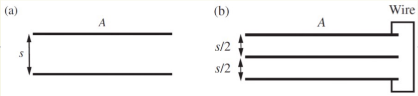
第二周 3月13日 周五作业.

1. *Inserting a plate* **

If the capacitance in Fig. 3.36(a) is C, what is the capacitance in Fig. 3.36(b), where a third plate is inserted and the outer plates are connected by a wire?



解. 由题目可知 (a) $C=\frac{\varepsilon_0 A}{s}$

(b): $C_1=\frac{\varepsilon_0 A}{\frac{s}{2}}=\frac{2\varepsilon_0 A}{s}$

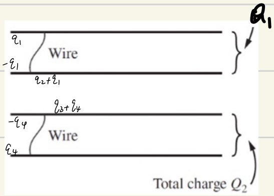
(b)中电容等效于两个C_1并联.

$C_b=C_1+C_1=\frac{4\varepsilon_0 A}{s}$

得 $C_b=\frac{4\varepsilon_0 A}{s}$

2. *Two pairs of plates* **

Four conducting plates lie parallel to each other, as shown in Fig. 3.38. The spacings between them are arbitrary (but small compared with the lateral dimensions). The top two plates are connected by a wire so that they are at the same potential, and likewise for the bottom two. A total charge Q_1 resides on the top two plates, and a total charge Q_2 on the bottom two. What is the charge on each of the four plates?



解 设自上而下各个极板带电量为 q_1 q_2 q_3 q_4.

则 $Q_1 = q_1 + q_2$ $Q_2 = q_3 + q_4$

令 σ_i 表示第 i 个极板的面电荷密度

则有 $\sigma_1 = \sigma_4$ $\sigma_2 = -\sigma_3$

即 $q_1 = q_4$ $q_2 = -q_3$

得 $q_1 = \frac{Q_1+Q_2}{2}$ $q_2 = \frac{Q_1-Q_2}{2}$ $q_3 = \frac{Q_2-Q_1}{2}$ $q_4 = \frac{Q_2+Q_1}{2}$

3. *Capacitance of a spheroid* **

Here is the exact formula for the capacitance C of a conductor in the form of a prolate spheroid of length $2a$ and diameter $2b$:

$$C = \frac{8\pi\epsilon_0 a\epsilon}{\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)}, \qquad \text{where} \qquad \epsilon = \sqrt{1-\frac{b^2}{a^2}}. \qquad (3.42)$$

First verify that the formula reduces to the correct expression for the capacitance of a sphere if $b \to a$. Now imagine that the spheroid is a charged water drop. If this drop is deformed at constant volume and constant charge Q from a sphere to a prolate spheroid, will the energy stored in the electric field increase or decrease? (The volume of the spheroid is $(4/3)\pi ab^2$.)

解、当 $b\to a$ 时 $\epsilon = \sqrt{1-\frac{b^2}{a^2}} = 0.$

则由洛必达法则:

$$\lim_{b\to a} \frac{8\pi\varepsilon_0 a\epsilon}{\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)} = \frac{8\pi\varepsilon_0 a}{\frac{1-\epsilon}{1+\epsilon}\cdot\frac{2}{(1-\epsilon)^2}} = 4\pi\varepsilon_0 a$$

即该公式还原了球体电容器 电容表达式.

设变化前半径为 r 则 $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi ab^2$ 即 $r = \sqrt[3]{ab^2}$

$E_1 = \frac{Q^2}{2C_1}$ 其中 $C_1 = 4\pi\varepsilon_0 r$

得 $E_1 = \frac{Q^2}{8\pi\varepsilon_0}\cdot\frac{1}{\sqrt[3]{ab^2}}$ (球体电容能量)

$E_2 = \frac{Q^2}{2C_2} = \frac{Q^2}{16\pi\varepsilon_0 a\epsilon}\cdot\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$ (长球体电容能量).

$\frac{E_2}{E_1} = \frac{\sqrt[3]{ab^2}}{2a\epsilon}\cdot\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$ $\quad \epsilon = \sqrt{1-\frac{b^2}{a^2}} \in [0,1)$

$b^2 = a^2(1-\epsilon^2)$ 则 $\sqrt[3]{ab^2} = a\cdot\sqrt[3]{1-\epsilon^2}$

$\frac{E_2}{E_1} = \frac{\sqrt[3]{1-\epsilon^2}}{2\epsilon}\cdot\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) < 1$ 恒成立.

则 $E_2 < E_1$ 即从球体变为长球体能量减小.

4. *Maximum energy storage between cylinders* **

We want to design a cylindrical vacuum capacitor, with a given radius a for the outer cylindrical shell, that will be able to store the greatest amount of electrical energy per unit length, subject to the constraint that the electric field strength at the surface of the inner cylinder may not exceed E_0. What radius b should be chosen for the inner cylindrical conductor, and how much energy can be stored per unit length?

解. 设圆柱形电容器长为l. 带电量为Q

则由高斯定理 $E\cdot l\cdot 2\pi r = \frac{Q}{\varepsilon_0}$

则 $E=\frac{Q}{2\pi\varepsilon_0 l}\frac{1}{r}$ $b<r<a$

$\Delta\varphi = \int_b^a E\cdot dr = \int_b^a \frac{Q}{2\pi\varepsilon_0 l}\frac{1}{r}dr = \frac{Q}{2\pi\varepsilon_0 l}\cdot\ln\frac{a}{b}$

$C=\frac{Q}{\Delta\varphi}=\frac{2\pi\varepsilon_0 l}{\ln\frac{a}{b}}$

单位长度储能为 $\frac{U}{l} = \frac{\frac{Q^2}{2C}}{l} = \frac{Q^2\ln\frac{a}{b}}{4\pi\varepsilon_0 l^2}$

即b越小单位长度储能越多.

由于内圆柱表面电场强度不超过E_0 即 $E_0 \geq \frac{Q}{2\pi\varepsilon_0 l}\frac{1}{b}$

则 $b\geq \frac{Q}{2\pi\varepsilon_0 l E_0} = \frac{\lambda}{2\pi\varepsilon_0 E_0}$ $\lambda=\frac{Q}{l}$表示单位长度电容器带电量.

当 $b=\frac{\lambda}{2\pi\varepsilon_0 E_0}$ 时单位长度储能最大. $\left(\frac{U}{l}\right)_{max} = \frac{\lambda^2\cdot\ln\frac{2\pi\varepsilon_0 a E_0}{\lambda}}{4\pi\varepsilon_0}$

$\lambda=\frac{Q}{l}$表示单位长度电容器带电量.

5. *Force and energy for two plates* **

Calculate the electrical force that acts on one plate of a parallel-plate capacitor. The potential difference between the plates is 10 volts, and the plates are squares 20 cm on a side with a separation of 3 cm. If the plates are insulated so the charge cannot change, how much external work could be done by letting the plates come together? Does this equal the energy that was initially stored in the electric field?

解: $C=\frac{\varepsilon_0 A}{s}$ $\quad Q=C\cdot U$

$F=\frac{Q^2}{2\varepsilon_0 A}=\frac{\varepsilon_0 A U^2}{2s^2}$ $\quad A=0.04\,m^2$ $\quad U=10V$ $\quad s=0.03m$

得 $F=1.97\times10^{-8}N$ $\quad U_0=\frac{1}{2}CU^2=5.90\times10^{-10}J$

在两极板靠拢过程中F不变.

$W=F\cdot s=5.91\times10^{-10}J$

即对外做功等于最初系统内部电势能。