数学物理方法
第二次作业

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三月 25 日

1

证明:

1.1

在三维球坐标中，利用坐标变换 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，其坐标变换的 Jacobe 行列式为

$$J = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} = r^2\sin\theta.$$

所以 δ 函数在球坐标下的表达式为

$$\delta\left(\vec{r}-\vec{r}_0\right) = \frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0) = \frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$$

1.2

$$\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|} = -\nabla\cdot\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$$

令 $\vec{l} = \vec{r}-\vec{r}_0$，则 $\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} = \frac{\hat{l}}{l^2}$。当 $\hat{l} \neq \vec{0}$ 时，利用求坐标可知

$$\nabla\cdot\frac{\hat{l}}{l^2} = \frac{1}{l^2}\frac{\partial}{\partial l}\left(l^2\frac{1}{l^2}\right) = 0.$$

当 $\hat{l} = \vec{0}$ 时，存在

$$\iiint_{R\to 0}\nabla\cdot\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\,dV = \oiint\frac{\hat{l}}{l^2}\cdot d\vec{a} = 4\pi$$

所以

$$\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|} = -\nabla\cdot\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} = -4\pi\delta(\vec{r}-\vec{r}_0)$$

2

2.1

由复数欧拉公式 $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$，可知

$$\int_{-\infty}^{+\infty}\frac{\cos(\omega t)}{\omega^2+a^2}\,d\omega = Re\left(\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}\,d\omega\right)$$

由留数定理可知:

$$\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}\,d\omega = \frac{\pi}{a}e^{-a|t|}$$

即

$$\int_{-\infty}^{+\infty}\frac{\cos(\omega t)}{\omega^2+a^2}\,d\omega = Re\left(\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}\,d\omega\right) = \frac{\pi}{a}e^{-a|t|}$$

2.2

2.2.1

由定义可知:

$$F(k_1,k_2,k_3)=\sqrt{2\pi}\iiint\frac{1}{r}e^{-i2\pi\vec{r}\cdot\vec{k}}\,dx\,dy\,dz$$

利用求坐标变换, 其 Jacobe 行列式为 $(J=r^2\sin\theta)$。则原积分可化为:

$$\begin{aligned}F(k_1,k_2,k_3)&=\sqrt{2\pi}\int_0^{2\pi}\,d\varphi\int_0^{\pi}\,d\theta\int_0^{\infty}r\sin\theta e^{-i2\pi kr\cos\theta}\,dr\\&=-\frac{1}{\sqrt{2\pi}k^2}\int_0^{\pi}\frac{\sin\theta}{\cos^2\theta}\,d\theta=\sqrt{\frac{2}{\pi}}\frac{1}{k^2}\end{aligned}$$

2.2.2

由定义可知:

$$G(\vec{k})=\frac{1}{2\pi\sqrt{2\pi}}\iiint\sqrt{\frac{\pi}{2}}\frac{\delta(r-a)}{r}e^{-i\vec{r}\cdot\vec{k}}\,dx\,dy\,dz$$

利用求坐标变换, 其 Jacobe 行列式为 $r^2\sin\theta$。则原积分可化为:

$$\begin{aligned}G(\vec{k})&=\frac{1}{4\pi}\int_0^{2\pi}\,d\varphi\int_0^{\pi}\,d\theta\int_0^{\infty}\delta(r-a)r\sin\theta e^{-ikr\cos\theta}\,dr\\&=\frac{1}{2}\int_0^{\pi}a\sin\theta e^{-ika\cos\theta}\,d\theta=\frac{\sin ak}{k}\end{aligned}$$

3

因为 f(t) 的拉普拉斯变换存在, 设为 F(p), 则有:

$$F(p)=\int_0^{+\infty}f(t)e^{-pt}\,dt$$

又因为 f(t) 为周期函数, 周期为 a, 即 f(t+a)=f(t), 令 u=t+a, 则有:

$$F(p)=\int_0^{+\infty}f(t)e^{-pt}\,dt=\int_0^{+\infty}f(t+a)e^{-pt}\,dt=e^{ap}\int_a^{+\infty}f(u)e^{-pu}\,du$$

所以

$$F(p)\left(1-e^{-ap}\right)=\int_0^{a}f(t)e^{-pt}\,dt$$

即:

$$F(p)=\frac{1}{1-e^{-ap}}\int_0^{a}f(t)e^{-pt}\,dt$$

4

令 λ 为轻绳质量线密度, 选取坐标 x 处一段长度为 Δx 微元为研究对象, 由于轻绳以 ω 角速度匀速转动, 则 x 处对于 x 到 l 轻绳的力恰好提供圆周运动向心力, 则有:

$$F(x)=\int_x^{l}\lambda\omega^2x\,dx=\frac{1}{2}\lambda\omega^2(l^2-x^2)$$

对于研究对象，u 增大为正方向，由牛顿第二定律可知:

$$\lambda\Delta x\frac{\partial^2 u}{\partial t^2} = -F(x)\sin(\alpha_1) - F(x+\Delta x)\sin(\alpha_2)$$

由于角度很小，可以利用小角度近似有:

$$\sin(\alpha_1) = \left.\frac{\partial u}{\partial x}\right|_x,\ \sin(\alpha_2) = -\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}$$

所以得到

$$\lambda\Delta x\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\lambda\omega^2\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}(l^2-(x+\Delta x)^2) - \left.\frac{\partial u}{\partial x}\right|_x(l^2-x^2)\right]$$

两侧同时除以 $\lambda\Delta x$，并令 $\Delta x \to 0$，可得振动方程为:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\omega^2\frac{\partial}{\partial x}\left[(l^2-x^2)\frac{\partial u}{\partial x}\right]$$

5

由题意可知，在 x=0 与 x=l 处由恒定的热流进入，则边界条件为:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = q_0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = -q_0$$

6

由题意可知: 弦振动过程中两端为自由端，则方程及边界条件可写为: $u_x(0,t)=0$, $u_x(L,t)=0$，可以得到:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi(x) = \begin{cases} \frac{h}{c}x, 0 < x \leqslant c, \\ \frac{h}{L-c}(L-x), c < x < L, \end{cases} \\ u_t(x,0) = 0 \\ u_x(0,t) = 0,\ u_x(L,t) = 0 \end{cases}$$

7

待求问题利用叠加原理可以转化为如下两个问题: 对于 $(-\infty < x < +\infty, t > 0)$

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \\ u(x,0) = e^{-2x^2}, u_t(x,0) = \sin(x) \end{cases}\qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \cos(\omega t)\cos(x) \\ u(x,0) = 0, u_t(x,0) = 0 \end{cases}$$

对于问题一: 其通解可以写成 $u(x,t) = f_1(x+at) + f_2(x-at)$，带入初始边界条件可以得到:

$$\begin{cases} f_1(x) + f_2(x) = e^{-2x^2} \\ af_1'(x) - af_2'(x) = \sin(x) \end{cases}$$

可以解得:

$$\begin{cases} f_1(x) = \frac{1}{2}e^{-2x^2} + \frac{1}{2a}(1-\cos(x)) + \frac{C}{2} \\ f_2(x) = \frac{1}{2}e^{-2x^2} - \frac{1}{2a}(1-\cos(x)) - \frac{C}{2} \end{cases}$$

所以问题一的解可以写为:

$$u_1(x,t)=f_1(x+at)+f_2(x-at)=\frac{1}{2}\left(e^{-2(x+at)^2}+e^{-2(x-at)^2}\right)-\frac{1}{2a}\left(\cos(x+at)-\cos(x-at)\right)$$

对于问题二：利用齐次化定理，找到一个函数 w(x,t;τ) 满足如下方程:

$$\begin{cases}\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2},\ t>\tau>0\\ w|_{t-\tau=0}=0,\quad w_t|_{t-\tau=0}=\cos(\omega t)\cos(x)\end{cases}$$

利用达朗贝尔公示，可得

$$\begin{aligned}w(x,t;\tau)&=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\cos(\omega t)\cos(\xi)\,d\xi\\&=\frac{1}{4a}\{\sin[(\omega-a)\tau+x+at]+\sin[(\omega-a)\tau-x+at]-\sin[(\omega+a)\tau-x-at]-\sin[(\omega+a)\tau+x-at]\}\end{aligned}$$

所以问题二的解为:

$$\begin{aligned}u_2(x,t)&=\int_0^t w(x,t;\tau)\,d\tau\\&=\frac{1}{2(\omega^2-a^2)}\left[\cos(at+x)+\cos(at-x)\right]-\frac{1}{2(\omega^2-a^2)}\left[\cos(\omega t+x)+\cos(\omega t-x)\right]\end{aligned}$$

所以原问题的解为:

$$\begin{aligned}u(x,t)=&u_1(x,t)+u_2(x,t)\\=&\frac{1}{2}\left(e^{-2(x+at)^2}+e^{-2(x-at)^2}\right)-\frac{1}{2a}\left(\cos(x+at)-\cos(x-at)\right)\\&+\frac{1}{2(\omega^2-a^2)}\left[\cos(at+x)+\cos(at-x)\right]-\frac{1}{2(\omega^2-a^2)}\left[\cos(\omega t+x)+\cos(\omega t-x)\right]\end{aligned}$$

8

由题意可知，端点 x=0 处不受垂直方向的力，对系统进行偶延拓得:

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, & -\infty<x<\infty,t>0,\\ u(x,0)=\sin(x),u_t(x,0)=0, & 0\leqslant x<\infty,\\ u(x,0)=\sin(-x),u_t(x,0)=0, & -\infty<x<0,\\ u_x(0,t)=0, & t>0.\end{cases}$$

当 $x-at\geqslant 0,t>0$ 时，由达朗贝尔公式可得:

$$u(x,t)=\frac{1}{2}\left[\sin(x+at)+\sin(x-at)\right]$$

当 $x-at<0,t>0,x\geqslant 0$ 时，$-(at-x)\geqslant 0$，由达朗贝尔公式可得:

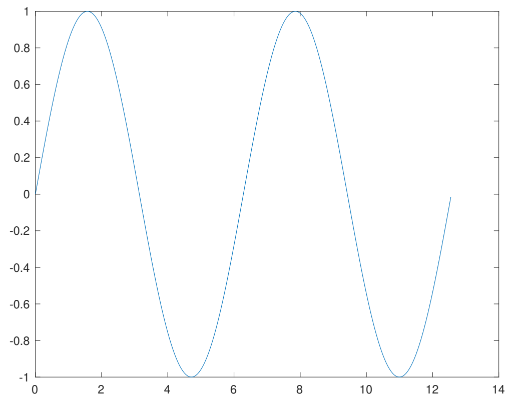
$$u(x,t)=\frac{1}{2}\left[\sin(x+at)+\sin(at-x)\right]$$

综上所述，初始位置为 $\sin(x)$，初始速度为 0 的边界条件下，u 的解为:

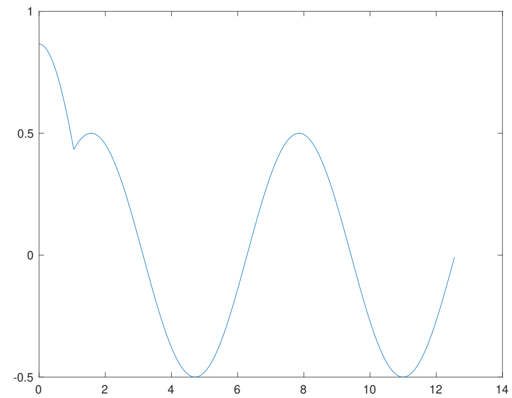
$$u(x,t)=\begin{cases}\frac{1}{2}\left[\sin(x+at)+\sin(x-at)\right], & \text{当}x-at\geqslant 0,t>0\text{时},\\ \frac{1}{2}\left[\sin(x+at)+\sin(at-x)\right], & \text{当}x-at<0,t>0,x\geqslant 0\text{时}.\end{cases}$$

画图如下 $(0 \leqslant x \leqslant 2\pi)$:

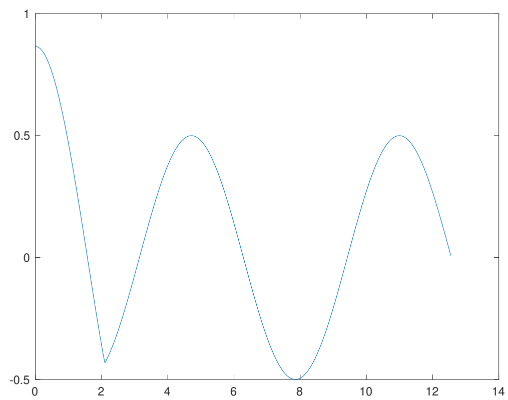
n=0:



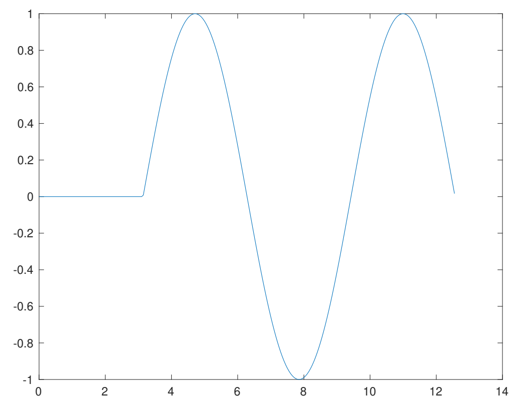
n=1:



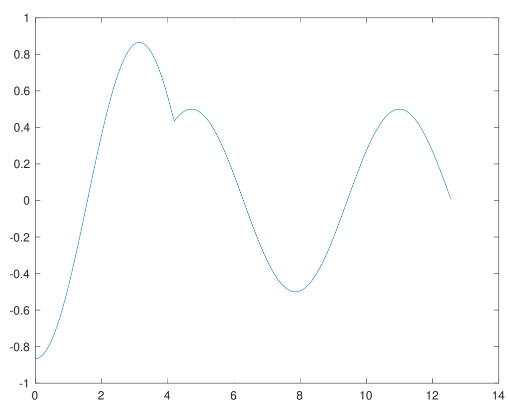
n=2:



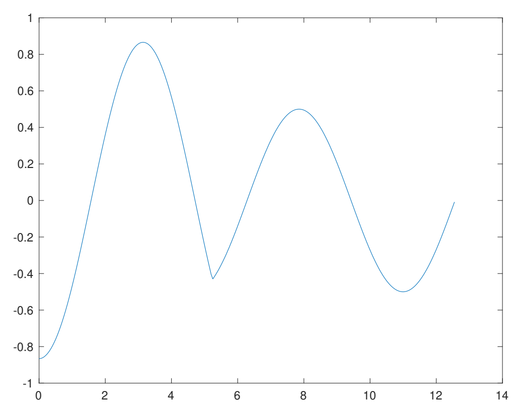
n=3:



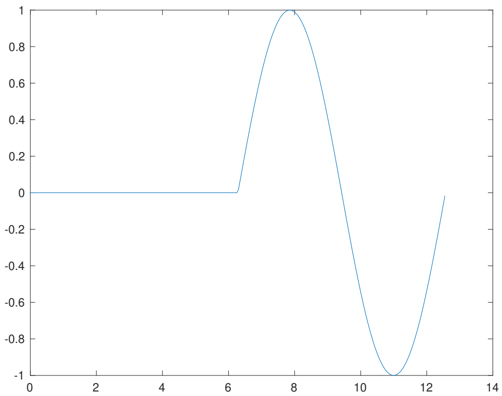
n=4:



n=5:



n=6:



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
2
4
6
8
10
12
14

a