介观物理第四次作业

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Problem I.10 Derive the diamagnetic part in Eq. (68).

$$\sigma_{\alpha\beta}(\omega) = \frac{1}{i\omega}\left[\Pi_{\alpha\beta}(\omega) - \frac{ne^2}{m}\delta_{\alpha\beta}\right]. \tag{68}$$

Proof. 抗磁电流密度为:

$$\mathbf{j}_D = -\frac{e^2}{m}\mathbf{A}(\mathbf{r})\rho(\mathbf{r}).$$

可以将电子密度近似为 $n = \langle\rho(\mathbf{r})\rangle$，则有:

$$\mathbf{j}_D = -\frac{ne^2}{m}\mathbf{A}(\mathbf{r}).$$

利用

$$\mathbf{E}(t) = -\frac{\partial \mathbf{A}(t)}{\partial t};\quad \mathbf{E}(\omega) = i\omega\mathbf{A}(\omega).$$

可以得到:

$$\mathbf{j}_D(\omega) = -\frac{ne^2}{m}\frac{1}{i\omega}\mathbf{E}(\omega).$$

写成分量形式，即:

$$j_{D\alpha}(\omega) = -\frac{1}{i\omega}\frac{ne^2}{m}\delta_{\alpha\beta}E_\beta(\omega).$$

即响应函数抗磁项为:

$$\sigma_{\alpha\beta}{}_D = -\frac{1}{i\omega}\frac{ne^2}{m}\delta_{\alpha\beta}.$$

□

Problem I.11 Write down the Kramers-Kronig relation for $\sigma_1(\omega)$ and $\sigma_2(\omega)$.

Proof.

$$\begin{cases} \sigma_1(\omega) = \displaystyle\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\sigma_2(\omega')P\frac{1}{\omega'-\omega}; \\ \sigma_2(\omega) = -\displaystyle\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\sigma_1(\omega')P\frac{1}{\omega'-\omega}. \end{cases}$$

□

Problem I.12 Verify Eq. (70b). Hint: Use Eq. (68).

$$\sigma(\omega \pm i0^+) = \pm\sigma_1(\omega) + i\sigma_2(\omega). \tag{70b}$$

Proof. 对于直流电，$\omega=0$，电导是一个有限值，即

$$\Pi_{\alpha\beta}(\mathbf{q}=0,\omega=0)=i\int_{-\infty}^{t}dt'\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle=\frac{ne^2}{m}\delta_{\alpha\beta}$$

即有 $\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle$ 为纯虚数。考虑电流电流耦合项拆分成实部和虚部 $\Pi_{\alpha\beta}(\omega)=\Pi^{(1)}_{\alpha\beta}(\omega)+i\Pi^{(2)}_{\alpha\beta}(\omega)$:

$$\begin{aligned}\Pi_{\alpha\beta}(\omega)=&i\int_{-\infty}^{t}dt'e^{i\omega(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\\=&i\int_{-\infty}^{t}dt'[\cos\omega(t-t')+i\sin\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\\=&i\int_{-\infty}^{t}dt'\cos[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle-\int_{-\infty}^{t}dt'\sin[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle.\end{aligned}$$

即有:

$$\begin{cases}\Pi^{(1)}_{\alpha\beta}(\omega)=i\int_{-\infty}^{t}dt'\cos[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle;\\ \Pi^{(2)}_{\alpha\beta}(\omega)=i\int_{-\infty}^{t}dt'\sin[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle.\end{cases}$$

从而可以得到电导率的实部和虚部分别为:

$$\begin{cases}\sigma_1(\omega)=\frac{1}{\omega}\Pi^{(2)}_{\alpha\beta}(\omega)=\frac{i}{\omega}\int_{-\infty}^{t}dt'\sin[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle;\\ \sigma_2(\omega)=\frac{1}{\omega}\left[\frac{ne^2}{m}\delta_{\alpha\beta}-\Pi^{(1)}_{\alpha\beta}\right]=\frac{1}{\omega}\left[\frac{ne^2}{m}\delta_{\alpha\beta}-i\int_{-\infty}^{t}dt'\cos[\omega(t-t')]\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right].\end{cases}$$

可以计算:

$$\begin{aligned}\sin[(\omega+i0^+)(t-t')]=&\sin[\omega(t-t')]\cosh[0^+(t-t')]+i\cos[\omega(t-t')]\sinh[0^+(t-t')];\\ \cos[(\omega+i0^+)(t-t')]=&\cos[\omega(t-t')]\cosh[0^+(t-t')]-i\sin[\omega(t-t')]\sinh[0^+(t-t')].\end{aligned}$$

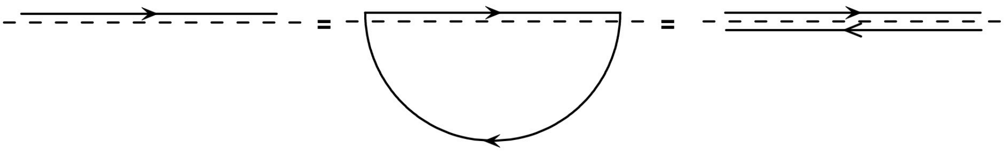
在原始电导率表达式中，考虑 $\sigma(\omega\pm0^+)$ 如下:

$$\begin{aligned}\sigma_{\alpha\beta}(\omega+i0^+)=&\frac{1}{i(\omega+i0^+)}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'e^{i(\omega+i0^+)(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\\=&\frac{1}{i\omega-0^+}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'e^{i\omega(t-t')}e^{-0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\\=&-\frac{i\omega+0^+}{\omega^2}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'\cos[\omega(t-t')]e^{-0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\\&+\frac{i\omega+0^+}{\omega^2}\int_{-\infty}^{t}dt'\sin[\omega(t-t')]e^{-0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle;\\ \sigma_{\alpha\beta}(\omega-i0^+)=&\frac{1}{i(\omega-i0^+)}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'e^{i(\omega-i0^+)(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\\=&\frac{1}{i\omega+0^+}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'e^{i\omega(t-t')}e^{0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\end{aligned}$$

$$
\begin{aligned}
=&-\frac{i\omega-0^+}{\omega^2}\left[-\frac{ne^2}{m}\delta_{\alpha\beta}+i\int_{-\infty}^{t}dt'\cos[\omega(t-t')]e^{0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle\right]\\
&+\frac{i\omega+0^+}{\omega^2}\int_{-\infty}^{t}dt'\sin[\omega(t-t')]e^{0^+(t-t')}\langle[j_{P\alpha}(t),j_{P\beta}(t')]\rangle;
\end{aligned}
$$

□

Problem I.13 Derive Eq. (79) using Eq. (70b). Hint: Consider a contour deformation as follows:



$$\sigma(\omega\pm i0^+)=\pm\sigma_1(\omega)+i\sigma_2(\omega). \tag{70b}$$

$$\frac{ne^2}{m}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega0^+}\sigma(\omega)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\sigma_1(\omega)+i\sigma_2(\omega)]=\int_{0}^{\infty}\frac{d\omega}{\pi}\sigma_1(\omega). \tag{79}$$

Proof. 注意到由于复平面下半平面无穷远处模收敛，所以从 $-\infty$ 到 ∞ 的积分可以拓展到复平面下半平面，形成闭合回路积分。下半平面无穷远处的积分，可以连续变换至位于实轴下方无穷小距离处从正无穷到负无穷的积分。即:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega0^+}\sigma(\omega)=&\oint\frac{d\omega}{2\pi}e^{-i\omega0^+}\sigma(\omega)\\
=&\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sigma(\omega+i0^+)+\int_{\infty}^{-\infty}\frac{d\omega}{2\pi}\sigma(\omega-i0^+)\\
=&\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\sigma(\omega+i0^+)-\sigma(\omega-i0^+)]\\
=&\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\sigma_1(\omega)\\
=&2\int_{0}^{\infty}\frac{d\omega}{\pi}\sigma_1(\omega)\\
=&\frac{ne^2}{m}.
\end{aligned}
$$

其中，利用了

$$\sigma(\omega+i0^+)-\sigma(\omega-i0^+)=\sigma_1(\omega)+i\sigma_2(\omega)-[-\sigma_1(\omega)+i\sigma_2(\omega)]=2\sigma_1(\omega).$$

□