PROBABILITY AND STATISTICS I
HOMEWORK IV

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Since X follows the Geometric distribution, by the definition we could get that

$$p_X(x) = P(X = x) = (1-p)^{x-1}\,p \qquad for\ x = 1, 2, \ldots$$

While a<1, we have that

$$F(a) = 0$$

While $a \geqslant 1$, then we could calculate the c.d.f. is

$$F(a) = P(x \leqslant a) = \sum_{x=1}^{\lfloor a \rfloor} (1-p)^{x-1}\,p = 1 - (1-p)^{\lfloor a \rfloor}$$

So the c.d.f. of X is

$$F(x) = \begin{cases} 0, & for\ x < 1, \\ 1 - (1-p)^{\lfloor x \rfloor}, & for\ x \geqslant 1. \end{cases}$$

2

We have that X_1 and X_2 have the joint p.d.f. is

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & for\ 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0, & otherwise. \end{cases}$$

2.1

To calculate the marginal p.d.f. of X_1 and X_2, we just need to calculate two integrals

$$f(x_1) = \int_{-\infty}^{+\infty} f(x_1, x_2)\,dx_2 = \int_0^1 (x_1 + x_2)\ dx_2 = x_1 + \frac{1}{2}$$

$$f(x_2) = \int_{-\infty}^{+\infty} f(x_1, x_2)\,dx_1 = \int_0^1 (x_1 + x_2)\ dx_1 = x_2 + \frac{1}{2}$$

So the marginal p.d.f. of X_1 is

$$f(x_1) = \begin{cases} x_1 + \frac{1}{2}, & for\ 0 < x_1 < 1, \\ 0, & otherwise. \end{cases}$$

We could also get the p.d.f. of X_2 is

$$f(x_2) = \begin{cases} x_2 + \frac{1}{2}, & for\ 0 < x_2 < 1, \\ 0, & otherwise. \end{cases}$$

2.2

To calculate the probability $P(X_1 + X_2 \leqslant 1)$, we just need to calculate that

$$\int_0^1 dx_2 \int_0^{1-x_2} (x_1 + x_2)\,dx_1 = \frac{1}{3}$$

So the probability

$$P(X_1 + X_2 \leqslant 1) = \frac{1}{3}$$

2.3

While $0 < a < 1,\ and\ 0 < b < 1$, we could calculate the c.d.f. of X_1 and X_2 is

$$F_{X_1,X_2}(a,b) = \int_{-\infty}^{a} dx_1 \int_{-\infty}^{b} f(x_1,x_2)\,dx_2 = \frac{1}{2}ab(a+b)$$

So the c.d.f. of X_1 and X_2 is

$$F_{X_1,X_2}(a,b) = \begin{cases} 1, & for\ a > 1\ and\ b > 1, \\ \frac{1}{2}a(1+a), & for\ 0 < a \leqslant 1\ and\ b > 1, \\ \frac{1}{2}b(1+b), & for\ a > 1\ and\ 0 < b \leqslant 1, \\ \frac{1}{2}ab(a+b), & for\ 0 < a \leqslant 1\ and\ 0 < b \leqslant 1, \\ 0, & otherwise. \end{cases}$$

The c.d.f. of X_1 is

$$F_{X_1}(a) = \int_{-\infty}^{a} f(x_1)\,dx_1 = \begin{cases} 0, & for\ a \leqslant 0, \\ \frac{1}{2}a(a+1), & for 0 < a \leqslant 1, \\ 1, & for\ a > 1. \end{cases}$$

The c.d.f. of X_2 is

$$F_{X_2}(b) = \int_{-\infty}^{b} f(x_2)\,dx_2 = \begin{cases} 0, & for\ b \leqslant 0, \\ \frac{1}{2}b(b+1), & for 0 < b \leqslant 1, \\ 1, & for\ b > 1. \end{cases}$$

While $a, b \in (0,1)$, we could easily get that

$$F_{X_1,X_2}(a,b) \neq F_{X_1}(a)F_{X_2}(b)$$

So X_1 and X_2 are not independent.

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3.1

Since a point (X,Y) is chosen at random from the circle S defined as follows

$$S = \left\{(x,y) : x^2 + y^2 \leqslant 1\right\}$$

we have that the p.d.f. of X and Y is

$$f(x,y) = \begin{cases} c, & for\ (x,y) \in S, \\ 0, & otherwise. \end{cases}$$

To calculate the joint p.d.f. of X and Y, we know that $\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} f(x,y)\,dy = 1$. According to the geometric meaning of integral, we could get that $c\pi = 1$, so $c = \frac{1}{\pi}$. So the joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & for\ (x,y) \in S, \\ 0, & otherwise. \end{cases}$$

To determine the marginal p.d.f. of X, we only need to calculate the integral

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & for\ x \in [-1,1], \\ 0, & otherwise. \end{cases}$$

To determine the marginal p.d.f. of X, we only need to calculate the integral

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2}, & for\ y \in [-1,1], \\ 0, & otherwise. \end{cases}$$

3.2

It is clear that

$$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$$

So we have $F_{X,Y}(x,y) \neq F_X(x)F_Y(y)$, which shows that X and Y are not independent.

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If F(x,y) = 1 for $x + 2y \geqslant 1$ and F(x,y) = 0 elsewhere is a c.d.f. of two random variables we could get the magrinal c.d.f. of X is

$$F_X(x) = \lim_{y\to+\infty} F_{X,Y}(x,y) = 1,\ for\ x + 2y \geqslant 1$$

and $F_X(x) = 0$ elsewhere.

So there exists $x \to -\infty$, satisfy that $F_X(x) = 1$, which is in contradiction with that if F(a) is a c.d.f., there is $\lim_{a\to-\infty} F(a) = 0$, which shows the assumption is wrong, so F(x,y) = 1 for $x + 2y \geqslant 1$ and F(x,y) = 0 elsewhere cannot be a (cumulative) distribution function of two random variables.

5

To prove $f(x_1, x_2)$ is a joint p.d.f. of two continuous-type random variables X_1 and X_2, we only need to prove that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_1\,dx_2 = 1$.

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $0 < r < \infty,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$, so we have

$$f(x_1,x_2) = h(r,\theta) = \frac{2g(r)}{\pi r}$$

Then the intergal could be written as

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_1\,dx_2 = \int_0^{\infty} dr \int_0^{\frac{\pi}{2}} \frac{2g(r)}{\pi r} r\,dr\,d\theta = \int_0^{\infty} g(r)\,dr = 1$$

Which shows that

$$f(x_1,x_2) = \begin{cases} \frac{2g(\sqrt{x_1^2+x_2^2})}{\pi\sqrt{x_1^2+x_2^2}}, & for\ 0 < x < \infty,\ 0 < x_2 < \infty \\ 0, & otherwise. \end{cases}$$

is a joint p.d.f. of two continuous-type random variables X_1 and X_2.

6

Since the problem says for a and b from 1 to 5, the joint probability $P(X = a, Y = b)$ is either 0 or $\frac{1}{14}$ and the incomplete table shows $p_X(2) = \frac{5}{14}$ and $p_Y(1) = \frac{5}{14}$, we could get that

$$P(X = 2, Y = j) = \frac{1}{14},\ for\ j = 1,2,3,4,5;\quad P(X = i, Y = 1) = \frac{1}{14},\ for\ i = 1,2,3,4,5.$$

Since $p_X(1) = \frac{1}{14}$ and $P(X = 1, Y = 1) = \frac{1}{14}$, and $p_Y(5) = \frac{1}{14}$ and $P(X = 2, Y = 5) = \frac{1}{14}$ as well, we could get

$$P(X = 1, Y = j) = 0,\ for\ j = 2, 3, 4, 5;\quad P(X = i, Y = 5) = 0,\ for\ i = 1, 3, 4, 5.$$

Since $p_X(3) = \frac{4}{14}$ and $P(X = 3, Y = 1) = \frac{1}{14}, P(X = 3, Y = 5) = 0$, and $p_Y(2) = \frac{4}{14}$ and $P(X = 1, Y = 2) = 0, P(X = 2, Y = 2) = \frac{1}{14}$ as well, we could get

$$P(X = 3, Y = j) = \frac{1}{14},\ for\ j = 2, 3, 4;\quad P(X = i, Y = 2) = \frac{1}{14},\ for\ i = 3, 4, 5.$$

Since $p_X(4) = \frac{2}{14}$ and $P(X = 4, Y = 1) = \frac{1}{14}, P(X = 4, Y = 2) = \frac{1}{14}, P(X = 4, Y = 5) = 0$, and $p_X(5) = \frac{2}{14}$ and $P(X = 5, Y = 1) = \frac{1}{14}, P(X = 5, Y = 2) = \frac{1}{14}, P(X = 5, Y = 5) = 0$ as well, we could get

$$P(X = 4, Y = j) = \frac{1}{14},\ for\ j = 3, 4;\quad P(X = 5, Y = j) = \frac{1}{14},\ for\ j = 3, 4.$$

So we could fill the table as following:

	a					
b	1	2	3	4	5	$p_Y(b)$
1	1/14	1/14	1/14	1/14	1/14	5/14
2	0	1/14	1/14	1/14	1/14	4/14
3	0	1/14	1/14	0	0	2/14
4	0	1/14	1/14	0	0	2/14
5	0	1/14	0	0	0	1/14
$p_X(a)$	1/14	5/14	4/14	2/14	2/14	1

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7.1

Since the joint probabilities P(X=a,Y=b) of the discrete random variables X and Y are given by the following table

	a		
b	-1	0	1
4	$\eta - \frac{1}{16}$	$\frac{1}{4} - \eta$	0
5	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$
6	$\eta + \frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4} - \eta$

So we have that $0 \leqslant \eta - \frac{1}{16} \leqslant 1$, $0 \leqslant \frac{1}{4} - \eta \leqslant 1$, $0 \leqslant \eta + \frac{1}{16} \leqslant 1$, $0 \leqslant \frac{1}{4} - \eta \leqslant 1$. So we could calculate that

$$\frac{1}{16} \leqslant \eta \leqslant \frac{1}{4}$$

7.2

According to the table, we could calculate the c.d.f. of X and Y is

$$F_{X,Y}(x,y)=\begin{cases}\eta-\frac{1}{16}, & for\ -1\leqslant x<0,\ 4\leqslant y<5,\\ \frac{3}{16}, & for\ x\geqslant 0,\ 4\leqslant y<5,\\ \eta+\frac{1}{16}, & for\ -1\leqslant x<0,\ 5\leqslant y<6,\\ \frac{1}{2}, & for\ 0\leqslant x<1,\ 5\leqslant y<6,\\ \frac{5}{8}, & for\ x\geqslant 1,\ \ 5\leqslant y<6,\\ 2\eta+\frac{1}{8}, & for\ -1\leqslant x<0,\ y\geqslant 6,\\ \eta+\frac{5}{8}, & for\ 0\leqslant x<1,\ y\geqslant 6,\\ 1, & for\ x\geqslant 1,\ y\geqslant 6,\\ 0, & otherwise.\end{cases}$$

We could calculate the c.d.f. of X is

$$F_X(x)=\begin{cases}2\eta+\frac{1}{8}, & for\ -1\leqslant x<0,\\ \eta+\frac{5}{8}, & for\ 0\leqslant x<1,\\ 1, & for\ x\geqslant 1,\\ 0, & otherwise.\end{cases}$$

The c.d.f. of Y is

$$F_Y(y)=\begin{cases}\frac{3}{16}, & for\ 4\leqslant y<5,\\ \frac{5}{8}, & for\ 5\leqslant y<6,\\ 1, & for\ y\geqslant 6,\\ 0, & otherwise.\end{cases}$$

If X and Y are independent, there must be $F_{X,Y}(x,y)=F_X(x)F_Y(y)$. While $x\geqslant 0,\ 4\leqslant y<5$, if $F_{X,Y}(x,y)=F_X(x)F_Y(y)$, there is $\eta=\frac{3}{8}$, but $\eta\leqslant\frac{1}{4}$, so there is not a value of η for X and Y are independent.

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8.1

To determine the magrinal (cumulative) distribution functions of X and Y, we just need to calculate two limits.

The marginal c.d.f. of X is

$$F_X(x)=\lim_{y\to\infty}F_{X,Y}(x,y)=\lim_{y\to\infty}\left(1-e^{-2x}-e^{-y}+e^{-(2x+y)}\right)=1-e^{-2x}\quad for\ x>0$$

and $F_X(x)=0\ otherwise.$

The marginal c.d.f. of Y is

$$F_Y(y)=\lim_{x\to\infty}F_{X,Y}(x,y)=\lim_{x\to\infty}\left(1-e^{-2x}-e^{-y}+e^{-(2x+y)}\right)=1-e^{-y}\quad for\ y>0$$

and $F_Y(y)=0\ otherwise.$

8.2

To calculate the joint p.d.f. of X and Y, we just need to calculate the derivation. The p.d.f. of X and Y is

$$f_{X,Y}(x,y)=\frac{d^2}{dx\,dy}F_{X,Y}(x,y)=2e^{-(2x+y)}\quad for\ x>0,\ y>0$$

and $f_{X,Y}(x,y) = 0,\ otherwise$

8.3

The marginal p.d.f. of X is

$$f_X(x) = \frac{dF_X(x)}{dx} = 2e^{-2x} \quad for\ x > 0$$

and $f_X(x) = 0,\ otherwise.$

The marginal p.d.f. of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = e^{-y} \quad for\ y > 0$$

and $f_Y(y) = 0,\ otherwise.$

8.4

It is clearly that $F_{X,Y}(x,y) = F_X(x)F_Y(y)$, so X and Y are independent.

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9.1

By the definition, we have the joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = g_X(x \mid y)f_Y(y) = \frac{(2y)^x}{x!}e^{-3y}\ for\ x = 0,1,2\ldots,\ y > 0$$

and $f_{X,Y}(x,y) = 0,\ otherwise.$

To calculate the p.m.f. of X, we only need to calculate the intergal

$$p_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \frac{2^x}{x!3^{x+1}}\int_0^{+\infty}(3y)^x e^{-3y}\,d(3y) = \frac{2^x}{3^{x+1}},\ for\ x = 0,1,2\ldots$$

and $p_X(x) = 0,\ elsewhere.$

9.2

By the definition, the conditional p.d.f. $g_Y(y \mid 0)$ of Y given X = 0 is

$$g_Y(y \mid 0) = \frac{f_{X,Y}(0,y)}{p_X(0)} = 3e^{-3y},\ if\ y > 0$$

$g_Y(y \mid 0) = 0,\ elsewhere$

9.3

By the definition, the conditional p.d.f. $g_Y(y \mid 1)$ of Y given X = 1 is

$$g_Y(y \mid 1) = \frac{f_{X,Y}(1,y)}{p_X(1)} = 9ye^{-3y},\ if\ y > 0$$

$g_Y(y \mid 1) = 0,\ elsewhere$

9.4

Let $g_Y(y \mid 1) > g_Y(y \mid 0)$, we get $y > \frac{1}{3}$, which says that for $y > \frac{1}{3}$, $g_Y(y \mid 1) > g_Y(y \mid 0)$. It agrees that the more calls i see, the higher i should think the rate is.

10

10.1

According to the question, we could assume that the joint p.d.f. of X and Y is

$$f_{X,Y}(a,b) = \begin{cases} c, & for\ a \geqslant 0,\ b \leqslant 1,\ b \geqslant a, \\ 0, & elsewhere. \end{cases}$$

To calculate the constant c, we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \frac{c}{2} = 1$$

so c=2.

Then the joint c.d.f. of X and Y is

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} dx \int_{-\infty}^{b} f_{X,Y}(x,y)\,dy = \begin{cases} 1, & for\ a \geqslant 1,\ b \geqslant 1, \\ 2a - a^2, & for\ 0 \leqslant a < 1,\ b \geqslant 1, \\ b^2, & for\ 0 \leqslant b \leqslant a < 1, \\ 2ab - a^2, & for\ 0 \leqslant a < b < 1, \\ 0, & elsewhere. \end{cases}$$

10.2

The joint p.d.f. of X and Y is

$$f_{X,Y}(a,b) = \begin{cases} 2, & for\ a \geqslant 0,\ b \leqslant 1,\ b \geqslant a, \\ 0, & elsewhere. \end{cases}$$

10.3

By the definition, when x is between 0 and 1, we have

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{x}^{1} 2\,dy = 2 - 2x,\ for\ 0 < x < 1$$

By the definition, when y is between 0 and 1, we have

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_{0}^{y} 2\,dx = 2y,\ for\ 0 < y < 1$$

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Since U = min{X,Y}, V = max{X,Y}, we could get the triangle region S:$\{(x,y) \mid 0 \leqslant x \leqslant a,\ x \leqslant y \leqslant a\}$. The area is $\frac{a^2}{2}$. Due to X,Y are uniformly distributed, U,V are uniformly distributed. So we have the p.d.f. of U and V is

$$f_{U,V}(u,v) = \begin{cases} \frac{2}{a^2}, & if\ 0 \leqslant u \leqslant a,\ u \leqslant v \leqslant a, \\ 0, & elsewhere. \end{cases}$$

So the c.d.f. of U and V is

$$F_{U,V}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{U,V}(x,y)\,dx\,dy = \frac{2}{a^2} \int_{0}^{u} dx \int_{x}^{v} dy = \frac{u(2v-u)}{a^2} = \frac{v^2 - (v-u)^2}{a^2}\ for\ 0 \leqslant u \leqslant v \leqslant a$$