}-1] - f'_{(v+y)}\cdot[g'_{(u-x)} + uv^2f'_{(ux)}g'_{(v^2y)}]$

$[1-xf'_{(ux)}][1 + uv^2f'_{(ux)}g'_{(v^2y)} - 2yvg'_{(v^2y)} - 2yuv^3f'_{(ux)}g'_{(v^2y)}g'_{(v^2y)} + v^2f'_{(v+y)}g'_{(u-x)}g'_{(v^2y)} - v^2g'_{(u-x)}g'_{(v^2y)}$

12.(1) $\begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$

解：两侧对 x 求偏导数得.

$$1 = f'_u \cdot \frac{\partial u}{\partial x} + f'_v \frac{\partial v}{\partial x}$$

$$0 = g'_u \frac{\partial u}{\partial x} + g'_v \frac{\partial v}{\partial x}$$

得 $\frac{\partial u}{\partial x} = \frac{g'_v}{f'_u g'_v - f'_v g'_u}$

$\frac{\partial v}{\partial x} = \frac{-g'_u}{f'_u g'_v - f'_v g'_u}$

两侧对 y 求偏导数得.

$$0 = f'_u \frac{\partial u}{\partial y} + f'_v \frac{\partial v}{\partial y}$$

$$1 = g'_u \frac{\partial u}{\partial y} + g'_v \frac{\partial v}{\partial y}$$

$\frac{\partial u}{\partial y} = \frac{f'_v}{f'_v g'_u - f'_u g'_v}$

$\frac{\partial v}{\partial y} = \frac{-f'_u}{f'_v g'_u - f'_u g'_v}$

(2). $\begin{cases} x = e^u + u\sin v \\ y = e^u - u\cos v \end{cases}$

解：两侧对 x 求偏导.

$$\begin{cases} 1 = e^u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\sin v + u\cos v \frac{\partial v}{\partial x} \\ 0 = e^u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\cos v + u\sin v \frac{\partial v}{\partial x} \end{cases}$$

得 $\frac{\partial u}{\partial x} = \frac{\sin v}{1 + e^u(\sin v - \cos v)}$

$\frac{\partial v}{\partial x} = \frac{\cos v - e^u}{u[1 + e^u(\sin v - \cos v)]}$

两边对 y 求偏导

$$\begin{cases} 0 = e^u \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\sin v + u\cos v \frac{\partial v}{\partial y} \\ 1 = e^u \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\cos v + u\sin v \frac{\partial v}{\partial y} \end{cases}$$

$\frac{\partial u}{\partial y} = \frac{\cos v}{e^u(\cos v - \sin v) - 1}$

$\frac{\partial v}{\partial y} = \frac{-(e^u + \sin v)}{u[e^u(\cos v - \sin v) - 1]}$

13. $u=f(x,y,z)$ $\varphi(x^2,e^y,z)=0$ $y=\sin x$

解: $u=f(x,y,z)$ 两侧对 x 求偏导数

$$\frac{du}{dx}=f'_x+f'_y\frac{\partial y}{\partial x}+f'_z\frac{\partial z}{\partial x}$$

$\varphi(x^2,e^y,z)=0$ 两侧对 x 求偏导数得:

$$\varphi'_{(x^2)}\cdot 2x+\varphi'_{(e^y)}e^y\frac{\partial y}{\partial x}+\varphi'_z\frac{\partial z}{\partial x}=0$$

其中 $\frac{\partial y}{\partial x}=\cos x$ 得 $\frac{\partial z}{\partial x}=-\frac{1}{\varphi'_z}\cdot\left(2x\varphi'_{x^2}+e^y\cos x\,\varphi'_{e^y}\right)$

得 $\frac{du}{dx}=f'_x+f'_y\cos x-\frac{f'_z}{\varphi'_z}\left(2x\varphi'_{x^2}+e^y\cos x\,\varphi'_{e^y}\right)$

16. $u=u(x,y)$ $u=f(x,y,z,t)$ $g(y,z,t)=0$ $h(z,t)=0$

解: $g(y,z,t)=0$ $h(z,t)=0$ 对y求偏导得:

$$g'_y+g'_z\frac{dz}{dy}+g'_t\frac{dt}{dy}=0$$

$$h'_z\frac{dz}{dy}+h'_t\frac{dt}{dy}=0$$

$$\frac{dt}{dy}=-\frac{h'_z}{h'_t}\frac{dz}{dy}$$

$$g'_y=\left(\frac{h'_z g'_t}{h'_t}-g'_z\right)\frac{dt}{dy}$$

得 $\frac{dz}{dy}=\frac{g'_y h'_t}{g'_t h'_z-g'_z h'_t}$ $\frac{dt}{dy}=\frac{-g'_y h'_z}{g'_t h'_z-g'_z h'_t}$

则 $\frac{\partial u}{\partial x}=f'_x$ $\frac{\partial u}{\partial y}=f'_y+f'_z\frac{dz}{dy}+f'_t\frac{dt}{dy}$

$$=f'_y+\frac{f'_z g'_y h'_t-f'_t g'_y h'_z}{g'_t h'_z-g'_z h'_t}$$

第五周 周四作业 4月2日

习题 9.4

3. $f(x,y) = \sin(\pi x) + \cos(\pi y)$

解：$f_x' = \pi\cos(\pi x)$ $\quad f_y' = -\pi\sin(\pi y)$ $\quad\therefore f(x,y)$ 的偏导数在 R^2 上连续.

令 $(x_1, y_1) = (-\frac{1}{2}, 1)$ $\quad (x_2, y_2) = (\frac{1}{2}, 0)$ $\quad$ 则 $(\frac{\theta}{2}, \frac{1-\theta}{2})$ $\theta\in(0,1)$ 位于 (x_1, y_1) (x_2, y_2) 线段上

由微分中值定理 $\quad f(\frac{1}{2}, 0) - f(-\frac{1}{2}, 1) = f_x'(\frac{\theta}{2}, \frac{1-\theta}{2}) - f_y'(\frac{\theta}{2}, \frac{1-\theta}{2})$

即 $\quad 4 = \pi\left[\cos\frac{\pi}{2}\theta + \sin\frac{\pi}{2}(1-\theta)\right]$

$$\frac{4}{\pi} = \cos\frac{\pi}{2}\theta + \sin\frac{\pi}{2}(1-\theta)$$

4. (2). $f(x,y) = \sqrt{1-x^2-y^2}$

解：$f_x' = \dfrac{-x}{\sqrt{1-x^2-y^2}}$ $\quad f_y' = \dfrac{-y}{\sqrt{1-x^2-y^2}}$

$f_{xx}'' = \dfrac{y^2-1}{(1-x^2-y^2)^{3/2}}$ $\quad f_{xy}'' = \dfrac{-xy}{(1-x^2-y^2)^{3/2}}$ $\quad f_{yy}'' = \dfrac{x^2-1}{(1-x^2-y^2)^{3/2}}$

$f_{xxx}''' = \dfrac{3x(y^2-1)}{(1-x^2-y^2)^{5/2}}$ $\quad f_{xxy}''' = \dfrac{y(y^2-2x^2-1)}{(1-x^2-y^2)^{5/2}}$ $\quad f_{xyy}''' = \dfrac{x^3-2xy^2-x}{(1-x^2-y^2)^{5/2}}$ $\quad f_{yyy}''' = \dfrac{3y(x^2-1)}{(1-x^2-y^2)^{5/2}}$

$f_{xxxx}^{(4)} = \dfrac{3(y^2-1)(1+4x^2-y^2)}{(1-x^2-y^2)^{7/2}}$ $\quad f_{xxxy}^{(4)} = \dfrac{3xy(3y^2-2x^2-3)}{(1-x^2-y^2)^{7/2}}$ $\quad f_{xxyy}^{(4)} = \dfrac{6x^4-21x^2y^2-6x^2-y^2+2y^4-1}{(1-x^2-y^2)^{7/2}}$ $\quad f_{xyyy}^{(4)} = \dfrac{14x^3y-16xy^3-14xy}{(1-x^2-y^2)^{7/2}}$

$f_{yyyy}^{(4)} = \dfrac{3(x^2-1)(9y^2-x^2+1)}{(1-x^2-y^2)^{7/2}}$

$$f(0,0) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{24}(-3x^4 - 6x^2y^2 - 3y^4)$$

$$= 1 - \frac{1}{2}(x^2+y^2) - \frac{1}{8}(x^4+2x^2y^2+y^4) + o(x^4+y^4)$$

(3). $f(x,y) = \frac{1}{1-x-y+xy}$

$f_x' = \frac{1-y}{(1-x-y+xy)^2}$ $\quad f_y' = \frac{1-x}{(1-x-y+xy)^2}$

$f_{xx}'' = \frac{2(1-y)^2}{(1-x-y+xy)^3}$ $\quad f_{xy}'' = \frac{1-x-y+xy}{(1-x-y+xy)^3} = \frac{1}{(1-x-y+xy)^2}$ $\quad f_y'' = \frac{2(1-x)^2}{(1-x-y+xy)^3}$

$f_{xxx}^{(3)} = \frac{3!\,(1-y)^3}{(1-x-y+xy)^4}$ $\quad f_{xxy}^{(3)} = \frac{2(1-y)(1-x-y+xy)}{(1-x-y+xy)^4} = \frac{2(1-y)}{(1-x-y+xy)^3}$ $\quad f_{xyy}^{(3)} = \frac{2(1-x)}{(1-x-y+xy)^3}$ $\quad f_{yyy}^{(3)} = \frac{3!\,(1-x)^3}{(1-x-y+xy)^4}$

$\vdots$

$f_{x\cdots x}^{(n)} = \frac{n!\,(1-y)^n}{(1-x-y+xy)^{n+1}}$ $\quad \cdots\cdots \quad$ $f_{y\cdots y}^{(n)} = \frac{n!\,(1-y)^n}{(1-x-y+xy)^{n+1}}$

$$f(x,y) = \sum_{m=0}^{n} \frac{1}{m!} \sum_{l=0}^{m} C_m^l\, x^l\, y^{m-l} \frac{\partial^m}{\partial x^l\, \partial y^{m-l}} f(0,0) + o(x^n + y^n)$$

(4) $f(x,y) = \arctan \frac{1+x+y}{1-x+y}$

$f(0,0) = \frac{\pi}{4}$

$f_x' = \frac{1+y}{(1+y)^2 + x^2}$ $\quad f_y' = \frac{-x}{(1+y)^2 + x^2}$

$f_{xx}'' = \frac{-2x(1+y)}{[(1+y)^2 + x^2]^2}$ $\quad f_{xy}'' = \frac{x^2 - (1+y)^2}{[(1+y)^2 + x^2]^2}$ $\quad f_{yy}'' = \frac{2x(1+y)}{[(1+y)^2 + x^2]^2}$

$\therefore f(x,y) = \frac{\pi}{4} + x - xy + o(x^2 + y^2)$

(6). $f(x,y) = \frac{\cos x}{\cos y}$

$f(0,0) = 1$

$f_x' = -\frac{\sin x}{\cos y}$ $\quad f_y' = \frac{\cos x \sin y}{\cos^2 y}$

$f_{xx}'' = -\frac{\cos x}{\cos y}$ $\quad f_{xy}'' = -\frac{\sin x \sin y}{\cos^2 y}$ $\quad f_{yy}'' = \frac{\cos x\,(\cos^2 y + 2\sin^2 y)}{\cos^3 y}$

$\therefore f(x,y) = 1 - \frac{1}{2}x^2 + \frac{1}{2}y^2 + o(x^2 + y^2)$

5. 解: $z^3-2xz+y=0$ $z(1,1)=1$

两侧对x求偏导数 $3z^2z'_x-2z-2xz'_x=0$ ① 得 $z'_x=\frac{2z}{3z^2-2x}$ $z'_x(1,1)=2$

两侧对x求二阶偏导数得 $3z^2z''_{xx}+6z(z'_x)^2-2z'_x-2z'_x-2xz''_{xx}=0$ 得 $z''_{xx}(1,1)=-16$

两侧对y求偏导数得 $3z^2z'_y-2xz'_y+1=0$ ② 得 $z'_y(1,1)=-1$

①式两侧对y求偏导数得 $6zz'_yz'_x+3z^2z''_{xy}-2z'_y-2xz''_{xy}=0$ 得 $z''_{xy}=10$

②式两侧对y求偏导数得 $6z(z'_y)^2+3z^2z''_{yy}-2xz''_{yy}=0$ 得 $z''_{yy}(1,1)=-6$

$\therefore z=1+2(x-1)-(y-1)-8(x-1)^2+10(x-1)(y-1)-3(y-1)^2+o[(x-1)^2+(y-1)^2]$

7. (1) $f(x,y)=xy+\frac{50}{x}+\frac{20}{y}$ $(x>0,\ y>0)$

$\frac{\partial f}{\partial x}=y-\frac{50}{x^2}$ $\frac{\partial f}{\partial y}=x-\frac{20}{y^2}$ 令 $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$ 得 $x=5$ $y=2$

$\frac{\partial^2 f}{\partial x^2}=\frac{100}{x^3}$ $\frac{\partial^2 f}{\partial x\partial y}=1$ $\frac{\partial^2 f}{\partial y^2}=\frac{40}{y^3}$

则 $Q(h,k)=\frac{4}{5}h^2+2hk+5k^2$

$\Delta=\frac{4}{5}\times5-1=3>0$ 则 $(5,2)$ 是极小值点. $f(5,2)=30$ 则极小值为30.

(4). $(x^2+y^2)^2=a^2(x^2-y^2)$ $a^2=\frac{(x^2+y^2)^2}{x^2-y^2}$

解: 两侧对x求导: $2(x^2+y^2)(2x+2y\,y'_x)=a^2(2x-2y\cdot y'_x)$

得 $2(x^2-y^2)(x+y\,y'_x)=(x^2+y^2)(x-y\,y'_x)$

当$y=y(x)$取极值时 $y'_x=0$ 得 $y^2=\frac{1}{3}x^2$ $y'=\frac{2}{3}x=0$ 得 $x=0$ $y''=\frac{2}{3}>0$

$\therefore$ 极小值为0

10 (2). $u=x+y+z$ $\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ $\quad x>0\ \ y>0\ \ z>0$

解: $u=x+y+\frac{xy}{xy-x-y}$

$u_x'=1+\frac{-y^2}{(xy-x-y)^2}$ $\quad u_y'=1+\frac{-x^2}{(xy-x-y)^2}$ $\quad$ 令 $u_x'=u_y'=0$ 得 $(x,y)=(1,1)$ 或 $(x,y)=(3,3)$

$u_{xx}''=\frac{2y^2(y-1)}{(xy-x-y)^3}$ $\quad u_{xy}''=\frac{2xy}{(xy-x-y)^3}$ $\quad u_{yy}''=\frac{2x^2(x-1)}{(xy-x-y)^3}$

当 $(x,y)=(1,1)$ 时 $\quad Q(h,k)=-4hk$ $\quad \Delta=-16<0$ $\quad \therefore Q(h,k)$ 是不定的. 即 $(x,y)=(1,1)$ 不是极值点.

当 $(x,y)=(3,3)$ 时 $\quad Q(h,k)=\frac{4}{3}h^2+\frac{4}{3}hk+\frac{4}{3}k^2$ $\quad \Delta=(\frac{4}{3})^2-(\frac{2}{3})^2>0$ $\quad \therefore (3,3)$ 为极小值点, 极小值为 9

14. $u=xyz$ $\quad x+y+z=0$ $\quad x^2+y^2+z^2=1$

解: $z=-x-y$ $\quad x^2+y^2+(x+y)^2=1$ 得 $\quad y^2+xy+x^2-\frac{1}{2}=0$ $\quad y^2+xy=\frac{1}{2}-x^2$

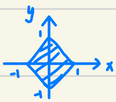
则 $u=-xy(x+y)=-x(y^2+xy)=-x\cdot(\frac{1}{2}-x^2)=x^3-\frac{1}{2}x$

$u_x'=3x^2-\frac{1}{2}=0$ 得 $\quad x=\frac{\sqrt{6}}{6}$ 或 $x=-\frac{\sqrt{6}}{6}$

$u_x''=6x$ $\quad u_x''(\frac{\sqrt{6}}{6})=\sqrt{6}>0$ $\quad u_x''(-\frac{\sqrt{6}}{6})=-\sqrt{6}<0$ $\quad \therefore x=\frac{\sqrt{6}}{6}$ 为极小值点, $x=-\frac{\sqrt{6}}{6}$ 为极大值点.

$u(\frac{\sqrt{6}}{6})=-\frac{\sqrt{6}}{18}$ $\quad u(-\frac{\sqrt{6}}{6})=\frac{\sqrt{6}}{18}$ $\quad \therefore$ 极小值为 $-\frac{\sqrt{6}}{18}$ 极大值 $\frac{\sqrt{6}}{18}$

11 (2). $z=x^2-xy+y^2$ $\quad \{(x,y)\mid |x|+|y|\le 1\}$



解: $z_x'=2x-y$ $\quad z_y'=2y-x$ $\quad$ 令 $z_x'=0$ $z_y'=0$ 得 $x=0,\ y=0$

$z_{xx}''=2$ $\quad z_{xy}'=-1$ $\quad z_{yy}''=2$

则 $Q(h,k)=2h^2-2hk+2k^2$ $\quad \Delta=4-1=3>0$ $\quad \therefore (0,0)$ 为极小值点, 也即最小值点.

$\therefore z_{min}=z(0,0)=0$ $\quad z_{max}=z(0,1)=1$ $\quad \therefore$ 最小值 0 最大值 1.

(4). $z=x^2y(4-x-y)$ $\{(x,y) \mid x\geq 0, y\geq 0, x+y\leq 6\}$

解: $z_x'=2xy(4-x-y)-x^2y=xy(8-3x-2y)$ $z_y'=x^2(4-x-y)-x^2y=x^2(4-x-2y)$

令 $z_x'=z_y'=0$ 得 $x=0$ $y\in[0,6]$ 或 $(x,y)=(4,0)$ 或 $(x,y)=(2,1)$

$z_{xx}''=y(8-6x-2y)$ $z_{xy}''=x(8-3x-4y)$ $z_{yy}''=-2x^2$

①. 当 $(x,y)=(4,0)$ 时 $Q(h,k)=-32hk-32k^2$ $\Delta=-16^2<0$ ∴ $(4,0)$ 不是极值点

②. 当 $(x,y)=(2,1)$ 时 $Q(h,k)=-6h^2-8hk-8k^2$ $\Delta=48-16>0$ 又∵ $A=-6<0$ ∴ $(2,1)$ 为极大值点

③. 当 $x=0$ $y\in[0,6]$ 时 $z=0$

∴ $z_{max}=z(2,1)=4$ $z_{min}=z(6,0)=-72$

14. 证明: $f(x,y)=3x^2y-x^4-2y^2$

$f_x'=6xy-4x^3$ $f_y'=3x^2-4y$ 当 $(x,y)=(0,0)$ 时 $f_x'(0,0)=f_y'(0,0)=0$ ∴ $(0,0)$ 为驻点

$f_{xx}''=6y-12x^2$ $f_{xy}''=6x$ $f_{yy}''=-4$

则 $Q(h,k)=-4k^2$ $\Delta=0$ ∴ 无法判断 $(0,0)$ 是否为极值点

①. 当过 $(0,0)$ 直线 l 不存在斜率时: l 为 $x=0$

则 $g(y)=-2y^2$ $g'(y)=-2y$ 当 $y>0$ 时 $g'(y)<0$ $g(y)$ 单调递减. 当 $y<0$ 时 $g'(y)>0$ $g(y)$ 单调递增.

∴ $y=0$ 为 $g(y)$ 极大值点.

②. 当过 $(0,0)$ 的直线 l 存在斜率时 l: $y=kx$

$h(x)=3kx^3-x^4-2k^2x^2$ $h'(x)=9kx^2-4x^3-4k^2x$ $\lim\limits_{x\to 0^+} h'(x)<0$ $h(x)$ 单调递减. $\lim\limits_{x\to 0^-} h'(x)>0$ $h(x)$ 单调递增

∴ $x=0$ 为 $h(x)$ 极大值点,

即 $(0,0)$ 为过 $(0,0)$ 直线的极大值点.

16. 解：设相邻两棱 棱长为 x, y 则第三条棱长为 $(3a-x-y)$

当体积最大时，平行六面体任意两条相交棱垂直.

则 $V = xy\cdot(3a-x-y)$

由基本不等式 $xy(3a-x-y) \le \left[\frac{x+y+(3a-x-y)}{3}\right]^3 = a^3$

当且仅当 $x=y=3a-x-y=a$ 时等号成立.

∴最大体积为 a^3

20. 解：设 $(x_0, y_0, z_0) \in \frac{x^2}{4}+y^2+z^2=1$ 即 $\frac{x_0^2}{4}+y_0^2+z_0^2=1$

$d = \frac{|x_0+y_0+2z_0-9|}{\sqrt{1+1+4}}$ 由几何关系得 椭球面与平面无交点. 则 $|x_0+y_0+2z_0-9| = -x_0-y_0-2z_0+9$

令 $f(x,y,z) = -x-y-2z+9$

①. $z=\sqrt{1-\frac{x^2}{4}-y^2}$ 时: $f(x,y) = -x-y-\sqrt{4-x^2-4y^2}+9$

$f_x' = -1+\frac{x}{\sqrt{4-x^2-4y^2}}$ $f_y' = -1+\frac{4y}{\sqrt{4-x^2-4y^2}}$ 令 $f_x'=f_y'=0$ 得 $(x,y)=(\frac{4}{3}, \frac{1}{3})$

$f_{xx}'' = \frac{4-4y^2}{(4-x^2-4y^2)^{3/2}}$ $f_{xy}'' = \frac{4xy}{(4-x^2-4y^2)^{3/2}}$ $f_{yy}'' = \frac{4(4-x^2)}{(4-x^2-4y^2)^{3/2}}$

当 $(x,y)=(\frac{4}{3},\frac{1}{3})$ 时 $Q(h,k) = \frac{3}{2}h^2+\frac{3}{2}hk+\frac{15}{4}k^2$ $\Delta = \frac{3}{2}\times\frac{15}{4}-(\frac{3}{4})^2 = \frac{81}{16} > 0$ ∴$(\frac{4}{3},\frac{1}{3})$为极小值点.

$f(\frac{4}{3},\frac{1}{3}) = -\frac{4}{3}-\frac{1}{3}-\frac{4}{3}+9 = 6$ $z=\frac{2}{3}$ $1-\frac{4}{9}-\frac{1}{9}$

②. $z=-\sqrt{1-\frac{x^2}{4}-y^2}$ 时 $f(x,y) = -x-y+\sqrt{4-x^2-4y^2}+9$

$f_x' = -1-\frac{x}{\sqrt{4-x^2-4y^2}}$ $f_y' = -1-\frac{4y}{\sqrt{4-x^2-4y^2}}$ 令 $f_x'=f_y'=0$ 得 $(x,y)=(-\frac{4}{3}, -\frac{1}{3})$

$f_{xx}'' = \frac{-4+4y^2}{(4-x^2-4y^2)^{3/2}}$ $f_{xy}'' = \frac{-4xy}{(4-x^2-4y^2)^{3/2}}$ $f_{yy}'' = \frac{4(x^2-4)}{(4-x^2-4y^2)^{3/2}}$

当 $(x,y)=(-\frac{4}{3},-\frac{1}{3})$ 时 $Q(h,k) = -\frac{3}{2}h^2-\frac{3}{2}hk-\frac{15}{4}k^2$ $\Delta = (-\frac{3}{2})\times(-\frac{15}{4})-(\frac{3}{4})^2 > 0$ $-\frac{3}{2}<0$ ∴$(-\frac{4}{3},-\frac{1}{3})$为极大值点.

$f(-\frac{4}{3},-\frac{1}{3}) = \frac{4}{3}+\frac{1}{3}+\frac{4}{3}+9 = 12$ $z=-\frac{2}{3}$

综上所述 $\frac{x^2}{4}+y^2+z^2=1$ 上距 $x+y+2z=9$ 最远点 $(-\frac{4}{3}, -\frac{1}{3}, -\frac{2}{3})$ 最近点 $(\frac{4}{3}, \frac{1}{3}, \frac{2}{3})$

21. 曲面S: $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$

(1). 证明: 令 $f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}$.

则 $f'_x=\frac{1}{2\sqrt{x}}$ $f'_y=\frac{1}{2\sqrt{y}}$ $f'_z=\frac{1}{2\sqrt{z}}$ 则曲面在 (x,y,z)处 法向量为 $\vec{n}=\left(\frac{1}{2\sqrt{x}},\frac{1}{2\sqrt{y}},\frac{1}{2\sqrt{z}}\right)$

则在 (x_0,y_0,z_0) 处切平面为 $\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$

即 $\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{a}$

则截距式方程为 $\frac{x}{\sqrt{ax_0}}+\frac{y}{\sqrt{ay_0}}+\frac{z}{\sqrt{az_0}}=1$

则截距和 $\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=a$

(2). $V=\frac{1}{3}\left(\frac{1}{2}\cdot\sqrt{ax_0}\cdot\sqrt{ay_0}\right)\sqrt{az_0}=\frac{1}{6}a\sqrt{a}\cdot\sqrt{x_0y_0z_0}$

$\sqrt{z_0}=\sqrt{a}-\sqrt{x_0}-\sqrt{y_0}$

令 $g(x,y)=\sqrt{x}\cdot\sqrt{y}\cdot(\sqrt{a}-\sqrt{x}-\sqrt{y})\le\left[\frac{\sqrt{x}+\sqrt{y}+(\sqrt{a}-\sqrt{x}-\sqrt{y})}{3}\right]^3=\frac{a\sqrt{a}}{27}$

当且仅当 $\sqrt{x}=\sqrt{y}=\sqrt{a}-\sqrt{x}-\sqrt{y}=\frac{\sqrt{a}}{3}$ 时等号成立.

$\therefore V_{max}=\frac{1}{6}a\sqrt{a}\cdot\frac{a\sqrt{a}}{27}=\frac{a^3}{162}$ 此时切面方程为: $\frac{3x}{a}+\frac{3y}{a}+\frac{3z}{a}=1$