PROBABILITY AND STATISTICS I
HOMEWORK XII

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1

First, since X_i are random sample from a gamma distribution with unknown parameters α and β, we could get the p.d.f. of X_i is

$$f(x_i;\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}x_i^{\alpha-1}e^{-\beta x_i},\ for x_i>0,$$

and X_i are independent, the likelihood function is

$$L(\alpha,\beta)=\prod_{i=1}^n\frac{\beta^\alpha}{\Gamma(\alpha)}x_i^{\alpha-1}e^{-\beta x_i}=\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n\left(\prod_{i=1}^n x_i\right)^{\alpha-1}exp\left(-\beta\sum_{i=1}^n x_i\right).$$

The logarithmic likelihood function is

$$l(\alpha,\beta)=\log L(\alpha,\beta)=n\alpha\log\beta-n\log(\Gamma(\alpha))+(\alpha-1)\sum_{i=1}^n\log x_i-\beta\sum_{i=1}^n x_i.$$

We notice the derivative of l with respect to β is

$$\frac{dl}{d\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^n x_i.$$

By equating it to zero, we get the M.L.E. of α/β is

$$\frac{\sum_{i=1}^n x_i}{n}.$$

2

First, since $(X_i)_{i=1}^{21}$ are random sample from an exponential distribution with mean $\mu>0$, we could get the p.d.f. of X_i is

$$f(x_i;\mu)=\frac{1}{\mu}exp(-\frac{x_i}{\mu}),\ for\ x_i>0,$$

and X_i are independent, the likelihood function is

$$L(\mu)=\prod_{i=1}^{21}\frac{1}{\mu}exp(-\frac{x_i}{\mu})=\mu^{-21}exp(-\frac{\sum_{i=1}^{21}x_i}{\mu}).$$

The logarithmic likelihood function is

$$l(\mu)=\log L(\mu)=-21\log\mu-\frac{\sum_{i=1}^{21}x_i}{\mu}.$$

According to the question, we have that there is an integer $j\in\{1,2,\dots,21\}$, such that

$$\frac{\sum_{i=1,i\neq j}^{21}x_i}{20}=6,\ and\ x_j>15.$$

We could determine that

$$\frac{dl}{d\mu}=-\frac{21}{\mu}+\frac{\sum_{i=1}^{21}x_i}{\mu^2}=\frac{\sum_{i=1}^{21}x_i-21\mu}{\mu^2}.$$

By equating it to zero, we have that

$$\mu=\frac{\sum_{i=1}^{21}x_i}{21}=\frac{120+x_j}{21}>\frac{45}{7}.$$

And the logarithmic likelihood function is

$$l(\mu) = -21\log\mu - 21,$$

which is a decrease function of μ. Thus, to maximize the logarithmic likelihood function, we need to choose the smallest μ. Since $\mu > \frac{45}{7}$, the M.L.E. of μ is

$$\hat{\mu} = \frac{45}{7}.$$

3

Suppose that $(X_i)_{i=1}^n$ form a random sample from an exponential distribution with unknown parameter $\beta > 0$. Then we could get the p.d.f. of X_i is

$$f(x_i;\beta) = \beta e^{-\beta x_i},\ for\ x_i > 0$$

Using the method of moments to get the estimator of β.

By the theorem, the mean of the exponential distribution is $1/\beta$. Which means that the first moment is

$$\mu_1 = \mathbf{E}(X) = \frac{1}{\beta}.$$

Then, we could get that

$$\beta = \frac{1}{\mu_1}.$$

Next, we set

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^n x_i$$

to obtain estimator

$$\hat{\beta} = \frac{1}{\hat{\mu_1}} = \frac{n}{\sum_{i=1}^n x_i}.$$

Using the method of maximum likelihood estimation to get the M.L.E. of β.

First, since X_i form a random sample from an exponential distribution with unknown parameter β and X_i are independent, the likelihood function is hence

$$L(\beta) = \prod_{i=1}^n \beta e^{-\beta x_i} = \beta^n exp(-\beta\sum_{i=1}^n x_i).$$

The logarithmic likelihood function is

$$l(\beta) = \log L(\beta) = n\log\beta - \beta\sum_{i=1}^n x_i.$$

Differentiation yields that

$$\frac{d}{d\beta}l(\beta) = \frac{n}{\beta} - \sum_{i=1}^n x_i.$$

By equating it to zero, we obtain that the M.L.E. is

$$\hat{\beta} = \frac{n}{\sum_{i=1}^n x_i}.$$

Which is the same as using the method of moments estimator.

4

Suppose that $(X_i)_{i=1}^n$ form a random sample from a Poisson distribution with unknown parameter $\lambda > 0$. Then we could get the p.m.f. of X_i is

$$f(x_i; \beta) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}, \ for\ x_i = 0, 1, 2, \ldots.$$

Using the method of moments to get the estimator of λ.

By the theorem, the mean of the Poisson distribution is λ. Which means that the first moment is

$$\mu_1 = \mathbf{E}(X) = \lambda.$$

Then, we could get that

$$\lambda = \mu_1.$$

Next, we set

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

to obtain the estimator

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Using the method of maximum likelihood estimation to get the M.L.E. of λ.

First, since X_i form a random sample from a Poisson distribution with unknown parameter λ and X_i are independent, the likelihood function is hence

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = (e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i})/(\prod_{i=1}^{n} x_i!).$$

The logarithmic likelihood function is

$$l(\lambda) = \log L(\lambda) = -n\lambda + \log\lambda \sum_{i=1}^{n} x_i - \log(\prod_{i=1}^{n} x_i!).$$

Differentiation yields that

$$\frac{d}{d\lambda} l(\lambda) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda}.$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\lambda}$ is

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Which is the same as the method of moment estimator.

5

By the theorem, the posterior p.d.f. of θ is

$$h(\theta \mid \mathbf{x}) = \frac{f(x_1 \mid \theta) \ldots f(x_3 \mid \theta) h(\theta)}{g_n(\mathbf{x})},$$

where $g_n(\mathbf{x})$ is the marginal joint p.d.f. of $(X_i)_{i=1}^3$. Since X follows the uniform distribution $\mathrm{U}(\theta, \theta+1)$ and the prior distribution of θ is uniform distribution U(10,16), we could say that $h(\theta \mid \mathbf{x})$ is a constant, which means that the posterior p.d.f. of θ is uniform distribution.

Since X follows the uniform distribution $U(\theta, \theta+1)$, there must be that

$$\theta < X < \theta + 1.$$

Then we could solve that

$$X - 1 < \theta < X.$$

Let $X_{(j)}$ be the j-th minimum among $(X_i)_{i=1}^3$. So that there is $X_{(3)} - 1 < \theta < X_{(1)}$. Which means that

$$11.1 < \theta < 11.7.$$

So the support of the conditional distribution of θ given $(X_i)_{i=1}^3$ is (11.1,11.7).

Since the prior distribution of θ is U(10,16), we could get the posterior distribution of θ is uniform distribution U(11.1,11.7), and the p.d.f. of θ is

$$f(\theta \mid \mathbf{x}) = \frac{5}{3},\ for\ 11.1 < \theta < 11.7,$$

zero elsewhere.

6

6.1

According to the question, the conditional p.m.f. of X_i is

$$\mathbf{P}(X_i = x_i \mid \theta) = \theta(1-\theta)^{x_i},\ for\ x_i = 0, 1, 2, \ldots.$$

The prior p.d.f. of θ is

$$h(\theta) = 1,\ for\ 0 < \theta < 1,$$

zero elsewhere.

Then we need to determine the marginal joint p.m.f. of $(X_i)_{i=1}^n$. Since X_i and θ are independent, by the theorem, we have that

$$g_n(\mathbf{x}) = \int_{-\infty}^{+\infty} g_n(\mathbf{x} \cap \theta)\, d\theta = \int_0^1 \prod_{i=1}^n f(x_i \mid \theta) h(\theta)\, d\theta = \int_0^1 \theta^n (1-\theta)^{\sum_{i=1}^n x_i}\, d\theta.$$

By the definition of Beta Function, we could get the intergal is

$$g_n(\mathbf{x}) = \frac{\Gamma(n+1)\Gamma(1+\sum_{i=1}^n x_i)}{\Gamma(n+2+\sum_{i=1}^n x_i)}.$$

Thus, by the theorem, we could get the posterior distribution of θ is

$$h(\theta \mid \mathbf{x}) = \frac{f(x_1 \mid \theta) \ldots f(x_n \mid \theta) h(\theta)}{g_n(\mathbf{x})} = \frac{\Gamma(n+2+\sum_{i=1}^n x_i)}{\Gamma(n+1)\Gamma(1+\sum_{i=1}^n x_i)} \theta^n (1-\theta)^{\sum_{i=1}^n x_i},\ for\ 0 < \theta < 1,$$

zero elsewhere

6.2

If we have the observation $x_1 = 4, x_2 = 3, x_3 = 1, x_4 = 6$, we could get the posterior distribution of θ is

$$h(\theta \mid \mathbf{x}) = \frac{19!}{4!14!} \theta^4 (1-\theta)^{14},\ for\ 0 < \theta < 1,$$

zero elsewhere. To get the Bayes' estimate of θ, we need to minimize $(\theta - h(\theta \mid \mathbf{x}))^2$.

By the theorem, we have the Bayes' estimate of θ is

$$\hat{\theta} = \mathbf{E}(\theta) = \int_0^1 \theta h(\theta \mid \mathbf{x})\, d\theta = \frac{1}{4}.$$

7

Since $(X_i)_{i=1}^n$ are random sample from $N(\mu, \sigma^2)$ and S_n^2 is the sample variance, by the theorem, we have that $\frac{(n-1)S_n^2}{\sigma^2}$ follows $\chi^2(n-1)$ distribution. Let $X = (n-1)S_n^2/\sigma^2$.

Since (n-1)>0, we have that

$$\mathbf{P}(S_n^2/\sigma^2 \leqslant 15) = \mathbf{P}(X \leqslant 15(n-1)) \geqslant 0.95.$$

According to the table, we could get that the smallest n is 2.

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