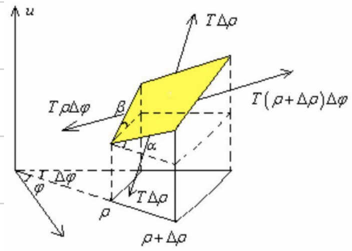
第四次作业 董建宇 2019511017

1. 解：在极坐标系中选取如图微元受力分析：



由牛顿第二定律得 $\rho_m \cdot \rho\Delta\varphi\Delta\rho \cdot \frac{\partial^2 u}{\partial t^2} = T(\rho+\Delta\rho)\Delta\varphi \cdot \sin\alpha - T\rho\Delta\varphi \sin\alpha + T\Delta\rho\sin\beta|_{\varphi+\Delta\varphi} - T\Delta\rho\sin\beta|_{\varphi}$ ①

当 α, β 很小时有 $\sin\alpha = \tan\alpha = \frac{\Delta u}{\Delta\rho}$ $\sin\beta = \tan\beta = \frac{1}{\rho}\frac{\Delta u}{\Delta\varphi}$ ①式两侧同时除以 $\Delta\varphi, \Delta\rho$ 并取极限

$\Delta\rho \to 0, \Delta\varphi \to 0$ 则有 $\rho_m \rho \frac{\partial^2 u}{\partial t^2} = T\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + T \frac{1}{\rho}\frac{\partial^2 u}{\partial\varphi^2}$

即 $\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho_m} \cdot \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial t^2}\right] = \frac{T}{\rho_m}\nabla^2 u$

化为直角坐标系得 $\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho_m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ 边界条件为 $u|_{r=R} = 0$ R为弹性圆膜半径。

2. 解：
$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} - b^2 u & (0 < x < L, t > 0) \\ u|_{t=0} = \sqrt{2}\, c\cos\frac{\pi}{4L}x & (0 \le x \le L) \\ \frac{\partial u}{\partial x}\Big|_{t=0} = 0 \quad u|_{x=L} = c & (t > 0) \end{cases}$$

令 $u = v + c$ 则 v 满足齐次初始条件 $\frac{\partial v}{\partial x}\Big|_{t=0} = \frac{\partial u}{\partial x}\Big|_{t=0} = 0$ $v|_{x=L} = u|_{x=L} - c = 0$ $(t>0)$

则
$$\begin{cases} \frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2} - b^2 v - b^2 c & (0 < x < L, t > 0) \\ v|_{t=0} = \sqrt{2}\, c\cos\frac{\pi}{4L}x - c & (0 \le x \le L) \\ \frac{\partial v}{\partial x}\Big|_{x=0} = 0 \quad v|_{x=L} = 0 & (t > 0) \end{cases}$$

设 $v(x,t)$ 可以写成 $v(x,t) = X(x)T(t)$

选择 $X''(x) + \lambda X(x) = 0$，$X'(0) = X(L) = 0$ 的基矢函数集

$\left\{\cos\frac{(2n+1)\pi}{2L}x\right\}$ $n = 0, 1, 2, \cdots$

则 $v(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos\frac{(2n+1)\pi}{2L}x$，代入泛定方程与初始条件可得：

$$\begin{cases} \sum_{n=0}^{\infty}\left\{T_n'(t) + a^2\left[\frac{(2n+1)\pi}{2L}\right]^2 T_n(t) + b^2 T_n(t)\right\}\cos\frac{(2n+1)\pi}{2L}x = -b^2 c \\ \sum_{n=0}^{\infty} T_n(0)\cos\frac{(2n+1)\pi}{2L}x = \sqrt{2}\, c\cos\frac{\pi}{4L}x - c \end{cases}$$

则有 $T_n'(t) + \left\{b^2 + \left[\frac{(2n+1)\pi a}{2L}\right]^2\right\}T_n(t) = \frac{2}{L}\int_0^L -b^2 c\cos\frac{(2n+1)\pi}{2L}x\,dx = \frac{(-1)^{n+1}4b^2 c}{(2n+1)\pi}$

$$T_n(0) = \frac{2}{L}\int_0^L\left(\sqrt{2}\, c\cos\frac{\pi}{4L}x - c\right)\cos\frac{(2n+1)\pi}{2L}x\,dx = \frac{2}{L}\left\{(-1)^n\frac{c}{2}\left[\frac{4L}{(4n+3)\pi} + \frac{4L}{(4n+1)\pi}\right] - (-1)^n c\frac{2L}{(2n+1)\pi}\right\}$$

$$= (-1)^n \cdot 4c \cdot \left[\frac{1}{(4n+3)\pi} + \frac{1}{(4n+1)\pi} - \frac{1}{(2n+1)\pi}\right]$$

即 $$T_n(t) = \left((-1)^n 4c\left[\frac{1}{(4n+3)\pi} + \frac{1}{(4n+1)\pi} - \frac{1}{(2n+1)\pi}\right] + (-1)^n\frac{4b^2 c}{(2n+1)\pi\left\{b^2 + \left[\frac{(2n+1)\pi a}{2L}\right]^2\right\}}\right)e^{-\left\{b^2 + \left[\frac{(2n+1)\pi a}{2L}\right]^2\right\}t} + (-1)^{n+1}\frac{4b^2 c}{(2n+1)\pi\left\{b^2 + \left[\frac{(2n+1)\pi a}{2L}\right]^2\right\}}$$

所以 $u(x,t) = v(x,t) + c = \sum_{n=0}^{\infty} T_n(t)\cos\frac{(2n+1)\pi}{2L}x + c$

其中 $T_n(t)$ 由上式给出。

3. (1). $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(2x+\lambda)y=0$ 则有 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+(\frac{2}{x}+\frac{\lambda}{x^2})y=0$

$k(x)=e^{\int\frac{1}{x}dx}=Cx$ （C为常数） 则 Sturm-Liouville 型方程的标准形式为

$\frac{d}{dx}(Cx\frac{dy}{dx})+Cx\cdot\frac{2}{x}y+\lambda\cdot(Cx\cdot\frac{1}{x^2})y=0$ 即 $\frac{d}{dx}(x\frac{dy}{dx})+2y+\lambda\cdot\frac{1}{x}y=0$

(2). $x(1-x)\frac{d^2y}{dx^2}+(a-bx)\frac{dy}{dx}-\lambda y=0$ 则有 $\frac{d^2y}{dx^2}+\frac{a-bx}{x(1-x)}\frac{dy}{dx}-\frac{\lambda}{x(1-x)}y=0$

$\int\frac{a-bx}{x(1-x)}dx=\int[a\cdot(\frac{1}{x}+\frac{1}{1-x})-\frac{b}{1-x}]dx=a\ln|x|-a\ln|x-1|+b\ln|x-1|+\ln C$ C为常数

$k(x)=e^{\int\frac{a-bx}{x(1-x)}dx}=\frac{e^{a\ln|x|}\cdot e^{b\ln|x-1|}}{e^{a\ln|x-1|}}\cdot e^{\ln C}=C\cdot|x|^a|x-1|^{b-a}$ 则 Sturm-Liouville 型方程的标准形式为

$\frac{d}{dx}(C|x|^a|x-1|^{b-a}\frac{dy}{dx})+\frac{\lambda}{x(x-1)}C|x|^a|x-1|^{b-a}y=0$ 即 $\frac{d}{dx}(|x|^a|x-1|^{b-a}\frac{dy}{dx})+\lambda\cdot\frac{|x|^a|x-1|^{b-a}}{x(x-1)}y=0$

4.解. ①当 $\lambda<0$ 时，$X''(x)+\lambda X(x)=0$ 通解为 $X(x)=Ae^{kx}+Be^{-kx}$ 其中 $k=\sqrt{-\lambda}$ 满足边界条件 $\alpha_1X(0)+\beta_1X'(0)=0$ $\alpha_2X(l)+\beta_2X'(l)=0$

则有 $\alpha_1(A+B)+\beta_1k(A-B)=0$ $\alpha_2(Ae^{kl}+Be^{-kl})+\beta_2k(Ae^{kl}-Be^{-kl})=0$ $(\alpha_2-\beta_2k)-\frac{(\alpha_1-\beta_1k)(\alpha_2+\beta_2k)}{\alpha_1+\beta_1k}e^{2kl}=0$

即 $(\alpha_1+\beta_1k)(\alpha_2-\beta_2k)-(\alpha_1-\beta_1k)(\alpha_2+\beta_2k)e^{2kl}=0$ (*)

即本征值问题的本征值为 $\lambda_n=-k_n^2$ k_n为(*)式的根。解为 $X_n(x)=A_ne^{k_nx}+B_ne^{-k_nx}$，$k_n$为(*)式的根。

②当 $\lambda=0$ 时，通解为 $X(x)=C+Dx$，满足边界条件 $\alpha_1X(0)+\beta_1X'(0)=0$，$\alpha_2X(l)+\beta_2X'(l)=0$ 的解为：

$\begin{cases}\alpha_1C+\beta_1D=0\\ \alpha_2(C+Dl)+\beta_2D=0\end{cases}$ 则 当 $\alpha_1(\alpha_2l+\beta_2)-\alpha_2\beta_1\neq0$ 时，解为 $\begin{cases}C=0\\D=0\end{cases}$

当 $\alpha_1(\alpha_2l+\beta_2)-\alpha_2\beta_1=0$ 时，解为 $\begin{cases}C=-\frac{\beta_1}{\alpha_1}t\\D=t\end{cases}$ $t\in R$

$\therefore\lambda_0=0$ $X(x)=t(x-\frac{\beta_1}{\alpha_1})$，当 $\alpha_1(\alpha_2l+\beta_2)-\alpha_2\beta_1=0$ 时； $\lambda_0=0$，$X(x)=0$，当 $\alpha_1(\alpha_2l+\beta_2)-\alpha_2\beta_1\neq0$ 时.

③当 $\lambda>0$ 时，通解为 $X(x)=E\cos kx+F\sin kx$ 其中 $k=\sqrt{\lambda}$，满足边界条件 $\alpha_1X(0)+\beta_1X'(0)=0$ $\alpha_2X(l)+\beta_2X'(l)=0$

则有 $E\alpha_1+F\beta_1k=0$ $\alpha_2(E\cos kl+F\sin kl)+\beta_2k(-E\sin kl+F\cos kl)=0$

即 $(\alpha_1\alpha_2+\beta_1\beta_2k^2)\sin kl-(\alpha_1\beta_2+\alpha_2\beta_1)k\cos kl=0$ (**) $k=\sqrt{\lambda}$

即本征值问题的本征值为 $\lambda_n=k_n^2$ k_n为(**)式的根。解为 $X_n(x)=E_n\cos k_nx+F_n\sin k_nx$，$k_n$为(**)式的根。

该问题中 $k(x)=1$ 则 Q因子为：

$Q=k(0)[X_n(0)X_m'(0)-X_m(0)X_n'(0)]-k(l)[X_n(l)X_m'(l)-X_m(l)X_n'(l)]=X_n(0)\cdot[-\frac{\alpha_1}{\beta_1}X_m(0)]-X_m(0)\cdot[-\frac{\alpha_1}{\beta_1}X_n(0)]-X_n(l)\cdot[-\frac{\alpha_2}{\beta_2}X_m(l)]+X_m(l)[-\frac{\alpha_2}{\beta_2}X_n(l)]=0$

所以本征函数 $X_n(x)$ 是正交的。

$\rho(\xi)=1$，则 当 $\lambda<0$ 时，归一因子为

$N_n=\int_0^lX_n^2(x)dx=\int_0^l(A_n^2e^{2k_nx}+2A_nB_n+B_n^2e^{-2k_nx})dx=\frac{A_n^2}{2k_n}(e^{2k_nl}-1)+2A_nB_nl+\frac{B_n^2}{2k_n}(1-e^{2k_nl})$，$k_n$为(*)式的根

当 $\lambda=0$ 时，归一因子为 $N_n(x)=\int_0^l(C+Dx)^2dx=\frac{1}{3}D^2l^3+CDl^2+C^2l$ 其中 C.D在②中给出

当 $\lambda>0$ 时，归一因子为

$N_n=\int_0^lX_n^2(x)dx=\int_0^l(A_n^2\cos^2k_nx+B_n^2\sin^2k_nx+2A_nB_n\sin k_nx\cos k_nx)dx=\frac{A_n^2+B_n^2}{2}l+\frac{A_n^2-B_n^2}{4k_n}\sin2k_nl+\frac{A_nB_n}{2k_n}(1-\cos2k_nl)$

k_n为(**)式的根。

5. 证明: $\begin{cases} \dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] + [\lambda\rho(x) - q(x)]y = 0 \\ y(b) = a_{11}y(a) + a_{12}y'(a), \quad y'(b) = a_{21}y(a) + a_{22}y'(a) \end{cases}$

假定 λ_m, λ_n 为两不同特征值, 对应的本征函数为 $y_m(x), y_n(x)$, 则其分别满足

$y_m(b) = a_{11}y_m(a) + a_{12}y_m'(a)$, $y_n(b) = a_{11}y_n(a) + a_{12}y_n'(a)$, $y_m'(b) = a_{21}y_m(a) + a_{22}y_m'(a)$, $y_n'(b) = a_{21}y_n(a) + a_{22}y_n'(a)$

其Q因子为: $Q = p(a)[y_n(a)y_m'(a) - y_m(a)y_n'(a)] - p(b)[y_n(b)y_m'(b) - y_m(b)y_n'(b)]$

$= p(a)[1 - (a_{11}a_{22} - a_{12}a_{21})][y_n(a)y_m'(a) - y_m(a)y_n'(a)]$

当 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 1$ 时 Q因子等于0, 即当 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1$ 时, 对应不同本征值的本征函数正交。

6. $\begin{cases} \dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < +\infty \\ u|_{t=0} = \varphi(x) \end{cases}$

(1) 关于 t 做拉普拉斯变换得 $pU(x,p) - \varphi(x) = a^2\dfrac{d^2U(x,p)}{dx^2} + F(x,p)$ $\quad F(x,p) = \int_0^{+\infty} f(x,t)e^{-pt}dt$

关于 x 做傅里叶变换得 $pU(\omega,p) - \Phi(\omega) = -a^2\omega^2U(\omega,p) + H(\omega,p)$ $\quad H(\omega,p) = \int_{-\infty}^{+\infty} F(x,p)e^{-i\omega x}dx$

则 $U(\omega,p) = \dfrac{\Phi(\omega) + H(\omega,p)}{p + a^2\omega^2}$ 做拉普拉斯逆变换得 $U(\omega,t) = \Phi(\omega)e^{-a^2\omega^2 t} + \mathcal{L}^{-1}\left(\dfrac{H(\omega,p)}{p+a^2\omega^2}\right)$

做傅里叶逆变换得 $u(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\Phi(\omega)e^{-a^2\omega^2 t} + \mathcal{L}^{-1}\left(\dfrac{H(\omega,p)}{p+a^2\omega^2}\right)\right]e^{i\omega x}d\omega$

$= \int_{-\infty}^{+\infty}\varphi(s)\dfrac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-s)^2}{4a^2t}}ds + \int_{-\infty}^{+\infty}\int_0^t f(\xi,\tau)\dfrac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\tau\, d\xi$

(2). 关于 x 做傅立叶变换得 $\begin{cases} \dfrac{dU(\omega,t)}{dt} = -a^2\omega^2U(\omega,t) + F(\omega,t) \\ U(\omega,0) = \Phi(\omega), \quad F(\omega,t) = \int_{-\infty}^{+\infty} f(x,t)e^{-i\omega x}dx, \quad \Phi(\omega) = \int_{-\infty}^{+\infty}\varphi(x)e^{-i\omega x}dx \end{cases}$

一阶常微分方程加定解条件可得 $U(\omega,t) = \left[\int_0^t F(\omega,\tau)e^{a^2\omega^2\tau}d\tau + \Phi(\omega)\right]e^{-a^2\omega^2 t}$

则 $u(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^t F(\omega,\tau)e^{a^2\omega^2\tau}d\tau + \Phi(\omega)\right]e^{-a^2\omega^2 t}\cdot e^{i\omega x}d\omega$

$= \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi(\omega)e^{-a^2\omega^2 t}\cdot e^{i\omega x}d\omega + \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^t F(\omega,\tau)e^{a^2\omega^2\tau}d\tau\; e^{-a^2\omega^2 t}\right]e^{i\omega x}d\omega$

$= \int_{-\infty}^{+\infty}\varphi(s)\left[\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-a^2\omega^2 t}\cdot e^{i\omega(x-s)}d\omega\right]ds + \int_0^t\int_{-\infty}^{+\infty}f(\xi,\tau)\dfrac{1}{2\pi}\left[\int_{-\infty}^{+\infty}e^{-a^2\omega^2(t-\tau)}\cdot e^{i\omega(x-\xi)}d\omega\right]d\xi\, d\tau$

$= \int_{-\infty}^{+\infty}\varphi(s)\dfrac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-s)^2}{4a^2t}}ds + \int_0^t\int_{-\infty}^{+\infty}f(\xi,\tau)\cdot\dfrac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi\, d\tau$

7. 解: $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = \varphi(x), \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$

关于x做傅里叶变换，得 $\begin{cases} \dfrac{d^2 U(\omega,t)}{dt^2} + c^2\omega^2 U(\omega,t) = 0 \\ U(\omega,0) = A(\omega), \quad \dfrac{dU(\omega,t)}{dt}\Big|_{t=0} = B(\omega), \quad A(\omega) = \int_{-\infty}^{+\infty} \varphi(x) e^{-i\omega x} dx \quad B(\omega) = \int_{-\infty}^{+\infty} \psi(x) e^{-i\omega x} dx \end{cases}$

对t做拉普拉斯变换.得 $p^2 U(\omega,p) - pU(\omega,0) - \dfrac{dU(\omega,t)}{dt}\Big|_{t=0} + c^2\omega^2 U(\omega,p) = 0$

$$U(\omega,p) = \frac{pA(\omega) + B(\omega)}{p^2 + c^2\omega^2} = A(\omega)\frac{p}{p^2+c^2\omega^2} + B(\omega)\frac{1}{p^2+c^2\omega^2}$$

做拉普拉斯逆变换得 $U(\omega,t) = A(\omega)\cos(c\omega t) + \dfrac{B(\omega)}{c\omega}\sin(c\omega t)$

做傅里叶逆变换得 $u(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[A(\omega)\cos(c\omega t) + \dfrac{B(\omega)}{c\omega}\sin(c\omega t)\right] e^{i\omega x} d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\cos(c\omega t)\cdot\int_{-\infty}^{+\infty}\varphi(s) e^{-i\omega s} ds + \sin(c\omega t)\cdot\frac{1}{c\omega}\int_{-\infty}^{+\infty}\psi(s) e^{-i\omega s} ds\right] e^{i\omega x} d\omega$$

代入 $\cos(c\omega t) = \dfrac{e^{ic\omega t} + e^{-ic\omega t}}{2}$ $\quad \sin(c\omega t) = \dfrac{e^{ic\omega t} - e^{-ic\omega t}}{2i}$

得 $u(x,t) = \dfrac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \dfrac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)\, d\xi$