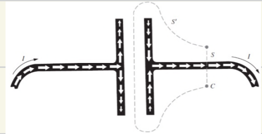
第六周周三作业 4月8日

1. 9.13 *Displacement-current flux* *

The flux of the real current through the surface S in Fig. 9.4 is simply I. Verify explicitly that the flux of the displacement current, $\mathbf{J}_{\rm d} \equiv \epsilon_0(\partial \mathbf{E}/\partial t)$, through the surface S' also equals I. What about the sign of the flux? As usual, work in the approximation where the spacing between the capacitor plates is small.



解：设电流为I时，两极板电压为u，间距为l 截面积A 电容为$C=\frac{\varepsilon_0 A}{l}$ 两极板间距很小，则外部电场强度忽略不计

则 $E=\frac{u}{d}$ 在极短时间 dt内 $dE=\frac{du}{l}=-\frac{I\,dt}{\varepsilon_0 A}$

则 $J_d \equiv \varepsilon_0 \frac{\partial E}{\partial t}=-\frac{I}{A}$ 则通过S'的位移电流为 $I'=J_d\cdot A=-I$

即通过S'位移电流大小也为I，负号表示位移电流从S'流入SS'闭合曲面.

2.

9.14 *Sphere with a hole* **

A current I flows along a wire toward a point charge, causing the charge to increase with time. Consider a spherical surface S centered at the charge, with a tiny hole where the wire is, as shown in Fig. 9.14. The circumference C of this hole is the boundary of the surface S. Verify that the integral form of Maxwell's equation,

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \int_S \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \right) \cdot d\mathbf{a}, \quad (9.61)$$

is satisfied.

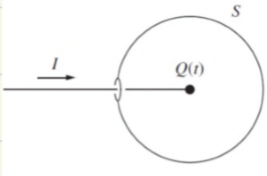


Figure 9.14.

解: $Q(t) = Q_0 + It$ Q_0 为初始带电量.

$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q(t)}{R^2} \hat{R}$ 则 $\frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi\varepsilon_0} \frac{I}{R^2} \hat{R}$ $\hat{R}$ 表示沿球半径方向的单位向量.

则 $\int_S (\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J})\, d\vec{a} = \mu_0\varepsilon_0 \left|\frac{\partial \vec{E}}{\partial t}\right| \cdot 4\pi R^2 = \mu_0 I$

$\int_C \vec{B} \cdot d\vec{s} = \mu_0 I$ 由安培环路定理可知.

则 $\int_C \vec{B} \cdot d\vec{s} = \int_S (\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J})\, d\vec{a}$

3.

9.16 *Changing flux from a moving charge* **

In terms of the electric field $\mathbf{E}$ of a point charge moving with constant velocity $\mathbf{v}$, the Lorentz transformation gives the magnetic field as $\mathbf{B} = (\mathbf{v}/c^2) \times \mathbf{E}$. Verify that Maxwell's equation in integral form, $\int \mathbf{B} \cdot d\mathbf{s} = (1/c^2)(d\Phi_E/dt)$, holds for the circle shown in Fig. 9.16. (We can therefore think of the magnetic field as being induced by the changing electric field of the moving charge.) *Hint:* Indicate geometrically the new electric flux that passes through the circle after the charge has moved a small distance to the right.

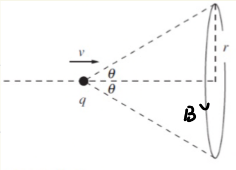


Figure 9.16.

解: $\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}$ 不妨设$q>0$

令 $\theta = \frac{\pi}{2}$ 即此时刻点电荷恰好处于半径为r的圆环圆心。

则 $\int \vec{B} \cdot d\vec{s} = \int \left(\frac{\vec{v}}{c^2} \times \vec{E}\right) \cdot d\vec{s} = \int \frac{v}{c^2} E_r \, ds = \frac{v}{c^2} E r l = \frac{1}{c^2} \frac{E r l \cdot dx}{dt} = \frac{1}{c^2} \frac{\partial \Phi_E}{\partial t}$

即 $\int \vec{B} \cdot d\vec{s} = \frac{1}{c^2} \frac{\partial \Phi_E}{\partial t}$ 得证.

4. 9.18 *Associated B field* *

If the electric field in free space is $\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \sin[(2\pi/\lambda)(z + ct)]$ with $E_0 = 20$ volts/m, then the magnetic field, not including any static magnetic field, must be what?

解: 在真空中 $\mathrm{curl}\,\vec{B} = \varepsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}$

得 $\nabla\times\vec{B} = \frac{E_0}{c}\frac{2\pi}{\lambda}(\hat{x}+\hat{y})\cos[\frac{2\pi}{\lambda}(z+ct)]$

$(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y)\hat{x} + (\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z)\hat{y} + (\frac{\partial}{\partial x}B_z - \frac{\partial}{\partial y}B_x)\hat{z} = \frac{E_0}{c}\frac{2\pi}{\lambda}(\hat{x}+\hat{y})\cos[\frac{2\pi}{\lambda}(z+ct)]$

即 $\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y = \frac{E_0}{c}\frac{2\pi}{\lambda}\cos[\frac{2\pi}{\lambda}(z+ct)]$

$\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z = \frac{E_0}{c}\frac{2\pi}{\lambda}\cos[\frac{2\pi}{\lambda}(z+ct)]$

$\frac{\partial}{\partial x}B_z - \frac{\partial}{\partial y}B_x = 0$

得 $\vec{B} = \frac{E_0}{c}\sin[\frac{2\pi}{\lambda}(z+ct)]\hat{x} - \frac{E_0}{c}\sin[\frac{2\pi}{\lambda}(z+ct)]\hat{y}$

5\.

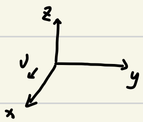
9.26 *An electromagnetic wave* **

Here is a particular electromagnetic field in free space:

$$E_x = 0, \quad E_y = E_0 \sin(kx + \omega t), \quad E_z = 0; \qquad (9.64)$$
$$B_x = 0, \quad B_y = 0, \quad B_z = -(E_0/c)\sin(kx + \omega t).$$

(a) Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

(b) Suppose $\omega = 10^{10}\ \mathrm{s}^{-1}$ and $E_0 = 1$ kV/m. What is the wavelength? What is the energy density in joules per cubic meter, averaged over a large region? From this calculate the power density, the energy flow in joules per square meter per second.



解: (a). 在真空中 $\mathrm{curl}\,\vec{B} = \nabla\times\vec{B} = \frac{\partial}{\partial y}B_z \cdot \hat{x} - \frac{\partial}{\partial x}B_z\,\hat{y} = \frac{E_0}{c}k\cos(kx+\omega t)\,\hat{y}$

$$\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\varepsilon_0\,\omega E_0\cos(kx+\omega t)\,\hat{y}$$

$$\mathrm{curl}\,\vec{E} = \nabla\times\vec{E} = -\frac{\partial}{\partial z}E_y\cdot\hat{x} + \frac{\partial}{\partial x}E_y\,\hat{z} = kE_0\cos(kx+\omega t)\,\hat{z}$$

$$-\frac{\partial\vec{B}}{\partial t} = \frac{\omega E_0}{c}\cos(kx+\omega t)\,\hat{z}$$

$$\mathrm{div}\,\vec{E} = \nabla\cdot\vec{E} = 0 \qquad \mathrm{div}\,\vec{B} = 0$$

由 Maxwell 方程组 $\mathrm{curl}\,\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ $\mathrm{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ 得 $\omega = ck$

即当 $\omega = ck$ 时, 题中电磁场满 Maxwell 方程.

(b). $T = \frac{2\pi}{\omega}$ $\lambda = cT = c\frac{2\pi}{\omega}$ 代入数据: $\lambda = 0.188\,m$ 即波长为 0.188 m.

$$U = \frac{1}{2x_0}\left[\int_{-x_0}^{x_0}\frac{1}{2}\varepsilon_0 E^2(1\cdot dx) + \int_{-x_0}^{x_0}\frac{1}{2}\frac{1}{\mu_0}B^2(1\,dx)\right] = \frac{1}{2}\varepsilon_0 E_0^2\frac{1}{2x_0}\int_{-x_0}^{x_0}2\sin^2(kx+\omega t)\,dx = \frac{1}{2}\varepsilon_0E_0^2\frac{1}{2x_0}\int_{-x_0}^{x_0}[1-\cos(2kx+2\omega t)]\,dx$$

$$= \frac{1}{2}\varepsilon_0E_0^2\frac{1}{2x_0}\left\{2x_0 - \frac{1}{2k}[\sin(2kx_0+2\omega t) - \sin(-2kx_0+2\omega t)]\right\}$$

$$= \frac{1}{2}\varepsilon_0E_0^2\left[1 - \frac{\cos(2\omega t)\sin(2kx_0)}{2kx_0}\right]$$

当 $x_0 \to +\infty$ $u = \frac{1}{2}\varepsilon_0E_0^2 = 4.43\times10^{-6}\ J/m^3$ $P = Uc = 1.33\times10^{3}\ J/(m^2\cdot s)$

则大区域内平均每立方米能量密度为 $4.43\times10^{-6}\ J/m^3$ 功率密度 $1.33\times10^{3}\ J/(m^2\cdot s)$