第二周 3月11日作业

1. *A thick shell* **

(a) A spherical shell with charge Q uniformly distributed throughout its volume has inner radius R_1 and outer radius R_2. Calculate (and make a rough plot of) the electric field as a function of r, for $0 \le r \le \infty$.

(b) What is the potential at the center of the shell? You can let $R_2 = 2R_1$ in this part of the problem, to keep things from getting too messy. Give your answer in terms of $R \equiv R_1$.

解: (a). 其体电荷密度 $\rho = \dfrac{Q}{\frac{4}{3}\pi(R_2^3-R_1^3)} = \dfrac{3Q}{4\pi(R_2^3-R_1^3)}$

当 $0 \le r \le R_1$ 时 $E=0$

当 $R_1 < r \le R_2$ 时 $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\rho\cdot\int_{R_1}^{r}4\pi x^2 dx}{r^2} = \dfrac{\rho}{3\varepsilon_0}\dfrac{r^3-R_1^3}{r^2} = \dfrac{Q}{4\pi\varepsilon_0(R_2^3-R_1^3)}\cdot\left(r-\dfrac{R_1^3}{r^2}\right)$

当 $r > R_2$ 时 $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}$

$$E = \begin{cases} 0, & 0 \le r \le R_1; \\ \dfrac{Q}{4\pi\varepsilon_0(R_2^3-R_1^3)}\left(r-\dfrac{R_1^3}{r^2}\right), & R_1 < r \le R_2; \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}, & r > R_2. \end{cases}$$

(b). 由于在 $0 \le r \le R_1$ 区域内无电场存在，即此区域内电势处处相等

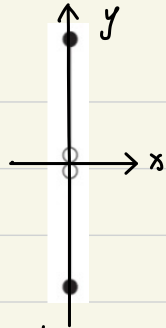
$$\Delta\varphi_1 = -\int_{+\infty}^{R_2}\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\Big|_{+\infty}^{R_2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R_2}$$

$$\Delta\varphi_2 = -\int_{R_2}^{R_1}\frac{Q}{4\pi\varepsilon_0(R_2^3-R_1^3)}\left(r-\frac{R_1^3}{r^2}\right)dr = -\frac{Q}{4\pi\varepsilon_0(R_2^3-R_1^3)}\left(\frac{1}{2}r^2+\frac{R_1^3}{r}\right)\Big|_{R_2}^{R_1}$$

$$= \frac{Q}{4\pi\varepsilon_0(R_2^3-R_1^3)}\left(\frac{1}{2}R_2^2+\frac{R_1^3}{R_2}-\frac{3}{2}R_1^2\right)$$

$\varphi_0 = \Delta\varphi_1 + \Delta\varphi_2$ 代入 $R_2 = 2R_1 = 2R$ 得

$$\varphi_0 = \frac{9Q}{56\pi\varepsilon_0}\frac{1}{R}$$



Linear quadrupole **

2. Consider a "linear quadrupole" consisting of two adjacent dipoles oriented oppositely and placed end to end; see the left quadrupole in Fig. 2.16. There is effectively a point charge $-2q$ at the center. By adding up the electric fields from the charges, find the electric field at a distant point (a) along the axis and (b) along the perpendicular bisector.

解. (a) 沿y轴方向 以向外为正方向. 设相邻两异性电荷间距为l

则 $E_y = \frac{1}{4\pi\varepsilon_0}\frac{q}{(y-l)^2} + \frac{1}{4\pi\varepsilon_0}\frac{q}{(y+l)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}2$

$= \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[\frac{1}{(1-\frac{l}{y})^2}+\frac{1}{(1+\frac{l}{y})^2} - 2\right]$

当 $y \gg l$ 时 $\frac{l}{y} \ll 1$ 则 $(1-\frac{l}{y})^{-2} = 1+\frac{2l}{y}+3\cdot\frac{l^2}{y^2}$

$(1+\frac{l}{y})^{-2} = 1-\frac{2l}{y}+3\frac{l^2}{y^2}$

则 $E_y = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\cdot 6\frac{l^2}{y^2} = \frac{3ql^2}{2\pi\varepsilon_0}\frac{1}{y^4}$ 当$y>0$时沿y轴正方向,当$y<0$时沿y轴负方向

(b) 沿x轴方向 以向外为正方向,相邻两异性电荷间距为l.

则 $E_x = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2+l^2}\cdot\frac{x}{\sqrt{x^2+l^2}}\,2 - \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}\cdot 2$

$= \frac{q}{2\pi\varepsilon_0}\left[\frac{x}{(x^2+l^2)^{\frac{3}{2}}} - \frac{1}{x^2}\right]$

当 $x \gg l$ 时 $\frac{l}{x} \ll 1$ 则 $\frac{x}{(x^2+l^2)^{\frac{3}{2}}} = \frac{1}{x^2(1+\frac{l^2}{x^2})^{\frac{3}{2}}}$

$E_x = \frac{q}{2\pi\varepsilon_0}\frac{1}{x^2}\cdot\left[\frac{1}{(1+\frac{l^2}{x^2})^{\frac{3}{2}}}-1\right] = -\frac{3ql^2}{4\pi\varepsilon_0}\frac{1}{x^4}$

即 $E_x = \frac{3ql^2}{4\pi\varepsilon_0}\frac{1}{x^4}$ 当$x>0$时沿x轴负方向

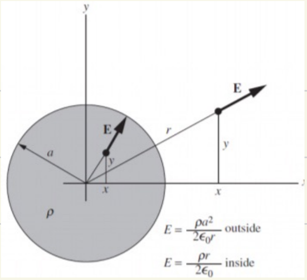
当$x<0$时沿x轴正方向。

3. *E and ϕ for a cylinder* **

For the cylinder of uniform charge density in Fig. 2.26:

(a) show that the expression there given for the field inside the cylinder follows from Gauss's law;

(b) find the potential ϕ as a function of r, both inside and outside the cylinder, taking $\phi = 0$ at $r = 0$.



解. (a). 在圆柱体内部 $E=\frac{\rho r}{2\varepsilon_0}$ 沿半径方向，沿圆柱轴向无电场强度

$$E\cdot l\cdot 2\pi r = \frac{\rho r}{2\varepsilon_0}\cdot l\cdot 2\pi r = \frac{1}{\varepsilon_0}(\rho\cdot\pi r^2\cdot l) = \frac{Q}{\varepsilon_0}$$

Q表示 l长度上所带电荷量

即 $\Phi=\frac{Q}{\varepsilon_0}$ 圆柱内部电场表达式遵循高斯定律

(b). 在$r=0$处令 $\varphi=0$

则 $\varphi(r) = -\int_0^r E(r)$

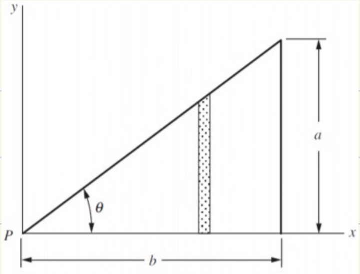
当$0<r\leq a$时 $\varphi(r) = -\int_0^r \frac{\rho t}{2\varepsilon_0}dt = -\frac{\rho}{2\varepsilon_0}\ \frac{1}{2}t^2\Big|_0^r = -\frac{\rho r^2}{4\varepsilon_0}$

当 $r>a$ 时 $\varphi(r) = -\int_0^a \frac{\rho t}{2\varepsilon_0}dt - \int_a^r \frac{\rho a^2}{2\varepsilon_0 t}dt = -\frac{\rho a^2}{4\varepsilon_0} - \frac{\rho a^2}{2\varepsilon_0}\cdot\ln\frac{r}{a}$

即 $$\varphi(r)=\begin{cases} -\frac{\rho r^2}{4\varepsilon_0}, & 0<r\leq a; \\ -\frac{\rho a^2}{4\varepsilon_0}-\frac{\rho a^2}{2\varepsilon_0}\cdot\ln\frac{r}{a}, & r>a. \end{cases}$$

4. *Right triangle ϕ* **

The right triangle shown in Fig. 2.48 with vertex P at the origin has base b, altitude a, and uniform density of surface charge σ. Determine the potential at the vertex P. First find the contribution of the vertical strip of width dx at x. Show that the potential at P can be written as $\phi_P = (\sigma b/4\pi\epsilon_0)\ln[(1+\sin\theta)/\cos\theta]$.



解. 直角三角形斜边方程为 $y=\frac{a}{b}x$

在x处, dx宽度垂直条带贡献设为$d\varphi$

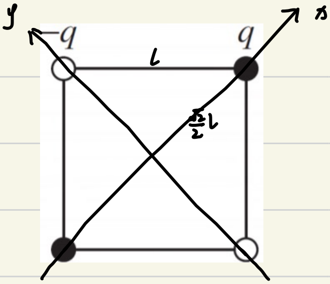
则 $d\varphi=\int_0^y \frac{1}{4\pi\varepsilon_0}\frac{\sigma\cdot dx\cdot du}{\sqrt{x^2+u^2}} = \frac{\sigma dx}{4\pi\varepsilon_0}\cdot\int_0^y\frac{du}{\sqrt{x^2+u^2}} = \frac{\sigma dx}{4\pi\varepsilon_0}\int_0^{\frac{a}{b}}\frac{d(\frac{u}{x})}{\sqrt{1+\frac{u^2}{x^2}}}$

$=\frac{\sigma dx}{4\pi\varepsilon_0}\cdot\ln(\sqrt{1+\tan^2\theta}+\tan\theta) = \frac{\sigma dx}{4\pi\varepsilon_0}\cdot\ln\frac{1+\sin\theta}{\cos\theta}$

则 $\varphi_P=\int_0^b d\varphi = \frac{\sigma dx}{4\pi\varepsilon_0}\ln(\frac{1+\sin\theta}{\cos\theta}) = \frac{\sigma b}{4\pi\varepsilon_0}\ln\frac{1+\sin\theta}{\cos\theta}$

5. *Square quadrupole* **

Consider a square quadrupole consisting of two adjacent dipoles oppositely oriented and placed side by side to form a square, as shown in Fig. 2.16. If the side length is ℓ, find the electric field at a large distance r along the diagonal containing the two positive charges. Be careful to take into account *all* quantities that are second order in ℓ/r.



解. 建立如图参考系：

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{(r-\frac{\sqrt{2}}{2}l)^2}+\frac{1}{4\pi\varepsilon_0}\frac{q}{(r+\frac{\sqrt{2}}{2}l)^2}-\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2+\frac{1}{2}l^2}\cdot 2\cdot\frac{r}{\sqrt{r^2+\frac{1}{2}l^2}}$$

$$=\frac{q}{4\pi\varepsilon_0}\left[(r-\tfrac{\sqrt{2}}{2}l)^{-2}+(r+\tfrac{\sqrt{2}}{2}l)^{-2}-2\frac{r}{(r^2+\frac{1}{2}l^2)^{\frac{3}{2}}}\right]$$

当 $r>>l$ 时 $\frac{l}{r}<<1$ $\quad (r-\frac{\sqrt{2}}{2}l)^{-2}=\frac{1}{r^2}(1+\sqrt{2}\frac{l}{r}+\frac{3\sqrt{2}}{2}\frac{l^2}{r^2})$

$$(r+\tfrac{\sqrt{2}}{2}l)^{-2}=\frac{1}{r^2}(1-\sqrt{2}\frac{l}{r}+\frac{3\sqrt{2}}{2}\frac{l^2}{r^2})$$

$$r\cdot(r^2+\tfrac{1}{2}l^2)^{-\frac{3}{2}}=\frac{1}{r^2}(1-\frac{3}{4}\frac{l^2}{r^2})$$

则 $E=\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}(3\sqrt{2}+\frac{3}{2})\frac{l^2}{r^2}=\frac{(6\sqrt{2}+3)q}{8\pi\varepsilon_0}\frac{l^2}{r^4}$

当 $r>0$ 时，方向沿 x 轴正方向

当 $r<0$ 时，方向沿 x 轴负方向.

Zero curl $*$

6. Consider the electric field, $\mathbf{E} = (2xy^2 + z^3, 2x^2y, 3xz^2)$. We have ignored a multiplicative factor with units of $\mathrm{V/m^4}$ necessary to make the units correct. Show that curl $\mathbf{E} = 0$, and then find the associated potential function $\phi(x, y, z)$.

解.

$$\nabla \times E = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(2xy^2 + z^3\,\vec{i} + 2x^2y\,\vec{j} + 3xz^2\,\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+z^3 & 2x^2y & 3xz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\,3xz^2 - \frac{\partial}{\partial z}\,2x^2y\right)\vec{i} + \left[\frac{\partial}{\partial z}(2xy^2+z^3) - \frac{\partial}{\partial x}\,3xz^2\right]\vec{j} + \left[\frac{\partial}{\partial x}\,2x^2y - \frac{\partial}{\partial y}(2xy^2+z^3)\right]\vec{k}$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

即 $\vec{E}$ 的旋度为 $\vec{0}$

$$\vec{E} = -\nabla\varphi = -\frac{\partial}{\partial x}\varphi\,\vec{i} - \frac{\partial}{\partial y}\varphi\,\vec{j} - \frac{\partial}{\partial z}\varphi\,\vec{k}$$

则 $-\dfrac{\partial\varphi}{\partial x} = 2xy^2 + z^3$

$-\dfrac{\partial\varphi}{\partial y} = 2x^2y$

$-\dfrac{\partial\varphi}{\partial z} = 3xz^2$

得 $\varphi(x, y, z) = -x^2y^2 - xz^3 + C$

C为常数.

7. *Curls and divergences* ∗

Calculate the curl and the divergence of each of the following vector fields. If the curl turns out to be zero, try to discover a scalar function ϕ of which the vector field is the gradient.

(a) $\mathbf{F} = (x+y, -x+y, -2z)$;

(b) $\mathbf{G} = (2y, 2x+3z, 3y)$;

(c) $\mathbf{H} = (x^2-z^2, 2, 2xz)$.

解. (a) 旋度: $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -x+y & -2z \end{vmatrix} = [\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(-x+y)]\vec{i} + [\frac{\partial}{\partial z}(x+y) - \frac{\partial}{\partial x}(-2z)]\vec{j}$
$[\frac{\partial}{\partial x}(-x+y) - \frac{\partial}{\partial y}(x+y)]\vec{k}$

$$= 0\vec{i} + 0\vec{j} - 2\vec{k} = (0, 0, -2)$$

散度 $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(-x+y) + \frac{\partial}{\partial z}(-2z) = 1+1-2 = 0$

(b) 旋度 $\nabla\times\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 2x+3z & 3y \end{vmatrix} = [\frac{\partial}{\partial y}3y - \frac{\partial}{\partial z}(2x+3z)]\vec{i} + (\frac{\partial}{\partial z}2y - \frac{\partial}{\partial x}3y)\vec{j}$
$+[\frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}2y]\vec{k}$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

散度 $\nabla\cdot\vec{G} = \frac{\partial}{\partial x}2y + \frac{\partial}{\partial y}(2x+3z) + \frac{\partial}{\partial z}3y = 0$ C为常数

$\nabla\varphi_G = \vec{G}$ 得 $-\frac{\partial\varphi}{\partial x} = 2y$; $-\frac{\partial\varphi}{\partial y} = 2x+3z$; $-\frac{\partial\varphi}{\partial z} = 3y$ 即 $\varphi = -2xy - 3yz + C$

(c) 旋度 $\nabla\times\vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-z^2 & 2 & 2xz \end{vmatrix} = [\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}2]\vec{i} + [\frac{\partial}{\partial z}(x^2-z^2) - \frac{\partial}{\partial x}2xz]\vec{j}$
$+[\frac{\partial}{\partial x}2 - \frac{\partial}{\partial y}(x^2-z^2)]\vec{k}$

$$= 0\vec{i} - 4z\vec{j} + 0\vec{k} = (0, -4z, 0)$$

散度 $\nabla\cdot\vec{H} = \frac{\partial}{\partial x}(x^2-z^2) + \frac{\partial}{\partial y}2 + \frac{\partial}{\partial z}2xz = 2x+0+2x = 4x$

8. *E and ϕ for a slab* **

A rectangular slab with uniform volume charge density ρ has thickness 2ℓ in the x direction and infinite extent in the y and z directions. Let the x coordinate be measured relative to the center plane of the slab. For values of x both inside and outside the slab:

(a) find the electric field $E(x)$ (you can do this by considering the amount of charge on either side of x, or by using Gauss's law);

(b) find the potential $\phi(x)$, with ϕ taken to be zero at $x = 0$;

(c) verify that $\rho(x) = \epsilon_0 \nabla \cdot \mathbf{E}(x)$ and $\rho(x) = -\epsilon_0 \nabla^2 \phi(x)$.

解. (a) 当 $-l \le x \le l$ 时. 由高斯定理 $2E\,\Delta S = \frac{1}{\varepsilon_0}\rho \cdot 2x \cdot \Delta S$

得 $E(x) = \frac{\rho x}{\varepsilon_0}$

当 $x<-l$ 或 $x>l$ 时. 由高斯定理 $2E \cdot \Delta S = \frac{1}{\varepsilon_0}\rho \cdot 2l \cdot \Delta S$

得 $E(x) = \frac{\rho l}{\varepsilon_0}$

(b). 当 $-l \le x \le l$ 时 $\varphi(x) = -\int_0^x E(t)dt = -\frac{\rho x^2}{2\varepsilon_0}$

当 $x>l$ 时 $\varphi(x) = -\int_0^l E(t)\,dt - \int_l^x E(t)dt = -\frac{\rho l^2}{2\varepsilon_0} - \frac{\rho l}{\varepsilon_0}(x-l) = \frac{\rho l}{2\varepsilon_0}(l-2x)$

同理 当 $x<-l$ 时 $\varphi(x) = \frac{\rho l}{2\varepsilon_0}(l+2x)$

$$\varphi(x) = \begin{cases} -\frac{\rho x^2}{2\varepsilon_0}, & -l \le x \le l; \\ \frac{\rho l}{2\varepsilon_0}(l-2|x|), & |x|>l. \end{cases}$$

(c). 当 $|x|>l$ 时 $\rho(x) = \varepsilon_0 \cdot 0 = 0$ 即 $\rho(x) = \varepsilon_0 \nabla \cdot \vec{E}(x)$

当 $|x| \le l$ 时 $\rho(x) = \varepsilon_0 \cdot \frac{\partial}{\partial x}\frac{\rho x}{\varepsilon_0} = \rho$ 即 $\rho(x) = \varepsilon_0 \nabla \cdot \vec{E}(x)$

得 $\rho(x) = \varepsilon_0 \nabla \cdot \vec{E}(x)$ 成立

当 $|x|>l$ 时 $-\varepsilon_0 \cdot \nabla^2\phi = -\varepsilon_0 \nabla\cdot\left[\frac{\partial}{\partial x}\frac{\rho l}{2\varepsilon_0}(l-2|x|)\right]\vec{i} = 0 = \rho(x)$

当 $|x| \le l$ 时 $-\varepsilon_0 \nabla^2\phi = -\varepsilon_0 \nabla\cdot\left(\frac{\partial}{\partial x} -\frac{\rho x^2}{2\varepsilon_0}\right)\vec{i} = \rho = \rho(x)$

得 $\rho(x) = -\varepsilon_0 \nabla^2\phi(x)$ 成立

Divergence of the curl **

If **A** is any vector field with continuous derivatives, div (curl **A**) = 0 or, using the "del" notation, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. We shall need this theorem later. The problem now is to prove it. Here are two different ways in which that can be done.

9.

(a) (Uninspired straightforward calculation in a particular coordinate system.) Using the formula for ∇ in Cartesian coordinates, work out the string of second partial derivatives that $\nabla \cdot (\nabla \times \mathbf{A})$ implies.

(b) (With the divergence theorem and Stokes' theorem, no coordinates are needed.) Consider the surface S in Fig. 2.52, a balloon almost cut in two which is bounded by the closed curve C. Think about the line integral, over a curve like C, of any vector field. Then invoke Stokes and Gauss with suitable arguments. (The reasoning also works if the curve C is a very tiny loop on the surface.)

Figure 2.52.

解. (a) 令 $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \left(\frac{\partial}{\partial y}a_3 - \frac{\partial}{\partial z}a_2\right)\vec{i} + \left(\frac{\partial}{\partial z}a_1 - \frac{\partial}{\partial x}a_3\right)\vec{j} + \left(\frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1\right)\vec{k}$$

$$\nabla\cdot(\nabla\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}a_3 - \frac{\partial}{\partial z}a_2\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}a_1 - \frac{\partial}{\partial x}a_3\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1\right) = 0$$

(b) 如图 2.52. 在封闭曲面构造一闭合曲线C 在曲线任意一处 $d\vec{l}$ 其邻近有 $-d\vec{l}$

则有 $\oint_C \vec{A}\cdot d\vec{\lambda} = 0$

By Stokes Thm: $\oint_C \vec{A}\cdot d\vec{\lambda} = \oiint_S \nabla\times\vec{A}\cdot d\vec{s}$

By Gauss Thm: $\oiint_S \nabla\times\vec{A}\cdot d\vec{s} = \iiint_V \nabla(\nabla\times\vec{A})\, dV$

即 $\iiint_V \nabla\cdot(\nabla\times\vec{A})\, dV = \oint_C \vec{A}\cdot d\vec{\lambda} = 0$

则 $\nabla\cdot(\nabla\times\vec{A}) = 0$