数学物理方法 II
第一次作业

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1

在极坐标系中任意一点 P 的坐标可以写为（r，θ）
在极坐标系下的速度可以写为:

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{dr}{dt}\vec{e_r}\ +\ r\frac{d\theta}{dt}\vec{e_\theta}=\dot{r}\vec{e_r}\ +\ r\dot{\theta}\vec{e_\theta}$$

极坐标系中加速度可以写为:

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2r}{dt^2}\vec{e_r}\ +\ \frac{dr}{dt}\frac{d\vec{e_r}}{dt}\ +\ \frac{dr}{dt}\frac{d\theta}{dt}\vec{e_\theta}\ +\ r\frac{d^2\theta}{dt^2}\vec{e_\theta}\ +\ r\frac{d\theta}{dt}\frac{d\vec{e_\theta}}{dt}$$

其中，两个单位向量对时间导数为:

$$\frac{d\vec{e_r}}{dt}=\frac{d\theta}{dt}\vec{e_\theta},\frac{d\vec{e_\theta}}{dt}=-\frac{d\theta}{dt}\vec{e_r}$$

所以加速度为:

$$\vec{a}=(\ddot{r}-r\dot{\theta}^2)\vec{e_r}\ +\ (r\ddot{\theta}+2\dot{r}\dot{\theta})\vec{e_\theta}$$

2

因为 $\vec{r}=\vec{x}-\vec{x'}=(x-x')\vec{e_x}+(y-y')\vec{e_y}+(z-z')\vec{e_z}$, 则有 $\mathrm{r}=|\vec{r}|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

2.1

$\bigtriangledown\frac{1}{r}=-\frac{(x-x')\vec{e_x}+(y-y')\vec{e_y}+(z-z')\vec{e_z}}{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{3}{2}}}=-\frac{\vec{r}}{r^3}$

2.2

$\bigtriangledown\times\frac{\vec{r}}{r^3}=\begin{vmatrix}\vec{e_x} & \vec{e_y} & \vec{e_z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x-x'}{r^3} & \frac{y-y'}{r^3} & \frac{z-z'}{r^3}\end{vmatrix}$
$=[\frac{-3(z-z')(y-y')}{r^5}+\frac{3(z-z')(y-y')}{r^5}]\vec{e_x}+[\frac{-3(z-z')(x-x')}{r^5}+\frac{3(z-z')(x-x')}{r^5}]\vec{e_y}+[\frac{-3(x-x')(y-y')}{r^5}+\frac{3(x-x')(y-y')}{r^5}]\vec{e_z}$
$=\vec{0}$
即: $\bigtriangledown\times\frac{\vec{r}}{r^3}=\vec{0}$

2.3

$$\bigtriangledown\times\vec{r}=\begin{vmatrix}\vec{e_x} & \vec{e_y} & \vec{e_z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-x' & y-y' & z-z'\end{vmatrix}=\vec{0}$$

3

设球坐标表示为 (r,θ,φ)，对应的直角坐标为 (x,y,z)，则有:

$$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta$$

对上列三个等式进行微分，得到:

$$dx=\sin\theta\cos\varphi dr+r\cos\theta\cos\varphi d\theta-r\sin\theta\sin\varphi d\varphi$$

$$dy=\sin\theta\sin\varphi dr+r\cos\theta\sin\varphi d\theta+r\sin\theta\cos\varphi d\varphi$$

$$dz=\cos\theta dr-r\sin\theta d\theta$$

反解出 dr, $d\theta$, $d\varphi$ 得到:

$$dr=\sin\theta\cos\varphi dx+\sin\theta\sin\varphi dy+\cos\theta dz$$

$$d\theta=\frac{\cos\theta\cos\varphi}{r}dx+\frac{\cos\theta\sin\varphi}{r}dy-\frac{\sin\theta}{r}dz$$

$$d\varphi=-\frac{\sin\varphi}{r\sin\varphi}dx+\frac{\cos\varphi}{r\sin\theta}dy$$

则有:

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}=\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}=\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2}=\frac{\partial r}{\partial x}(\frac{\partial}{\partial x})\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}(\frac{\partial}{\partial x})\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}(\frac{\partial}{\partial x})\frac{\partial}{\partial\varphi}$$

$$=\sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2}+\frac{\cos^2\theta\cos^2\varphi}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$+\frac{2\sin\theta\cos\theta\cos^2\varphi}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{2\sin\varphi\cos\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi}-\frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}$$

$$+\frac{\cos^2\theta\cos^2\varphi+\sin^2\theta}{r}\frac{\partial}{\partial r}+\frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}-\frac{2\sin^2\theta\cos\theta\cos^2\varphi-\cos\theta\sin^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial y^2}=\frac{\partial r}{\partial y}(\frac{\partial}{\partial y})\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}(\frac{\partial}{\partial y})\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}(\frac{\partial}{\partial y})\frac{\partial}{\partial\varphi}$$

$$=\sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2}+\frac{\cos^2\theta\sin^2\varphi}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}$$

$$+\frac{2\sin\theta\cos\theta\sin^2\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi}+\frac{2\sin\varphi\cos\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi}+\frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta\partial\varphi}$$
$$+\frac{\cos^2\theta\sin^2\varphi+\cos^2\varphi}{r}\frac{\partial}{\partial r}-\frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}-\frac{2\sin^2\theta\cos\theta\sin^2\varphi-\cos\theta\cos^2\varphi}{r^2\sin\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial z^2}=\frac{\partial r}{\partial z}(\frac{\partial}{\partial z})\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}(\frac{\partial}{\partial z})\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}(\frac{\partial}{\partial z})\frac{\partial}{\partial\varphi}$$
$$=\cos^2\theta\frac{\partial^2}{\partial z^2}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

在笛卡尔坐标系中，拉普拉斯算符表示为:

$$\bigtriangledown^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

则进一步可以计算得拉普拉斯算符在球坐标下的表示形式为:

$$\bigtriangledown^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

4

由题意可知，垂直于细杆方向的截面上每一点的力与位移相同，则以细杆最左端为原点，细杆方向为 x 轴。选取 (x,x+Δx) 为研究对象，由 Hooke 定律可知:

$$S\rho(x)\Delta x\frac{\partial^2u(\overline{x},t)}{\partial t^2}=Y(x)Su_x|_{x+\Delta x}-Y(x)Su_x|_x+f_0(\overline{x},t)\Delta x$$

$\overline{x}$ 为研究对象质心的坐标。令 $\Delta x\to 0$，进一步化简可得:

$$\frac{\partial}{\partial t}\left(\rho(x)\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial x}\left(Y(x)\frac{\partial u}{\partial x}\right)+\frac{f_0(x,t)}{S}$$

4.1

当 x=0 处固定时，边界条件为

$$u(0,t)=0$$

4.2

当 x=0 处受 G(t) 的横向外力时，边界条件为

$$\frac{\partial^2u}{\partial t^2}=\frac{Y(0)}{\rho(0)}\frac{\partial^2u}{\partial x^2}+\frac{G(t)}{\rho(0)S}$$

5

$\cos^2 x$ 的傅立叶变换为:

$$F(\omega)=\int_{-\infty}^{+\infty}\cos^2(x)e^{-i\omega x}\,dx=\frac{1}{2}\int_{-\infty}^{+\infty}(1+\cos 2x)e^{-i\omega x}\,dx=\pi\delta(\omega)+\frac{1}{2}\pi[\delta(\omega+2)+\delta(\omega-2)]$$

因为 $\cos^2 x$ 为偶函数，则其的傅立叶级数中 $b_n(n\geqslant 1,\ n\in N^*)$ 为 0。

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx=\frac{1}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\cos nx\,dx=\begin{cases}2, & n=\frac{1}{2}\\ 0, & n\neq 2\ and\ n\in N^*\end{cases}$$

则其傅立叶展开为:

$$f(x)=\frac{1}{2}+\frac{1}{2}cos(2x)$$

6

定义在 (0,∞) 上的 f(t) 为

$$f(t)=\begin{cases}h & (0<t<T),\\ 0 & (T<t).\end{cases}$$

6.1

当边界条件为 f'(0) = 0 时，可以对 f(t) 进行偶延拓，将 f(t) 展成傅立叶积分为

$$F(\omega)=2\int_{0}^{+\infty}f(t)\cos(\omega t)\,dt=2\int_{0}^{T}h\cos(\omega t)\,dt=\frac{2h}{\omega}\sin(\omega T)$$

$$f(t)=\frac{1}{2\pi}\int_{0}^{+\infty}F(\omega)e^{i\omega t}\,d\omega=\int_{0}^{+\infty}\frac{h}{\pi\omega}\sin(\omega T)e^{i\omega t}\,d\omega$$

6.2

当边界条件为 f(0) = 0 时，可以对 f(t) 进行奇延拓，将 f(t) 展成傅立叶积分为

$$F(\omega)=-2i\int_{0}^{+\infty}f(t)\sin(\omega t)\,dt=-2i\int_{0}^{T}h\sin(\omega t)\,dt=\frac{2ih}{\omega}\left(\cos(\omega T)-1\right)$$

$$f(t)=\frac{1}{2\pi}\int_{0}^{+\infty}F(\omega)e^{i\omega t}\,d\omega=\int_{0}^{+\infty}\frac{ih}{\pi\omega}\left(\cos(\omega T)-1\right)e^{i\omega t}\,d\omega$$

7

求解 $\frac{d^2T}{dt^2}+\omega^2a^2T=g(t)$，可以先求解 $\frac{d^2T}{dt^2}+\omega^2a^2T=0$。解得:

$$T(t)=C_1\cos(\omega at)+C_2\sin(\omega at)\quad C_1\ and\ C_2\ are\ constants$$

原非齐次方程的解为齐次方程的通解加非齐次方程的特解，即原方程的解可以写为

$$T(t)=C_1\cos(\omega at)+C_2\sin(\omega at)+T_0(t)$$

然后利用常数变易法，假设非齐次方程 $\frac{d^2T}{dt^2}+\omega^2a^2T=g(t)$ 也具有形如

$$T_0(t)=C_3(t)\cos(\omega at)+C_4(t)\sin(\omega at)$$

的特解，但是 $C_3(t)$ $C_4(t)$ 为待定函数。带入求解方程，令

$$C_3'(t)\cos(\omega at)+C_4'\sin(\omega at)=0$$

可得:

$$C_3(t)=-\int_{t_0}^{t}\frac{\sin(\omega ax)g(x)}{\omega a}\,dx,\quad C_4(t)=\int_{t_0}^{t}\frac{\cos(\omega ax)g(x)}{\omega a}\,dx,\quad t_0\ is\ a\ constant$$

则原方程的解为

$$T(t)=\left(-\int_{t_0}^{t}\frac{\sin(\omega ax)g(x)}{\omega a}\,dx+C_1\right)\cos(\omega at)+\left(\int_{t_0}^{t}\frac{\cos(\omega ax)g(x)}{\omega a}\,dx+C_2\right)\sin(\omega at)$$

8

假设拉盖尔方程 $t\frac{d^2y}{dt^2}+(1-t)\frac{dy}{dt}+\lambda y=0$ 的解为:

$$y=\sum_{k=0}^{n}a_kt^k$$

则有

$$\frac{dy}{dt}=\sum_{k=1}^{n}ka_kt^{k-1}$$

$$\frac{d^2y}{dt^2}=\sum_{k=2}^{n}k(k-1)a_kt^{k-2}$$

代入原方程，可得

$$\sum_{k=2}^{n}k(k-1)a_kt^{k-1}+\sum_{k=1}^{n}ka_kt^{k-1}-\sum_{k=1}^{n}ka_kt^k+\sum_{k=0}^{n}\lambda a_kt^k=0$$

则有

$$a_{k+1}=\frac{k-\lambda}{(k+1)^2}a_k$$

所以

$$a_n = \frac{(n-1-\lambda)(n-2-\lambda)...(1-\lambda)(-\lambda)}{(n!)^2} a_0$$

使上式在 n 有限的条件下成立，即要求存在一个正整数 N，当 $n \geqslant N$ 时，t^n 系数为 $a_n = 0$。所以当 λ 取非负整数时可以使方程的解为多项式。