第五次作业 董建宇 2019511017

1. 解. $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & , -\infty < x < +\infty , \quad t>0 \\ u(x,0)=\varphi(x) , \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) & , -\infty < x < +\infty \end{cases}$

对于泛定方程关于 x 做傅立叶变换得 $\begin{cases} \dfrac{d^2 U(w,t)}{dt^2} = -a^2 w^2 U(w,t) , \\ U(w,0) = A(w) = \int_{-\infty}^{+\infty} \varphi(x) e^{-iwx} dx , \quad \dfrac{dU}{dt}\Big|_{t=0} = B(w) = \int_{-\infty}^{+\infty} \psi(x) e^{-iwx} dx \end{cases}$

则常微分方程解为 $U(w,t) = A(w)\cos(awt) + \dfrac{B(w)}{aw}\sin(awt)$

(a) 当 $\psi(x)=0$ 时, 有 $B(w) = \int_{-\infty}^{+\infty} 0\cdot e^{-iwx} dx = 0$, 则 $U(w,t) = A(w)\cos(awt)$

做傅立叶逆变换得: $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [A(w)\cos(awt)] e^{iwx} dw$ 　　其中 $\cos(awt) = \dfrac{e^{iawt}+e^{-iawt}}{2}$

则 $u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} [A(w)(e^{iawt}+e^{-iawt})] e^{iwx} dw = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \varphi(\xi) e^{-iw\xi} d\xi\right]\cdot(e^{iawt}+e^{-iawt}) e^{iwx} dw$

$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\left[\int_{-\infty}^{+\infty} e^{iawt}\cdot e^{-iw(\xi-x)} dw + \int_{-\infty}^{+\infty} e^{-iawt} e^{-iw(\xi-x)} dw\right] d\xi$

关于 w 的积分为 e^{iawt} 和 e^{-iawt} 的傅立叶变换 令 $X=\xi-x$ 则有:

$\int_{-\infty}^{+\infty} e^{iawt} e^{-iwX} dw = 2\pi\delta(X-at) \quad \int_{-\infty}^{+\infty} e^{-iawt} e^{-iwX} dw = 2\pi\delta(X+at)$

则 $u(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} \varphi(\xi)[\delta(\xi-x-at)+\delta(\xi-x+at)] d\xi = \varphi(x) * \left\{\frac{1}{2}[\delta(x-at)+\delta(x+at)]\right\} = \varphi(x) * K_1(x,t)$

即积分核为 $K_1(x,t) = \begin{cases} \frac{1}{2}[\delta(x-at)+\delta(x+at)] , & t>0 \\ 0 , & t\le 0 \end{cases}$

物理意义为: K_1 为初始时刻位于 $x=0$ 的振幅点源, 在 t 时刻引起的振幅分布.

(b). 当 $\psi(x)$ 不为零时, $B(w) = \int_{-\infty}^{+\infty} \psi(x) e^{-iwx} dx$ 　　$U(w,t) = A(w)\cos(awt) + \dfrac{B(w)}{aw}\sin(awt)$

做傅立叶逆变换得 $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[A(w)\cos(awt) + \frac{B(w)}{aw}\sin(awt)\right] e^{iwx} dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(w)\cos(awt) e^{iwx} dw + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{B(w)}{aw}\sin(awt) e^{iwx} dw$

其中, 由(a)问可知 $\frac{1}{2\pi}\int_{-\infty}^{+\infty} A(w)\cos(awt) e^{iwx} dw = \varphi(x) * \left[\frac{\delta(x+at)+\delta(x-at)}{2}\right]$

接下来考虑 $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{B(w)}{aw}\sin(awt) e^{iwx} dw = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{B(w)}{iaw}(e^{iawt}-e^{-iawt}) e^{iwx} dw$

$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{iaw}\left[\int_{-\infty}^{+\infty} \psi(s) e^{-isw} ds\right]\cdot(e^{iawt}-e^{-iawt}) e^{iwx} dw$

$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \psi(s)\left[\int_{-\infty}^{+\infty} \frac{1}{iaw} e^{iawt}\cdot e^{-iw(s-x)} dw - \int_{-\infty}^{+\infty} \frac{1}{iaw} e^{-iawt}\cdot e^{-iw(s-x)} dw\right] ds$ 　　①

令 $X=s-x$, 则可计算得 $\int_{-\infty}^{+\infty} \frac{1}{iaw} e^{iawt}\cdot e^{-iwX} dw = \frac{1}{ia}\int_{-\infty}^{+\infty} \frac{1}{w} e^{-iw(X-at)} dw = -\frac{\pi}{a}\,\mathrm{sgn}(X-at)$

$\int_{-\infty}^{+\infty} \frac{1}{iaw} e^{-iawt}\cdot e^{-iwX} dw = \frac{1}{ia}\int_{-\infty}^{+\infty} \frac{1}{w} e^{-iw(X+at)} dw = -\frac{\pi}{a}\,\mathrm{sgn}(X+at)$

则 ①式 $= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \psi(s)\cdot\frac{\pi}{a}[\mathrm{sgn}(X+at)-\mathrm{sgn}(X-at)] ds = \psi(s) * \left\{\frac{1}{4a}[\mathrm{sgn}(x+at)-\mathrm{sgn}(x-at)]\right\}$

即积分核为 $K_2(x,t) = \begin{cases} \frac{1}{4a}[\mathrm{sgn}(x+at)-\mathrm{sgn}(x-at)] , & t>0 \\ 0 , & t\le 0 \end{cases}$

$u(x,t) = \varphi(x) * K_1(x,t) + \psi(x) * K_2(x,t)$

物理意义为: K_1 为初始时刻位于 $x=0$ 的振幅点源, 在 t 时刻引起的振幅分布.

K_2 为初始时刻位于 $x=0$ 的速度点源在 t 时刻引起的振幅分布

2. 解: $(\nabla^2+k^2)G(\vec{r},\vec{r}')=-\frac{1}{\varepsilon_0}\delta(\vec{r}-\vec{r}')$ 因为无界空间, 则可令 $\vec{x}=\vec{r}-\vec{r}'$

则原式可化为 $(\nabla^2+k^2)G(\vec{x})=-\frac{1}{\varepsilon_0}\delta(\vec{x})$ 记 $g(\vec{\omega})=\iiint G(\vec{x})e^{-i\vec{\omega}\cdot\vec{x}}\cdot d\vec{x}$

关于 $\vec{x}$ 做傅里叶变换得 $(\omega^2+k^2)g(\vec{\omega})=-\frac{1}{\varepsilon_0}$

则有 $g(\vec{\omega})=-\frac{1}{\varepsilon_0}\frac{1}{\omega^2+k^2}$

进行傅里叶逆变换, 取 ω_z 取为位矢 $\vec{x}$ 方向.

则有 $G(x)=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint\frac{1}{\omega^2+k^2}e^{i\vec{\omega}\cdot\vec{x}}d\vec{\omega}$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{e^{i\omega x\cos\theta}}{\omega^2+k^2}\cdot\omega^2\sin\theta\,d\omega\,d\theta\,d\varphi$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{+\infty}\frac{\omega^2}{\omega^2+k^2}d\omega\int_0^{\pi}e^{i\omega x\cos\theta}\sin\theta\,d\theta\int_0^{2\pi}d\varphi$$

$$=-\frac{1}{2\pi^2\varepsilon_0}\int_0^{+\infty}\frac{\omega}{\omega^2+k^2}\cdot\frac{\sin\omega x}{x}d\omega=-\frac{1}{2\pi^2\varepsilon_0 x}\int_0^{+\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}d\omega$$

下面计算 $I=\int_0^{+\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}d\omega$, 注意到被积函数 $\frac{\omega\sin(\omega x)}{\omega^2+k^2}$ 为偶函数, 则有 $I=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\omega\sin(\omega x)}{\omega^2+k^2}d\omega$

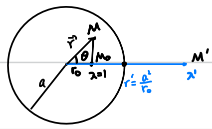
由于 $e^{i\omega x}=\cos(\omega x)+i\sin(\omega x)$, 则 $I=\mathrm{Im}\left(\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\omega}{\omega^2+k^2}e^{i\omega x}d\omega\right)$

由留数定理可知 $\int_{-\infty}^{+\infty}\frac{\omega}{\omega^2+k^2}e^{i\omega x}d\omega=2\pi i\,\mathrm{Res}[f(\omega),ik]=i\pi e^{-kx}$, 所以 $I=\frac{\pi}{2}e^{-kx}$

即 $G(x)=-\frac{1}{4\pi\varepsilon_0}\frac{1}{xe^{kx}}$, 则 $G(\vec{r},\vec{r}')=-\frac{1}{4\pi\varepsilon_0}\frac{e^{-k|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$

3. 解. 3.1 $\begin{cases}\nabla^2u=0, & (\rho<a)\\ u|_{\rho=a}=f(\varphi)\end{cases}$ 则格林函数对应的方程为 $\begin{cases}\nabla^2G(\vec{r},\vec{r}_0)=-\delta(\vec{r}-\vec{r}_0)\\ G|_{|\vec{r}_0|=a}=0\end{cases}$

则由电像法可得, $G(\vec{r},\vec{r}_0)$ 为在 $\vec{r}_0$ 处有一沿 z 轴无限长单位线电荷位于半径为 a 的沿 z 轴无限长接地导体。



由对称性可知, 像电荷位于 OM_0 延长线上, 设线电荷为 λ'

在满足边界条件 $G|_{r=a}=0$ 时, 任意一点 $M(r,\theta)$ 的电位为:

$$G(\vec{r},\vec{r}_0)=\frac{1}{2\pi}\ln\frac{1}{|MM_0|}+\frac{\lambda'}{2\pi}\ln\frac{1}{|MM'|}+\frac{1}{2\pi}\ln\frac{r_0}{a}$$

由余弦定理 $|MM_0|=\sqrt{r^2+r_0^2-2rr_0\cos\theta}$ $|MM'|=\sqrt{r'^2+r^2-2r'r\cos\theta}$

由于 $G(\vec{r},\vec{r}_0)$ 与 θ 无关, 则 $\frac{\partial G}{\partial\theta}\Big|_{r=a}=-\frac{1}{2\pi}\frac{r_0a\sin\theta}{a^2+r_0^2-2r_0r\cos\theta}-\frac{\lambda'}{2\pi}\frac{r'a\sin\theta}{a^2+r'^2-2r'a\cos\theta}=0$

即 $\lambda'r'(a^2+r_0^2-2r_0a\cos\theta)=-r_0(a^2+r'^2-2r'a\cos\theta)$, 对于任意 θ 等式都成立.

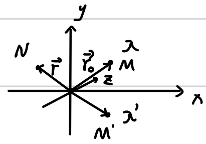
则有 $\lambda'=-1$, $r'=\frac{a^2}{r_0}$. 则 $G(\vec{r},\vec{r}_0)=\frac{1}{4\pi}\ln\frac{r^2+r'^2-2r'r\cos\theta}{r^2+r_0^2-2r_0r\cos\theta}+\frac{1}{2\pi}\ln\frac{r_0}{a}=\frac{1}{4\pi}\ln\left(\frac{r^2+r'^2-2r'r\cos\theta}{r^2+r_0^2-2r_0r\cos\theta}\cdot\frac{r_0^2}{a^2}\right)$

将 $r'=\frac{a^2}{r_0}$ 代入得 $G(\vec{r},\vec{r}_0)=\frac{1}{4\pi}\ln\left[\frac{r^2r_0^2+a^4-2a^2r_0r\cos\theta}{a^2(r^2+r_0^2-2r_0r\cos\theta)}\right]$

由于区域边界为以 $r_0=a$ 为半径的圆, 则 $\frac{\partial G}{\partial n_0}=\frac{\partial G}{\partial r_0}\Big|_{r_0=a}=\frac{1}{2\pi a}\frac{r^2-a^2}{r^2+a^2-2ar\cos\theta}$ 其中 $\theta=\varphi$

则有 $u(\vec{r})=-\int_c f(\varphi)\cdot\frac{\partial G}{\partial n_0}dl_0=\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\varphi)}{r^2+a^2-2ar\cos\varphi}d\varphi$

3.2 $\begin{cases} \nabla^2 u = 0, & y>0 \\ u|_{y=0} = f(x) \end{cases}$ 则格林函数对应的方程为 $\begin{cases} \nabla^2 G = -\delta(\vec{r}-\vec{r}_0) \\ G|_{y=0} = 0 \end{cases}$



由电像法可知，$G(\vec{r},\vec{r}_0)$ 为一沿 z 轴无限长单位线电荷在接地导体板上方，产生的电位。

由对称性可知，像电荷位于 $-\vec{r}_0$，像电荷线密度 $\lambda' = -\lambda$

任意一点 $N(x,y)$ 的电位为 $G(\vec{r},\vec{r}_0) = \frac{1}{2\pi}\ln\frac{1}{|MN|} - \frac{1}{2\pi}\ln\frac{1}{|M'N|}$，其中 $|MN| = \sqrt{(x_0-x)^2+(y_0-y)^2}$ $\quad |M'N| = \sqrt{(x_0-x)^2+(y_0+y)^2}$

则 $G(\vec{r},\vec{r}_0) = \frac{1}{4\pi}\ln\left[\frac{(x_0-x)^2+(y_0+y)^2}{(x_0-x)^2+(y_0-y)^2}\right]$，由于区域边界为 $y_0=0$，则 $\frac{\partial G}{\partial n_0} = \frac{\partial G}{\partial y_0}\Big|_{y_0=0} = \frac{1}{\pi}\frac{y}{(x_0-x)^2+y^2}$

则有 $u(\vec{r}) = -\int_c f(x_0)\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n_0}dl_0 = -\frac{y}{\pi}\int_{+\infty}^{-\infty}\frac{f(x_0)}{(x_0-x)^2+y^2}dx_0 = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(x_0)}{(x_0-x)^2+y^2}dx_0$

3.3. 由题意可得：$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{k}{\rho}\frac{\partial^2 u}{\partial x^2} + f_0\sin(\omega t)\delta(x-x_0) \\ u(0,t) = u(b,t) = 0,\ u(x,0)=0,\ \frac{\partial u}{\partial t}\big|_{t=0}=0 \end{cases}$ 则设 $u(x,t) = \int_0^b dx_1\int_0^{+\infty}dt_1\ f(x_1,t_1)\,G(x,t,x_1,t_1)$

设 $a^2 = \frac{k}{\rho}$

则 $G(x,t,x_0,t_0)$ 满足方程为：$\begin{cases} \frac{\partial^2 G}{\partial t^2} = a^2\frac{\partial^2 G}{\partial x^2} + \delta(x-x_1)\delta(t-t_1) \\ G(0,t,x_1,t_1) = G(b,t,x_1,t_1) = 0 \quad G(x,0,x_1,t_1) = 0 \quad \frac{\partial G}{\partial t}\big|_{t=0} = 0 \end{cases}$

设解为 $G(x,t,x_1,t_1) = X(x,x_1)T(t,t_1)$，可得到关于 x 的本征值问题 $X_n = B_n\sin(\frac{n\pi}{L}x)$，$\lambda_n = (\frac{n\pi}{L})^2$

得 $\sum_{n=1}^{\infty}(T_n'' + a^2\lambda_n T_n)\sin(\frac{n\pi}{L}x) = \delta(x-x_1)\delta(t-t_1)$，则有 $T_n'' + a^2\lambda_n T_n = \frac{2}{L}\int_0^L\delta(x-x_1)\delta(t-t_1)\sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\sin(\frac{n\pi}{L}x_1)\cdot\delta(t-t_1)$

关于 t 做拉普拉斯变换得 $p^2F(p) + a^2\lambda_n F(p) = \frac{2}{L}\sin(\frac{n\pi}{L}x_1)e^{-pt_1}$，即 $F(p) = \frac{2}{L}\sin(\frac{n\pi}{L}x_1)\cdot\frac{e^{-pt_1}}{p^2+a^2\lambda_n}$

做拉普拉斯逆变换得：$T_n(t) = \frac{2}{L}\sin(\frac{n\pi}{L}x_1)\cdot\frac{1}{a\sqrt{\lambda_n}}\cdot\mathcal{L}^{-1}\left(e^{-pt_1}\cdot\frac{a\sqrt{\lambda_n}}{p^2+a^2\lambda_n}\right) = \frac{2}{L}\sin(\frac{n\pi}{L}x_1)\frac{1}{a\sqrt{\lambda_n}}\int_0^t d\tau\,\delta(\tau-t_1)\sin[a\sqrt{\lambda_n}(t-\tau)]$

$$= \frac{2}{an\pi}\sin(\frac{n\pi}{L}x_1)\ \sin\left(\frac{an\pi}{L}(t-t_1)\right)$$

则格林函数为 $G(x,x_1,t,t_1) = \sum_{n=1}^{\infty}\frac{2}{an\pi}\sin(\frac{n\pi}{L}x_1)\cdot\sin(\frac{n\pi}{L}x)\cdot\sin\left(\frac{an\pi}{L}(t-t_1)\right)$

则 $u(x,t) = \int_0^L dx_1\int_0^{+\infty}dt_1\ G(x,x_1,t,t_1)\cdot f_0\cdot\sin(\omega t_1)\delta(x_1-x_0)$

$$= \frac{2f_0}{\pi a}\sum_{n=1}^{+\infty}\frac{1}{n}\sin(\frac{n\pi}{L}x_0)\sin(\frac{n\pi}{L}x)\int_0^{+\infty}\sin\left(\frac{an\pi}{L}(t-t_1)\right)\cdot\sin(\omega t_1)\,dt_1$$

3.4. $\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, & x>0 \\ u|_{x=0} = At, & u|_{t=0} = 0 \end{cases}$ 令 $w = u - At$，则 w 满足 $\begin{cases} \dfrac{\partial w}{\partial t} = a^2 \dfrac{\partial^2 w}{\partial x^2} + A \\ w|_{x=0} = 0, \quad w|_{t=0} = 0 \end{cases}$ (*)

即求解 w 的定解问题：设定解问题 $\begin{cases} \dfrac{\partial v}{\partial t} = a^2 \dfrac{\partial^2 v}{\partial x^2} \\ v|_{t=\tau} = A, \quad v|_{x=0} = 0 \end{cases}$ (**) 的解为 $v(x,t,\tau)$，则(*)式的解为 $w(x,t) = \int_0^t v(x,t,\tau)\,d\tau$. ①

下面验证①式为(*)式的解：$w(0,t) = \int_0^t v(0,t,\tau)\,d\tau = 0$，$w(x,0) = \int_0^0 v(x,0,\tau)\,d\tau = 0$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\int_0^t v(x,t,\tau)\,d\tau = \int_0^t \left[\frac{\partial}{\partial t} v(x,t,\tau)\right] d\tau + v(x,\tau) = a^2 \frac{\partial^2}{\partial x^2}\int_0^t v(x,t,\tau)\,d\tau + A = a^2 \frac{\partial^2 w}{\partial x^2} + A.$$

即①式为(*)式的解. 设(**)的解为 $v(x,t,\tau) = \int_0^{+\infty} A\,G(x,x_0,t,\tau)\,dx_0$

则有 $\begin{cases} \dfrac{\partial G}{\partial t} = a^2 \dfrac{\partial G}{\partial x^2} \\ G|_{t=\tau} = \delta(x-x_0), \quad G|_{x=0} = 0 \end{cases}$ 由于端点固定，可以奇延拓至全空间.

关于 x 做傅里叶变换 $F(w,t) = \int_{-\infty}^{+\infty} G(x,x_0,t)\cdot e^{-iwx}\,dx$

则有：$\begin{cases} \dfrac{\partial F(w,t)}{\partial t} = -a^2 w^2 F(w,t) \\ F(w,\tau) = e^{-iwx_0} \end{cases}$，解为 $F(w,t) = e^{-iwx_0}\cdot e^{-a^2 w^2 (t-\tau)}$

$X = \xi - x$

利用傅里叶逆变换得 $G(x,x_0,t,\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} e^{-iwx_0}\cdot e^{-a^2 w^2 (t-\tau)}\,dw = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\xi - x_0)\left[\int_{-\infty}^{+\infty} \exp(-w^2 a^2 (t-\tau))\cdot e^{-iwX}\,dw\right] d\xi$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\xi - x_0)\sqrt{\frac{\pi}{a^2(t-\tau)}}\cdot \exp\left[-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right] d\xi$$

$$= \frac{1}{2a\sqrt{\pi(t-\tau)}}\cdot \exp\left[-\frac{(x-x_0)^2}{4a^2(t-\tau)}\right]$$

则 $v(x,t,\tau) = \int_0^{+\infty} A\cdot G(x,x_0,t,\tau)\,dx_0$

则 $u(x,t) = At + w(x,t) = At + \int_0^t \int_0^{+\infty} A\,G(x,x_0,t,\tau)\,dx_0\,d\tau$

$$= At + A\int_0^t \int_0^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x-x_0)^2}{4a^2(t-\tau)}\right] dx_0\,d\tau$$

4.解: a. $u''+\frac{1-2\alpha}{z}u'+\left[(\beta\gamma z^{\gamma-1})^2+\frac{\alpha^2-\gamma^2\nu^2}{z^2}\right]u=0$ $(z>0)$ ①

令 $y=\frac{u}{z^\alpha}$, $x=\beta z^\gamma$, 则 $u=y\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha}{\gamma}}$ $z=\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}$

则 $u'=\frac{du}{dz}=\frac{du}{dx}\cdot\frac{dx}{dz}=\left[y'\left(\frac{x}{\beta}\right)^{\frac{\alpha}{\gamma}}+y\cdot\frac{\alpha}{\gamma}\cdot\frac{1}{\beta}\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha}{\gamma}-1}\right]\cdot\beta\cdot\gamma\cdot\left(\frac{x}{\beta}\right)^{\frac{\gamma-1}{\gamma}}$

$$=\beta\gamma\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha+\gamma-1}{\gamma}}y'+\alpha\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha-1}{\gamma}}\cdot y$$

$u''=\frac{du'}{dz}=\frac{du'}{dx}\cdot\frac{dx}{dz}=\left[\beta\gamma\left(\frac{x}{\beta}\right)^{\frac{\alpha+\gamma-1}{\gamma}}\cdot y''+(\alpha+\gamma-1)\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha-1}{\gamma}}y'+\alpha\left(\frac{x}{\beta}\right)^{\frac{\alpha-1}{\gamma}}y'+\alpha\cdot\frac{\alpha-1}{\beta\gamma}\left(\frac{x}{\beta}\right)^{\frac{\alpha-1}{\gamma}-1}y\right]\cdot\beta\gamma\cdot\left(\frac{x}{\beta}\right)^{\frac{\gamma-1}{\gamma}}$

$$=\beta^2\gamma^2\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha+2\gamma-2}{\gamma}}y''+(2\alpha+\gamma-1)\beta\gamma\left(\frac{x}{\beta}\right)^{\frac{\alpha+\gamma-2}{\gamma}}y'+\alpha(\alpha-1)\left(\frac{x}{\beta}\right)^{\frac{\alpha-2}{\gamma}}y$$

代入①式得: $\beta^2\gamma^2\left(\frac{x}{\beta}\right)^{\frac{\alpha+2\gamma-2}{\gamma}}y''+\beta\gamma^2\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha+\gamma-2}{\gamma}}y'-\alpha^2\left(\frac{x}{\beta}\right)^{\frac{\alpha-2}{\gamma}}y+\left[\beta^2\gamma^2\cdot\left(\frac{x}{\beta}\right)^{\frac{2\gamma-2}{\gamma}}+\alpha^2\cdot\left(\frac{x}{\beta}\right)^{-\frac{2}{\gamma}}-\gamma^2\nu^2\left(\frac{x}{\beta}\right)^{-\frac{2}{\gamma}}\right]\cdot y\cdot\left(\frac{x}{\beta}\right)^{\frac{\alpha}{\gamma}}=0$

即 $\beta^{\frac{2-\alpha}{\gamma}}\cdot\gamma^2\cdot x^{\frac{\alpha-2}{\gamma}+2}y''+\beta^{\frac{2-\alpha}{\gamma}}\gamma^2\cdot x^{\frac{\alpha-2}{\gamma}+1}y'+\left(\beta^{\frac{2-\alpha}{\gamma}}\cdot\gamma^2\cdot x^{\frac{\alpha-2}{\gamma}+2}-\beta^{\frac{2-\alpha}{\gamma}}\cdot\gamma^2\nu^2x^{\frac{\alpha-2}{\gamma}+2}\right)y=0$

同时除以 $\beta^{\frac{2-\alpha}{\gamma}}\cdot\gamma^2\cdot x^{\frac{\alpha-2}{\gamma}}$ 得 $x^2y''+xy'+(x^2-\nu^2)y=0$

则 y 的通解为: $y(x)=C_1J_\nu(x)+C_2Y_\nu(x)$, 则 $u(z)=C_1z^\alpha J_\nu(\beta z^\gamma)+C_2z^\alpha Y_\nu(\beta z^\gamma)$

b.1. $u''+az^bu=0$ 则有 $\begin{cases}1-2\alpha=0\\ \beta\gamma=\sqrt{a}\\ 2(\gamma-1)=b\\ \alpha^2-\gamma^2\nu^2=0\end{cases}$ 即 $\begin{cases}\alpha=\frac{1}{2}\\ \beta=\frac{2\sqrt{a}}{2+b}\\ \gamma=1+\frac{b}{2}\\ \nu=\pm\frac{1}{2+b}\end{cases}$

代入a.通解得 方程的解为: $u=C_1\cdot\sqrt{z}\cdot J_{\frac{1}{2+b}}\left(\frac{2\sqrt{a}}{2+b}z^{1+\frac{b}{2}}\right)+C_2\sqrt{z}\cdot Y_{\frac{1}{2+b}}\left(\frac{2\sqrt{a}}{2+b}z^{1+\frac{b}{2}}\right)$.

2. $zu''-3u'+zu=0$, 同时除以 z 得 $u''+\frac{-3}{z}u'+u=0$

则有 $\begin{cases}1-2\alpha=-3\\ \gamma-1=0\\ \beta\gamma=1\\ \alpha^2-\gamma^2\nu^2=0\end{cases}$ 得 $\begin{cases}\alpha=2\\ \gamma=1\\ \beta=1\\ \nu=\pm2\end{cases}$

代入通解得 $u(z)=C_1z^2J_2(z)+C_2z^2Y_2(z)$.