PROBABILITY AND STATISTICS I
HOMEWORK II

Jianyu Dong 2019511017

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1

Let A_0 = { A shopper chooses brand A on his first and second purchase and brand B on his third and fourth purchase }

Let A_1 = { The shopper choose brand A on his first purchase }

Let A_2 = { The shopper choose brand A on his second purchase }

Let A_3 = { The shopper choose brand B on his third purchase }

Let A_4 = { The shopper choose brand B on his fourth purchase }

Then we have :

$$A_0 = A_1 \cap A_2 \cap A_3 \cap A_4$$

$$P(A_1) = \frac{1}{2}, P(A_2|A_1) = \frac{1}{3}, P(A_3|A_1 \cap A_2) = \frac{2}{3}, P(A_4|A_1 \cap A_2 \cap A_3) = \frac{1}{3}$$

$$P(A_0) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times P(A_4|A_1 \cap A_2 \cap A_3) = \frac{1}{27}$$

So the probability of both his first and second purchases will be brand A and both his third and fourth purchases will be brand B is $\frac{1}{27}$

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If after swapping n times the black ball is still in box A, there must be even times we pick the black ball and swap it with a white ball.

Let C = {After swapping n times, the black ball is still in box A}

(1).If n is odd, there could be 0,2,4,6...(n-1) times we pick the black ball and swap it with a white ball. Then the probability of C is :

$$P(C) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} (\frac{1}{3})^{2k} \times (\frac{2}{3})^{n-2k}$$

(2).If n in even, there could be 0,2,4,6...n times we pick the black ball and swap it with a white ball. Then the probability of C is :

$$P(C) = \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} (\frac{1}{3})^{2k} \times (\frac{2}{3})^{n-2k}$$

We can calculate that:

$$P(C) = \frac{1+3^n}{2 \times 3^n}$$

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4

Let A_0 = { The first and third cards are spades, the second and fourth cards are red} Let A_1 = { The first card is spade.}

Let A_2 = { The second card is red}

Let A_3 = { The third card is spade}

Let A_4 = { The fourth card is red}

We have that:

$$P(A_1) = \frac{13}{52},\ P(A_2 \mid A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{26}{51},$$

$$P(A_3 \mid A_1 \cap A_2) = \frac{P(A_3 \cap A_1 \cap A_2)}{P(A_1 \cap A_2)} = \frac{12}{50},\ P(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{P(A_4 \cap A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2 \cap A_3)} = \frac{25}{49}$$

Then we could get that:

$$P(A_0) = P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2) \times P(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{13}{833}$$

Let $B_0 = \{$ The first and third cards are red, the second and fourth cards are spades$\}$ Let $B_1 = \{$ The first card is red.$\}$

Let $B_2 = \{$ The second card is spade$\}$

Let $B_3 = \{$ The third card is red$\}$

Let $B_4 = \{$ The fourth card is spade$\}$

We have that:

$$P(B_1) = \frac{26}{52},\ P(B_2 \mid B_1) = \frac{P(B_1 \cap B_2)}{P(B_1)} = \frac{13}{51},$$

$$P(B_3 \mid B_1 \cap B_2) = \frac{P(B_3 \cap B_1 \cap B_2)}{P(B_1 \cap B_2)} = \frac{25}{50},\ P(B_4 \mid B_1 \cap B_2 \cap B_3) = \frac{P(B_4 \cap B_1 \cap B_2 \cap B_3)}{P(B_1 \cap B_2 \cap B_3)} = \frac{12}{49}$$

Then we could get that:

$$P(B_0) = P(B_1 \cap B_2 \cap B_3 \cap B_4) = P(B_1) \times P(B_2 \mid B_1) \times P(B_3 \mid B_1 \cap B_2) \times P(B_4 \mid B_1 \cap B_2 \cap B_3) = \frac{13}{833}$$

Since A_0 and B_0 are disjoint, we can get the probability of the spades and red cards alternate is

$$P(A_0) + P(B_0) = \frac{26}{833}$$

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Since C_1 and C_2 are independent events, according to the definition of independent events we can get:

5.1

$$P(C_1 \cap C_2) = P(C_1) \times P(C_2) = 0.6 \times 0.3 = 0.18$$

5.2

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = 0.6 + 0.3 - 0.18 = 0.72$$

5.3

Since C_1 and C_2 are independent events and $P(C_2) + P(C_2^c) = 1$, we have C_1 and C_2^c are independent events and $P(C_2^c) = 0.7$.

Then we have:

$$P(C_1 \cap C_2^c) = P(C_1) \times P(C_2^c) = 0.6 \times 0.7 = 0.42$$

$$P(C_1 \cup C_2^c) = P(C_1) + P(C_2^c) - P(C_1 \cap C_2^c) = 0.6 + 0.7 - 0.42 = 0.88$$

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Let A_1 = {Player A win in his first throw}
Let A_2 = {Player A win in his second throw}
Let A_3 = {Player A win in his third throw}
Let B_1 = {Player B win in his first throw}
Let B_2 = {Player B win in his second throw}
Let B_3 = {Player B win in his third throw}

Since player A and B play a sequence of independent games, according to the question stem, we can easily get that:

$$P(A\ win\ the\ game) = P(A_1)+P(A_1^c \cap B_1^c \cap A_2)+P(A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap A_3) = \frac{1}{6}+\frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}+\frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}\times\frac{2}{6}\times\frac{5}{6} = \frac{169}{324}$$

Since either A or B will win the game, we have:

$$P(B\ win\ the\ game) = 1 - P(A\ win\ the\ game) = 1 - \frac{169}{324} = \frac{155}{324}$$

In a word, the probability of player A win the game is $\frac{169}{324}$, the probability of player B win the game is $\frac{155}{324}$

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Let A = { At least one six in four independent casts of a six-sided die }, then A^c = { There is no six in four independent casts of a six-sided die }.

We can easily get that:

$$P(A^c) = (\frac{5}{6})^4 = \frac{625}{1296}$$

$$P(A) = 1 - P(A^c) = \frac{671}{1296} \approx 0.5177$$

Let B = { At least a pair of sixs in 24 independent casts of a pair of dice}, then B^c = { There is no pair of sixs in 24 independent casts of a pair of dice}

We could caculate that:

$$P(B^c) = (\frac{35}{36})^{24}$$

$$P(B) = 1 - P(B^c) = 1 - (\frac{35}{36})^{24} \approx 0.4914$$

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Let Ω = { The person draw two cards from an ordinary deck of cards without replacement }, and Ω is a simple sample space. We can get that:

$$|\Omega| = \binom{52}{2} = 1326$$

Let A = { The person get two cards in same suit}. We have:

$$A \subset \Omega\ and\ \ |A| = \binom{4}{1} \times \binom{13}{2} = 312$$

Then, we have that:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{17}$$

If let the bet is fair, there must be that:

$$b \times P(A) = 1$$

So b = 4.25

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According to the definition of conditional probability, we have that:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

We know that P($A \cap B$) = P(A) + P(B) - P($A \cup B$) = P(A) + 1 - P($A \cup B$) - P(B^c) $\geq$ P(A) - P(B^c), which uses 1 - P($A \cup B$) ≥ 0

We get that:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \geq \frac{P(A) - P(B^c)}{P(A)} \geq 1 - \frac{P(B^c)}{P(A)}$$

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Since A and B are mutually exclusive, we get that $A \cap B^c =$ A.

According to the definition of conditional probability, we have:

$$P(A \mid B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A)}{1 - P(B)}$$

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If $A \subset B$ and P(B)>0, we can get that $A \cap B = A$.

According to the multiplication rule, we have:

$$P(A) = P(A \cap B) = P(A \mid B) \times P(B) \leq P(A \mid B)$$

Using P(B) $\leq$ 1, we prove that P(A) $\leq$ P(A | B)

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According to the definition of conditional probability, we have:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

We know that P($A \cap B$) = P(A) + P(B) - P($A \cup B$) $\leq$ P(A) = p. Which says that:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \leq \frac{p}{1 - \varepsilon}$$

We know that P($A \cap B$) = P(A) + P(B) - P($A \cup B$) $\geq$ P(A) + P(B) -1 = p -ε. Which says that:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq \frac{p - \varepsilon}{1 - \varepsilon}$$

In summary, there is:

$$\frac{p - \varepsilon}{1 - \varepsilon} \ \leq \ P(A \mid B) \ \leq \ \frac{p}{1 - \varepsilon}$$

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For any event B, if P(A) = 0, we can easily get that P($A \cap B$) = 0, which says that P($A \cap B$) = P(A)$\times$ P(B) = 0.Then A is independent with any event B has been proven.

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According to the definition of conditional probability, we have:

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

If $\{B_j\}_{j=1}^k$ is a partition of Ω, we have that:

$$P(A \cap C) = \sum_{j=1}^{k} P(A \cap C \cap B_j)$$

Then

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \sum_{j=1}^{k} \frac{P(B_j \cap C)}{P(C)} \times \frac{P(A \cap C \cap B_j)}{P(B_j \cap C)} = \sum_{j=1}^{k} P(B_j \mid C) \times P(A \mid B_j \cap C)$$

Which proves that if $\{B_j\}_{j=1}^k$ is a partition of Ω, for any A and C:

$$P(A \mid C) = \sum_{j=1}^{k} P(B_j \mid C) \times P(A \mid B_j \cap C)$$

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Since 0 < P(A) < 1, 0 < P(B) < 1, according to the definition of conditional probability, we have:

$$P(A \mid B) + P(A^c \mid B^c) = \frac{P(A \cap B)}{P(B)} + \frac{P(A^c \cap B^c)}{P(B^c)} = 1 \qquad (*)$$

We know that

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$

$$P(B^c) = 1 - P(B)$$

Then $(*)$ change into:

$$P(A \cap B) = P(A) \times P(B)$$

Which says that A and B are independent.

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If P(A) > 0 and P(B) > 0, A and B are independent, we have P($A \cap B$) = P(A) $\times$ P(B) > 0. While P($\varnothing$) = 0, so $A \cap B \neq \varnothing$.