第三周 周三作业 3.18

1. *Spherical resistor* **

(a) The region between two concentric spherical shells is filled with a material with resistivity ρ. The inner radius is r_1, and the outer radius r_2 is many times larger (essentially infinite). Show that the resistance between the shells is essentially equal to $\rho/4\pi r_1$.

(b) Without doing any calculations, dimensional analysis suggests that the above resistance should be proportional to ρ/r_1, because ρ has units of ohm-meters and r_1 has units of meters. But is this reasoning rigorous?

解 (a). $R = \int_{r_1}^{r_2} \rho \frac{dr}{4\pi r^2} = \frac{\rho}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{\rho}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$

当 $r_2 >> r_1$ 时 $R = \frac{\rho}{4\pi r_1}$

(b) 不进行计算情况下，根据量纲的关系，认为电阻正比于 $\frac{\rho}{r_1}$ 是合理的。

但是这种思考方式不严谨，因为题目中仍有另一个量纲与 r_1 相同的物理量 r_2

考虑到 $r_2 >> r_1$ 即 $r_2 \to +\infty$ 或 $r_1 \to 0$.

① 当 $r_2 \to +\infty$ 时：任何与 r_2 有关的表达式均为 0 或 ∞. 所以我们有电阻的可能取值为 0；∞；或正比于 $\frac{\rho}{r_1}$ 的表达式。但电阻显然不等于 0. 则只可能为后两种情况。

②. 当 $r_1 \to 0$ 时. 与①类比. 电阻可能为 ∞ 或正比于 $\frac{\rho}{r_2}$。

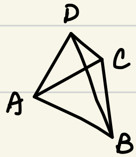
综合①②. 电阻正比于 $\frac{\rho}{r_1}$ 符合题意。

2. *Tetrahedron resistance* **

A tetrahedron has equal resistors R along each of its six edges. Find the equivalent resistance between any two vertices. Do this by:

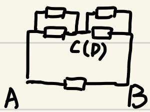
(a) using the symmetry of the tetrahedron to reduce it to an equivalent resistor;

(b) laying the tetrahedron flat on a table, hooking up a battery with an emf $\mathcal{E}$ to two vertices, and writing down the four loop equations. It's easy enough to solve this system of equations by hand, but it's even easier if you use a computer.



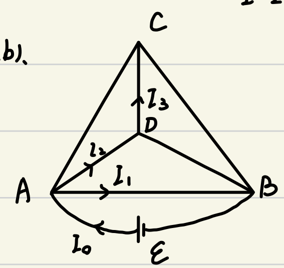
解(a) 考虑AB两点间等效电阻. 由四面体的对称性易知导线CD上无电流.

即原电路图可化为



则 $\frac{1}{R_0}=\frac{1}{R}+\frac{1}{\frac{R}{2}+\frac{R}{2}}$ 得 $R_0=\frac{R}{2}$

(b)、



$\varepsilon - I_1 R = 0$ ①

$-I_2R - I_3R + I_2R = 0$ ②

$-I_2R - I_2R + I_1R = 0$ ③

$I_0 = I_1 + 2I_2$ ④

联立①②③④得

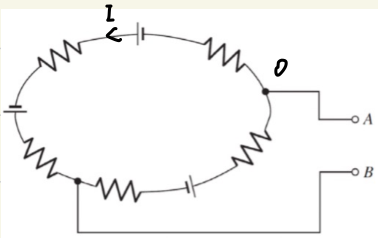
$I_3 = 0$ $I_1 = 2I_2$ $I_1 = \frac{1}{2}I_0$

$\varepsilon = \frac{1}{2}I_0R$

由 $\varepsilon = I_0 \cdot R_{AB}$ 可知 $R_{AB} = \frac{R}{2}$ 即任意两点间等效电阻为 $\frac{R}{2}$

3. *Battery/resistor loop* **

In the circuit shown in Fig. 4.57, all five resistors have the same value, 100 ohms, and each cell has an electromotive force of 1.5 V. Find the open-circuit voltage and the short-circuit current for the terminals A and B. Then find $\mathcal{E}_{eq}$ and R_{eq} for the Thévenin equivalent circuit.

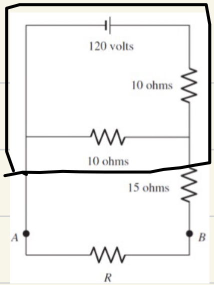


解：AB断路时，环路电流为 $I = \frac{3\varepsilon}{5R} = 9\times10^{-3}A$.

$\varepsilon_{eq} = 2\varepsilon - 3IR$ 代入数据 ε_{eq} 为 0.3V

R_{eq} 等效于 2R 和 3R 的并联. $\frac{1}{R_{eq}} = \frac{1}{2R} + \frac{1}{3R}$

得 $R_{eq} = \frac{6}{5}R$



4. *Maximum power via Thévenin* **

A resistor R is to be connected across the terminals A and B of the circuit shown in Fig. 4.58. For what value of R will the power dissipated in the resistor be greatest? To answer this, construct the Thévenin equivalent circuit and then invoke the result from Exercise 4.39. How much power will be dissipated in R?

解. 将框中区域等效于一个新的电源. 设其电动势为 ε 内阻为 r.

$\varepsilon = \frac{R_1}{2R}\varepsilon_0$ 得 $\varepsilon = 60V$ $\quad \frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_1}$ 得 $r = 5\Omega$

由闭合电路欧姆定律 $\varepsilon = I(r+R_2+R)$

$$P = I^2R = \frac{\varepsilon^2}{(r+R_2+R)^2}R$$

$$\frac{dP}{dR} = \varepsilon^2\left[\frac{1}{(r+R_2+R)^2} - 2R\frac{1}{(r+R_2+R)^3}\right] = \frac{\varepsilon^2(r+R_2-R)}{(r+R_2+R)^3}$$

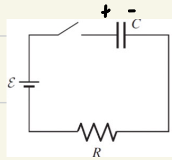
即当 $R = r+R_2$ 时 P 取到极大值.

即 $R = 20\ \Omega$ 时 R 上消耗功率最大.

此时 $P = 45W$

Charging a capacitor **

5\. A battery is connected to an RC circuit, as shown in Fig. 4.42. The switch is initially open, and the charge on the capacitor is initially zero. If the switch is closed at $t = 0$, find the charge on the capacitor, and also the current, as functions of time.



解. $C = \frac{Q}{V}$ $\quad I = \frac{dQ}{dt}$ $\quad I = \frac{\varepsilon - V}{R}$

得 $\frac{dQ}{dt} = \frac{\varepsilon}{R} - \frac{Q}{RC}$

$dt = \frac{RC}{\varepsilon C - Q} dQ$ 两侧不定积分得 $-t = RC \ln(\varepsilon C - Q) + C_1$ C_1为常数.

当 $t=0$ 时 $Q=0$ 得 $C_1 = -RC \ln(\varepsilon C)$

$-\frac{t}{RC} = \ln\left(\frac{\varepsilon C - Q}{\varepsilon C}\right)$ $\quad 1 - \frac{Q}{\varepsilon C} = e^{-\frac{t}{RC}}$

得 $Q = \varepsilon C \left(1 - e^{-\frac{t}{RC}}\right)$

$I = \frac{dQ}{dc} = \frac{\varepsilon}{R} e^{-\frac{t}{RC}}$