介观物理第九次作业

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Problem III.3 Order of magnitude estimation: For a Fermi energy of E_F and N electrons in the metallic ring, the spacing between the bands will be on the order of E_F/N. Suppose that $N = 10^4$, estimate the level spacing of a typical ladder and the corresponding temperature scale. Then find the physical condition under which the persistent current can be observed in the presence of inelastic scattering.

Hint: The width of the Fermi tail should be small compared to the level spacing in Fig. 17.

Proof. 为了进行数量级估计，可以取费米能为 $E_F = 5eV$，则能隙大小约为:

$$E_{gap} \approx \frac{E_F}{N} = 5 \times 10^{-4} eV.$$

对应温度为:

$$T = \frac{E_{gap}}{k_B} = 5.80K.$$

即温度高于 $5.80K$，且圆环尺寸为介观尺度，约小于 $10^{-6}m$，可以观察到非弹性散射存在条件下的持续的电流。 □

Problem III.5 Check Eqs. (193) with the unitary condition $SS^\dagger = \mathbb{I}$.

$$T_i = \sum_j T_{ij}, \quad R_i = \sum_j R_{ij}; \tag{193a}$$

$$\sum_i T_i = \sum_i (1 - R_i), \quad \sum_i T_i' = \sum_i (1 - R_i'); \tag{193b}$$

$$R_i' + T_i = 1, \quad R_i + T_i' = 1; \tag{193c}$$

Proof. S 矩阵为:

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.$$

其中 r, r'; t, t' 都是 $N \times N$ 矩阵。则可以计算:

$$SS^\dagger = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} r^\dagger & t^\dagger \\ t'^\dagger & r'^\dagger \end{pmatrix} = \begin{pmatrix} rr^\dagger + t't'^\dagger & rt^\dagger + t'r'^\dagger \\ tr^\dagger + r't'^\dagger & tt^\dagger + r'r'^\dagger \end{pmatrix} = \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N \end{pmatrix}.$$

其中 $\mathbb{I}_N$ 是 $N \times N$ 的单位矩阵。当具有时间反演对称性时，S 是一个对称矩阵，即 $S = S^T$，从而有

$$r = r^T; \quad r' = r'^T; \quad t' = t^T; \quad t = t'^T.$$

即矩阵元满足如下关系;

$$r_{ij} = r_{ji}; \quad r'_{ij} = r'_{ji}.$$

根据 $rr^\dagger + t't'^\dagger = \mathbb{I}_N$，从而可以计算:

$$\sum_m r_{im}r^*_{im} + \sum_n t'_{in}t'^*_{in} = R_i + T'_i = 1.$$

类似的，根据 $tt^\dagger + r'r'^\dagger = \mathbb{I}_N$，从而可以计算:

$$\sum_m t_{im}t^*_{im} + \sum_n r'_{in}r'^*_{in} = T_i + R'_i = 1.$$

即验证了式 (193c)。

利用 $t' = t^T$，则有 $t'^\dagger = t^*$，从而有:

$$rr^\dagger + t^T t^* = \mathbb{I}_N.$$

两侧取迹，则有:

$$\sum_i \sum_j R_{ij} + \sum_m \sum_n T_{nm} = \sum_i R_i + \sum_n T_n = \sum_k 1.$$

所有求和指标均从 1 求和至 N。其中由于求和项有限，可以交换 m 和 n 的求和顺序，移项可将方程重新写为:

$$\sum_i T_i = \sum_i (1 - R_i).$$

类似的，利用 $t = t'^T$，则有 $t^\dagger = t'^*$，从而有:

$$r'r'^\dagger + t'^T t'^* = \mathbb{I}_N.$$

同样的，两侧取迹，交换求和顺序，移项可以得到:

$$\sum_i T'_i = \sum_i (1 - R'_i).$$

即验证了方程 (193b)。 $\square$