# PROBABILITY AND STATISTICS I
# HOMEWORK XI

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# 1

## 1.1

First, since each $X_i$ has the p.d.f.

$$p(x_i;\theta) = \sqrt{\theta}x^{\sqrt{\theta}-1},\ 0 < x < 1,\ \theta > 0$$

and $X_i$ are independent, the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \sqrt{\theta}x_i^{\sqrt{\theta}-1} = \theta^{\frac{n}{2}} \left(\prod_{i=1}^{n} x_i\right)^{\sqrt{\theta}-1}$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = \frac{n}{2}\log\theta + (\sqrt{\theta}-1)\log(\prod_{i=1}^{n} x_i).$$

Then determine the differential coefficient of l

$$\frac{d}{d\theta}l(\theta) = \frac{n}{2}\frac{1}{\theta} + \frac{1}{2\sqrt{\theta}}\log(\prod_{i=1}^{n} x_i).$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\theta}$ is

$$\left(\frac{n}{\log(\prod_{i=1}^{n} x_i)}\right)^2$$

and so the maximum likelihood estimator is

$$T(X_1, X_2, \ldots, X_n) = \left(\frac{n}{\log(\prod_{i=1}^{n} x_i)}\right)^2.$$

Thus, the problem is solved.

## 1.2

First, since each $X_i$ has the p.d.f.

$$p(x_i;\theta) = \theta c^{\theta} x_i^{-(\theta+1)},\ x_i > c, c > 0, and\ \theta > 1$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \theta c^{\theta} x_i^{-(\theta+1)} = \theta^n c^{n\theta} \left(\prod_{i=1}^{n} x_i\right)^{-(\theta+1)}.$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = n\log\theta + n\theta\log c - (\theta+1)\log(\prod_{i=1}^{n} x_i)$$

Then determine the differential coefficient of l

$$\frac{d}{d\theta}l(\theta) = \frac{n}{\theta} + n\log c - \log(\prod_{i=1}^{n} x_i).$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\theta}$ is

$$\frac{n}{\log(\prod_{i=1}^{n} x_i) - n\log c}$$

and so the maximum likelihood estimator is

$$T(X_1, X_2, \ldots, X_n) = \frac{n}{\log(\prod_{i=1}^{n} x_i) - n\log c}$$

Thus, the problem is solved.

## 1.3

First, since each $X_i$ has the p.d.f.

$$p(x_i; \theta, \mu) = \frac{1}{\theta} e^{-\frac{x_i - \mu}{\theta}}, \ x > \mu, \theta > 0$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta, \mu) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i - \mu}{\theta}} = \theta^{-n} exp(\frac{n\mu}{\theta} - \frac{\sum_{i=1}^{n} x_i}{\theta}).$$

The logarithmic likelihood function is

$$l(\theta, \mu) = \log L(\theta, \mu) = -n\log\theta + \frac{n\mu}{\theta} - \frac{\sum_{i=1}^{n} x_i}{\theta}.$$

The partial derivatives of $l(\theta, \mu)$ are

$$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} - \frac{n\mu}{\theta^2} + \frac{\sum_{i=1}^{n} x_i}{\theta^2}$$

and

$$\frac{\partial l}{\partial \mu} = \frac{n}{\theta} > 0$$

To maximize the likelihood function, we need to choose $\mu$ as large as possible. Let $X_{(j)}$ be the j-th minimum among $(X_i)_{i=1}^{n}$. Since x>$\mu$, the maximum likelihood estimator of $\mu$ is

$$\hat{\mu} = X_{(1)}.$$

Then let $\frac{\partial l}{\partial \theta} = 0$, we could get the maximum likelihood estimator of $\theta$ is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} - x_{(1)}.$$

## 1.4

First, since each $X_i$ has the p.d.f.

$$p(x_i; \theta) = \frac{1}{2\theta} e^{-|x_i|/\theta}, \ \theta > 0.$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} e^{-|x_i|/\theta} = \frac{1}{(2\theta)^n} exp(-\frac{\sum_{i=1}^{n} |x_i|}{\theta}).$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = -n \log 2\theta - \frac{\sum_{i=1}^{n} |x_i|}{\theta}.$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} |x_i|}{\theta^2}.$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\theta}$ is

$$\frac{\sum_{i=1}^{n} |x_i|}{n}$$

and so the maximum likelihood estimator is

$$T(X_1, X_2, \ldots, X_n) = \frac{\sum_{i=1}^{n} |x_i|}{n}.$$

Thus, the problem is solved.

## 1.5

First, since each $X_i$ has the p.d.f.

$$p(x_i; \theta) = 1,\ \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} 1 = 1.$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = 0.$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = 0$$

for any $\theta \in R$. But there must be that

$$\theta - \frac{1}{2} < x_i < \theta + \frac{1}{2},\ for\ i = 1, 2, \ldots, n.$$

Then we could let $X_{(j)}$ be the j-th minimum among $(X_i)_{i=1}^{n}$. So we get

$$\theta - \frac{1}{2} < x_{(1)},\ \theta + \frac{1}{2} > x_{(n)}.$$

Then we could determine that

$$x_{(n)} - \frac{1}{2} < \theta < x_{(1)} + \frac{1}{2}.$$

Since the likelihood function is constant, to maximize the likelihood function, the M.L.E of $\theta$ could be any statistic $u(X_1, X_2, \ldots, X_n)$ which satisfies that

$$x_{(n)} - \frac{1}{2} < u(X_1, X_2, \ldots, X_n) < x_{(1)} + \frac{1}{2}.$$

### 1.6

First, since each $X_i$ has the p.d.f.

$$p(x_i;\theta_1,\theta_2)=\frac{1}{\theta_2-\theta_1},\ \theta_1<x<\theta_2$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta_1,\theta_2)=\prod_{i=1}^{n}\frac{1}{\theta_2-\theta_1}=\frac{1}{(\theta_2-\theta_1)^n}.$$

The logarithmic likelihood function is

$$l(\theta)=\log L(\theta)=-n\log(\theta_2-\theta_1).$$

The partial derivatives of $l(\theta_1,\theta_2)$ are

$$\frac{dl}{d\theta_1}=\frac{n}{\theta_2-\theta_1}>0$$

and

$$\frac{dl}{d\theta_2}=-\frac{n}{\theta_2-\theta_1}<0.$$

So to maximize the likelihood function L, we need to choose $\theta_1$ as large as possible and choose $\theta_2$ as small as possible.

Let $X_{(j)}$ be the j-th minimum among $(X_i)_{i=1}^n$. Then by the relation, we get the maximum likelihood estimators are

$$\hat{\theta_1}=X_{(1)},\ \hat{\theta_2}=X_{(n)}.$$

## 2

### 2.1

Sicen X follows the uniform distribution on $(\theta,2\theta)$ for $\theta>0$, we could get the mean and variance of X is $\mathbf{E}(X)=\frac{3}{2}\theta$ and $\mathbf{Var}(X)=\mathbf{E}(X^2)-\mathbf{E}(X)^2=\frac{1}{12}\theta^2$. Let $T=\frac{2}{3}\overline{X}_n$ be a statistic. We could determine that

$$\mathbf{E}(T)=\frac{2}{3}\mathbf{E}(\overline{X}_n)=\frac{2}{3n}\sum_{i=1}^{n}\mathbf{E}(X_i)=\theta.$$

Thus, $\hat{\theta}=\frac{2}{3}\overline{X}_n$ is an unbiasedness estimator.

We get that $\lim_{n\to\infty}\mathbf{E}(\hat{\theta})=\theta$. Then we could determine that

$$\lim_{n\to\infty}\mathbf{Var}(\hat{\theta})=\lim_{n\to\infty}\frac{4}{9n^2}\sum_{i=1}^{n}\mathbf{Var}(X_i)=\lim_{n\to\infty}\frac{\theta^2}{27n}=0$$

By the theorem, we have that $\hat{\theta}=\frac{2}{3}\overline{X}_n$ is a consistent estimator.

So in a word, $\hat{\theta}=\frac{2}{3}\overline{X}_n$ is an unbiasedness estimator and consistent estimator.

### 2.2

First, since each $X_i$ has the p.d.f.

$$f(x;\theta)=\frac{1}{\theta},\ 0<\theta<x<2\theta$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \theta^{-n}, \ \theta > 0.$$

To let the likelihood function as large as possible, we need to let $\theta$ as small as possible. Let $X_{(j)}$ be the j-th minimum among $(X_i)_{i=1}^n$. So there are

$$\frac{X_{(n)}}{2} < \theta < X_{(1)}.$$

So the M.L.E. of $\theta$ is $\hat{\theta} = \frac{X_{(n)}}{2}$.
We could determine that

$$\mathbf{E}(\hat{\theta}) = \frac{1}{2}\mathbf{E}(X_{(n)})$$

Then we need to determine the p.d.f. of $X_{(n)}$. The c.d.f. of $X_{(n)}$ is

$$F(x_{(n)}) = \prod_{i=1}^{n} P(X_i \leqslant x_{(n)}) = \left(\frac{x_{(n)}}{\theta} - 1\right)^n, \ \theta < x_{(n)} < 2\theta.$$

Then we could determine that the p.d.f. of $X_{(n)}$ is

$$f(x_{(n)}) = \frac{dF}{dx_{(n)}} = \frac{n}{\theta}\left(\frac{x_{(n)}}{\theta} - 1\right)^{n-1}, \ \theta < x_{(n)} < 2\theta.$$

So we have the expectation of $X_{(n)}$ is

$$\mathbf{E}(X_{(n)}) = \int_{\theta}^{2\theta} x_{(n)} f(x_{(n)})\, dx_{(n)} = 2\theta - \frac{\theta}{n+1}.$$

And we could determine that

$$\mathbf{E}(X_{(n)}^2) = \int_{\theta}^{2\theta} x_{(n)}^2 f(x_{(n)})\, dx_{(n)} = 4\theta^2 - \frac{4\theta^2}{n+1} + \frac{2\theta^2}{(n+1)(n+2)}.$$

So that

$$\mathbf{E}(\hat{\theta}) = \theta - \frac{\theta}{2(n+1)} \neq \theta$$

Thus, is is not unbiased.
Then we have

$$\lim_{n\to\infty} \mathbf{E}(\hat{\theta}) = \frac{1}{2}\lim_{n\to\infty} \mathbf{E}(X_{(n)}) = \frac{1}{2}\lim_{n\to\infty}\left(2\theta - \frac{\theta}{n+1}\right) = \theta,$$

and

$$\lim_{n\to\infty} \mathbf{Var}(\hat{\theta}) = \frac{1}{4}\lim_{n\to\infty} \mathbf{Var}(X_{(n)}) = \frac{1}{4}\lim_{n\to\infty}\left(\mathbf{E}(X_{(n)}^2) - \mathbf{E}(X_{(n)})^2\right) = \frac{1}{4}\lim_{n\to\infty}\frac{n\theta^2}{(n+1)^2(n+2)} = 0$$

Which means that the M.L.E. of $\theta$ is consistent.

# 3

Since $(X_i)_{i=1}^n$ is a random sample from $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $\theta \in (0, \infty)$, we could get the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^3}{2} x_i^2 e^{-\theta x_i} = \frac{\theta^{3n}}{2^n}(\prod_{i=1}^{n} x_i)^2 exp(-\theta(\sum_{i=1}^{n} x_i)).$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = 3n\log\theta - n\log 2 + 2\sum_{i=1}^{n}\log x_i - \theta(\sum_{i=1}^{n} x_i).$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = \frac{3n}{\theta} - \sum_{i=1}^{n} x_i.$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\theta}$ is

$$\frac{3n}{\sum_{i=1}^{n} x_i}$$

and so the maximum likelihood estimator is

$$T(X_1, X_2, \ldots, X_n) = \frac{3n}{\sum_{i=1}^{n} x_i}.$$

# 4

Assume that there exist the M.L.E. of $\mu$ and $\sigma^2$.
Since X $\sim$ N($\mu, \sigma^2$) and we only have one observation, we could get the likelihood function is

$$L(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\left(\frac{x_1 - \mu}{\sigma}\right)^2\right).$$

and $\sigma$ must be positive. The logarithmic likelihood function is

$$l(\mu, \sigma^2) = \log L(\mu, \sigma^2) = -\log(\sqrt{2\pi}\sigma) - \frac{(x_1 - \mu)^2}{2\sigma^2}$$

The partial derivatives of l($\mu, \sigma$) are

$$\frac{\partial l}{\partial \mu} = \frac{x_1 - \mu}{\sigma^2}$$

and

$$\frac{\partial l}{\partial \sigma} = \frac{(x_1 - \mu)^2 - \sigma^2}{\sigma^3}.$$

Set both equal to zero and $\sigma > 0$, we get that $\mu = x_1$ and $\sigma$ has no solution. So the assumption is wrong. Which means that the M.L.E. of $\mu$ and $\sigma^2$ doesn't exist.

# 5

Since $Y_1 < Y_2 < \cdots < Y_n$ are the order statistics of a random sample from the distribution with p.d.f.

$$f(x;\theta) = 1,\ \ for\ \theta - \frac{1}{2} \leqslant x \leqslant \theta + \frac{1}{2}, -\infty < \theta < \infty,$$

there must be that $Y_i \in (\theta - \frac{1}{2}, \theta + \frac{1}{2})$ for all i=1,2,...,n. Then we get that

$$\theta - \frac{1}{2} \leqslant Y_1,\ \ \theta + \frac{1}{2} \geqslant Y_n.$$

Thus, we could determine that

$$Y_n - \frac{1}{2} \leqslant \theta \leqslant Y_1 + \frac{1}{2}.$$

Since $X_i$ are independent, we could get the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} 1 = 1,\ for\ Y_n - \frac{1}{2} \leqslant \theta \leqslant Y_1 + \frac{1}{2},$$

which is a constant. To maximize the likelihood function, the estimate $\hat{\theta}$ could be any real number $\in [Y_n - \frac{1}{2}, Y_1 + \frac{1}{2}]$. Which means that every estimate $u(X_1, X_2, \ldots, X_n)$ such that

$$Y_n - \frac{1}{2} \leqslant u(X_1, X_2, \ldots, X_n) \leqslant Y_1 + \frac{1}{2}$$

is a M.L.E. of $\theta$.
Since $Y_1 + \frac{1}{2} \geqslant Y_n - \frac{1}{2}$, we have $Y_1 + 1 - Y_n \geqslant 0$.
We could calculate that

$$\frac{4Y_1 + 2Y_n + 1}{6} - Y_n + \frac{1}{2} = \frac{2}{3}(Y_1 + 1 - Y_n) \geqslant 0$$

and

$$Y_1 + \frac{1}{2} - \frac{4Y_1 + 2Y_n + 1}{6} = \frac{1}{3}(Y_1 + 1 - Y_n) \geqslant 0.$$

So we get

$$Y_n - \frac{1}{2} \leqslant \frac{4Y_1 + 2Y_n + 1}{6} \leqslant Y_1 + \frac{1}{2},$$

which means that $(4Y_1 + 2Y_n + 1)/6$ is a statistic.
We could calculate that

$$\frac{Y_1 + Y_n}{2} - Y_n + \frac{1}{2} = \frac{1}{2}(Y_1 + 1 - Y_n) \geqslant 0$$

and

$$Y_1 + \frac{1}{2} - \frac{Y_1 + Y_n}{2} = \frac{1}{2}(Y_1 + 1 - Y_n) \geqslant 0.$$

So we get that

$$Y_n - \frac{1}{2} \leqslant \frac{Y_1 + Y_n}{2} \leqslant Y_1 + \frac{1}{2},$$

which means that $(Y_1 + Y_n)/2$ is a statistic.
We could calculate that

$$\frac{2Y_1 + 4Y_n - 1}{6} - Y_n + \frac{1}{2} = \frac{1}{3}(Y_1 + 1 - Y_n) \geqslant 0$$

and

$$Y_1 + \frac{1}{2} - \frac{2Y_1 + 4Y_n - 1}{6} = \frac{2}{3}(Y_1 + 1 - Y_n) \geqslant 0,$$

which means that $(2Y_1 + 4Y_n - 1)/6$ is a statistic.
In a word, $(4Y_1 + 2Y_n + 1)/6$, $(Y_1 + Y_n)/2$ and $(2Y_1 + 4Y_n - 1)/6$ are three statistics. So uniqueness is not, in general, a property of a M.L.E..

# 6

Since each $X_i$ has the p.d.f.

$$f(x_i; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, x > 0$$

and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}} = \theta^{-n} exp\left(-\frac{\sum_{i=1}^{n} x_i}{\theta}\right).$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = -n\log\theta - \frac{\sum_{i=1}^n x_i}{\theta}.$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2}.$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\theta}$ is

$$\frac{\sum_{i=1}^n x_i}{n}$$

and so the maximum likelihood estimator is

$$T(X_1, X_2, \ldots, X_n) = \frac{\sum_{i=1}^n X_i}{n}.$$

Then we could determine that

$$\mathbf{P}(X \leqslant 2) = \int_0^2 \frac{1}{\hat{\theta}} e^{-\frac{x}{\hat{\theta}}} = 1 - exp\left(-\frac{2n}{\sum_{i=1}^n x_i}\right)$$

# 7

Since $(X_i)_{i=1}^{55}$ is a random sample from the Poisson distribution, we could get the p.d.f. of X is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$$

zero elsewhere and $X_i$ are independent, the likelihood function is hence

$$L(\lambda) = \prod_{i=1}^{55} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-55\lambda}\lambda^{\sum_{i=1}^{55} x_i}/(\prod_{i=1}^{55} x_i!).$$

The logarithmic likelihood function is

$$l(\lambda) = \log L(\lambda) = -55\lambda + (\sum_{i=1}^{55} x_i)\log\lambda - \log(\prod_{i=1}^{55} x_i!).$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = -55 + \frac{\sum_{i=1}^{55} x_i}{\lambda}.$$

By equating it to zero, we obtain that the maximum likelihood estimate $\hat{\lambda}$ is

$$\frac{\sum_{i=1}^{55} x_i}{55}$$

Based on the data given by the table, we could calculate that

$$\hat{\lambda} = \frac{\sum_{i=1}^{55} x_i}{55} = \frac{116}{55}.$$

Then we could get that

$$\mathbf{P}(X = 2) = \frac{e^{-\hat{\lambda}}\hat{\lambda}^2}{2!} \approx 0.27$$

## 8

Since $(X_i)_{i=1}^n$ is a random sample from the Poisson distribution with unknown parameter $\theta \in (0, 2]$, we could get the p.d.f. of X is

$$p(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}, \; for \; x = 0, 1, 2, \ldots$$

zero elsewhere and $X_i$ are independent, the likelihood function is hence

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = e^{-n\theta}\theta^{\sum_{i=1}^n x_i}/(\prod_{i=1}^{n} x_i!).$$

The logarithmic likelihood function is

$$l(\theta) = \log L(\theta) = -n\theta + (\sum_{i=1}^{n} x_i)\log\theta - \log(\prod_{i=1}^{n} x_i!).$$

Then determine the differential coefficient of l

$$\frac{dl}{d\theta} = -n + \frac{\sum_{i=1}^n x_i}{\theta}.$$

Then we need to talk about the relationship of the value of $\overline{X}_n$ and 2.

If $2 \geqslant \overline{X}_n$, to maximize the likelihood function, we need to let $\frac{dl}{d\theta} = 0$, then we get that the maximum likelihood estimate of $\theta$ is

$$\hat{\theta} = \overline{X}_n.$$

If $2 < \overline{X}_n$, we get that $\frac{dl}{d\theta} > 0$ all the time. So $l(\theta)$ increases with $\theta$ increasing. To maximize the likelihood function, we need to get the largest value as the estimate of $\theta$. Which means that the maximum likelihood estimate of $\theta$ is

$$\hat{\theta} = 2$$

In a word, the maximum likelihood estimate of $\theta$ is

$$\hat{\theta} = \min\{\overline{X}_n, 2\}.$$

## 9

If $\theta = 1$, we have the p.d.f. of X is

$$f(x;\theta=1) = \frac{1}{\sqrt{2\pi}}exp(-\frac{x^2}{2}), \; for \; -\infty < x < \infty.$$

Since $(X_i)_{i=1}^n$ are independent, we could get the likelihood function is

$$L_1 = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}}exp(-\frac{x_i^2}{2}) = (2\pi)^{-\frac{n}{2}}exp(-\frac{\sum_{i=1}^n x_i^2}{2}).$$

The logarithmic likelihood function is

$$l_1 = \log L_1 = -\frac{n}{2}\log(2\pi) - \frac{\sum_{i=1}^n x_i^2}{2}.$$

If $\theta = 2$, we have the p.d.f. of X is

$$f(x,\theta=2) = \frac{1}{\pi(1+x^2)}, \; for \; -\infty < x < \infty.$$

Since $(X_i)_{i=1}^n$ are independent, we could get the likelihood function is

$$L_2 = \prod_{i=1}^n \frac{1}{\pi(1+x_i^2)} = \pi^{-n} \prod_{i=1}^n (1+x_i^2)^{-1}.$$

The logarithmic likelihood function is

$$l_2 = \log L_2 = -n \log \pi - \sum_{i=1}^n \log(1+x_i^2).$$

To determine the maximum likelihood estimate of $\theta$, we need to let the likelihood function as large as possible. Which means that let the logarithmic likelihood function as large as possible.

If $l_1 > l_2$, which means that

$$\sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n \log(1+x_i^2) < n(\log \pi - \log 2),$$

we have that the M.L.E of $\theta$ is $\hat{\theta} = 1$.

If $l_1 < l_2$, which means that

$$\sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n \log(1+x_i^2) > n(\log \pi - \log 2),$$

we have that the M.L.E of $\theta$ is $\hat{\theta} = 2$.

If $l_1 = l_2$, which means that

$$\sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n \log(1+x_i^2) = n(\log \pi - \log 2),$$

we have that the M.L.E. of $\theta$ are $\hat{\theta} = 1$, $\hat{\theta} = 2$.