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1. (1) $\int x J_2(x)\,dx = \int x\left[-J_0(x)+\frac{2}{x}J_1(x)\right]dx = -\int xJ_0(x)\,dx + 2\int J_1(x)dx$

$= -xJ_1(x) - 2J_0(x) + C$　C为常数.

(2) $\int x^4 J_1(x)\,dx = \int x^2\, d[x^2 J_2(x)] = x^4 J_2(x) - 2\int x^3 J_2(x)dx = x^4J_2(x) - 2x^3J_3(x) + C$　C为常数.

(3). $\int_0^R J_0(x)\cos x\,dx = xJ_0(x)\cos x\Big|_0^R - \int_0^R x\cdot[J_0'(x)\cos x - J_0(x)\sin x]\,dx = RJ_0(R)\cos R + \int_0^R [xJ_1(x)\cos x + xJ_0(x)\sin x]\,dx$

其中 $\int_0^R[xJ_1(x)\cos x + xJ_0(x)\sin x]\,dx = \int_0^R xJ_1(x)\,d(\sin x) + \int_0^R xJ_0(x)\sin x\,dx = xJ_1(x)\sin x\Big|_0^R - \int_0^R xJ_0(x)\sin x\,dx + \int_0^R xJ_0(x)\sin x\,dx = RJ_1(R)\sin R$

则 $\int_0^R J_0(x)\cos x\,dx = RJ_0(R)\cos R + RJ_1(R)\sin R$

(4). $3J_0'(x) + 4J_0^{(3)}(x) = -3J_1(x) + 4J_0^{(3)}(x)$

$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta$　　$J_0^{(3)}(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin^3\theta\, d\theta$

利用积化和差得: $\sin(x\sin\theta)\cdot\sin^3\theta = \frac{1}{2}[\cos(x\sin\theta-\theta) - \cos(x\sin\theta+\theta)]\cdot\frac{1}{2}(1-\cos2\theta)$

$= \frac{1}{4}[\cos(x\sin\theta-\theta) - \cos(x\sin\theta+\theta)] - \frac{1}{8}[\cos(x\sin\theta-3\theta) + \cos(x\sin\theta+\theta)] + \frac{1}{8}[\cos(x\sin\theta-\theta) + \cos(x\sin\theta+3\theta)]$

则有 $J_0^{(3)}(x) = \frac{1}{\pi}\int_0^\pi\left[\frac{3}{8}\cos(x\sin\theta-\theta) - \frac{3}{8}\cos(x\sin\theta+\theta) - \frac{1}{8}\cos(x\sin\theta-3\theta) + \frac{1}{8}\cos(x\sin\theta+3\theta)\right]d\theta$

$= \frac{3}{8}J_1(x) - \frac{3}{8}J_{-1}(x) - \frac{1}{8}J_3(x) + \frac{1}{8}J_{-3}(x) = \frac{3}{4}J_1(x) - \frac{1}{4}J_3(x)$

则 $3J_0'(x) + 4J_0^{(3)}(x) = -J_3(x)$

2. 解. $f(x) = \begin{cases} 1, & 0\le x\le 1 \\ 0, & x\in(-\infty,0)\cup(1,+\infty)\end{cases}$

设 $f(x) = \sum_{m=1}^{+\infty} A_m J_0(\mu_{0m}x)$

$A_m = \frac{2}{J_1^2(\mu_{0m})}\int_0^1 xJ_0(\mu_{0m}x)\,dx = \frac{2}{J_1^2(\mu_{0m})}\frac{J_1(\mu_{0m})}{\mu_{0m}} = \frac{2}{\mu_{0m}J_1(\mu_{0m})}$

即 $1 = \sum_{m=0}^{+\infty}\frac{2}{\mu_{0m}J_1(\mu_{0m})}J_0(\mu_{0m}x)$　$0<x<1$. 其中μ_{0m}为零阶贝塞尔函数 $J_0(x)$ 的零点.

3. 解: 由题意得: $\begin{cases} \frac{\partial u}{\partial t} = \frac{k}{c\rho}\nabla^2 u \\ u|_{t=0} = u_0\left[1-\left(\frac{\rho}{a}\right)^2\right],\ \frac{\partial u}{\partial\rho}\Big|_{\rho=a} = 0,\ \frac{\partial u}{\partial z}\Big|_{z=0} = 0,\ \frac{\partial u}{\partial z}\Big|_{z=h} = 0\end{cases}$

记 $l^2 = \frac{k}{c\rho}$, 选取柱坐标系, 则有 $\frac{\partial u}{\partial t} = l^2\left(\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}\right)$

由初始条件及边界条件可知, u与φ无关, 即 $\frac{\partial^2 u}{\partial\varphi^2} = 0$, 则有 $\frac{\partial u}{\partial t} = l^2\left(\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{\partial^2 u}{\partial z^2}\right)$

设 $u(\rho,z,t) = S(\rho,z)T(t)$, 则有 $\frac{T'}{l^2T} = \frac{\frac{\partial^2 S}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial S}{\partial\rho} + \frac{\partial^2 S}{\partial z^2}}{S} = -\beta$　β为常数.

则 $T(t) + l^2\beta T'(t) = 0$　　$\frac{\partial^2 S}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial S}{\partial\rho} + \frac{\partial^2 S}{\partial z^2} + \beta S = 0$

设 $S(\rho,z) = R(\rho)\cdot Z(z)$, 则 $R''Z + \frac{1}{\rho}R'Z + RZ'' + \beta RZ = 0$, 即 $\frac{R'' + \frac{1}{\rho}R'}{R} = -\frac{Z''}{Z} - \beta = -\lambda^2$

则有 $T(t) + l^2\beta T'(t) = 0$;　$R'' + \frac{1}{\rho}R' + \lambda^2 R = 0$;　$Z'' - (\lambda^2-\beta)Z = 0$

则 $T(t) = e^{-l^2\beta t}$;　$R(\rho) = AJ_0(\lambda\rho) + BY_0(\lambda\rho)$;　$Z = C\cos(\sqrt{\beta-\lambda^2}\,z + \varphi)$　当 $\beta>\lambda^2$ 时

$Z'' + (\beta-\lambda^2)Z = 0$　　$Z = C\sin(\sqrt{\beta-\lambda^2}\,z + \varphi)$

$\sqrt{\lambda^2-\beta}\ z) + D\cosh(\sqrt{\lambda^2-\beta}\ z)$

$$\frac{\partial Z}{\partial z}\Big|_{z=0}=C\sqrt{\beta-\lambda^2}\cos\varphi \qquad \varphi=\frac{\pi}{2}$$

$$\frac{\partial Z}{\partial z}\Big|_{z=0}\ C\sqrt{\beta-\lambda^2}\cdot\cos(\sqrt{\beta-\lambda^2}h+\varphi)$$

$$\sqrt{\beta-\lambda^2}h=n\pi$$

由于当 $t\to+\infty$ 时，u 为有限值，则 $\beta\geq0$；当 $\rho\to0$ 时，u 为有限值，则 $B=0$

由边界条件 $\frac{\partial R}{\partial\rho}\Big|_{\rho=a}=-A\lambda J_1(\lambda a)=0$ 即 λa 为一阶贝塞尔方程的根，即 $\lambda_m=\frac{\mu_{1m}}{a}$ $(m=1,2,3\cdots)$

$\frac{\partial Z}{\partial z}=C\cdot\sqrt{\lambda^2-\beta}\cosh(\sqrt{\lambda^2-\beta}z)+D\sqrt{\lambda^2-\beta}\sinh(\sqrt{\lambda^2-\beta}z)$ $\quad\frac{\partial Z}{\partial z}\Big|_{z=0}=\frac{\partial Z}{\partial z}\Big|_{z=h}=0$ 得 $\varphi=0$，$\sqrt{\beta-\lambda^2}h=n\pi$ $\quad n=0,1,2,\cdots$ $\quad\beta_{mn}=\lambda_m^2+(\frac{n\pi}{h})^2$

则有 $u(\rho,z,t)=\sum_{m=1}^{\infty}A_mJ_0(\lambda_m\rho)\cdot e^{-l^2\beta_mt}$ 。代入初始条件 $u_0[1-(\frac{\rho}{a})^2]=\sum_{m=1}^{\infty}A_mJ_0(\lambda_m\rho)$

记 $y=\frac{\rho}{a}$ 则 $u_0(1-y^2)=\sum_{m=1}^{\infty}A_mJ_0(\mu_{1m}y)$，记 $z=\frac{\mu_{1m}}{\mu_{0m}}y$，则 $y=\frac{\mu_{0m}}{\mu_{1m}}z$

所以有 $u_0(1-\frac{\mu_{0m}^2}{\mu_{1m}^2}z^2)=\sum_{m=1}^{\infty}A_mJ_0(\mu_{0m}z)$

则有 $A_m=\frac{2}{J_1^2(\mu_{0m})}\int_0^1zJ_0(\mu_{0m}z)u_0(1-\frac{\mu_{0m}^2}{\mu_{1m}^2}z^2)dz=\frac{2u_0}{J_1^2(\mu_{0m})}\int_0^1zJ_0(\mu_{0m}z)dz-\frac{2u_0}{J_1^2(\mu_{0m})}\frac{\mu_{0m}^2}{\mu_{1m}^2}\int_0^1z^3J_0(\mu_{0m}z)dz$

其中 $\int_0^1zJ_0(\mu_{0m}z)dz=\frac{1}{\mu_{0m}}\int_0^1d(zJ_1(\mu_{0m}z))=\frac{J_1(\mu_{0m})}{\mu_{0m}}$ $\qquad\int_0^1z^3J_0(\mu_{0m}z)dz=\frac{J_1(\mu_{0m})}{\mu_{0m}}-2\frac{J_2(\mu_{0m})}{\mu_{0m}^2}$

则 $A_m=\frac{2u_0}{\mu_{0m}J_1(\mu_{0m})}(1-\frac{\mu_{0m}^2}{\mu_{1m}^2})+4u_0\frac{J_2(\mu_{0m})}{\mu_{1m}^2J_1^2(\mu_{0m})}$

则 $u(\rho,z,t)=\sum_{m=1}^{\infty}A_mJ_0(\lambda_m\rho)e^{-l^2\beta_mt}$，$\lambda_m=\frac{\mu_{1m}}{a}$，$\beta_m=\lambda_m^2$，$l^2=\frac{k}{c\rho}$

4. 解：由题意可知：
$$\begin{cases}\frac{\partial u}{\partial t}=a^2\nabla^2u & (a<r<r_0)\\ u|_{t=0}=f(r)\cos\theta, & u|_{r=r_0}=0\end{cases}$$

选取球坐标系，得 $\frac{\partial u}{\partial t}=a^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\varphi^2}\right]$

由初始条件及变界条件可知，u 与 φ 无关，即 $\frac{\partial^2u}{\partial\varphi^2}=0$，即 $\frac{\partial u}{\partial t}=a^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})\right]$

设解为 $u(r,\theta,t)=S(r,\theta)T(t)$，则有 $S(r,\theta)T'(t)=a^2T(t)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial S}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial S}{\partial\theta})\right]$

即 $\frac{T'(t)}{a^2T(t)}=\frac{1}{S(r,\theta)}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial S}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial S}{\partial\theta})\right]=-\beta$ 。则有 $T(t)=e^{-a^2\beta t}$。因为 $t\to+\infty$ 时，u 为有限值，则 $\beta\geq0$，则 $\beta=k^2$

设 $S(r,\theta)=R(r)\Theta(\theta)$，则有 $\frac{r^2R''+2rR'}{R}+\beta r^2=-\frac{\Theta''+\frac{\cos\theta}{\sin\theta}\Theta'}{\Theta}=n(n+1)$

即 $r^2R''+2rR'+(k^2r^2-n(n+1))R$ ①，$\quad\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+n(n+1)\Theta=0$ ②

②的解为 $\Theta(\theta)=P_n(\cos\theta)$

对于①式，令 $R(r)=\frac{w(r)}{\sqrt{r}}$，则 $R=\frac{w}{\sqrt{r}}$，$R'=\frac{w'}{\sqrt{r}}-\frac{w}{2\sqrt{r^3}}$，$R''=\frac{w''}{\sqrt{r}}-\frac{w'}{\sqrt{r^3}}+\frac{w}{4\sqrt{r^5}}$，则①式化为 $r^2w''+rw'+[k^2r^2-(n+\frac{1}{2})^2]w=0$ ③.

要使③式存在有界解，则 $k^2=\lambda_{nm}^2=\left(\frac{\mu_{n+\frac{1}{2},m}}{r_0}\right)^2$，其中 $\mu_{n+\frac{1}{2},m}$ 表示 $(n+\frac{1}{2})$ 阶贝塞尔函数的第 m 个零点.

则③式的解为 $w_{nm}(r)=J_{n+\frac{1}{2}}(\lambda_{nm}r)$ $\quad n=0,1,2,\cdots\quad m=1,2,3\cdots$ $\quad$ 则 $R(r)=\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(\lambda_{nm}r)$

则 $u(r,\theta,t)=\sum_{n=0}^{+\infty}\left\{P_n(\cos\theta)\cdot\left[\sum_{m=1}^{+\infty}\frac{A_{nm}}{\sqrt{r}}J_{n+\frac{1}{2}}(\lambda_{nm}r)\right]\cdot e^{-a^2\beta_{nm}t}\right\}$，初始条件为 $u(r,\theta)=f(r)\cos\theta$.

由于 $P_n(\cos\theta)$ 的正交性易知当 $n\neq1$ 时 $A_n=0$，则 $u(r,\theta)=\cos\theta\cdot\frac{1}{\sqrt{r}}\left[\sum_{m=1}^{+\infty}A_{1m}J_{\frac{3}{2}}(\lambda_{1m}r)\right]=f(r)\cos\theta$

则 $\sum_{m=1}^{\infty}A_{1m}\sqrt{\frac{2\lambda_{1m}}{\pi}}j_1(\lambda_{1m}r)=f(r)$。则两侧同时乘 $r^2j_1(\lambda_{1m}r)$ 在 $[0,r_0]$ 上积分得

$A_{1m}\cdot\sqrt{\frac{2\lambda_{1m}}{\pi}}\cdot\frac{r_0^3}{2}j_2^2(\mu_{\frac{3}{2},m})=\int_0^{r_0}r^2j_1(\lambda_{1m}r)f(r)dr$，则 $A_{1m}=\sqrt{\frac{\pi}{2\lambda_{1m}}}\cdot\frac{2}{r_0^3}\frac{1}{j_2^2(\mu_{\frac{3}{2},m})}\cdot\int_0^{r_0}r^2j_1(\lambda_{1m}r)f(r)dr$

$u(r,\theta,t)=\sum_{m=1}^{+\infty}\frac{A_{1m}}{\sqrt{r}}J_{n+\frac{1}{2}}(\lambda_{1m}r)\cdot\cos\theta\cdot e^{-a^2\beta_{1m}t}$ $\quad\lambda_{1m}=\frac{\mu_{\frac{3}{2},m}}{r_0}$ $\quad\beta_{1m}=\lambda_{1m}^2$ $\quad A_{1m}$ 如上，$\mu_{\frac{3}{2},m}$ 为 $\frac{3}{2}$ 阶贝塞尔函数第 m 个正零点.

5. 解：$f(x)=\begin{cases} x^2, & 0\le x\le 1; \\ 0, & -1\le x\le 0. \end{cases}$

设 $f(x)=\sum_{n=0}^{\infty} a_n P_n(x)$，两侧同时乘 P_n 并在 $[-1,1]$ 上积分得：

$\frac{2}{2n+1}a_n=\int_{-1}^{1} f(x)P_n(x)dx$，即 $a_n=\frac{2n+1}{2}\int_0^1 x^2 P_n(x)dx$. 当 $n=0$ 时 $a_0=\frac{1}{2}\int_0^1 x^2 dx=\frac{1}{6}$

当 $n-1=0$ 时 即 $n=1$ 时 $a_1=\frac{3}{2}\int_0^1 x^2\cdot x\ dx=\frac{3}{8}$

当 $n=2$ 时 $a_2=\frac{5}{2}\int_0^1 x^2\cdot\frac{1}{2}(3x^2-1)dx=\frac{1}{3}$

当 $n\ge 3$ 时 利用递推关系：$(2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x)$，则

$a_n=\frac{1}{2}\int_0^1 x^2[P_{n+1}'(x)-P_{n-1}'(x)]dx=\frac{1}{2}\left[\int_0^1 x^2 d(P_{n+1}(x))-\int_0^1 x^2 d(P_{n-1}(x))\right]=\frac{1}{2}\left[x^2P_{n+1}(x)\Big|_0^1-\int_0^1 2xP_{n+1}(x)dx-x^2P_{n-1}(x)\Big|_0^1+\int_0^1 2xP_{n-1}(x)\right]$

$=\frac{1}{2}[P_{n+1}(1)-P_{n-1}(1)]-\left[\int_0^1 xP_{n+1}(x)dx-\int_0^1 xP_{n-1}(x)dx\right]$

则 $[2(n+1)+1]P_{n+1}(x)=P_{n+2}'(x)-P_n'(x)$ $\quad [2(n-1)+1]P_{n-1}(x)=P_n'(x)-P_{n-2}'(x)$

则 $\int_0^1 xP_{n+1}(x)dx=\frac{1}{2n+3}\left[\int_0^1 x\,d(P_{n+2}(x))-\int_0^1 x\,d(P_n(x))\right]=\frac{1}{2n+3}\left[xP_{n+2}(x)\Big|_0^1-\int_0^1 P_{n+2}(x)dx-xP_n(x)\Big|_0^1+\int_0^1 P_n(x)dx\right]$

$=\frac{1}{2n+3}\left\{[P_{n+2}(1)-P_n(1)]-\left[\int_0^1 P_{n+2}(x)dx-\int_0^1 P_n(x)dx\right]\right\}$

$\int_0^1 xP_{n-1}(x)dx=\frac{1}{2n-1}\left\{[P_n(1)-P_{n-2}(1)]-\left[\int_0^1 P_n(x)dx-\int_0^1 P_{n-2}(x)dx\right]\right\}$

$\int_0^1 P_{n+2}(x)dx=\frac{1}{2(n+2)+1}[P_{n+3}(1)-P_{n+1}(1)]$ $\quad \int_0^1 P_n(x)dx=\frac{1}{2n+1}[P_{n+1}(1)-P_{n-1}(1)]$ $\quad \int_0^1 P_{n-2}(x)dx=\frac{1}{2(n-2)+1}[P_{n-1}(1)-P_{n-3}(1)]$

则有 $\int_0^1 xP_{n+1}(x)dx=\frac{1}{2n+3}[P_{n+2}(1)-P_n(1)]-\frac{1}{(2n+3)(2n+5)}[P_{n+3}(1)-P_{n+1}(1)]+\frac{1}{(2n+1)(2n+3)}[P_{n+1}(1)-P_{n-1}(1)]=0$

$\int_0^1 xP_{n-1}(x)dx=\frac{1}{2n-1}[P_n(1)-P_{n-2}(1)]-\frac{1}{(2n-1)(2n+1)}[P_{n+1}(1)-P_{n-1}(1)]+\frac{1}{(2n-3)(2n-1)}[P_{n-1}(1)-P_{n-3}(1)]=0$

则 $n\ge 3$ 时 $a_n=0$. 所以展开式为 $f(x)=\frac{1}{6}P_0(x)+\frac{3}{8}P_1(x)+\frac{1}{3}P_2(x)$

6. $m=2$ 的连带勒让德函数为 $P_l^2(x)=(1-x^2)P_l^{(2)}(x)$

设 $x(1-x^2)=\sum_{l=2}^{+\infty} C_l P_l^2(x)$，则 两侧同时乘 $P_l^2(x)$ 在 $[-1,1]$ 上积分得

$I_l=\int_{-1}^{1} x(1-x^2)P_l^2(x)=C_l\cdot\frac{(l+2)!}{(l-2)!}\cdot\frac{2}{2l+1}$，则 $C_l=\frac{2l+1}{2}\cdot\frac{(l-2)!}{(l+2)!}\int_{-1}^{1} x(1-x^2)P_l^2(x)=\frac{2l+1}{2}\frac{(l-2)!}{(l+2)!}\cdot I_l$

计算 I_l：$I_l=\int_{-1}^{1} x(1-x^2)^2\frac{1}{2^l l!}\frac{d^{l+2}}{dx^{l+2}}(x^2-1)^l dx=\frac{1}{2^l l!}\int_{-1}^{1} x(1-x^2)^2\cdot d\left[\frac{d^{l+1}(x^2-1)^l}{dx^{l+1}}\right]=\frac{1}{2^l l!}\left\{x(1-x^2)^2\frac{d^{l+1}(x^2-1)^l}{dx^{l+1}}\Big|_{-1}^{1}-\int_{-1}^{1}(5x^4-6x^2+1)\cdot d\left[\frac{d^l(x^2-1)^l}{dx^l}\right]\right\}$

$=\frac{1}{2^l l!}\left\{-(5x^4-6x^2+1)\cdot\frac{d^l(x^2-1)^l}{dx^l}\Big|_{-1}^{1}+\int_{-1}^{1}(20x^3-12x+1)\,d\left[\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}\right]\right\}=\frac{1}{2^l l!}\left[(20x^3-12x+1)\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}\Big|_{-1}^{1}-\int_{-1}^{1}(60x^2-12)\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}dx\right]$

易知 当 $l\ge 2$ 时 $(20x^3-12x+1)\frac{d^{l-1}(x^2-1)^l}{d^{l-1}}\Big|_{-1}^{1}=0$，则 $I_l=-\frac{1}{2^l l!}\int_{-1}^{1}(60x^2-12)\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}dx$

当 $l=2$ 时 $I_2=-\frac{1}{8}\int_{-1}^{1}(60x^2-12)\cdot 4x(x^2-1)=0$

当 $l\ge 3$ 时 $I_l=-\frac{1}{2^l l!}\int_{-1}^{1}(60x^2-12)\cdot d\left[\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}\right]=-\frac{1}{2^l l!}\left[(60x^2-12)\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}\Big|_{-1}^{1}-\int_{-1}^{1}120x\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}dx\right]$

当 $l\ge 3$ 时有 $(60x^2-12)\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}\Big|_{-1}^{1}=0$，则 $I_l=\frac{120}{2^l l!}\int_{-1}^{1} x\frac{d^{l-2}(x^2-1)^l}{dx^{l-2}}dx$

当 $l=3$ 时有 $I_3=\frac{5}{2}\int_{-1}^{1} x\ 3(x^2-1)^2\cdot 2x\,dx=\frac{16}{7}$

当 $l\ge 4$ 时有 $I_l=\frac{120}{2^l l!}\int_{-1}^{1} x\cdot d\left[\frac{d^{l-3}(x^2-1)^l}{dx^{l-3}}\right]=\frac{120}{2^l l!}\left[x\frac{d^{l-3}(x^2-1)^l}{dx^{l-3}}\Big|_{-1}^{1}-\int_{-1}^{1}\frac{d^{l-3}(x^2-1)^l}{dx^{l-3}}dx\right]=0$

综上所述 $C_l=\begin{cases}\frac{1}{15}, & l=3\\ 0, & l=1,2,4,5,6,\cdots\end{cases}$ 则 $x(1-x^2)=\frac{1}{15}P_3^2(x)$

7 (1). 设 $(\sin\theta-2\cos^2\theta)\cos^2\varphi=\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}A_l^m Y_l^m(\theta,\varphi)$ $\quad Y_l^m(\theta,\varphi)=P_l^{|m|}(\cos\theta)e^{im\varphi}$

则 $A_l^m=\oint(\sin\theta-2\cos^2\theta)\cos^2\varphi\, Y_l^{m*}(\theta,\varphi)\ \sin\theta\, d\theta\, d\varphi$

$$=\frac{1}{(N_l^m)^2}\int_0^{2\pi}\cos^2\varphi\, e^{-im\varphi}d\varphi\int_0^{\pi}\sin\theta(\sin\theta-2\cos^2\theta)P_l^{|m|}(\cos\theta)\,d\theta$$ 其中 $(N_l^m)^2=\frac{4\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}$

其中 $\int_0^{2\pi}\cos^2\varphi\, e^{-im\varphi}d\varphi=\int_0^{2\pi}\frac{1}{2}(1+\cos2\varphi)\cdot(\cos m\varphi-\sin m\varphi)\,d\varphi=\begin{cases}\pi, & m=0\\ \frac{\pi}{2}, & m=\pm2\\ 0, & m\neq0\text{或}\pm2\end{cases}$

当 $m=0$ 时 $l=0$, $A_0^0=\pi\frac{1}{(N_0^0)^2}\int_{-1}^{1}(\sqrt{1-x^2}-2x^2)P_0^0(x)\,dx$

当 $m=2$ 时 $l=2,3,4\cdots$ $\quad A_l^2=\frac{1}{(N_l^2)^2}\frac{\pi}{2}\int_{-1}^{1}(\sqrt{1-x^2}-2x^2)P_l^2(x)\,dx$

当 $m=-2$ 时 $l=2,3,4\cdots$ $\quad A_l^{-2}=\frac{1}{(N_l^{-2})^2}\frac{\pi}{2}\int_{-1}^{1}(\sqrt{1-x^2}-2x^2)P_l^2(x)\,dx$

则 $(\sin\theta-2\cos^2\theta)\cos^2\varphi=A_0^0+\sum_{l=2}^{+\infty}\left[A_l^2Y_l^2(\theta,\varphi)+A_l^{-2}Y_l^{-2}(\theta,\varphi)\right]$

(2). 设 $(1-2\sin\theta)\cos\theta\cos\varphi=\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}A_l^mY_l^m(\theta,\varphi)$ $\quad Y_l^m(\theta,\varphi)=P_l^{|m|}(\cos\theta)e^{im\varphi}$

则 $A_l^m=\int_0^{2\pi}\int_0^{\pi}Y_l^{m*}(\theta,\varphi)\,(1-2\sin\theta)\cos\theta\cos\varphi\cdot\sin\theta\cdot d\theta\, d\varphi$

$$=\frac{1}{(N_l^m)^2}\int_0^{2\pi}\cos\varphi\, e^{-im\varphi}d\varphi\int_0^{\pi}(1-2\sin\theta)\cos\theta\cdot P_l^{|m|}(\cos\theta)\sin\theta\, d\theta$$ 其中 $(N_l^m)^2=\frac{4\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}$

其中 $\int_0^{2\pi}\cos\varphi\, e^{-im\varphi}d\varphi=\begin{cases}\pi & m=\pm1\\ 0 & m\neq\pm1\end{cases}$

当 $m=1$ 时 $l=1,2,3,\cdots$ $\quad A_l^1=\frac{1}{(N_l^1)^2}\pi\int_{-1}^{1}(1-2\sqrt{1-x^2})\,x\,P_l^1(x)\,dx$

当 $m=-1$ 时 $l=1,2,3\cdots$ $\quad A_l^{-1}=\frac{1}{(N_l^{-1})^2}\pi\int_{-1}^{1}(1-2\sqrt{1-x^2})\,x\,P_l^1(x)\,dx$

则 $(1-2\sin\theta)\cos\theta\cos\varphi=\sum_{l=1}^{+\infty}\left[A_l^1Y_l^1(\theta,\varphi)+A_l^{-1}Y_l^{-1}(\theta,\varphi)\right]$

8.解 (1). 由题意得：$\begin{cases} \nabla^2 u=0 \\ u|_{\theta=\frac{\pi}{2}}=0 \qquad u|_{r=r_0}=u_0 \end{cases}$

由泛定方程及边界条件可知, 温度分布与φ无关（球坐标系）. 则 $u=u(r,\theta)$

设 u 的解具有 $u(r,\theta)=R(r)\Theta(\theta)$的形式,

则有 $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}R(r)\right]\Theta(\theta)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)\right]R(r)=0$

分离变量得 $\frac{1}{R(r)}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}R(r)\right]=-\frac{1}{\sin\theta\,\Theta(\theta)}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)\right]=l(l+1)$

则 $R(r)=Ar^l+\frac{B}{r^{l+1}}$ $\quad\Theta(\theta)=P_l(\cos\theta)$, 则 $u(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

因为 $r=0$ 时 $u(0,\theta)=0$, 则 $B_l=0$. $\quad u(r_0,\frac{\pi}{2})=\sum_{l=0}^{+\infty}A_l r_0^l P_l(0)=0$ 则 当 $l=2k$, $k=0,1,2,\cdots$ 时 $A_l=0$

因为 $u(r_0,\theta)=\sum_{l=0}^{\infty}A_l r_0^l P_l(\cos\theta)=u_0$ 则 $A_l r_0^l\cdot\frac{1}{2l+1}=u_0\left[\int_0^1 P_l(x)dx-\int_{-1}^0 P_l(x)dx\right]=-\frac{u_0}{l}P_{l+1}(0)$, 当 $l=2k+1$ 时 $A_l=-\frac{(2l+1)u_0}{l r_0^l}P_{l+1}(0)$

则 $u(r,\theta)=\sum_{k=0}^{+\infty}-\frac{(4k+3)u_0}{(2k+1)r_0^{(2k+1)}}P_{2k+2}(0)\cdot r^{2k+1}\cdot P_{2k+1}(\cos\theta)$

(2). $\begin{cases} \nabla^2 u=0 \\ u|_{r=r_0}=u_0 \qquad \frac{\partial u}{\partial\theta}\Big|_{\theta=0}=0 \end{cases}$

由(1)可知 $u(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$

则 $u|_{r=r_0}=\sum_{l=0}^{\infty}A_l r_0^l P_l(\cos\theta)=u_0$. 则有: 当 $l\neq0$ 时 $A_l=0$, $A_0=u_0$ 则 $u(r,\theta)=u_0$

$\frac{\partial u}{\partial\theta}\Big|_{\theta=0}=0$ 成立. 则 稳定温度分布为 $u(r,\theta)=u_0$

9. 解 $\begin{cases} \nabla^2 u = 0 \\ u|_{r=r_0} = 4\sin^2\theta(\cos\varphi\sin\varphi + \frac{1}{2}) \end{cases}$

设 $u(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$，则有 $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}R(r)\right]Y(\theta,\varphi) + \left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y(\theta,\varphi)\right]R(r) = 0$

则有 $\frac{1}{R(r)}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}R(r)\right] = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = l(l+1)$

则 $R_l(r) = Ar^l + \frac{B}{r^{l+1}}$，$Y_l^m(\theta,\varphi) = P_l^m(\cos\theta)e^{im\varphi}$，$l = 0,1,2,\cdots$ $m = 0,\pm1,\pm2,\cdots\pm l$

则 $u(r,\theta,\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left(A_l^m r^l + \frac{B_l^m}{r^{l+1}}\right)Y_l^m(\theta,\varphi)$.

(1). 球的内部 $\because r\to0$ 时 $u<+\infty$. 则 $B_l=0$，$u(r,\theta,\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}A_l^m r^l Y_l^m$.

代入边界条件 则有: $\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}A_l^m r_0^l Y_l^m(\theta,\varphi) = 4\sin^2\theta(\cos\varphi\sin\varphi + \frac{1}{2}) = 4\sin^2\theta\cos\varphi\sin\varphi + 2\sin^2\theta$

其中 $4\sin^2\theta\cos\varphi\sin\varphi = 4\cdot\frac{1}{3}P_2^2(\cos\theta)\cdot\frac{1}{2}\cdot\frac{e^{i2\varphi}-e^{-i2\varphi}}{2i} = \frac{1}{3i}\left[Y_2^2(\theta,\varphi) - Y_2^{-2}(\theta,\varphi)\right]$

$2\sin^2\theta = 2(1-\cos^2\theta) = 2\cdot\frac{2}{3}\left[P_0^0(\cos\theta) - P_2^0(\cos\theta)\right] = \frac{4}{3}\left[Y_0^0(\theta,\varphi) - Y_2^0(\theta,\varphi)\right]$

则 $A_0^0 = \frac{4}{3}$ $A_2^0 = \frac{1}{r_0^2}\frac{4}{3}$ $A_2^2 = -\frac{i}{3r_0^2}$ $A_2^{-2} = \frac{i}{3r_0^2}$ 其余 $A_l^m = 0$

得 $u(r,\theta,\varphi) = \frac{4}{3} + \frac{4r^2}{3r_0^2}P_2(\cos\theta) + \frac{ir^2}{3r_0^2}Y_2^{-2}(\theta,\varphi) - \frac{ir^2}{3r_0^2}Y_2^2(\theta,\varphi)$

(2). 球的外部 $\because r\to+\infty$ 时 $u<+\infty$ 则 $A_l^m=0$ $u(r,\theta,\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{B_l^m}{r^{l+1}}Y_l^m(\theta,\varphi)$

$u(r,\theta,\varphi)$ 在 $r=r_0$ 处连续得: $A_l^m r_0^l = \frac{B_l^m}{r_0^{l+1}}$ 则 $B_l^m = A_l^m r_0^{2l+1}$

与(1)同理，代入边界条件后只有 $\begin{cases}l=0\\m=0\end{cases}$ $\begin{cases}l=2\\m=-2\end{cases}$ $\begin{cases}l=2\\m=0\end{cases}$ $\begin{cases}l=2\\m=2\end{cases}$ 四项.

$B_0^0 = A_0^0 r_0 = \frac{4}{3}r_0$ $B_2^{-2} = A_2^{-2}\cdot r_0^5 = \frac{i}{3}r_0^3$ $B_2^0 = A_2^0 r_0^5 = \frac{4}{3}r_0^3$ $B_2^2 = A_2^2 r_0^5 = -\frac{i}{3}r_0^3$

则 $u(r,\theta,\varphi) = \frac{4}{3}\frac{r_0}{r} + \frac{4}{3}\frac{r_0^3}{r^3}P_2(\cos\theta) + \frac{i}{3}\frac{r_0^3}{r^3}Y_2^{-2}(\theta,\varphi) - \frac{i}{3}\frac{r_0^3}{r^3}Y_2^2(\theta,\varphi)$

代入边界条件. $\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{B_l^m}{r_0^{l+1}}Y_l^m(\theta,\varphi) = 4\sin^2\theta(\cos\varphi\sin\varphi + \frac{1}{2}) = 4\sin^2\theta\cos\varphi\sin\varphi + 2\sin^2\theta$

与(1)同理可知. B