# PROBABILITY AND STATISTICS I
# HOMEWORK X

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# 1

By the definition, we have the sample mean and sample variance are independent with i. Thus, for any constants c and d, we could get that

$$\begin{aligned}\sum_{i=1}^{n}(x_i-c)(y_i-d) &= \sum_{i=1}^{n}(x_i-\overline{x}+\overline{x}-c)(y_i-\overline{y}+\overline{y}-d)\\ &= \sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})+(\overline{x}-c)\sum_{i=1}^{n}(y_i-\overline{y})+(\overline{y}-d)\sum_{i=1}^{n}(x_i-\overline{x})+n(\overline{x}-c)(\overline{y}-d)\end{aligned}$$

We could easily get that

$$\sum_{i=1}^{n}(x_i-\overline{x})=\sum_{i=1}^{n}x_i-n\overline{x}=0,\ \ \sum_{i=1}^{n}(y_i-\overline{y})=\sum_{i=1}^{n}y_i-n\overline{y}=0$$

So that we could get that

$$\sum_{i=1}^{n}(x_i-c)(y_i-d)=\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})+n(\overline{x}-c)(\overline{y}-d)$$

# 2

Since $X_i$ form a random sample with mean value $\mu$ and variance $\sigma^2$, we could get that

$$\mathbf{E}(X_i)=\mu,\ \ \mathbf{Var}(X_i)=\mathbf{E}(X_i^2)-\mathbf{E}(X_i)^2=\sigma^2.$$

Since $\overline{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $T=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$, we could get that

$$\mathbf{E}(\overline{X}_n)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{E}(X_i)=\mu,\ \ \mathbf{Var}(\overline{X}_n)=\mathbf{E}(\overline{X}_n^2)-\mathbf{E}(\overline{X}_n)^2=\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{Var}(X_i)=\frac{\sigma^2}{n}.$$

So we could determine that

$$\mathbf{E}(T)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{E}(X_i^2-2X_i\overline{X}_n+\overline{X}_n^2)=\frac{1}{n}\sum_{i=1}^{n}(\mathbf{E}(X_i^2)-2\mathbf{E}(X_i\overline{X}_n)+\mathbf{E}(\overline{X}_n^2)).$$

We have get that

$$\sum_{i=1}^{n}\mathbf{E}(X_i^2)=n(\sigma^2+\mu^2),\ \ \sum_{i=1}^{n}\mathbf{E}(\overline{X}_n^2)=\sigma^2+n\mu^2.$$

By the definition, we could calculate that

$$\mathbf{E}(X_i\overline{X}_n)=\mathbf{E}(\frac{1}{n}\sum_{j=1}^{n}X_iX_j)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}(X_iX_j)=\frac{1}{n}((n-1)\mu^2+\mu^2+\sigma^2)=\mu^2+\frac{\sigma^2}{n}$$

Thus, we could get that

$$\sum_{i=1}^{n}\mathbf{E}(X_i\overline{X}_n)=n\mu^2+\sigma^2$$

So we get that

$$\mathbf{E}(T)=\frac{1}{n}(n(\sigma^2+\mu^2)+\sigma^2+n\mu^2-2(n\mu^2+\sigma^2))=\frac{n-1}{n}\sigma^2.$$

If we use this as an estimator for the variance, this is biased.
An unbiased estimator is

$$T' = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

We could get that

$$\mathbf{E}(T') = \mathbf{E}(\frac{n}{n-1}T) = \sigma^2.$$

Which means that T' is an unbiased estimator.

# 3

Since $X_i$ are random samples from U(-1,1), we could get the p.d.f. of $X_i$ is

$$f_{X_i}(x_i) = \frac{1}{2},\ for\ -1 < x_i < 1$$

zero elsewhere. Thus, we could get the expectation and variance of $X_i$ are

$$\mathbf{E}(X_i) = \int_{-1}^{1}\frac{1}{2}x_i\,dx_i = 0,\ \mathbf{Var}(X_i) = \int_{-1}^{1}\frac{1}{2}(x_i-0)^2 = \frac{1}{3}.$$

So we could get that

$$\mathbf{E}(\overline{X}) = 0,\ \mathbf{Var}(\overline{X}) = \frac{1}{n}\mathbf{Var}(X_i) = \frac{1}{3n}.$$

# 4

By the definition, we have the sample variance is

$$S_2^2 = \frac{1}{2-1}\sum_{i=1}^{2}(X_i - \overline{X})^2.$$

We have that $\overline{X} = \frac{X_1+X_2}{2}$, so that

$$S_2^2 = (\frac{X_1-X_2}{2})^2 + (\frac{X_2-X_1}{2})^2 = \frac{1}{2}(X_1-X_2)^2.$$

By the definition, we have that

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

We have that

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

So that

$$S_n^2 = \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\left(\sum_{j\neq i}(X_i - X_j)\right)^2$$

We could calculate that

$$\left(\sum_{j\neq i}(X_i - X_j)\right)^2 = \sum_{j\neq i}(X_i-X_j)^2 + \sum_{j\neq i,k\neq i,k\neq j}(X_i-X_j)(X_i-X_k)$$

Then we add all i=1...n, and we have $(X_i - X_j)^2 = (X_j - X_i)^2$, so that we get

$$\sum_{i=1}^{n} \left( \sum_{j \neq i} (X_i - X_j) \right)^2 = (2 + n - 2) \sum_{i<j} (X_i - X_j)^2 = n \sum_{i<j} (X_i - X_j)^2.$$

Which uses that

$$(X_i - X_j)(X_i - X_k) + (X_j - X_i)(X_j - X_k) = X(i - X_j)(X_i - X_k + X_k - X_j) = (X_i - X_j)^2.$$

So we have prove that

$$S_n^2 = \frac{1}{n(n-1)} \sum_{i<j} (X_i - X_j)^2.$$

# 5

Since $X_i$ form a random sample from an exponential distribution with parameter $\lambda$, we could get that

$$\mu_1 = \mathbf{E}(X) = \frac{1}{\lambda}.$$

So that we get

$$\lambda = \frac{1}{\mu_1}.$$

Next, we set

$$\hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$$

to obtain estimator

$$\hat{\lambda} = \frac{1}{\hat{\mu}_1} = \frac{n}{\sum_{i=1}^{n} X_i}.$$

Since $X_i$ are independent, we could get the joint p.d.f. of $X_i$ is

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}, \ for \ x_i > 0,$$

zero elsewhere. Let $T = n / \sum_{i=1}^{n} X_i$, we could get that

$$\mathbf{E}(T) = \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{n\lambda^n}{x_1 + \cdots + x_n} e^{-\lambda(x_1 + \cdots + x_n)} \, dx_1 \ldots \, dx_n.$$

Let

$$y_1 = \frac{x_1 + \cdots + x_n}{n}, y_i = x_i, \ for \ i = 2, 3, \ldots, n.$$

So we could get that

$$x_1 = ny_1 - \sum_{i=2}^{n} y_i, x_j = y_j, \ for \ j = 2, 3, \ldots, n.$$

So the Jacobe is $J = n$, then the intergal could be written as

$$\begin{aligned} \mathbf{E}(T) &= \int_0^{+\infty} \int_0^{ny_1} \int_0^{ny_1 - y_2} \cdots \int_0^{ny_1 - \sum_{i=2}^{n-1} y_i} \frac{n\lambda^n}{y_1} e^{-\lambda n y_1} \, dy_1 \, dy_2 \, dy_3 \ldots \, dy_n \\ &= \frac{(n\lambda)^n}{(n-1)!} \int_0^{+\infty} y_1^{n-2} e^{-\lambda n y_1} \, dy_1 \end{aligned}$$

Using integration by parts, we could get that

$$\mathbf{E}(T) = \frac{(n\lambda)^n}{(n-1)!} \frac{(n-2)!}{(n\lambda)^{n-1}} = \frac{n}{n-1} \lambda \neq \lambda$$

So this estimator is biased.

# 6

Since $(X_i)_{i=1}^n$ are random samples from the uniform distribution U(a,b), let X follows the uniform distribution on (a,b), we could get the p.d.f. of X is

$$f_X(x) = \frac{1}{b-a},\ for\ a < x < b,$$

zero elsewhere. So the first and second moments are

$$\mu_1 = \mathbf{E}(X) = \frac{a+b}{2},\ \mu_2 = \mathbf{E}(X^2) = \frac{a^2+b^2+ab}{3}.$$

Then we could determine that

$$a = \mu_1 \mp \sqrt{3(\mu_2 - \mu_1^2)},\ b = \mu_1 \pm \sqrt{3(\mu_2 - \mu_1^2)}.$$

Since there must be a<b, we could get

$$a = \mu_1 - \sqrt{3(\mu_2 - \mu_1^2)},\ b = \mu_1 + \sqrt{3(\mu_2 - \mu_1^2)}.$$

Based on the data $(x_i)_{i=1}^n$, we could calculate the sample moments

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^n x_i,\ \hat{\mu_2} = \frac{1}{n}\sum_{i=1}^n x_i^2.$$

So the estimators of a and b are

$$\hat{a} = \hat{\mu_1} - \sqrt{3(\hat{\mu_2} - \hat{\mu_1}^2)},\ \hat{b} = \hat{\mu_1} + \sqrt{3(\hat{\mu_2} - \hat{\mu_1}^2)};$$

We could determine that

$$\hat{a} = \frac{1}{n}\sum_{i=1}^n X_i - \sqrt{\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2}$$

Then we get that

$$\mathbf{E}(\hat{a}) = \frac{a+b}{2} - \mathbf{E}(\sqrt{\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2})$$

Assume that $\mathbf{E}(\hat{a}) = a$, so we could get that

$$\mathbf{E}(\sqrt{\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2}) = \frac{b-a}{2}$$

Then we could get that

$$\mathbf{Var}(\sqrt{\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2}) = \mathbf{E}(\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2) - (\frac{b-a}{2})^2 = -\frac{1}{4n}(b-a)^2 < 0.$$

Since there must be that $\mathbf{Var}(\sqrt{\frac{3}{n}\sum_{i=1}^n (X_i - \overline{X})^2}) \geqslant 0$, the assumption is wrong. So $\mathbf{E}(\hat{a}) \neq a$. Similarly, we could get that $\mathbf{E}(\hat{b}) \neq b$ Which means that the estimators are biased.

## 7

Since X follows uniform distribution on $(0,\theta)$, we could get the p.d.f. of X is

$$f_X(x) = \frac{1}{\theta},\ for\ 0 < x < \theta,$$

zero elsewhere. So the first moment is

$$\mu_1 = \mathbf{E}(X) = \frac{\theta}{2}.$$

Then we could determine that

$$\theta = 2\mu_1.$$

Based on the data $(x_i)_{i=1}^9$, we could get that

$$\hat{\mu_1} = \frac{\sum_{i=1}^9 x_i}{9} = \frac{119}{90}.$$

So that we could determine that

$$\hat{\theta} = 2\hat{\mu_1} = \frac{119}{45}.$$

## 8

Since $X_i$(i=1,...,n) are random samples from a distribution with p.d.f.

$$p(x,\theta) = \frac{2}{\theta^2}(\theta - x),\ for\ 0 < x < \theta, \theta > 0.$$

We could get the first moment is

$$\mu_1 = \mathbf{E}(X) = \int_0^\theta \frac{2}{\theta^2} x(\theta - x)\, dx = \frac{\theta}{3}.$$

Then we could determine that

$$\theta = 3\mu_1.$$

Based on the data $(x_i)_{i=1}^n$, we could get that

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^n x_i.$$

Then the estimator of $\theta$ is

$$\hat{\theta} = 3\hat{\mu_1} = \frac{3}{n}\sum_{i=1}^n x_i.$$

## 9

Since $(X_i)_{i=1}^n$ follows the binomial distribution with parameters n and p, we could get the mean and variance of X are

$$\mathbf{E}(X) = np,\ \mathbf{Var}(X) = np(1-p).$$

Then we could get the first and second moments of X are

$$\mu_1 = \mathbf{E}(X) = np,\ \mu_2 = \mathbf{E}(X^2) = np(1-p) + (np)^2.$$

So we could determine that

$$p = 1 - \frac{\mu_2 - \mu_1^2}{\mu_1},\ n = \frac{\mu_1^2}{\mu_1^2 + \mu_1 - \mu_2}.$$

Next, baesd on the data, we set

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i,\ \hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

to obtain estimators

$$\hat{p} = 1 - \frac{\hat{\mu}_2 - \hat{\mu}_1^2}{\hat{\mu}_1},\ \hat{n} = \frac{\hat{\mu}_1^2}{\hat{\mu}_1^2 + \hat{\mu}_1 - \hat{\mu}_2}.$$

a