数学物理方法
第三次作业

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1.1

二阶偏微分方程为

$$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$$

则 A=1,B=2,C=5,D=1,E=2，即 $B^2-AC=-1<0$，则特征方程有两个共轭复数解。特征方程为

$$\left(\frac{dy}{dx}\right)^2-4\frac{dy}{dx}+5=0$$

则特征方程的解为 $y=(2\pm i)x$，令 $\xi=2x-y$，$\eta=x$，则可得

$$\begin{aligned}
a&=\left(\frac{\partial\xi}{\partial x}\right)^2+4\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+5\left(\frac{\partial\xi}{\partial y}\right)^2=1\\
b&=\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+2\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+5\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0\\
c&=\left(\frac{\partial\eta}{\partial x}\right)^2+4\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+5\left(\frac{\partial\eta}{\partial y}\right)^2=1\\
d&=\frac{\partial^2\xi}{\partial x^2}+4\frac{\partial^2\xi}{\partial x\partial y}+5\frac{\partial^2\xi}{\partial y^2}+\frac{\partial\xi}{\partial x}+2\frac{\partial\xi}{\partial y}=0\\
e&=\frac{\partial^2\eta}{\partial x^2}+4\frac{\partial^2\eta}{\partial x\partial y}+5\frac{\partial^2\eta}{\partial y^2}+\frac{\partial\eta}{\partial x}+2\frac{\partial\eta}{\partial y}=1\\
f&=0\\
g&=0
\end{aligned}$$

则该二阶偏微分方程标准形式为:

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=-\frac{\partial u}{\partial\eta}$$

1.2

二阶偏微分方程为

$$\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\frac{\partial u}{\partial y}=0$$

则 A=1,B=0,C=y,D=0,E=$\frac{1}{2}$，则特征方程为

$$\left(\frac{dy}{dx}\right)^2+y=0$$

$\Delta = B^2 - AC = -y$。

①当 y>0 时: 特征方程的解为 $2\sqrt{y} \mp ix = C$,C 为常数，则令 $\xi = 2\sqrt{y}, \eta = x$, 则可得

$$\begin{aligned}
a &= \left(\frac{\partial \xi}{\partial x}\right)^2 + y\left(\frac{\partial \xi}{\partial y}\right)^2 = 1 \\
b &= \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + y\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0 \\
c &= \left(\frac{\partial \eta}{\partial x}\right)^2 + y\left(\frac{\partial \eta}{\partial y}\right)^2 = 1 \\
d &= \frac{\partial^2 \xi}{\partial x^2} + y\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2}\frac{\partial \xi}{\partial y} = 0 \\
e &= \frac{\partial^2 \eta}{\partial x^2} + y\frac{\partial^2 \eta}{\partial y^2} + \frac{1}{2}\frac{\partial \eta}{\partial y} = 0 \\
f &= 0 \\
g &= 0
\end{aligned}$$

则当 $y > 0$ 时，该二阶偏微分方程标准形式为:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0.$$

②当 y=0 时: 在 x 轴上，u 可以取任意函数，只需满足

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2}\frac{\partial u}{\partial y}.$$

③当 y<0 时: 特征方程的解为 $-2\sqrt{-y} \pm x = C$,C 为常数，则令 $\xi = -2\sqrt{-y} + x, \eta = -2\sqrt{-y} - x$, 可得

$$\begin{aligned}
a &= \left(\frac{\partial \xi}{\partial x}\right)^2 + y\left(\frac{\partial \xi}{\partial y}\right)^2 = 0 \\
b &= \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + y\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = -2 \\
c &= \left(\frac{\partial \eta}{\partial x}\right)^2 + y\left(\frac{\partial \eta}{\partial y}\right)^2 = 0 \\
d &= \frac{\partial^2 \xi}{\partial x^2} + y\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{2}\frac{\partial \xi}{\partial y} = 0 \\
e &= \frac{\partial^2 \eta}{\partial x^2} + y\frac{\partial^2 \eta}{\partial y^2} + \frac{1}{2}\frac{\partial \eta}{\partial y} = 0 \\
f &= 0 \\
g &= 0
\end{aligned}$$

则当 y<0 时，该二阶偏微分方程标准形式为

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

1.3

二阶偏微分方程为

$$\frac{\partial^2 u}{\partial x^2} - 2\cos x\frac{\partial^2 u}{\partial x \partial y} - (3 + \sin^2 x)\frac{\partial^2 u}{\partial y^2} - y - \frac{\partial u}{\partial y} = 0$$

则 A=1,B=$-\cos x$,C=$-(3+\sin^2 x)$,D=0,E=-1,F=-1,G=y，则特征方程为

$$\left(\frac{dy}{dx}\right)^2+2\cos x\frac{dy}{dx}-(3+\sin^2 x)=0$$

$\Delta=B^2-AC=4>0$，特征方程的解为 $y=-\sin x\pm 2x$，令 $\xi=\sin x-2x+y,\eta=\sin x+2x+y$，可得

$$\begin{aligned}
&a=\left(\frac{\partial\xi}{\partial x}\right)^2-2\cos x\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}-(3+\sin^2 x)\left(\frac{\partial\xi}{\partial y}\right)^2=0\\
&b=\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}-\cos x\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)-(3+\sin^2 x)\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=-8\\
&c=\left(\frac{\partial\eta}{\partial x}\right)^2-2\cos x\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-(3+\sin^2 x)\left(\frac{\partial\eta}{\partial y}\right)^2=0\\
&d=\frac{\partial^2\xi}{\partial x^2}-2\cos x\frac{\partial^2\xi}{\partial x\partial y}-(3+\sin^2 x)\frac{\partial^2\xi}{\partial y^2}-\frac{\partial\xi}{\partial y}=-\sin x-1\\
&e=\frac{\partial^2\eta}{\partial x^2}-2\cos x\frac{\partial^2\eta}{\partial x\partial y}-(3+\sin^x)\frac{\partial^2\eta}{\partial y^2}-\frac{\partial\eta}{\partial y}=-\sin x-1\\
&f=0\\
&g=y
\end{aligned}$$

则二阶偏微分方程标准形式为

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=-\frac{(\sin x+1)}{16}\frac{\partial u}{\partial\xi}-\frac{(\sin x+1)}{16}\frac{\partial u}{\partial\eta}+y$$

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2.1

二阶微分方程为:

$$a_1(x)\frac{\partial^2 u}{\partial x^2}+b_1(y)\frac{\partial^2 u}{\partial y^2}+a_2(x)\frac{\partial u}{\partial x}+b_2(y)\frac{\partial u}{\partial y}=0$$

假设原方程有分离变量解:

$$u(x,y)=X(x)Y(y)$$

代入原方程，可得:

$$a_1(x)X''(x)Y(y)+b_1(y)X(x)Y''(y)+a_2(x)X'(x)Y(y)+b_2(y)X(x)Y'(y)=0$$

整理可得:

$$a_1(x)\frac{X''(x)}{X(x)}+a_2(x)\frac{X'(x)}{X(x)}=-b_1(y)\frac{Y''(y)}{Y(y)}-b_2(y)\frac{Y'(y)}{Y(y)}$$

左侧与 y 无关，右侧与 x 无关，则可令其等于一常数 $-\lambda$，则有

$$\begin{aligned}
a_1(x)X''(x)+a_2(x)X'(x)+\lambda X(x)=0\\
-b_1(y)Y''(y)-b_2(y)Y'(y)+\lambda Y(y)=0
\end{aligned}$$

2.2

二阶偏微分方程为:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=0$$

假设原方程有分离变量解:

$$u(\rho,\varphi)=S(\rho)\Phi(\varphi)$$

代入原方程，可得:

$$\Phi(\varphi)\left(S''(\rho)+\frac{S'(\rho)}{\rho}\right)+\frac{S(\rho)}{\rho^2}\Phi''(\varphi)=0$$

整理可得:

$$\frac{\rho^2 S''(\rho)+\rho S'(\rho)}{S(\rho)}=-\frac{\Phi''(\varphi)}{\Phi(\varphi)}$$

左侧与 φ 无关，右侧与 ρ 无关，则可令其等于一常数 $-\lambda$，则有

$$\begin{aligned}\rho^2 S''(\rho)+\rho S'(\rho)+\lambda S(\rho)=0\\ \Phi''(\varphi)-\lambda\Phi(\varphi)=0\end{aligned}$$

2.3

二阶偏微分方程为:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$$

假设原方程有分离变量解:

$$u(r,\theta)=R(r)\Theta(\theta)$$

代入原方程，可得:

$$\Theta(\theta)\frac{\partial}{\partial r}\left(r^2R'(r)\right)+R(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\Theta'(\theta)\right)=0$$

整理可得:

$$\frac{r^2R''(r)+2rR'(r)}{R(r)}=-\frac{\Theta''(\theta)+\cot\theta\Theta'(\theta)}{\Theta(\theta)}$$

左侧与 θ 无关，右侧与 r 无关，则可令其等于一常数 $-\lambda$，则有

$$\begin{aligned}r^2R''(r)+2rR'(r)+\lambda R(r)=0\\ \Theta''(\theta)+\cot\theta\Theta'(\theta)-\lambda\Theta(\theta)=0\end{aligned}$$

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由题意可知：方程及初始条件与边界条件

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}=a^2\dfrac{\partial^2u}{\partial x^2}(0<x<l,t>0)\\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0},\left.\dfrac{\partial u}{\partial x}\right|_{x=l}=0\\ u\mid_{t=0}=e^{-x^2},\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=2axe^{-x^2}\end{cases}$$

设方程的解具有 $u(x,t)=X(x)T(t)$ 的形式，则有

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

其中 λ 为一常数。
求解 X(x):

$$X''(x)+\lambda X(x)=0$$

由边界条件 $\frac{\partial u}{\partial x}\Big|_{x=0}, \frac{\partial u}{\partial x}\Big|_{x=l}=0$ 可知,X(x) 的本征值为 $\lambda_n=\left(\frac{n\pi}{l}\right)^2$, 则有

$$X_n(x)=A_n\cos\left(\frac{n\pi}{l}x\right),\ n=0,1,2,3,\ldots$$

求解 T(t):

$$T''(t)+\lambda_n a^2 T(t)=0$$

则

$$T$$

$$T_n(t)=B_n'\cos\left(\frac{n\pi a}{l}t\right)+C_n'\sin\left(\frac{n\pi a}{l}t\right),\ n=1,2,3,\ldots$$

则有

$$u_n(x,t)=X_n(x)T_n(t)=\left[B_n\cos\left(\frac{n\pi a}{l}t\right)+C_n\sin\left(\frac{n\pi a}{l}t\right)\right]\cos\left(\frac{n\pi}{l}x\right)$$

线性组合出一般解为:

$$u(x,t)=\sum_{n=0}^{\infty}\left[B_n\cos\left(\frac{n\pi a}{l}t\right)+C_n\sin\left(\frac{n\pi a}{l}t\right)\right]\cos\left(\frac{n\pi}{l}x\right)$$

代入初始条件:

$$B_0+\sum_{n=1}^{\infty}B_n\cos\left(\frac{n\pi}{l}x\right)=e^{-x^2}$$

$$\sum_{n=1}^{\infty}\frac{n\pi a}{l}C_n\cos\left(\frac{n\pi}{l}x\right)=2axe^{-x^2}$$

由傅立叶系数公式可得:

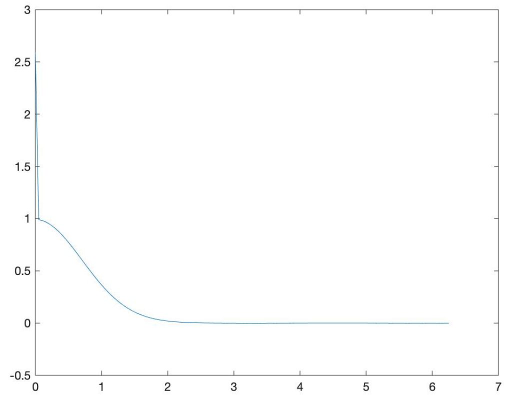
$$\begin{aligned}
B_0&=\frac{1}{l}\int_0^l e^{-x^2}\,dx\\
B_n&=\frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)\,dx,\ n=1,2,3,\ldots\\
C_0&=\frac{a}{l}\left(1-e^{-l^2}\right)\\
C_n&=\frac{4}{n\pi}\int_0^l xe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)\,dx,\ n=1,2,3,\ldots
\end{aligned}$$

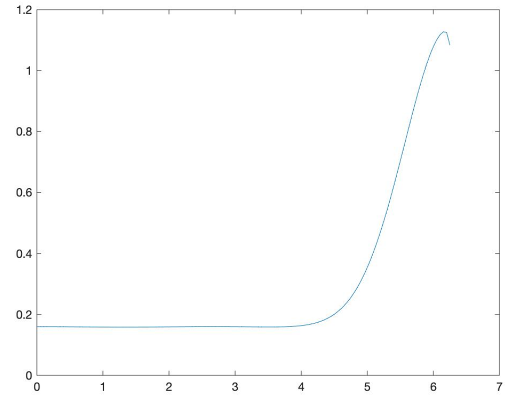
所以原方程的解为:

$$\begin{aligned}
u(x,t)=&\frac{1}{l}\int_0^l e^{-x^2}\,dx+\frac{a}{l}\left(1-e^{-l^2}\right)t\\
&+\sum_{n=1}^{\infty}\left[\cos\left(\frac{n\pi a}{l}t\right)\frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)\,dx+\sin\left(\frac{n\pi a}{l}t\right)\frac{4}{n\pi}\int_0^l xe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)\,dx\right]\cos\left(\frac{n\pi}{l}x\right)
\end{aligned}$$

取 l=1,a=1,n 加到 1000. 画图如下:

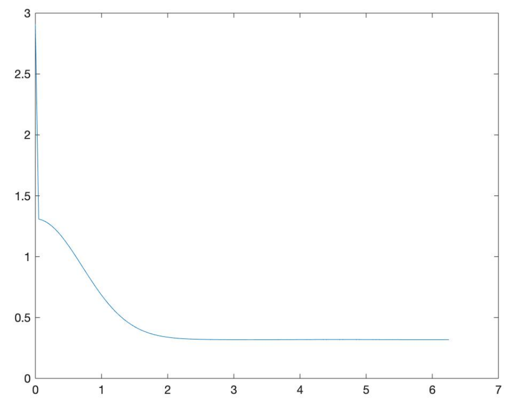
t=0: t=1:

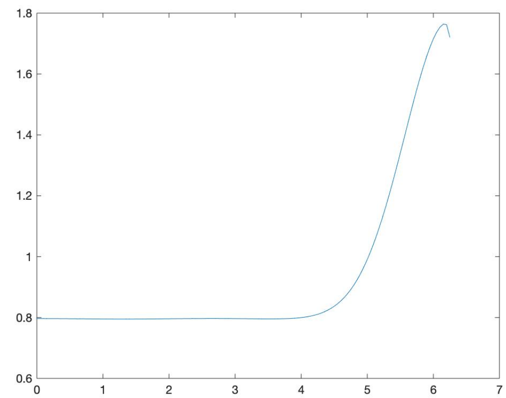




t=2:

t=5:





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由题意可知：方程及初始条件与边界条件

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}(0 < x < l, t > 0) \\ u\mid_{t=0} = \dfrac{bx(l-x)}{l^2} \\ u(0,t) = u(l,t) = 0 \end{cases}$$

假设方程的解具有 $u(x,t) = X(x)T(t)$ 的形式，则有

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

其中 λ 为一常数。
求解 X(x):

$$X''(x) + \lambda X(x) = 0$$

由边界条件 $u(0,t) = u(l,t) = 0$ 可知 X(x) 解的特征值为 $\lambda_n = \left(\frac{n\pi}{l}\right)^2$，则有

$$X_n(x) = A_n \sin\left(\frac{n\pi}{l} x\right),\ n = 1, 2, 3, \ldots$$

求解 T(t):

$$T'(t)+\lambda_n a^2 T(t)=0$$

则

$$T_n(t)=B_n exp\left[-\left(\frac{na\pi}{l}\right)^2 t\right]$$

则有

$$u_n(x,t)=X_n(x)T_n(t)=C_n exp\left[-\left(\frac{na\pi}{l}\right)^2 t\right]\sin\left(\frac{n\pi}{l}x\right)$$

线性组合出一般解为:

$$u(x,t)=\sum_{n=1}^{\infty}C_n exp\left[-\left(\frac{na\pi}{l}\right)^2 t\right]\sin\left(\frac{n\pi}{l}x\right)$$

代入初始条件:

$$u(x,0)=\frac{bx(l-x)}{l^2}=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi}{l}x\right)$$

利用傅立叶系数公式可知

$$C_n=\frac{2}{l}\int_0^l\frac{bx(l-x)}{l^2}\sin\left(\frac{n\pi}{l}x\right)dx=\begin{cases}\frac{8b}{(n\pi)^3}, & n=1,3,5,\dots\\ 0, & n=2,4,6\dots\end{cases}$$

令 n=2k-1, 则温度分布函数为

$$u(x,t)=\sum_{k=1}^{\infty}\frac{8b}{((2k-1)\pi)^3}exp\left[-\left(\frac{(2k-1)a\pi}{l}\right)^2 t\right]\sin\left(\frac{(2k-1)\pi}{l}x\right)$$

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选取球坐标系, 则有:

$$x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$$

则波动方程可化为:

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$$

由题意可知, u 不依赖于角向变量 θ,φ, 则波动方程可化为:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

注意到:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2}$$

则有:

$$\frac{\partial^2(ru)}{\partial t^2}=a^2\frac{\partial^2(ru)}{\partial r^2}$$

这是一个以 ru 为变量的一维波动方程, 其通解为

$$ru(r,t)=f_1(r+at)+f_2(r-at)$$

代入初始条件可得

$$\begin{cases}f_1(r)+f_2(r)=ru_0,\ r<R\\ f_1(r)+f_2(r)=0,\ r>R\\ af_1'(r)-af_2'(r)=0\end{cases}$$

则有

$$f_1(r)=\begin{cases}\frac{u_0}{2}r+C, & r<R\\ C, & r>R\end{cases}\quad f_2(r)=\begin{cases}\frac{u_0}{2}r-C, & r<R\\ -C, & r>R\end{cases}$$

所以振动方程的解为

$$u(r,t)=\frac{f_1(r+at)+f_2(r-at)}{r}$$

其中 $r=\sqrt{x^2+y^2+z^2}$，$f_1(r+at), f_2(r-at)$ 由上式给出。

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选取极坐标系，则有:

$$x=r\cos\theta, y=r\sin\theta$$

则波动方程可化为

$$\begin{cases}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=6r^2(1+\sin 2\theta), & 0<a<r<b<\infty\\ u\big|_{x^2+y^2=a^2}=1,\ \ \frac{\partial u}{\partial n}\big|_{x^2+y^2=b^2}=0\end{cases}$$

设方程的解可以写成

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n(r)\cos n\theta+B_n(r)\sin n\theta$$

则有

$$\sum_{n=0}^{\infty}\left[A_n''(r)+\frac{1}{r}A_n'(r)-\frac{n^2}{r^2}A_n(r)\right]\cos n\theta+\left[B_n''(r)+\frac{1}{r}B_n'(r)-\frac{n^2}{r^2}B_n(r)\right]\sin n\theta=6r^2(1+\sin 2\theta)$$

所以有

$$A_0''(r)+\frac{1}{r}A_0'(r)=6r^2$$

$$B_2''(r)+\frac{1}{r}B_2'(r)-\frac{4}{r^2}B_2(r)=6r^2$$

初始条件为

$$A_0(a)=1, B_n(a)=0, A_n'(b)=0, B_n'(b)=0$$

则方程的解为

$$u(r,\theta)=A_0(r)+B_2(r)\sin(2\theta)$$

设

$$A_0(r)=c_0+d_0\ln(r)+\frac{3}{8}r^4,\ B_2(r)=c_2r^2+d_2r^{-2}+r^4$$

则有

$$c_2a^2+d_2a^{-2}+a^4=0,\ c_0+d_0+\frac{3}{8}a^4=1$$

$$c_2b-d_2b^{-3}+2b^3=0,\ \frac{d_0}{b}+\frac{3}{2}b^3=0$$

可得

$$u(r,\theta)=1-\frac{3}{8}a^4+\frac{3}{2}b^4\ln(\frac{a}{r})+\frac{3}{8}r^4+\left[-\frac{a^6+2b^6}{a^4+b^4}r^2-\frac{a^4b^4(a^2-2b^2)}{a^4+b^4}r^{-2}+r^4\right]\cos(2\theta)$$

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7.1

由题意可知，振动方程为:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, t > 0, \\ u(x,0) = \sin(x), u_t(x,0) = 0, & 0 \leqslant x < l, \\ u_t(x,0) = 0, & -l < x < 0, \\ u(l,t) = 0, & t > 0. \end{cases}$$

假设方程的解具有 $u(x,t) = X(x)T(t)$ 的形式，则有

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

其中 λ 为一常数。
求解 X(x):

$$X''(x) + \lambda X(x) = 0$$

由边界条件 $u_x(0,t) = 0, u(l,t) = 0$ 可知 X(x) 的本征值为 $\lambda_n = \left[\frac{(2n+1)\pi}{2l}\right]^2$，则有

$$X_n(x) = A_n \cos\left[\frac{(2n+1)\pi}{2l} x\right],\ n = 0, 1, 2, 3, \ldots$$

求解 T(t):

$$T(t)'' + \lambda_n a^2 T(t) = 0$$

则:

$$T_n(t) = B_n' \cos\left[\frac{(2n+1)a\pi}{2l} t\right] + C_n' \sin\left[\frac{(2n+1)a\pi}{2l} t\right]$$

则一般解可以写为

$$u(x,t) = \sum_{n=0}^{\infty} \left\{ B_n \cos\left[\frac{(2n+1)a\pi}{2l} t\right] + C_n \sin\left[\frac{(2n+1)a\pi}{2l} t\right] \right\} \cos\left[\frac{(2n+1)\pi}{2l} x\right]$$

代入初始条件可知

$$\sum_{n=0}^{\infty} B_n \cos\left[\frac{(2n+1)a\pi}{2l} x\right] = \sin x$$
$$\sum_{n=1}^{\infty} \frac{(2n+1)a\pi}{2l} C_n \cos\left[\frac{(2n+1)\pi}{2l} x\right] = 0$$

所以可得

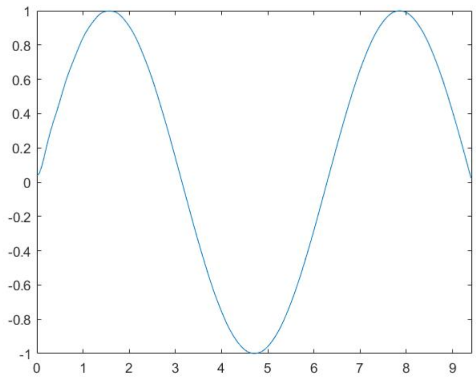
$$\begin{aligned} B_n &= \frac{2}{l} \int_0^l \sin x \cos\left(\frac{(2n+1)a\pi}{2l} x\right) \\ &= \frac{4}{2l + (2n+1)a\pi} \left[1 - \cos\left(\frac{2l + (2n+1)a\pi}{2l}\right)\right] + \frac{4}{2l - (2n+1)a\pi} \left[1 - \cos\left(\frac{2l - (2n+1)a\pi}{2l}\right)\right] \\ C_n &= 0 \end{aligned}$$

所以解为

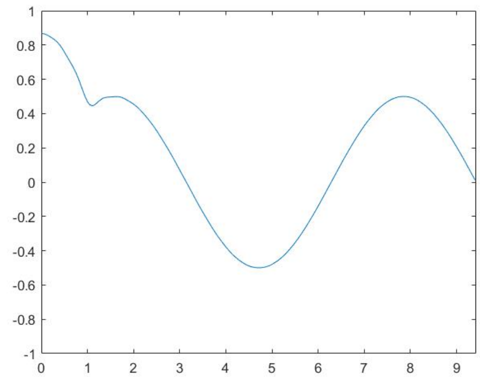
$$u(x,t) = \sum_{n=0}^{\infty} B_n \cos\left[\frac{(2n+1)a\pi}{2l} t\right] \cos\left[\frac{(2n+1)\pi}{2l} x\right]$$

其中 B_n 上式给出。当 a=1,l=3π，画图如下:

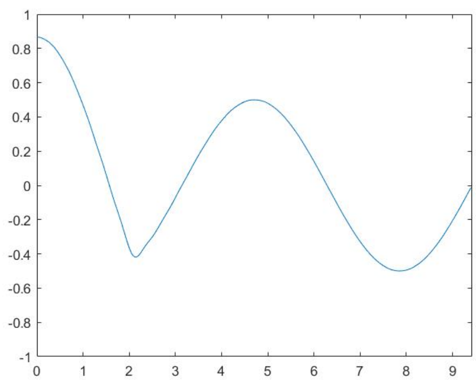
n=0:



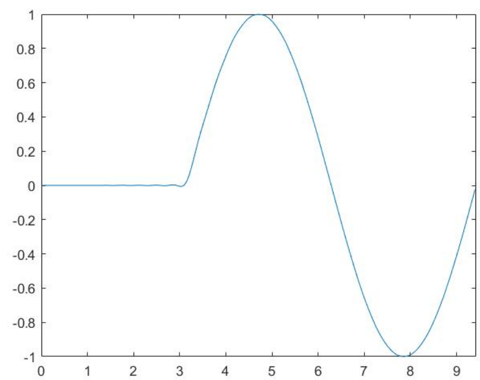
n=1:



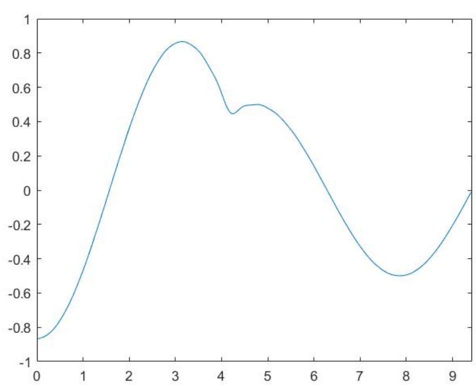
n=2:



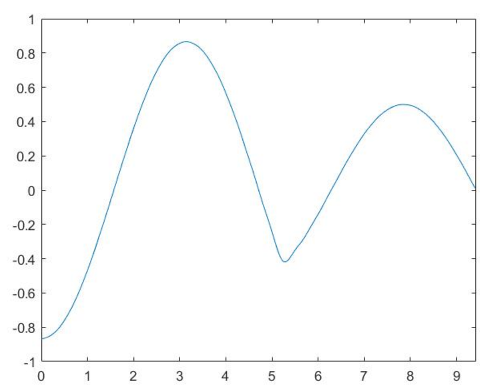
n=3:



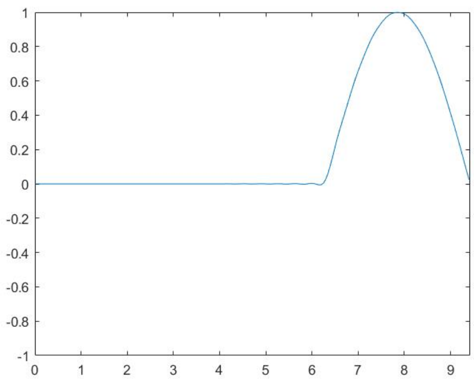
n=4:



n=5:



n=6:



7.2

由上一问结果可知，若在该初始条件 $(u(x,0)=\sin(x))$ 下在 x=l 处固定，振动的解与 l 无关，当 l 外推至无穷大时，结果近似为

$$u(x,t)=\begin{cases}\frac{1}{2}\left[\sin(x+at)+\sin(x-at)\right], & \text{当}x-at\geqslant 0,t>0\text{时},\\ \frac{1}{2}\left[\sin(x+at)+\sin(at-x)\right], & \text{当}x-at<0,t>0,x\geqslant 0\text{时}.\end{cases}$$

与半无界弦振动问题一致。