PROBABILITY AND STATISTICS I
HOMEWORK VI

Jianyu Dong 2019511017

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1

By the theorem, we have the mean of random variable X is

$$E(X) = M^{(1)}(0) = \left.\frac{dM(t)}{dt}\right|_{t=0} = \frac{1}{2}.$$

We also could determine that

$$E(X^2) = M^{(2)}(0) = \left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = 1$$

By the theorem, we could get the variance of X is

$$Var(x) = E(X^2) - E(X)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

2

By the definition, we could get the m.g.f. of Cauchy distribution is

$$M_X(t) = \int_{-\infty}^{+\infty} \frac{e^{xt}}{\pi(1+x^2)}\, dx.$$

If $t > 0$, we could easily get that when $x \to +\infty$, $\frac{e^{xt}}{\pi(1+x^2)} \to +\infty$. If $t < 0$, we could easily get that when $x \to -\infty$, $\frac{e^{xt}}{\pi(1+x^2)} \to +\infty$. Which means $M_X(t)$ does not exist for $t \neq 0$. Thus, there does not exist the moment generating function for Cauchy distribution.

3

Since X_1 and X_2 are independent random variables and each follows the Binomial distribution with parameters n_i and p, X_1 could be seen as the sum of n_1 independent and identical random variables following the Bernoulli distribution with parameter p. Let Y follows the Bernoulli distribution with parameter p, we could get the m.g.f. of Y is

$$M_Y(t) = E(e^{ty}) = pe^t + 1 - p$$

We have $X_1 = n_1 Y,\ X_2 = n_2 Y$, by the theorem, we could get the moment generating functions of X_1 and X_2 are

$$M_{X_1}(t) = \left(pe^t + 1 - p\right)^{n_1},\ M_{X_2}(t) = \left(pe^t + 1 - p\right)^{n_2}$$

Let $Z = X_1 + X_2$, by the theorem, we could determine the m.g.f. of Z is

$$M_Z(t) = M_{X_1}(t) M_{X_2}(t) = \left(pe^t + 1 - p\right)^{n_1+n_2}.$$

So $X_1 + X_2$ follows the Binomial distribution with parameters $n_1 + n_2$ and p.

4

Let $f(x) = x^2$, we have that $f^{(2)}(x) = 2 > 0$, which means that $f(x) = x^2$ is a convex function. Since the fourth moment of random variable X is finite, we could get that the second moment and the second central moment of X is finite. Using Jensen's Inequality we could get that

$$E\left(\left((X-\mu)^2\right)^2\right) = E\left(f\left((X-\mu)^2\right)\right) \geqslant f\left(E\left((X-\mu)^2\right)\right) = \left(E\left((X-\mu)^2\right)\right)^2 = \sigma^4$$

So we get that

$$E\left((X-\mu)^4\right) \geqslant \sigma^4$$

5

Since r is a one-to-one and continuous function defined on $I \subseteq R$ and a random variable X take values on I, we could get that the function r is strictly monotonous function. Without loss of generality,, we could assume r(x) is a monotone increasing function. By the definition, if m is the median of X, we could get

$$P(X < m) \leqslant \frac{1}{2},\ P(X \geqslant m) \geqslant \frac{1}{2}.$$

Then according to monotone increasing, we have that r(x)<r(m) if and only if x<m, and r(x)>r(m) if and only if x>m. So we get

$$P(r(X) < r(m)) = P(X < m) \leqslant \frac{1}{2},\ P(r(X) \geqslant r(m)) = P(X > m) \geqslant \frac{1}{2}.$$

In a similar way, if r(x) is a monotone decreasing function, we could also get that r(m) is the median of r(x). So r(m) is the median of r(X).

6

Since X follows Binomial distribution with parameters n and p, and Y follows Binomial distribution with parameters n and 1-p, by the definition, we could get the skewness of X and Y is

$$\frac{E((X-\mu)^3)}{\sigma^3} = \frac{n\left(p(1-p)^3 - (1-p)p^3\right)}{(np(1-p))^{\frac{3}{2}}},\ \frac{E((Y-\mu)^3)}{\sigma^3} = \frac{n\left((1-p)p^3 - p(1-p)^3\right)}{(n(1-p)p)^{\frac{3}{2}}}$$

Thus, it is obvious that the skewness of Y is the negative of the skewness of X.

7

7.1

By the theorem, we could get that to minimize $E((X-d)^2)$ d is the expectation of X. Thus

$$d = E(X) = \int_0^1 2x^2\,dx = \frac{2}{3}.$$

So $d = \frac{2}{3}$ minimizes $E((X-d)^2)$.

7.2

By the theorem, we could get that to minimize $E(|X-d|)$ d is the median of X. Thus we have

$$P(X < d) \leqslant \frac{1}{2},\ P(X \leqslant d) \geqslant \frac{1}{2}$$

So we get $d = \frac{\sqrt{2}}{2}$ minimizes $E(|X-d|)$.

8

To prove that m is the unique median of the distribution of X, we need to prove m is a median firstly. According to the question, we have $P(X < m) < \frac{1}{2}\ and\ P(X > m) < \frac{1}{2}$, and we also know that $P(X > m) + P(X \leqslant m) = 1$, so that

$$P(X < m) < \frac{1}{2}\ and\ P(X \leqslant m) \geqslant \frac{1}{2}.$$

Thus, we get m is a median of the distribution of X.

If X is a continuous random variable and m is a median of the distribution of X, we could get that

$$P(X < m) = \frac{1}{2},\ P(X > m) = \frac{1}{2}.$$

Which does not conform to the meaning of the question, so X is a discrete random variable and 0<P(X=m)<1.

Then we need to prove the median is unique. Assume there exists another median $m' \neq m$ of the distribution of X and without loss of generality, suppose m'>m. By the definition, we have $P(X < m') \leqslant \frac{1}{2}\ and\ P(X \leqslant m') \geqslant \frac{1}{2}$. While we have that

$$P(X < m') \geqslant P(X < m) + P(X = m) = 1 - P(X > m) > \frac{1}{2}$$

which means there is contradiction between the assumption and the question, so the assumption is wrong, so there is not another median of the distribution of X. So m is the unique median of the distribution of X.

9

By the definition, we have the m.g.f. of X is

$$M_X(t) = \int_0^{+\infty} e^{tx} x e^{-x} = \int_0^{+\infty} x e^{-(1-t)x}.$$

In order to guarantee the integral exists, 1-t must be positive which means t<1. Thus we could determine the integral

$$M_X(t) = \int_0^{+\infty} x e^{-(1-t)x} = \frac{1}{(1-t)^2}.$$

So the moment generating function of X is

$$M_X(t) = \frac{1}{(1-t)^2},\ for\ t < 1.$$

10

By the theorem, if we could find a probability function of X_1 which satisfy the m.g.f. is

$$M_{X_1}(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$$

this probability function must be the probability function of X. We could easily get that if

$$p_{X_1}(x) = \begin{cases} \frac{1}{6}, & for\ x = -2, \\ \frac{1}{3}, & for\ x = -1, \\ \frac{1}{4}, & for\ x = 1, \\ \frac{1}{4}, & for\ x = 2, \end{cases}$$

the m.g.f. of X_1 is

$$M_{X_1}(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$$

the same as X. Thus, the probability function of X must be the same as X_1. So we could easily determine that

$$P(|X| \leqslant 1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

11

By the theorem, if we could find a probability function of Y which satisfy the m.g.f. is

$$M_Y(t) = \frac{e^t}{3 - 2e^t}$$

this probability function must be the probability function of X. If Y follows Geometric distribution with parameter p, we could determine the m.g.f. of Y is

$$M_Y(t) = \sum_{y=1}^{+\infty} (1-p)^{y-1} p e^{ty} = \frac{pe^t}{1-(1-p)e^t}.$$

If p=$\frac{1}{3}$, we get that the moment generating function of Y is the same as X. So X follows Geometric distribution with parameter p, and the c.d.f. of X is

$$F_X(x) = \sum_{k=1}^{\lfloor x \rfloor} \left(1 - \frac{1}{3}\right)^{k-1} \frac{1}{3} = 1 - \left(\frac{2}{3}\right)^{\lfloor x \rfloor}.$$