PROBABILITY AND STATISTICS I
HOMEWORK III

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March, 13 2021

1

Let B_1 = {The original ball in the box is black}
Let B_2 = {The original ball in the box is white}
Information tells us that

$$P(B_1) = p, \quad P(B_2) = 1 - p$$

Let A = {The ball we picked is white}
We can easily get that

$$P(A \mid B_1) = 0.5, \quad P(A \mid B_2) = 1$$

Putting this into Baye's theorem implies that

$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{\sum_{j=1}^{2} P(B_j)P(A \mid B_j)} = \frac{p}{2-p}$$

So now the probability that the original ball in the box is black is $\frac{p}{2-p}$

2

Let B_1 = {The person have cancer}
Let B_2 = {The person doesn't have cancer}
Let A = {The test show positive}
Information from the question tells that

$$P(B_1) = 0.04, \quad P(B_2) = 0.96, \quad P(A \mid B_1) = 0.79, \quad P(A \mid B_2) = 1 - 0.95 = 0.05$$

Putting this into Baye's theorem implies that

$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{\sum_{j=1}^{2} P(B_j)P(A \mid B_j)} = \frac{79}{199}$$

3

The discrete random variable X follows the binomial distribution with parameters 10 and $\frac{1}{2}$, so that the probability mass function is

$$p_X(x) = P(X = x) = \binom{10}{x}(\frac{1}{2})^x(1 - \frac{1}{2})^{10-x} = \binom{10}{x}\frac{1}{1024} \qquad for\ x = 0, 1, ..., 10$$

4

The discrete random variable X follows the geometric distribution with parameter $\frac{1}{2}$, so that the probability mass function is

$$p_X(x) = P(X = x) = (1 - \frac{1}{2})^{x-1}\frac{1}{2} = (\frac{1}{2})^x$$

5

Since X have the p.m.f $p_X(x) = \left(\frac{1}{2}\right)^{|x|}$, for x = -1,-2,-3,...
If $Y = X^4,\ we\ have\ y = 1, 2^4, 3^4, \ldots,$ so the p.m.f of Y is

$$p_Y(y) = \left(\frac{1}{2}\right)^{\sqrt[4]{y}}$$

6

6.1

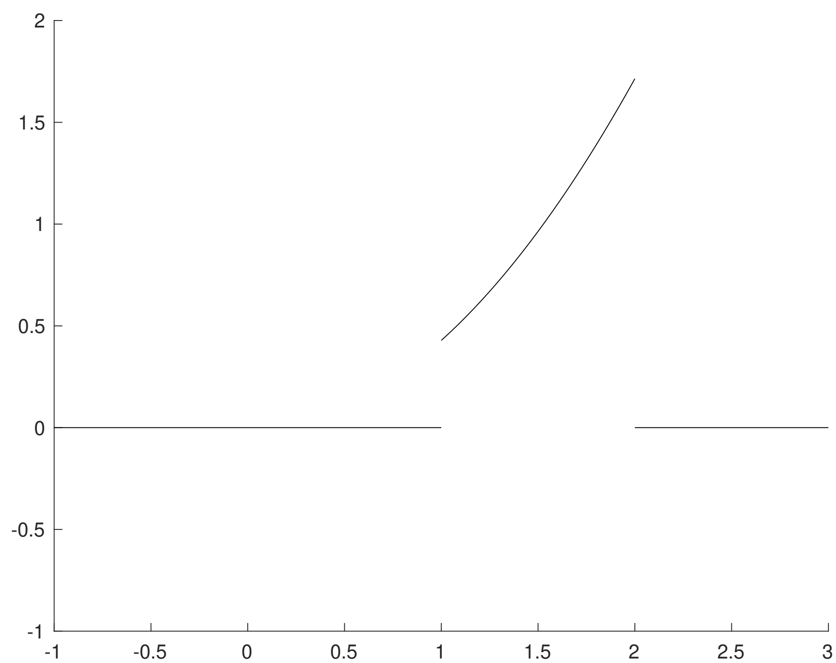
Since the probability density function of X is

$$f(x) = \begin{cases} cx^2, & for\ 1 \leqslant x \leqslant 2, \\ 0, & otherwise. \end{cases}$$

We have that

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{1}^{2} cx^2\,dx = \frac{7}{3}c = 1$$

So we get the value of c is $\frac{3}{7}$



6.2

We could calculate that

$$P(X > \frac{3}{2}) = \int_{\frac{3}{2}}^{+\infty} f(x)\,dx = \int_{\frac{3}{2}}^{2} \frac{3}{7}x^2\,dx = \frac{37}{56}$$

7

Assume there exists a number c satisfying the function given is a probability density function. If c = 0, f(x) = 0 for all x is a real number. So we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = 0$$

Which shows f(x) is not a probability density function, so c $\neq$ 0.
For c$\neq$ 0, we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^1 \frac{c}{x}\,dx = -c \lim_{x\to 0} \ln x$$

For all c$\neq$ 0, we know that

$$\lim_{x\to 0} |c \ln x| = \infty$$

So the assumption is false. So there doesn't exist any number c for the given function is a p.d.f.

8

Problem information tells us that the p.d.f of X is

$$f(x) = \begin{cases} \frac{1}{x^2}, & for\ x > 1 \\ 0, & otherwise. \end{cases}$$

If C_1 = {x : 1 < x < 2}, C_2 = {x : 4 < x < 5}, we know that $C_1 \cup C_2$ = {x : 1 < x < 2 or 4 < x < 5} and $C_1 \cap C_2 = \varnothing$, so there are

$$P(C_1 \cup C_2) = \int_1^2 \frac{1}{x^2}\,dx + \int_4^5 \frac{1}{x^2}\,dx = \frac{11}{20}, \qquad P(C_1 \cap C_2) = 0$$

9

Problem information tells us that the p.d.f of X is

$$f(x) = \begin{cases} \frac{x+2}{18} & for\ -2 < x < 4, \\ 0 & otherwise. \end{cases}$$

We can calculate that

$$P(|X| < 1) = P(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{x+2}{18}\,dx = \frac{2}{9}$$

$$P(X^2 < 9) = P(-3 < X < 3) = \int_{-3}^{3} f(x)\,dx = \int_{-2}^{3} \frac{x+2}{18}\,dx = \frac{25}{36}$$

10

If $f(x) = ce^{-x^2}$ is a p.d.f for x $\in$ R, we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} ce^{-x^2}\,dx = 1$$

So c = $\left(\int_{-\infty}^{+\infty} ce^{-x^2}\,dx\right)^{-1}$
In order to calculate above-mentioned integral, firstly we need to calculate

$$\int_{-R}^{+R} e^{-x^2}\,dx$$

Then we calculate

$$\left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2 = \int_{-R}^{R} e^{-x^2}\,dx \int_{-R}^{R} e^{-y^2}\,dy = \iint_{\substack{-R \leqslant x \leqslant R \\ -R \leqslant y \leqslant R}} e^{-x^2-y^2}\,dx\,dy \tag{1}$$

Since $e^{-x^2-y^2} > 0$ and containment relationship of integration area, we know that

$$\iint_{x^2+y^2\leqslant R^2} e^{-x^2-y^2}\,dx\,dy \leqslant \left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2 \leqslant \iint_{x^2+y^2\leqslant 2R^2} e^{-x^2-y^2}\,dx\,dy$$

With polar coordinate transformation, we get

$$\iint_{x^2+y^2\leqslant R^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^{R} e^{-r^2} r\,dr = \pi\left(1-e^{-R^2}\right)$$

$$\iint_{x^2+y^2\leqslant 2R^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}R} e^{-r^2} r\,dr = \pi\left(1-e^{-2R^2}\right)$$

So we get an inequality

$$\pi\left(1-e^{-R^2}\right) \leqslant \left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2 \leqslant \pi\left(1-e^{-2R^2}\right)$$

Let R $\to +\infty$, we get

$$\int_{-R}^{R} e^{-x^2}\,dx = \sqrt{\pi}$$

So if $f(x) = ce^{-x^2}$ is a p.d.f for x $\in$ R,

$$c = \frac{1}{\sqrt{\pi}}$$

11

Problem information tells us that the p.d.f of X is

$$f(x) = \begin{cases} 4x^3 & for\ 0 < x < 1, \\ 0 & otherwise. \end{cases}$$

So its c.d.f is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & x < 0, \\ x^4 & 0 \leqslant x < 1, \\ 1 & x \geqslant 1. \end{cases}$$

Assume the 20th percentile is x_0, since the c.d.f is continuous for $x \in R$, so we have $F(x_0) = 0.2$. We could calculate that

$$x_0 = \frac{1}{\sqrt[4]{5}}$$

12

Problem information tells us that the c.d.f of X is

$$F(x) = \begin{cases} 0 & \text{for x} < 0, \\ 1 - e^{-x} - xe^{-x} & \text{for } x \geqslant 0. \end{cases}$$

So we could calculate the p.d.f is

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} 0 & \text{for x} < 0, \\ xe^{-x} & \text{for } x \geqslant 0. \end{cases}$$

Assume the median is x_0, so we have $F(x_0) = 1 - e^{-x} - xe^{-x} = 0.5$, then we could calculate that $x_0 \approx 1.678$

13

Since the probability set function of X is

$$P(C) = \int_C e^{-x}\,dx$$

wehere support is C = (0,+∞). If $C_k = \{x : 2 - \frac{1}{k} < x \leqslant 3\}$, k = 1,2,3,...
We can easily get that $\lim_{k\to\infty} C_k = \{x : 2 < x \leqslant 3\}$, so we could calculate that

$$P(\lim_{x\to\infty} C_k) = \int_2^3 e^{-x}\,dx = \frac{e-1}{e^3}$$

Then

$$P(C_k) = \int_{2-\frac{1}{k}}^{3} e^{-x}\,dx = \sqrt[k]{e}e^{-2} - e^{-3}$$

We know that $\lim_{x\to\infty} \sqrt[x]{e} = 1$ So we have

$$\lim_{k\to\infty} P(C_k) = \lim_{k\to\infty} \sqrt[k]{e}e^{-2} - e^{-3} = \frac{e-1}{e^3}$$

Which shows that $\lim_{k\to\infty} P(C_k) = P(\lim_{x\to\infty} C_k)$

14

Since X follows the binomial distribution with parameters n and p = 1/2, we have

$$P(X = x) = \binom{n}{x}\left(\frac{1}{2}\right)^x\left(1 - \frac{1}{2}\right)^{n-x} = \binom{n}{x}\left(\frac{1}{2}\right)^n$$

When k = 0, we have

$$P(X \leqslant 0) = P(X = 0) = \frac{1}{2^n} \leqslant \frac{1}{2^n} \quad for\ all\ sufficiently\ large\ n$$

Assume when $k = t \geqslant 1$, there still is

$$P(X \leqslant t) \leqslant \frac{1}{2^n}\binom{n}{t}\frac{n-(t-1)}{n-(2t-1)} \quad for\ all\ sufficiently\ large\ n$$

Then when $k = t+1$, we could calculate that

$$P(X \leqslant t+1) = P(X \leqslant t) + P(X = t+1) \leqslant \frac{1}{2^n}\binom{n}{t}\frac{n-(t-1)}{n-(2t-1)} + \frac{1}{2^n}\binom{n}{t+1}$$

$$= \frac{1}{2^n}\binom{n}{t+1}\left(1 + \frac{(t+1)(n-t+1)}{(n-t)(n-2t+1)}\right)$$

To prove while $k = t+1$, P(k) still satisfy the inequality, we just need to prove that

$$1 + \frac{(t+1)(n-t+1)}{(n-t)(n-2t+1)} \leqslant \frac{n-t}{n-2t-1} = 1 + \frac{t+1}{n-2t-1}$$

So we need to prove

$$\frac{(t+1)(n-t+1)}{(n-t)(n-2t+1)} \leqslant \frac{t+1}{n-2t-1}$$

Since problem says $n \geqslant 2k$, we have that $n \geqslant 2(t+1)$, so that $n-t > 0$,$n-2t+1 > 0$,$n-2t-1 > 0$,$t+1 > 0$. Then we just need to prove that

$$(n-t)(n-2t+1) - (n-t+1)(n-2t-1) = \left(n^2 - 3nt + n + 2t^2 - t\right) - \left(n^2 - 3nt + 2t^2 - t - 1\right) = n+1 > 0$$

It is obvious that $n+1 > 0$, so we have proven that

$$1 + \frac{(t+1)(n-t+1)}{(n-t)(n-2t+1)} \leqslant \frac{n-t}{n-2t-1}$$

Which shows that

$$P(X \leqslant t+1) \leqslant 2^{-n} \binom{n}{t+1} \frac{n-((t+1)-1)}{n-(2(t+1)-1)}$$

In summary, we have proven that for a fixed k and all $n \leqslant 2k$, there exists

$$P(X \leqslant k) \leqslant 2^{-n} \binom{n}{k} \frac{n-(k-1)}{n-(2k-1)}$$

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